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Drift resonance in high density non-neutral plasmas

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Theoretical studies of the operation of crossed-field electron vacuum devices such as magnetrons and crossed-field amplifiers (CFA) have usually centered on their initial growth, taking this as an indication of their operating modes. In such an analysis one solves the equations for the density profile, the operating frequency, the growth rate, and other features of these devices. What one really obtains then are only the conditions for the device to turn on. The dominant interaction in this stage is a Rayleigh-type instability which initiates a quasilinear diffusion process whereby the electron density profile redistributes itself into a profile which will be in equilibrium with the ponderomotive-like forces produced by the growing rf fields. Eventually the rf fields will saturate and an operating device will settle into a stationary operating regime. This stage of a device's operation is called the "saturation stage." This latter stage involves a different set of physical interactions from the initiation stage. No longer is there a growth rate; rather the rf amplitudes have saturated and as a result, the ponderomotive-like forces have also vanished along with the quasilinear diffusion. In this saturation stage, we find that new rf modes appear. In fact, there are a total of five rf modes, two of which are the usual slow modes of the initiation stage, and three of which have fast oscillations in the vertical direction. One fast mode corresponds to a drift plasma oscillation while the other two fast modes are drift cyclotron modes. In this paper, we will describe how the drift plasma oscillation interacts and couples with the slow rf modes at the diocotron resonance. © 2006 American Institute of Physics. [DOI: 10.1063/1.2199227]

I. INTRODUCTION

Crossed-field devices such as magnetrons and crossed-field amplifiers (CFA) are important generators of high power microwaves. They are also relatively simple in their structure and operation. They consist of a vacuum region inside of which a high density electron plasma is created. The theory of their general operating range follows straightforwardly from electrostatics and magnetostatics.¹ However developing a theory for the rf oscillations has not been so direct. The genesis of the rf theory is the Buneman, Levy, Linson theory of 1966.² This theory is based on a linear instability which at best, for the operating mode, lies deep inside the electron sheath and does not correspond to the parameter regime used by the design engineers. The predicted instability is indeed there,^{3,4} however it seems to have nothing to do with the initiation of power generation in a CFA or magnetron. The dominant effect of the instability appears to be to simply diffuse the edge of the sheath with no growth in the generation of power. We now know that this instability is related to the "magnetron resonances," which occur when the local rf frequency, as seen by the electrons, $\omega_e(=\omega-kv_0)$, is equal in magnitude to the cyclotron frequency, Ω . (Here ω is the rf frequency, k is the rf wave vector, and v_0 is the $E \times B$ drift velocity.)

It is also now well known that a linear theory is insufficient to describe magnetrons and CFAs, and it is necessary to include ponderomotive effects in order to understand an op-

erating device. Davidson was the first to point out the importance of the diocotron resonance, which occurs whenever $\omega_e=0$, in the operation of this device. He showed that this resonance leads to the reshaping of any positive density gradients into negative density gradients.⁵ Also he had early on recognized that the quasilinear theory would be an important component for any understanding of magnetron operation. Following this, other works have also pursued the study of nonlinear interactions in magnetron operation, with the use of multiple scales.⁶⁻⁹

Let us note that typically such devices are initiated by first applying the ambient magnetic field (and for a CFA, also applying the rf input signal), and then secondly ramping up the dc voltage. Under these circumstances one finds that within a couple of cyclotron oscillations a near classical Brillouin sheath will form next to the cathode.^{4,10-13} Meanwhile, as the dc voltage rises, this near-Brillouin sheath slowly expands its thickness, typically taking hundreds of cyclotron oscillations for the dc voltage to ramp up to its full value. As the sheath expands, so does the value of the $E \times B$ drift velocity, v_0 , at the edge of the sheath. If the operating parameters are taken to be inside the device's operating range then at some time shortly before the voltage reaches its full value, the electron drift velocity will equal the phase velocity of some unstable rf mode in the slow-wave structure. At this point the electrons at the edge of the sheath will be strongly accelerated by the unstable rf mode, initiating it as a linear instability on a shear flow¹⁴ described by a Rayleigh-type equation for a shear flow.¹⁵ Once this linear instability initializes, the electrons at the edge of the sheath will undergo

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the Davidson “quasilinear” diffusion.^{5,6} The growth rate of this nonlinear instability is found to be directly proportional to the density gradient at the edge of the sheath,^{5,14} which for the near-Brillouin flow is very large. The second-order theory then predicts that the dc electron density profile will change and evolve. As it evolves, the density profile relaxes to a new density profile, one which is in equilibrium with the propagating and growing rf wave in the slow-wave structure and one which also extends all the way up to the anode.¹⁴ This instability of a near-Brillouin sheath is not surprising since earlier kinetic studies in magnetically insulated diodes^{16,17} had revealed that stochastic electron motion occurs as the Brillouin limit is approached. Thus electrons in a Brillouin flow can be easily perturbed and kicked out of the flow and into new states.

Knowing the ordering in the physical processes involved and due to the time scales involved, one can use an adiabatic approximation to construct an analytical qualitative description of the final reshaped density profiles in the presence of a slow-wave structure by means of the WKB approximation.¹⁸ To go beyond the WKB approximation one has to turn to numerical solutions^{5,6} which can be done in a self-consistent manner, producing reshaped density profiles which were in equilibrium with a rf wave propagating in the slow-wave structure.^{19,20} These solutions were also shown to agree quite well with the operating range of the device #T266, a CFA manufactured by CPI (Communications and Power Industries, Inc., formerly Varian Beverly Microwave Division) operating in the GHz range. A review and summary of this entire theory has been given.⁹

The above analysis was for the nonrelativistic cold-fluid equations in the electrostatic approximation. The key approximation in it was to assume only three modes; the background dc fields, only one frequency for the oscillating rf fields, and second-order dc ponderomotive terms generated by the oscillating rf fields. Many of the results of this analysis have since been corroborated by, or have corroborated, results from other approaches such as particle-in-cell (PIC) code results or results from a guiding center (GC) approach. We note that in comparing these other approaches with the cold-fluid approach, one must keep in mind that the resolution afforded by PIC codes is limited due to the relatively large grid size required as well as the relative low number of particles that can be handled. The approximations used in the guiding center approach are most closely satisfied for low density electron plasmas where the plasma frequency is much smaller than the electron cyclotron frequency, Ω .²¹ With regard to our cold-fluid approach, it needs to be noted that these solutions are obtained in the adiabatic approximation, are therefore equilibrium solutions and are valid in the limit of slowly growing rf fields. Also we note that Lau and Chernin²² has pointed out that eigenfunction expansions such as those which have been used in our cold-fluid approach have definite advantages over other methods.

Summarizing some of the results of other approaches, the second-order vertical drift^{6,9} has also been found in the GC approximation by Riyopoulos.²³ This is a very important feature about which we will have more to say later. This drift has also been observed in PIC code calculations⁴ as well.

Riyopoulos has also used the GC approximation to study the initiation of the instability of a classical Brillouin sheath^{7,24} which was then confirmed also in the cold-fluid approach.¹⁴ He has also showed that in order to fit experimental data, it is necessary for the electron density at the diocotron resonance to be taken on the order of one-half of the maximum Brillouin value^{8,25} with the density in the spokes being on the same order of magnitude. This is indeed what happens in the self-consistent cold-fluid theory^{18–20} as one can also see from figures given below. Along with this, it is to be noted that many PIC code results^{4,26} now show hub and spoke features similar to that obtained by our self-consistent cold-fluid treatment.^{9,18–20} We also point out that the ultralow noise feature of the CPI #T266 (Ref. 20) has also been found in other crossed-field devices.²⁷ We note that all these mechanisms used to create ultralow noise devices seem to have the consequence that the normally (assumed) laminar electron flow just above the cathode has been disturbed somehow into a more turbulent flow. Lastly, we note that except for the experimental knowledge of the operating regimes of magnetrons and CFAs, there has been no other corroborating evidence concerning an explanation as to why the operating voltage regime of these devices is experimentally limited to less than 20% or so above the Hartree value²⁸ and certainly never ever really approaching the higher Hull cutoff voltage.

However, the above theory applies only to what we shall call the “initiation stage” which is where the device is initializing and where the internal rf wave is still growing. The “saturation stage” follows the initiation stage and is the stage where the rf wave will have reached its maximum amplitude, have saturated and the device will be steadily delivering rf power. These two stages differ significantly in two distinctly different features. First, by the end of the initiation stage a second-order dc flow of electrons from the cathode to the anode will have come into existence. The second feature is that when one arrives at the saturation stage, the growth rate of the rf fields will have vanished.²⁹ [Note that consistent with this, in the initiation stage one does see the growth rate smoothly decreasing toward zero as the rf amplitude rises, as shown in Figs. 4(b) and 6(b) of Ref. 20.] These two features have significant physical and mathematical consequences. Physically, the quasilinear diffusion ceases in the saturation stage and the additional dc current between the cathode and the anode can no longer be neglected. Mathematically, one finds three new rf modes appearing and three narrow resonance layers located at the positions of the magnetron and diocotron resonances.³⁰ The latter are sufficiently narrow that a new scaling must be introduced in order to obtain solutions for the rf fields. Furthermore they are so narrow that one would not expect to see them in PIC code results. This is not to say that their effects are not contained in PIC code results, but only that the statistics of PIC codes do not have the resolution to see such narrow resonances.

In this paper, we will study the structure of the rf solutions about the diocotron resonance in the saturation stage which contains the major mechanism for the transfer of dc power into rf power. In an operating device, optimally this resonance will occur somewhere along the edge of the electron sheath where the density gradient is large and negative.

FIG. 1. The geometry and configuration of a planar magnetron or CFA. The magnetic field is directed into the paper and the electron drift velocity is to the right. The z axis is perpendicular to the paper, the x axis is parallel to the cathode surface, and the y axis is perpendicular to the cathode.

Using two multiple scales, we will obtain the inner and the outer solutions around this resonance to first order in an expansion parameter (taken to be the square root of the vertical dc velocity). From these solutions we will obtain the coefficients that asymptotically match the inner and outer solutions, thereby obtaining the mode conversion coefficients.

In Sec. II we will discuss the geometry used and give the nonrelativistic rf equations for the saturation stage in planar geometry and the electrostatic approximation.²⁹ In Sec. III we will quickly treat and dispense with the cyclotron modes since they will be noninteracting at the diocotron resonance. In Sec. IV we will obtain the “outer” solutions which are valid outside of, but near the diocotron resonance. In Sec. V we shall proceed to use a multiple scale expansion to solve for the “inner” solutions inside the diocotron resonance. An expansion will be found for the inner solutions and the solutions will be given in terms of Fresnel-type integrals. In Sec. VI we shall find the asymptotic form of the connection coefficients that connect the inner and outer solutions. We shall then use these coefficients to obtain the conversion coefficients which will give how any one mode, as it passes through the diocotron resonance, converts into the other modes in passing to the other side. In Sec. VII we will make some concluding remarks.

II. BASIC EQUATIONS AND WKB SOLUTIONS

The geometry and configuration that we shall be using is shown in Fig. 1 which is a diagram of a “planar magnetron.” At the bottom is the cathode which is an electron emitting material. It is located in the xz plane and at $y=0$. We take the ambient magnetic field to be directed along the negative z axis so that the electron drift velocity will be mainly in the positive x direction. (In the saturation stage this dc drift current will have a small vertical component in the y direction, as shown.) The electron cyclotron frequency is $\Omega = eB_0/(mc)$, where B_0 is the ambient magnetic field. Next to the cathode is the electron sheath which extends out for some distance from the cathode as shown. The anode is located at a distance ℓ above the cathode. On the anode is a slow-wave structure which is simply a wave guide along which the rf wave will propagate. This slow-wave structure is there to (i)

FIG. 2. (Color) The electron density for one period of a fully developed spoke structure in the initiation stage. The anode is in the foreground and the cathode toward the back. Taken from Fig. 5 of Ref. 9.

slow down the electromagnetic rf wave and (ii) to select the effective rf (electrostatic) wave vector, $\mathbf{k}$, the plasma will see and respond to.

During the initiation stage the diocotron interaction causes the sheath to readjust its profile from a pure Brillouin flow to one in equilibrium with the growing rf wave. The wavelength of the rf wave will determine the periodicity of the spoke-like structure which evolves from the sheath due to the nonlinear diffusion and will extend from the surface of the sheath out to the anode. A typical plot of this structure is seen in Fig. 2 which is a plot of the electron density vs x and y . At the back is the Brillouin-type sheath while in the front one can see the spoke of charge which carries the y component of the dc current to the anode. Note that the structure shown contains only two Fourier components in the x direction: a dc background and a single value of k in e^{ikx} representing the rf component.⁹

The basic cold-fluid equations coupled with Poisson’s equation that we will use are:

$$\partial_t n + \nabla \cdot (\mathbf{v}n) = 0, \quad (1)$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla \phi + \mathbf{v} \times \boldsymbol{\Omega} = 0, \quad (2)$$

$$\nabla^2 \phi = n. \quad (3)$$

In the above equations the fluid velocity is in the xy plane and all quantities are taken to be independent of the z coordinate. We have also simplified the notation by taking n to be the electron plasma frequency squared [$4\pi e^2 \rho/m$ where $e(m)$ is the electronic charge (mass) and ρ is the electron number density] and ϕ to be (e/m) times the electrostatic potential.

For the saturation stage one expands these equations significantly different from that for the initiation stage. Here we take

$$n(x, y, t) = n_0 + (n_1 e^{i(kx - \omega t)} + \text{c.c.}), \quad (4)$$

and similarly for the other quantities. In the above the subscript “0” will refer to a dc quantity while the subscript “1” will refer to a rf quantity. Also, k is the wave vector for the rf wave and ω is the rf frequency, both of which are real. Note

that we do not include any “second-order dc terms,” by which this expansion differs from that used in the initiation stage.⁶ The reason for this change in the form of the expansion for the saturation stage is to incorporate the dc vertical velocity into lowest order. By the end of the initiation stage, second-order corrections would typically no longer be small, and therefore in the saturation stage should be directly combined with the lower order dc terms into a general dc term. As a consequence the dc equations of the saturation stage will contain quadratic rf terms in addition to the usual dc terms of the initiation stage. Thus unlike the initiation stage, the dc drift velocity in the saturation stage will have both an x and y component, as:

$$\mathbf{v}_0 = v_{0x}(y)\hat{x} + v_{0y}(y)\hat{y}. \quad (5)$$

To obtain the equations for the saturation stage we insert expansions for the density, velocity and potential, as in (4), into (1)–(3) and expand. Note that in the saturations stage we will not use a small amplitude expansion; instead we only separate on the basis of Fourier components. First we have the dc components, which are:

$$\frac{d}{dy}(n_0 v_{0y} + n_1 v_{1y}^* + n_1^* v_{1y}) = 0, \quad (6)$$

$$v_{0y} \left(\frac{dv_{0x}}{dy} - \Omega \right) + \left[v_{1y}^* \frac{dv_{1x}}{dy} + \text{c. c.} \right] = 0, \quad (7)$$

$$v_{0y} \frac{dv_{0y}}{dy} + \Omega v_{0x} - \frac{d\phi_0}{dy} + \left[v_{1y}^* \left(\frac{dv_{1y}}{dy} - ikv_{1x} \right) + \text{c. c.} \right] = 0, \quad (8)$$

$$\frac{d^2 \phi_0}{dy^2} - n_0 = 0. \quad (9)$$

Note the quadratic ponderomotive-like rf terms present. Lastly, from the components of the rf fundamental, $e^{i(kx - \omega t)}$, of the same equations [which are also the linear perturbations of (1)–(3)] we obtain:

$$\frac{d}{dy}(n_1 v_{0y} + n_0 v_{1y}) + ikn_0 v_{1x} - i\omega_e n_1 = 0, \quad (10)$$

$$v_{1y} \frac{dv_{0x}}{dy} + v_{0y} \frac{dv_{1x}}{dy} - \Omega v_{1y} - i\omega_e v_{1x} - ik\phi_1 = 0, \quad (11)$$

$$\frac{d}{dy}(v_{0y} v_{1y}) - i\omega_e v_{1y} + \Omega v_{1x} - \frac{d\phi_1}{dy} = 0, \quad (12)$$

$$\frac{d^2 \phi_1}{dy^2} - k^2 \phi_1 - n_1 = 0, \quad (13)$$

where

$$\omega_e = \omega - kv_{0x}. \quad (14)$$

When v_{0y} is zero, Eqs. (10)–(13) become the same rf equations as those used for the initiation stage^{2,6} upon taking ω to be complex. In the initiation stage the vanishing of v_{0y} causes the derivatives of n_1 , v_{1x} , and v_{1y} to vanish from these

FIG. 3. The theoretical background density for the T266 from the initiation equations. The units of n_0 are $(\text{rad/ns})^2$ and of y are $m/100$. The cathode is to the left and the anode is located at $y=0.381$.

equations. Thus the above set of five ODEs of the saturation stage can be reduced to the second-order set of ODEs and three algebraic relations of the initiation stage.

When v_{0y} is nonzero as in the saturation stage, one then has all five derivatives of the rf quantities present and furthermore, Eqs. (10)–(13) can be solved for each derivative of the rf quantities. Doing so, one obtains

$$\frac{dn_1}{dy} = \frac{1}{v_{0y}^2} \left[(n_0 v_{1y} - n_1 v_{0y}) \left(\frac{dv_{0y}}{dy} - i\omega_e \right) - \frac{dn_0}{dy} v_{1y} v_{0y} - n_0 \frac{d\phi_1}{dy} + n_0 v_{1x} (\Omega - ikv_{0y}) \right], \quad (15)$$

$$\frac{dv_{1x}}{dy} = \frac{1}{v_{0y}} \left[v_{1y} \left(\Omega - \frac{dv_{0x}}{dy} \right) + i\omega_e v_{1x} + ik\phi_1 \right], \quad (16)$$

$$\frac{dv_{1y}}{dy} = \frac{1}{v_{0y}} \left[v_{1y} \left(i\omega_e - \frac{dv_{0y}}{dy} \right) - \Omega v_{1x} + \frac{d\phi_1}{dy} \right], \quad (17)$$

$$\frac{d^2 \phi_1}{dy^2} = k^2 \phi_1 + n_1. \quad (18)$$

This is a fifth-order set of nonsingular, linear ODEs for the rf quantities when v_{0y} is nonzero and all the dc quantities exist. By nonsingular we mean that the derivatives of the rf quantities always exist and are finite. (The equations are linear in the rf variables and in the saturation stage there are no denominator terms which could vanish and give rise to singularities.)

The above rf equations of the saturation stage differ from the rf equations of the initiation stage in two aspects. First, v_{0y} is nonzero in the saturation stage and is zero in the initiation stage and second, the growth-rate is zero (which means that ω is real) in the saturation stage and is nonzero (which means that ω is complex) in the initiation stage.

As an example of the solution in the *initiation* stage, is shown in Figs. 3–5 a typical numerical solution for the CPI device #T266. One notes in Fig. 3 that the background density consists of a strong Brillouin-type hub with an extended foot on which the rf density variations would oscillate. Fig-

FIG. 4. The theoretical parallel drift velocity for the T266 from the initiation equations. The units for v_{0x} are $m/(100 \text{ ns})$.

Figure 4 shows the parallel dc drift velocity while Fig. 5 shows the vertical dc velocity. One notes that the vertical dc velocity is rather small, typical about one-thousandth of the parallel velocity. (The spike in the vertical dc velocity is due to the drift-cyclotron resonance at $\omega_e = \Omega$ located just above the cathode.) Except where the drift-cyclotron resonance occurs, this vertical dc current is essentially constant. Here the important point is that v_{0y} is significantly smaller than the other dc variables. [The reader should note that at the present time there exist no self-consistent solutions of the saturation stage (6)–(13). That will be the object of future work. The above plots are for the initiation stage. However one would expect those of the saturation stage to be similar to these.]

Due to the smallness of v_{0y} , it becomes difficult and time-consuming to numerically integrate the rf equations of the saturation stage. One notes that three of the derivatives in Eqs. (15)–(18) are inversely proportional to v_{0y} or v_{0y}^2 . Thus it follows that in general large values of the derivatives can occur and do occur, as can be seen in Figs. 6 and 7 which shows a typical solution that occurs when one integrates (15)–(18) using the *initiation* stage dc values with sample initial data. As one can see, the density oscillations dominate while the velocity oscillations are an order of magnitude

FIG. 5. The theoretical vertical drift velocity for the T266 from the initiation equations. The units for v_{0y} are $m/(100 \text{ ns})$.

FIG. 6. The magnitude of the rf N_1 as a function of y from a direct numerical solution of Eqs. (15)–(18) for the background given in Figs. 3–5.

smaller. (The oscillations in the potential are even another order smaller.)

In Ref. 30 the WKB solutions of the above equations, (15)–(18), were presented. Assuming fast phase variations of the form

$$n_1 \approx A_n e^{i f^y \text{ ady}}, \text{ etc.}, \quad (19)$$

where A_n is a slowly varying amplitude and α is a fast WKB wave vector, it was found that α could take on five possible values of which two were zero and corresponded to the usual (slow) modes of the initiation stage. The other three modes were fast modes and their eigenvalues, α , and unnormalized eigenvectors, $[n_1, v_{1x}, v_{1y}, \phi_1]$, in lowest order were found to be

$$\alpha_1 = (\omega_e + \lambda)/v_{0y}; [-\alpha_1^2 n_0/k^2, -i\alpha_1(\Omega - v'_{0x})/k^2, \alpha_1 \lambda/k^2, n_0/k^2], \quad (20)$$

$$\alpha_2 = (\omega_e - \lambda)/v_{0y}; [-\alpha_2^2 n_0/k^2, -i\alpha_2(\Omega - v'_{0x})/k^2, -\alpha_2 \lambda/k^2, n_0/k^2], \quad (21)$$

FIG. 7. The magnitude of the rf v_{1x} as a function of y from a direct numerical solution of Eqs. (15)–(18) for the background given in Figs. 3–5. The insert shows that the solution is actually comprised of more than one fast mode, three in fact.

FIG. 8. (Color) The five rf modes for the background given in Figs. 3–5. Note that the three fast modes never cross and tend to have a fixed separation except near the resonance between the lower cyclotron mode and the slow modes which is near the cathode. The main resonance which is the diocotron resonance, occurs at the edge of the sheath ($y \approx 0.25$) and corresponds to the crossing of the slow modes ($\alpha=0$) and the drift mode ($\alpha = \omega_e/v_{0y}$).

$$\alpha_3 = \omega_e/v_{0y}; [-\alpha_3^2 \Omega(\Omega - v'_{0x})/k^2, i\alpha_3(\Omega - v'_{0x})/k^2, -i\Omega/k, \Omega(\Omega - v'_{0x})/k^2], \quad (22)$$

where

$$\lambda = \sqrt{\Omega^2 + n_0 - \Omega v'_{0x}}, \quad (23)$$

and v'_{0x} is the derivative of v_{0x} with respect to y . [In the dc initiation equations v'_{0x} is equal exactly to n_0/Ω in which case λ would be exactly equal to Ω . However from dc saturation Eq. (7), we see that in general this need not necessarily be true in the saturation stage.] A plot of these α s vs y for the initiation stage background is given in Figs. 3–5 is given in Fig. 8. Note that the typical values for α in a fast mode are in the hundreds, which is large, whereas the typical value for the dc quantities is unity. Thus as seen from the eigenvectors in (20)–(22) the density oscillations dominate in the fast modes and the potential oscillations are the weakest.

From the above eigenvalues, eigenvectors and the overall phase, which is

$$\varphi_j = kx - \omega t + \int^y \alpha_j dy, \quad (24)$$

one can determine the nature of these three fast modes. Following a fluid particle one finds that in the first two modes the fluid particle executes circular oscillations at the modified cyclotron frequency, λ , about the guiding center. These eigenmodes are thus the fluid analogy of the well-known “Slater orbits”¹ which are the single-particle cycloidal orbits for electrons in a Brillouin sheath. In the Eulerian frame one sees rapid oscillations in y , as already shown in Figs. 6 and 7. We shall refer to these modes as the “drift-cyclotron modes.”

For the third eigenmode we again have that the dominant component of the eigenvector is the density oscillation with the motion of the fluid particle being mainly in the parallel

direction and with the two velocity components in phase. The motion is thus a drifting back-and-forth motion and will therefore be referred to as the “drift mode.”

For any two modes when the α s are well separated, the modes are noninteracting. However when they cross they interact and energy can transfer from one mode to another. For the two slow modes, they are of course continually interacting and satisfy the well-known initiation equations⁶ except near the cyclotron resonances and the diocotron resonance. At the diocotron resonance, the slow modes cross the drift mode, interact and exchange energy with it.

For an operational device the diocotron mode must cross the slow modes near the Brillouin edge. This is the interaction which allows the device to generate rf power. Thus an understanding of the interactions at the crossing of the drift mode and the slow modes is of prime importance. Also of importance is when the second cyclotron mode crosses the zero modes. This occurs in the example shown in Figs. 3–5 and appears to be responsible for the ultralow noise operation of this device.²⁰ No known examples of when the first cyclotron mode crosses the slow modes are known nor is it known if this interaction would be of interest.

To treat this problem it is necessary to use multiscale methods to separate the fast variations from the slow ones. In the next few sections we shall use a multiscale expansion to obtain the rf solutions for all five modes in the region of the diocotron resonance, which is near $y \approx 0.25$ in Fig. 8.

III. THE DRIFT-CYCLOTRON MODES NEAR THE DIOCOTRON RESONANCE

Let us first handle the two drift-cyclotron modes near the diocotron resonance. We introduce the following notation for our rf eigenvector:

$$V = [n_1, v_{1x}, v_{1y}, \phi'_1, \phi_1]. \quad (25)$$

This eigenvector is to satisfy the fifth-order set of ODEs given by (15)–(18). From the WKB analysis given above we have that the solution for the drift-cyclotron modes near the diocotron resonance will be of the form given in (20) and (21). This is:

$$V_1 = \mathcal{N} \left[n_0 \frac{\omega_e + \lambda}{\lambda}, i(\Omega - v'_{0x}) \frac{v_{0y}}{\lambda}, -v_{0y}, -i \frac{n_0 v_{0y}}{\lambda}, -\frac{n_0 v_{0y}^2}{\lambda(\omega_e + \lambda)} \right] e^{i \int^y \alpha_1 dy}, \quad (26)$$

where the subscript 1 indicates that this is the eigenvector for the first drift-cyclotron mode. This expression also contains an overall amplitude factor which is found from the first-order correction to the WKB approximation. This amplitude factor, suitably normalized to be unitless, for this mode is

$$\mathcal{N} = \sqrt{\frac{\Omega^4}{k^3 v_{0y}^3 |\lambda|}}. \quad (27)$$

The eigenvector for the second drift-cyclotron mode will be the same as for the first drift-cyclotron mode except that λ is to be replaced by $-\lambda$ and α_1 replaced by α_2 .

As is obvious from the above, for $\omega_e \rightarrow 0$, no components are singular and α_1 and α_2 are well separated from the other three modes. Thus the WKB solution for the drift-cyclotron modes will simply continue across the diocotron resonance without any significant mixing or conversion. In the next section we treat the outer solutions for the remaining three modes and following that we will obtain the inner solutions.

IV. THE OUTER SOLUTIONS NEAR THE DIOCOTRON RESONANCE

The outer solutions are the solutions for the eigenvectors when one is approaching the diocotron resonance from outside. From the WKB solution for the drift mode, (22), we have the scalings for the various components in the eigenvector. Thus this mode will be of the form

$$V_{3o} = \left[p_1 \Omega^2, p_2 v_{0y}, p_3 \frac{kv_{0y}^2}{\Omega}, p_4 v_{0y} \Omega, p_5 v_{0y}^2 \right] e^{i \int y \alpha_3 dy}, \quad (28)$$

where the p_j s are unitless functions, of order unity, to be determined. Also the subscript 3o on the eigenvector simply denotes that this is the outer solution for the third eigenvector. Inserting (28) into (15)–(18) and taking the lowest order in v_{0y} for each equation, one readily obtains the solution

$$[p_1, p_2, p_3, p_4, p_5] = \frac{\Omega}{kv_{0y}} \left[1, -i \frac{\Omega}{\omega_e}, i \frac{\Omega^2}{\omega_e^2}, -i \frac{\Omega}{\omega_e}, -\frac{\Omega^2}{\omega_e^2} \right] + O(1), \quad (29)$$

where the prefactor $\frac{\Omega}{kv_{0y}}$ is the amplitude factor that comes from the WKB first-order correction for this mode. Then from (28) and (29) we construct the outer solution, which in lowest order is

$$V_{3o} = \mathcal{A}_{3o} \frac{\Omega}{kv_{0y}} \left[\Omega^2, -i \frac{\Omega v_{0y}}{\omega_e}, i \frac{k \Omega v_{0y}^2}{\omega_e^2}, -i \frac{\Omega^2 v_{0y}}{\omega_e}, -\frac{\Omega^2 v_{0y}^2}{\omega_e^2} \right] e^{i \int y \alpha_3 dy}, \quad (30)$$

where $\mathcal{A}_{3o}$ is a constant which we will specify later.

Next we address the outer solutions for the slow modes. In this case we can simply set $v_{0y}=0$ in (10)–(13) and proceed to solve. The resulting equations are the same as the rf equations for the initiation stage except that (i) there is no growth rate and (ii) we can no longer equate $\partial_y v_{0x}$ to n_0/Ω [see (8)]. Taking this into account we then have:

$$\partial_y v_{1x} = i v_{1y} \left\{ -k - \frac{n'_0}{\Omega \omega_e} + \frac{2k}{A} \left[\frac{(\lambda^2 - n_0)^2}{\Omega^2} + n_0 - \omega_e^2 \right] + \frac{2\lambda' \lambda}{\Omega \omega_e A} \right\}, \quad (31)$$

$$\partial_y v_{1y} = -\frac{A'}{A} v_{1y} - i k v_{1x}, \quad (32)$$

where A is

$$A = \lambda^2 - \omega_e^2. \quad (33)$$

Once solutions are obtained for v_{1x} and v_{1y} , then the other components can be obtained by:

$$n_1 = i v_{1y} \left(\frac{2k n_0}{A} - \frac{n'_0}{\omega_e} + \frac{2\lambda' \lambda n_0}{\omega_e A} \right), \quad (34)$$

$$\phi'_1 = v_{1x} \Omega - i v_{1y} \omega_e, \quad (35)$$

$$\phi_1 = -v_{1x} \frac{\omega_e}{k} + i v_{1y} \frac{v'_{0x} - \Omega}{k}. \quad (36)$$

Since the quantities v_{1x} and v_{1y} continue to interact on the slow scale, it is not possible to separate these two slow modes into separate eigenvectors. Rather we should simply represent them as a single eigenvector with two degrees of freedom. Thus with v_{1x} and v_{1y} satisfying (31) and (32), we take this eigenvector to be

$$[n_1, v_{1x}, v_{1y}, \phi'_1, \phi_1]_{45o} = v_{1x} \left[0, 1, 0, \Omega, -\frac{\omega_e}{k} \right] + v_{1y} \left[i \left(\frac{2k n_0}{A} - \frac{n'_0}{\omega_e} + \frac{2\lambda' \lambda n_0}{\omega_e A} \right), 0, 1, -i \omega_e, i \frac{v'_{0x} - \Omega}{k} \right], \quad (37)$$

where the subscript 45o indicates that this is the eigenvector for the 4,5 slow modes.

The above solutions are accurate to the next order in v_{0y} and are valid solutions providing one is not too near the diocotron resonance. Within a distance of $\mathcal{O}(\sqrt{\frac{v_{0y}}{k\Omega}})$ around the diocotron resonance, these two slow modes will interact with the drift mode. To match these two outer solutions to the inner solutions, we shall require the limit of these solutions as one approaches the diocotron resonance.

To obtain this limit we will scale the variables as follows. We take our scale parameter to be the small unitless quantity, ϵ , defined by

$$\epsilon = \sqrt{v_{0y} k / \Omega}, \quad (38)$$

where we will take the value of v_{0y} (see Fig. 5) to be that at the diocotron resonance. The scaled coordinate, z , will be defined from ω_e by

$$z = -(\omega_e / \Omega) / \epsilon, \quad (39)$$

so that $z=0$ at the resonance. Let ν be the unitless derivative of the parallel dc drift velocity, defined by

$$\nu = v'_{0x} / \Omega, \quad (40)$$

then we have

$$\partial_y = \frac{k \nu}{\epsilon} \partial_z. \quad (41)$$

Now if we take

$$\mu = \lambda / \Omega, \quad (42)$$

then it follows that

$$n_0 = \Omega^2(\mu^2 + \nu - 1), \quad (43)$$

$$n'_0 = k\Omega^2(2\mu_\chi\mu + \nu_\chi), \quad (44)$$

$$\lambda' = k\Omega\mu_\chi, \quad (45)$$

$$v''_{0x} = k\Omega\nu_\chi, \quad (46)$$

and so on. The subscript χ in the above indicates the derivative with respect to ky . This is so that we do not have to convert these dc quantities into functions of z since they are supposedly of order unity functions of ky . Whence we have introduced the variable $\chi=ky$ in order to have a unitless derivative of order unity for these quantities.

Let us comment that ν is essentially the scaled density profile at the diocotron resonance. Since this resonance usually occurs at or near the Brillouin edge, we can expect the derivative of ν to become large but not so large as to become of order $1/\epsilon$. Similarly since μ is the scaled value of λ , which is the “effective” cyclotron frequency inside the resonance, we can expect its derivative to be generally small.

First we scale the two slow modes and the equations that they satisfy, (31) and (32), so that we can obtain approximate solutions for v_{1x} and v_{1y} near the resonance. We do this so that we will have functional forms for the solution, required in the matching process. From (38)–(46), for this scaling, the ODE's (31) and (32) become

$$\partial_z p_3 + \epsilon \left(i \frac{p_2}{\nu} + 2 \frac{p_3 \mu_\chi}{\mu \nu} \right) + O(\epsilon^2) = 0, \quad (47)$$

$$\partial_z p_2 + i p_3 \frac{\beta_0}{z} - i \epsilon p_3 \left(\frac{1}{\nu} - \frac{1-\nu}{\mu^2} \right) + O(\epsilon^2) = 0, \quad (48)$$

where $p_2 = \frac{kv_{1x}}{\Omega}$ and $p_3 = \frac{kv_{1y}}{\Omega}$ are unitless functions of z . Also β_0 is a constant coefficient. It and two other coefficients (to appear later) are defined as

$$\beta_0 = 2 \frac{\nu-1}{\nu} \frac{\mu_\chi}{\mu} - \frac{\nu_\chi}{\nu}, \quad (49)$$

$$\beta_1 = \frac{2\nu-1}{\nu} \frac{\mu_\chi}{\mu} - \frac{\nu_\chi}{\nu}, \quad (50)$$

$$\beta_2 = \frac{2\nu-1}{\nu} \frac{\mu_\chi}{\mu}. \quad (51)$$

The general solution of (47) and (48), through order ϵ , may be verified to be

$$p_2 = C_4 - \frac{i}{2} \beta_0 C_5 \ln z^2 + O(\epsilon), \quad (52)$$

$$p_3 = C_5 - i \epsilon C_4 \frac{z}{\nu} - 2 \epsilon C_5 \frac{z \mu_\chi}{\mu \nu} - \epsilon C_5 \frac{\beta_0 z}{2\nu} (\ln z^2 - 2) + O(\epsilon^2), \quad (53)$$

where C_4 and C_5 are the constants of integration. From these solutions one may now construct the eigenvectors for these slow modes. From the above and (34)–(46), taking the eigen-

vectors to be the appropriate coefficients of C_4 and C_5 , we obtain

$$V_{4o} = \left[-\beta_0 \Omega^2, \frac{\Omega}{k}, -i \epsilon \frac{z \Omega}{k \nu}, \frac{\Omega^2}{k}, \epsilon \frac{\Omega^2 z}{k^2 \nu} \right], \quad (54)$$

$$V_{5o} = \left[-i \beta_0 \frac{\nu}{z} \Omega^2, -i \epsilon \beta_0 \frac{\Omega}{2k} \ln z^2, \epsilon \frac{\Omega}{k} - \epsilon^2 \beta_0 \frac{\Omega z}{2k \nu} \right. \\ \times (\ln z^2 - 2) - \epsilon^2 \frac{2 \Omega z \mu_\chi}{k \mu \nu}, \\ \left. -i \epsilon \beta_0 \frac{\Omega^2}{2k} \ln z^2, i \epsilon \frac{\Omega^2}{k^2 \nu} (1-\nu) - i \epsilon^2 \beta_0 \right. \\ \left. \times \frac{\Omega^2 z}{2k^2 \nu} (\ln z^2 - 2 + 2\nu) - 2i \epsilon^2 \frac{\Omega^2 z}{k^2 \mu \nu} (1-\nu) \mu_\chi \right], \quad (55)$$

where the error in each component is the next order in ϵ . Eigenvector 4 corresponds to the component of the slow mode where the potential vanishes at the resonance while eigenvector 5 corresponds to the component where, in lowest order, the potential is nonzero at the resonance. Thus we will refer to these two modes respectively as the “potential gradient mode” and the “potential mode” due to their structure at this resonance.

Upon taking the same limit for the third outer eigenvector, one obtains

$$V_{3o} = \left[\Omega^2, -i \frac{\epsilon \Omega}{z k}, i \frac{\epsilon^2 \Omega}{z^2 k}, -i \frac{\epsilon \Omega^2}{z k}, -\frac{\epsilon^2 \Omega^2}{z^2 k^2} \right] e^{i \int^y \alpha_3 dy}, \quad (56)$$

where we have taken $\mathcal{A}_{3o} = \epsilon^2$ so that the largest component of this eigenvector would be of order unity.

V. THE INNER SOLUTIONS AT THE DIOCOTRON RESONANCE

To construct inner solutions we start with the scalings for the quantities $[n_1, v_{1x}, v_{1y}, \phi_1]$ as given by (20) and (21) and by (38)–(46). Thus we take

$$V = [\Omega^2 \psi_1, \epsilon \Omega \psi_2 / k, \epsilon^2 \Omega \psi_3 / k, \epsilon \Omega^2 \psi_4 / k, \epsilon^2 \Omega^2 \psi_5 / k^2], \quad (57)$$

where the ψ s are other unitless functions of z to be determined. Substituting the above into (15)–(18) and using (41) to convert y derivatives to z derivatives as needed, one obtains, through first order in ϵ , the five scaled equations

$$\nu \partial_z \psi_1 = i(\psi_2 - z \psi_3)(1 - \mu^2 - \nu) - \epsilon \psi_3(2\mu_\chi\mu + \nu_\chi) \\ + (\psi_4 - \psi_2)(1 - \mu^2 - \nu)/\epsilon^2 + O(\epsilon^2), \quad (58)$$

$$\nu \partial_z \psi_2 = -iz \psi_2 + \psi_3(1 - \nu) + i \psi_5 + O(\epsilon^2), \quad (59)$$

$$\nu \partial_z \psi_3 = -iz \psi_3 + (\psi_4 - \psi_2)/\epsilon^2 + O(\epsilon^2), \quad (60)$$

$$\nu \partial_z \psi_4 = \psi_1 + O(\epsilon^2), \quad (61)$$

$$\nu \partial_z \psi_5 = \psi_4. \quad (62)$$

The last equation, being the definition of ψ_4 , is exact to all orders.

These equations reduce to a third-order system. Starting with (58) and (60), one deduces

$$\psi_4 = \psi_2 + \epsilon^2 p_4 + O(\epsilon^4), \quad (63)$$

where p_4 is another function of z to be determined. From (59) and using (63) to eliminate ψ_4 from (61), one obtains

$$\psi_1 = -iz\psi_2 + \psi_3(1 - \nu) + i\psi_5 + O(\epsilon^2). \quad (64)$$

Using this to eliminate ψ_1 from (58), we have

$$p_4 = i(z\psi_3 - \psi_2) - 2\epsilon\psi_3\mu_\chi/\mu + O(\epsilon^2). \quad (65)$$

Then upon eliminating p_4 from (60), it follows from (62) that

$$\psi_5 = i\psi_3 + C_5 + \epsilon p_5 + O(\epsilon^2), \quad (66)$$

where C_5 is a constant and p_5 is another unknown function of z . From (60) we obtain ψ_2

$$\psi_2 = i\nu\partial_z\psi_3 + 2i\epsilon\psi_3\mu_\chi/\mu + O(\epsilon^2). \quad (67)$$

Lastly we use the above to eliminate all variables except ψ_3 and p_5 from (59) and (62), to obtain the third-order system

$$\begin{aligned} \nu^2\partial_z^2\psi_3 + [iz + \epsilon(2\mu_\chi/\mu + \nu_\chi/\nu)]\nu\partial_z\psi_3 + 2i\epsilon\psi_3\mu_\chi/\mu \\ - \epsilon p_5 = C_5 + O(\epsilon^2), \end{aligned} \quad (68)$$

$$\nu\partial_z p_5 = 2i\psi_3\mu_\chi/\mu + O(\epsilon). \quad (69)$$

The solution of (68) and (69) for ψ_3 and p_5 may be verified to be

$$p_5 = C_4 \frac{iz^2}{\mu\nu} \mu_\chi + C_3 [iz e^{i\theta} + (i - z^2/\nu)F_1] \frac{\mu_\chi}{\mu\nu} - C_5 \frac{2z\mu_\chi}{\mu\nu^2} + O(\epsilon), \quad (70)$$

$$\begin{aligned} \psi_3 = C_4 \left[z + \epsilon \left(i\beta_0 - \beta_2 \frac{z^2}{\nu} \right) \right] + C_3 \left\{ e^{i\theta} + \frac{iz}{\nu} F_1 + \epsilon \left[-\frac{\beta_1}{\nu} \left(i \frac{z^2}{\nu} \right. \right. \right. \\ \left. \left. \left. + 1 \right) F_1 + z e^{i\theta} \right] \right\} + C_5 \frac{i}{\nu} \left\{ 1 + \epsilon \left[\frac{iz}{\nu^2} (\nu_\chi - \mu_\chi/\mu) \right. \right. \right. \\ \left. \left. \left. + \frac{\beta_0}{\nu} \left(\frac{z}{\nu} F_2 - i F_1^* e^{i\theta} \right) - i \frac{\beta_1 z}{\nu} \right] \right\} + O(\epsilon^2), \end{aligned} \quad (71)$$

where C_3 and C_4 are additional constants, F_1^* is the complex conjugate of F_1 and

$$\theta = \int_0^z \frac{z}{\nu} dz, \quad (72)$$

$$F_1 = \int_0^z e^{i\theta} dz, \quad (73)$$

$$F_2 = \int_0^z e^{i\theta} F_1^* dz. \quad (74)$$

In lowest order F_1 is a Fresnel integral. The other ψ s may be obtained from the above and (63)–(67).

With this we have three linearly independent inner solutions which correspond to the drift mode and the two slow modes. Letting the independent solutions be the coefficients of C_3 , C_4 , and C_5 , we have

$$\begin{aligned} V_{3i} = & \left\{ \left[-\nu e^{i\theta} + \epsilon\beta_0 F_1 + \epsilon z e^{i\theta} \frac{\nu_\chi}{\nu} \right] \Omega^2, \right. \\ & \times \left[-\epsilon F_1 + \epsilon^2 \beta_0 \frac{z}{\nu} F_1 - 2i\epsilon^2 \left(\beta_1 - \frac{\mu_\chi}{\mu} \right) \right] \frac{\Omega}{k}, \\ & \left[\epsilon^2 \left(e^{i\theta} + i F_1 \frac{z}{\nu} \right) - \epsilon^3 \beta_1 \left(\frac{z}{\nu} + F_1 \left(i \frac{z^2}{\nu^2} + \frac{1}{\nu} \right) \right) \right] \frac{\Omega}{k}, \left[-\epsilon F_1 \right. \\ & \left. + \epsilon^2 \beta_0 \frac{z}{\nu} F_1 - 2i\epsilon^2 \left(\beta_1 - \frac{\mu_\chi}{\mu} \right) \right] \frac{\Omega^2}{k}, \left[\epsilon^2 \left(i e^{i\theta} - F_1 \frac{z}{\nu} \right) \right. \\ & \left. + \epsilon^3 \left(\beta_1 - \frac{\mu_\chi}{\mu} \right) + \epsilon^3 \left(F_1 \frac{z^2 - i\nu}{\nu^2} - i \frac{z}{\nu} e^{i\theta} \right) \right] \frac{\Omega^2}{k^2} \right\}, \end{aligned} \quad (75)$$

$$\begin{aligned} V_{4i} = & \left\{ -i\epsilon\beta_0\nu\Omega^2, \left[i\epsilon\nu + 2i\epsilon^2 \frac{z\mu_\chi}{\mu\nu} (1 - \nu) \right] \frac{\Omega}{k}, \right. \\ & \left[\epsilon^2 z - \epsilon^3 \left(\beta_2 \frac{z^2}{\nu} - \beta_0 \right) \right] \frac{\Omega}{k}, \left[i\epsilon\nu + 2i\epsilon^2 \frac{z\mu_\chi}{\mu\nu} (1 - \nu) \right] \frac{\Omega^2}{k}, \\ & \left[i\epsilon^2 z + i\epsilon^3 \left(\frac{z^2\mu_\chi}{\mu\nu^2} (1 - \nu) - \beta_0 \right) \right] \frac{\Omega^2}{k^2} \right\}, \end{aligned} \quad (76)$$

$$\begin{aligned} V_{5i} = & \left\{ \frac{i}{\nu} \beta_0 F_1^* \Omega^2, \epsilon \left[\frac{2}{\nu} \left(\beta_1 - \frac{\mu_\chi}{\mu} \right) + i \frac{\beta_0}{\nu^2} F_2 \right] \frac{\Omega}{k}, \right. \\ & \times \left[i \frac{\epsilon}{\nu} + \frac{\epsilon^2}{\nu^2} \beta_0 \left(\frac{z}{\nu} F_2 - i F_1^* e^{i\theta} \right) - 2i\epsilon^2 \frac{z}{\nu^2} \left(\frac{\mu_\chi}{\mu} \right. \right. \\ & \left. \left. - \frac{\nu_\chi}{\nu} \right) \right] \frac{\Omega}{k}, \epsilon \left[\frac{2}{\nu} \left(\beta_1 - \frac{\mu_\chi}{\mu} \right) + i \frac{\beta_0}{\nu^2} F_2 \right] \frac{\Omega^2}{k}, \left[-\frac{\epsilon}{\nu} (1 - \nu) \right. \\ & \left. - 2\epsilon^2 \frac{z\nu_\chi}{\nu^3} + \frac{\epsilon^2}{\nu^2} \beta_0 \left(\frac{iz}{\nu} F_2 + F_1^* e^{i\theta} \right) \right] \frac{\Omega^2}{k^2} \right\}, \end{aligned} \quad (77)$$

where the subscript i refers to the inner solution. In the next section we will match the inner and outer solutions and determine the connection coefficients.

VI. THE CONNECTION COEFFICIENTS

In order to match the inner solutions to the outer solutions, we need the asymptotics of the inner solutions in the limit of z large. For this we need the asymptotics of θ , F_1 and F_2 . One may easily verify from (38)–(40) that the asymptotic of θ for large z , up to an additive constant, is just the phase of the drift mode, $\int^y \alpha_3 dy$. The asymptotics of F_1 will be required to first order in ϵ . Thus it will be necessary to include the χ dependence of ν in the evaluation of F_1 . However the asymptotics of F_2 will only be required to zeroth order in ϵ , in which case we may ignore the variation of ν with respect to χ .

As shown in the Appendix, in the limit of $|z|$ large, the asymptotics of these integrals are:

$$\begin{aligned} F_1 \rightarrow & \sqrt{\frac{\pi\nu_0}{2}} \frac{z}{|z|} e^{i\frac{\pi}{4}} - \frac{2i\epsilon}{3\nu_0} \nu_\chi + \left(\frac{i\nu}{z} + \frac{\epsilon}{z^2} \nu_\chi \right) e^{i\theta} + O(\epsilon^2) \\ & + O(1/z^3), \end{aligned} \quad (78)$$

$$F_2 \rightarrow \frac{-i}{2} \nu \left[i \frac{\pi}{2} + \gamma + \ln \left(\frac{4z^2}{\nu} \right) \right] + O \left(\frac{1}{z} \right), \quad (79)$$

where ν_0 is the value of ν at the resonance ($z=0$) and $\gamma=0.57721\dots$ is Euler's constant.

Let us evaluate the connection coefficients in the following manner. Start with a linear superposition of the three outer solutions below the resonance ($z < 0$) of the form

$$V = C_3^{(-)} V_{3o} + C_4^{(-)} V_{4o} + C_5^{(-)} V_{5o}, \quad (80)$$

where $C_n^{(-)}$ is the amplitude of the n th eigenvector and the superscript $(-)$ indicates that this is the amplitude below the resonance. We take the form of the interior solution to be

$$V = D_3 V_{3i} + D_4 V_{4i} + D_5 V_{5i}, \quad (81)$$

where D_n is the amplitude of the n th interior solution. Then upon inserting (78) and (79) into (75)–(77), for z large and negative, and upon comparing the result with (54)–(56), one finds that in lowest order of ϵ ,

$$D_3 = -\frac{C_3^{(-)}}{\nu_0} - \beta_0 C_5^{(-)} \sqrt{\frac{\pi}{2\nu_0}} e^{i\pi/4}, \quad (82)$$

$$D_4 = -\frac{C_3^{(-)}}{\nu_0} \sqrt{\frac{\pi}{2\nu_0}} e^{i\pi/4} - i \frac{C_4^{(-)}}{\nu_0} + \frac{C_5^{(-)}}{2\nu_0} \mathcal{E}, \quad (83)$$

$$D_5 = -i\nu_0 C_5^{(-)}, \quad (84)$$

where

$$\mathcal{E} = \beta_0 \left[\gamma + 1 + \ln \left(\frac{4}{\nu_0} \right) - i \frac{\pi}{2} \right] - 2 \frac{\nu_\chi}{\nu_0}. \quad (85)$$

Repeating the same for z large and positive, we take the outer solution to be the linear superposition

$$V = C_3^{(+)} V_{3o} + C_4^{(+)} V_{4o} + C_5^{(+)} V_{5o}, \quad (86)$$

where the superscript $(+)$ indicates that this coefficient is for $z > 0$ and large. Then we obtain

$$D_3 = -\frac{C_3^{(+)}}{\nu_0} + \beta_0 C_5^{(+)} \sqrt{\frac{\pi}{2\nu_0}} e^{i\pi/4}, \quad (87)$$

$$D_4 = \frac{C_3^{(+)}}{\nu_0} \sqrt{\frac{\pi}{2\nu_0}} e^{i\pi/4} - i \frac{C_4^{(+)}}{\nu_0} + \frac{C_5^{(+)}}{2\nu_0} \mathcal{E}, \quad (88)$$

$$D_5 = -i\nu_0 C_5^{(+)}. \quad (89)$$

Combining (82)–(84) with (87)–(89), upon eliminating the D s we then have

$$C_3^{(+)} = C_3^{(-)} + \beta_0 C_5^{(-)} \sqrt{2\pi\nu_0} e^{i\pi/4} + O(\epsilon), \quad (90)$$

$$C_4^{(+)} = C_4^{(-)} + C_3^{(-)} \sqrt{\frac{2\pi}{\nu_0}} e^{-i\pi/4} + \pi\beta_0 C_5^{(-)} + O(\epsilon), \quad (91)$$

$$C_5^{(+)} = C_5^{(-)} + O(\epsilon). \quad (92)$$

Some physical understanding to these results can be obtained by noting the following points. Recall that β_0 , (49), is essentially the negative of the logarithmic derivative of the

density with respect to χ , and as such, it is only significantly nonzero at the edge of the sheath. Away from the edge we may take β_0 to be essentially zero. Thus when $\beta_0 \approx 0$, we see that the amplitudes of the drift mode and potential mode would be transmitted across the resonance without change. Meanwhile the potential gradient mode would always be modified, picking up an additional part proportional to the amplitude of the drift mode. On the other hand when the resonance is near the edge such that β_0 is significantly nonzero, then the potential mode always modifies the other two modes, although itself is transmitted across the resonance unchanged.

Let us review the nature of these three modes in the outer region ($|z| \gg 1$). From (56) we see that the drift mode basically only has fast density oscillations as long as z is in the outer region. The potential gradient mode, (54), has a rf density “bulge” at the resonance, on the slow scale, proportional to β_0 . It also has a slow rf potential gradient and essentially a vanishing value for the potential at the resonance. On the other hand the potential mode, (55), is more complex, but basically is the only mode with a nonzero value for the rf potential at the resonance.

Now consider the following. Potentials are more difficult to shift than potential gradients, which could then be taken as an explanation for why the potential mode tends to be so resilient and transmitted across the resonance unchanged. (To justify this explanation, note that without work being done, one can shift the charge distribution inside the plasma such that one shifts the potential gradient but not the overall potential.) Consider when we have an absence of a density gradient at the resonance and thus $\beta_0 \approx 0$. The drift mode, containing essentially only fast density oscillations, would tend to simply transmit unchanged across the resonance. However as this drift mode crosses the resonance, since the WKB wave vector α_3 will cross zero here, its fast oscillations will become slower and slower and eventually reverse. Thus at the resonance, this mode would leave behind a non-zero average for the rf potential gradient, thereby contributing to the potential gradient mode. On the other hand since its density average is zero it would not significantly contribute to the potential mode. Now let the resonance be located near the edge of the sheath so that $\beta_0 \neq 0$. Here we have an additional term coming into play which is the coupling provided by the dc density gradient at the edge of the sheath as in (52) and (53). This dc density gradient, combining with the potential mode, then generates contributions to both the drift mode and the potential gradient mode.

VII. CONCLUSIONS

We have shown that in the saturation stage the rf component of the non-neutral plasma inside a magnetron becomes a fifth-order system. This fifth-order system consists of two slow modes, two fast drift-cyclotron modes (Slater orbits) and one fast drift mode. The three fast modes are not found in the initiation stage since their existence requires the dc drift velocity to have a component directed from the cathode to the anode. Although the current generated by this vertical dc velocity is only a small fraction of the total cur-

rent circulating in the device, it is exactly this smallness which creates a dramatic change in this structure of the rf solutions. In particular this vertical dc current causes resonance layers to occur at the position of any resonance in the plasma, inside of which mode conversions between these five modes can occur. These resonances were already in the initiation stage: (1) the diocotron resonance at $\omega_e=0$, and (2) the two cyclotron resonances at $\omega_e=\pm\Omega$. The structure of these resonances occurs due to the small parameter (vertical velocity) multiplying three of the derivatives of the fifth-order set of ODEs.³¹

As we have already seen in Fig. 6 these fast modes prevent one from obtaining rapid numerical solutions in this stage. Now that the structure and location of these resonances are known, and the structure and widths of the resonance layers, it will be possible to develop numerical code to rapidly solve this fifth-order system. For example if there was no magnetron resonance inside the plasma then the drift-cyclotron modes never cross the other resonances. In this case the fifth-order system can be reduced to a third-order system and two first-order systems resulting in a very significant decrease in the required computational time for the drift-cyclotron modes. In effect what one can do is to use the WKB approximation (followed by a perturbation expansion) to separate the fifth-order system into three first-order ODE's and one second-order ODE, each of which could be rapidly computed since the fast variations could be analytically separated out. The only problem would be in the region of any resonances where one would have to appropriately couple the ODEs and carefully numerically integrate through the resonance. However such can be done: one can separately propagate each fast mode from the cathode until it either reaches the anode or crosses the two slow modes. When the latter occurs one simply does a separate computation around the resonance.

Then there is the question of boundary conditions on the rf fields. In the initiation stage one only requires two boundary conditions: (i) the vanishing of the rf potential at the cathode and (ii) specifying the magnitude of the rf density oscillations at the anode. The former follows whenever the cathode is a smooth conductor and the latter is generally determined by the dc density profile. (Note that since the electron density can never be negative, the amplitude of the rf density oscillations must always be less than the dc electron density.²⁰ Taking these to be equal has the effect of coupling the rf amplitude at the anode to the dc amplitude at the anode.) For the saturation stage three additional boundary conditions will be required. It seems reasonable that one should take these three conditions to be the vanishing of all fast modes at the cathode. This would follow from recent results^{10–13,32} on the stability of various non-neutral flows. However it should also be mentioned that various recent methods for breaking up the laminar flow at the cathode, as in Ref. 27, could be found to be a method for creating non-zero sources of fast modes at the cathode.

In the usual situation of a smooth cathode with no magnetron resonance present, the rf structure in the saturation stage becomes simpler since only the diocotron resonance exists. Taking the fast modes to be zero at the cathode, it

follows that both drift-cyclotron modes will be absent. Also the drift mode will be zero between the cathode and the location of the diocotron resonance. Now we apply (90)–(92) and find that above the diocotron resonance the drift mode will exist with $C_3^{(+)} \propto \beta_0 C_5^{(-)}$. Thus the total rf field at the anode would then consist of a drift mode and two slow modes and the ponderomotive terms in (6)–(9) would consist of two parts: the usual contribution from the slow modes and an additional contribution from the drift-cyclotron mode created at the diocotron resonance from the slow modes.

How important are these three fast modes to the generation of the rf power of a magnetron? At the moment we cannot assess this quantitatively since we still are to obtain self-consistent saturation stage solutions for dc and the rf fields [i.e. satisfying (6)–(13)]. We do know that the growth rate vanishes when the device is in the saturation stage. We do know that if we start with the dc field of the initiation stage that we obtain the rf solutions discussed herein. We would expect the dc solution of the saturation stage, satisfying (6)–(9), to be qualitatively similar to that of the initiation stage as given in Figs. 3–5. We do know that these devices generally operate only when the diocotron resonance occurs at or near the middle of the density profile and where the density gradient is large. We do know that if the parameters are outside this range then the device will not operate. In the latter case $\beta_0 \ll 1$ and per (90)–(92), essentially no drift mode would be produced. What we do not know is the range of values for β_0 in (90). One can argue from (30) and (37) that for $\beta_0 \gg 1$, the amplitude of the rf electric field of the drift mode at the anode could even exceed that of the slow modes. It all depends on the sharpness of the edge of the electron sheath in the saturation stage. All these points are also consistent with the statement: a magnetron will only operate when a significant drift mode has been generated at the diocotron resonance from the slow modes. Thus it certainly appears that the drift mode is a key component in the operation of a magnetron. We will understand more about this after self-consistent solutions for the saturation stage are obtained following along the lines of the self-consistent calculations done for the initiation stage.⁹

In this paper we have detailed the five independent solutions in the region of the diocotron resonance. We have determined the conversion coefficients for the three modes concerned and have given a general explanation for their structure. In particular, the amplitude of the potential mode remains unaffected in crossing the resonance while the drift mode and the potential gradient mode pick up additional contributions to their amplitudes from the potential mode with the coupling being proportional to the dc density gradient. The drift mode will always add a contribution to the potential gradient mode due to the break in the potential gradient of the drift mode at the resonance.

In summary, we have studied the nature of the rf solutions of the steady-state magnetron when it is steadily delivering rf power, particularly the rf modes in the non-neutral plasma in the neighborhood of the diocotron resonance. We have found that in this neighborhood the ordinary differential equations for the rf fields are nonsingular and that nonsingular solutions do exist for the rf quantities. This resonance

FIG. 9. The contour in the complex plane for evaluating $F_1(z)$ in the limit of large z .

layer is relatively thin and in general the rf fields will rapidly vary as they pass through the resonance in a manner similar to what occurs at boundary layers. During the passage of the slow modes and/or the drift mode through this resonance, conversion from one mode to another will occur with the conversion coefficients being given by (90)–(92).

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APPENDIX: ASYMPTOTIC LIMITS OF INTEGRALS

In this Appendix we will derive the asymptotic limits of F_1 and F_2 for $|z| \rightarrow \infty$.

Starting with F_1 , note that we will need this result to order ϵ . Thus we cannot just simply replace F_1 by a Fresnel integral, but we must consider the first-order variations of ν with respect to χ . As long as ν does not significantly change over the region where $|z| < \approx 1/\sqrt{\epsilon}$, we may consider ν to be an analytic function of z and use a contour integral to evaluate the asymptotic value of F_1 . For clarity we introduce dummy variables w and u and rewrite (72) and (73) as

$$\theta(z) = \int_0^z \frac{udu}{\nu(u)}, \quad F_1(w) = \int_0^w e^{i\theta(z)} dz. \quad (\text{A1})$$

Consider the integral of $\int e^{i\theta(z)} dz$ around the contour shown in Fig. 9 for w large and positive. It can be seen that $F_1(w)$ is the integral along the real-axis segment of the contour. By assumption θ will be analytic inside the contour. Thus there will be no residue and the value of the contour integral will be zero. Thus the value of the real-axis segment will be the negative of the contributions along the other two segments.

Along the arc, the major contribution to the integral will come from the region of the arc next to the real axis. To evaluate the integral along this arc we expand $\theta(z)$ in a Taylor series, expanded about $z=w$. Then evaluation of this integral gives

$$\left[-\frac{i}{w} - \frac{\epsilon \nu_\chi}{w^2} + O\left(\frac{1}{w^3}\right) \right] e^{i\theta(w)} + O(\epsilon^2). \quad (\text{A2})$$

Along the ray the major contribution to that segment of the contour integral will come from the region next to the origin. Here we expand $\theta(z)$ in a Taylor series, expanded about z

$=0$, and the value of the integral is then found to be

$$-\sqrt{\frac{\pi \nu_0}{2}} e^{-i\pi/4} + \frac{2i\epsilon \nu_\chi}{3\nu_0} + O(\epsilon^2), \quad (\text{A3})$$

where ν_0 is the value of ν at the center of the resonance. Since F_1 will be the negative of the sum of these two integrals, we have that in the limit where z is large and positive,

$$F_1(z) \rightarrow \sqrt{\frac{\pi \nu_0}{2}} e^{-i\pi/4} - \frac{2i\epsilon \nu_\chi}{3\nu_0} + \left[\frac{i}{z} + \frac{\epsilon \nu_\chi}{z^2} + O\left(\frac{1}{z^3}\right) \right] e^{i\theta(z)} + O(\epsilon^2). \quad (\text{A4})$$

To evaluate the limit of $F_2(z)$ for large $|z|$, we only need the limit to order unity and thus we take ν to be a constant. In this case, from (74) and (A1), $F_2(z)$ may be given by

$$F_2(z) = \int_0^z du \int_0^u e^{-i\left(\frac{u^2-v^2}{2\nu}\right)} dv. \quad (\text{A5})$$

This double integral may be evaluated by transforming to a hyperbolic coordinate system. Let us take $u=w \cosh \varphi$ and $v=w \sinh \varphi$. Then the above integral can be transformed into

$$F_2(z) = I_1(z) + I_2(z), \quad (\text{A6})$$

where

$$I_1(z) = \int_0^z w e^{\frac{-iw^2}{2\nu}} \ln\left(\frac{z}{w}\right) dw, \quad (\text{A7})$$

$$I_2(z) = \int_0^z w e^{\frac{-iw^2}{2\nu}} \ln\left(1 + \sqrt{1 - \frac{w^2}{z^2}}\right) dw. \quad (\text{A8})$$

Integrating by parts on the first integral, to eliminate the $\ln(z/w)$ term gives

$$I_1(z) = \frac{i}{2} \nu \left(-i\frac{\pi}{2} - \int_0^{\frac{z^2}{2\nu}} \frac{1-e^{-t}}{t} dt \right), \quad (\text{A9})$$

where the last integral is listed in Ref. 33. In the limit of large $|z|$ this gives

$$I_1(z) = \frac{i}{2} \nu \left[-i\frac{\pi}{2} - \gamma - \ln\left(\frac{z^2}{\nu}\right) \right], \quad (\text{A10})$$

up to exponentially small terms. In the above, $\gamma = 0.57721\dots$ is Euler's constant.

For I_2 , we first transform variables from w to p where $w^2 = z^2(1-p^2)$ and then we perform an integration by parts to eliminate the $\ln$ term. This gives

$$I_2(z) = -i\nu \ln 2 + i\nu e^{-\frac{iz^2}{2\nu}} \int_0^1 \frac{e^{i\frac{z^2 p^2}{2\nu}}}{1+p} dp. \quad (\text{A11})$$

The major contribution to the remaining integral comes from the region near $p=0$, so we replace $\frac{1}{1+p}$ by $1 - \frac{p}{1+p}$ under the integral. When that is done, the first term becomes a Fresnel integral and can be evaluated as with $F_1(z)$ above. The second term, $\frac{p}{1+p}$, under the integral can be shown to be bounded by an order of $1/z^2$ for large $|z|$. Thus we have

$$I_2(z) = -i\nu \left[\ln 2 - \sqrt{\frac{\pi\nu}{2z^2}} e^{-\frac{iz^2}{2\nu}} e^{i\frac{\pi}{4}} + O\left(\frac{1}{z^2}\right) \right]. \quad (\text{A12})$$

Combining the above results for (A10) and (A12) into (A6) gives

$$F_2(z) \rightarrow \frac{-i}{2} \nu \left[i\frac{\pi}{2} + \gamma + \ln\left(\frac{4z^2}{\nu}\right) - e^{-\frac{iz^2}{2\nu}} e^{i\frac{\pi}{4}} \sqrt{\frac{2\pi\nu}{z^2}} \right] + O\left(\frac{1}{z^2}\right) + O(\epsilon), \quad (\text{A13})$$

for $|z|$ large.

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