Frames and Phase Retrieval

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FRAMES AND PHASE RETRIEVAL

by

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M.S. University of Central Florida, 2014
B.A. University of Florida, 2008

A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Summer Term
2019

Major Professor: Deguang Han

ABSTRACT

Phase retrieval tackles the problem of recovering a signal after loss of phase. The phase problem shows up in many different settings such as X-ray crystallography, speech recognition, quantum information theory, and coherent diffraction imaging. In this dissertation we present some results relating to three topics on phase retrieval. Chapters 1 and 2 contain the relevant background materials. In chapter 3, we introduce the notion of exact phase-retrievable frames as a way of measuring a frame's redundancy with respect to its phase retrieval property. We show that, in the d-dimensional real Hilbert space case, exact phase-retrievable frames can be of any lengths between $2d-1$ and $d(d+1)/2$, inclusive. The complex Hilbert space case remains open.

In chapter 4, we investigate phase-retrievability by studying maximal phase-retrievable subspaces with respect to a given frame. These maximal PR-subspaces can have different dimensions. We are able to identify the ones with the largest dimension and this can be considered as a generalization of the characterization of real phase-retrievable frames. In the basis case, we prove that if M is a k-dimensional PR-subspace then $|\text{supp}(x)| \geq k$ for every nonzero vector $x \in M$. Moreover, if $1 \leq k < [(d+1)/2]$, then a k-dimensional PR-subspace is maximal if and only if there exists a vector $x \in M$ such that $|\text{supp}(x)| = k$.

Chapter 5 is devoted to investigating phase-retrievable operator-valued frames. We obtain some characterizations of phase-retrievable frames for general operator systems acting on both finite and infinite dimensional Hilbert spaces; thus generalizing known results for vector-valued frames, fusion frames, and frames of Hermitian matrices.

Finally, in Chapter 6, we consider the problem of characterizing projective representations that admit frame vectors with the maximal span property, a property that allows for an algebraic recovering of the phase-retrieval problem. We prove that every irreducible projective representation of

a finite abelian group admits a frame vector with the maximal span property. All such vectors can be explicitly characterized. These generalize some of the recent results about phase-retrieval with Gabor (or STFT) measurements.

To my mother, Eveline.

ACKNOWLEDGMENTS

I want to thank my advisor, Dr. Deguang Han, for his support, guidance, and infinite patience. Deguang believed in me when I did not believe in myself. Our weekly meetings during the first four years were sometimes intimidating (when I was not prepared) but I was able to learn a lot from him by observing his habits and approach to research. He was kind and generous to me not only with his ideas but also with financial support so I could focus on my studies and my research. As I begin a new chapter of my academic career, I'm confident that the training that I have received from him has prepared me for future challenges.

I also thank Drs. Qiyu Sun, Dorin Dutkay, and Dingbao Wang for serving on my committee. Last, but not least, I thank my collaborators Joseph P. Brennan, Chuangxun Cheng, Lan Li, Youfa Li, and Wenchang Sun.

During my years at UCF I have made some friends and acquaintances. We went to the gym and played soccer; we complained about the department and our teaching duties; we joked; we discussed math. Thank you for making this journey worthwhile.

This dissertation was partially supported by the National Science Foundation grants DMS-1403400 and DMS-1712606 (PI: Deguang Han).

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CHAPTER 1: FRAMES AND THEIR BASIC PROPERTIES

The concept of a frame has its roots in the theory of nonharmonic Fourier series, which studies the expansion and completeness properties of sequences of complex exponentials $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ in $L^p[-\pi,\pi]$. It was first introduced by R.J. Duffin and A.C. Schaeffer in their 1952 paper titled *A Class of Nonharmonic Fourier Series*. They first defined a frame in the following manner

A set of functions $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a frame over an interval $(-\gamma,\gamma)$ if there exist positive constants A and B which depend exclusively on γ and the set of functions $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ such that

$$A \le \frac{\frac{1}{2\pi}\sum_n \left|\int_{-\gamma}^{\gamma} g(t)e^{i\lambda_n t}\,dt\right|^2}{\int_{-\gamma}^{\gamma}|g(t)|^2\,dt} \le B$$

for every function $g \in L^2(-\gamma,\gamma)$.

In that same paper, Duffin and Schaeffer extended this definition to separable Hilbert spaces. We now have a notion of frames in the context of Banach spaces. Frames have a wide range of applications in signal processing, image processing, and internet coding. In this chapter we will go over some of the basic properties of frames and present some well-known results. We begin with the definition of a frame in a separable Hilbert space.

1.1 Frames in Separable Hilbert Spaces

Definition 1.1.1. Let $\mathcal{H}$ be a separable Hilbert space. A sequence of vectors $\{f_n\}_{n=1}^{\infty}$ is a *frame* for $\mathcal{H}$ if there exist positive constants A and B such that for all $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2 \tag{1.1}$$

The numbers A and B are called **lower frame bound** and **upper frame bound** for $\{f_n\}_{n=1}^{\infty}$, respectively. The **optimal lower frame bound** is the supremum amongst all lower frame bounds, and the **optimal upper frame bound** is the infimum amongst all upper frame bounds. It is easily shown that the optimal frame bounds are frame bounds; that is, they are finite, positive and satisfy (1.1). The frame is called a **tight frame** if $A = B$. It is called a **Parseval frame** if $A = B = 1$. Every tight frame can be turned into a Parseval frame by scaling its elements by the square root of the frame bound. A frame is said to be **exact** if it is no longer a frame when any one element is removed from the sequence. A frame which is not exact is said to be **overcomplete**. We will only be concerned with Hilbert spaces that are either finite-dimensional or separable.

Note that some sequences satisfy the upper bound condition but not the lower bound condition. Sequences that satisfy the upper bound condition of (1.1) are called **Bessel sequences**.

We provide some examples of frames.

Example 1.1.2. Let $\mathcal{H}$ be a Hilbert space and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$.

i) $\{0, e_1, 0, 0, e_2, 0, 0, 0, e_3, ...\}$ is a Parseval frame for $\mathcal{H}$ with frame bounds $A = B = 1$. It is not exact.

ii) $\{e_1, e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, ...\}$ is a tight frame for $\mathcal{H}$ with frame bounds $A = B = 3$.

iii) $\{3e_1, e_2, e_3, ...\}$ is an exact frame for $\mathcal{H}$ with $A = 1$ and $B = 3$.

iv) $\{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, e_3/\sqrt{3}, ...\}$ is a Parseval frame for $\mathcal{H}$.

If $\{f_n\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ then its **analysis operator** $T : \mathcal{H} \to \ell^2(\mathbb{N})$ is defined by

$$T(f) = \{< f, f_n >\}_{n=1}^{\infty} \text{ for all } f \in \mathcal{H}.$$

Since T is a bounded operator, it has a bounded adjoint T^*, called the **synthesis operator**. Let $\{e_n\}_{n=1}^{\infty}$ be the standard orthonormal basis for $\ell^2(\mathbb{N})$. For every f in $\mathcal{H}$, the j-th coordinate of Tf is given by both $\langle f, f_j \rangle$ and $\langle Tf, e_j \rangle$. Hence for f in $\mathcal{H}$, we have

$$\begin{aligned} 0 &= \langle Tf, e_j \rangle - \langle f, f_j \rangle \\ &= \langle f, T^* e_j \rangle - \langle f, f_j \rangle \\ &= \langle f, T^* e_j - f_j \rangle. \end{aligned}$$

It follows that $T^* e_j = f_j$ for $j \in \mathbb{N}$. Therefore,

$$T^*(\{c_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} c_n f_n \quad \text{for all} \quad \{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}).$$

The following theorem provides a characterization of frames in separable Hilbert spaces using the analysis and synthesis operators. Its proof can be found in [Heil98].

Theorem 1.1.3. *Let $\mathcal{H}$ be a Hilbert space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of vectors in $\mathcal{H}$. The following are equivalent:*

i. $\{f_n\}_{n=1}^{\infty}$ *is a frame for* $\mathcal{H}$.

ii. *The synthesis operator* T^* *is well-defined for all* $\{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$. *It is bounded and onto* $\mathcal{H}$.

iii. *The analysis operator T is a bijective map between* $\mathcal{H}$ *and the closed subspace* $\mathrm{range}(T)$ *of* $\ell^2(\mathbb{N})$.

When we compose the analysis operator and the synthesis operator, we obtain the **frame operator**, defined as follows

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = T^*Tf = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n.$$

The proof of the following theorem can be found in [Chris03].

Theorem 1.1.4. *Let* $\mathcal{H}$ *be a Hilbert space and let* $\{f_n\}_{n=1}^{\infty}$ *be a frame with frame bounds* A *and* B *and frame operator* S. *Then the following holds:*

i. S *is bounded, invertible, self-adjoint, and positive.*

ii. $\{S^{-1}f_n\}_{n=1}^{\infty}$ *is a frame with frame operator* S^{-1} *and frame bounds* B^{-1}, A^{-1}.

iii. *If* A, B *are optimal frame bounds for* $\{f_n\}_{n=1}^{\infty}$, *then* B^{-1}, A^{-1} *are optimal frame bounds for* $\{S^{-1}f_n\}_{n=1}^{\infty}$.

iv. *For each* $f \in \mathcal{H}$,

$$f = \sum_{n=1}^{\infty} \langle f, S^{-1}f_n \rangle f_n = \sum_{n=1}^{\infty} \langle f, f_n \rangle S^{-1} f_n \tag{1.2}$$

v. $\{S^{-1/2} f_n\}_{n=1}^{\infty}$ *is a Parseval frame for* $\mathcal{H}$*.*

Equation 1.2 is called the **frame decomposition** (or **reconstruction formula**) and the numbers $\{\langle f, S^{-1} f_n\rangle\}_{n=1}^{\infty}$ are called **frame coefficients**. When the frame $\{f_n\}_{n=1}^{\infty}$ is overcomplete then there are many coefficients $\{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ such that

$$f = \sum_{n=1}^{\infty} c_n f_n.$$

The next proposition, which can be found in [Chris03], shows that the frame coefficients have minimal ℓ^2-norm amongst all such coefficients.

Proposition 1.1.5. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a frame for* $\mathcal{H}$ *and let* S *be its frame operator. If* f *is in* $\mathcal{H}$ *and* $f = \sum_{n=1}^{\infty} c_n f_n$ *for some coefficients* $\{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ *then*

$$\sum_{n=1}^{\infty} |\langle f, S^{-1} f_n\rangle|^2 \leq \sum_{n=1}^{\infty} |c_n|^2.$$

Proof. We will show that

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |\langle f, S^{-1} f_n\rangle|^2 + \sum_{n=1}^{\infty} |c_n - \langle f, S^{-1} f_n\rangle|^2. \tag{1.3}$$

Write

$$\{c_n\}_{n=1}^{\infty} = \{c_n\}_{n=1}^{\infty} - \{\langle f, S^{-1} f_n\rangle\}_{n=1}^{\infty} + \{\langle f, S^{-1} f_n\rangle\}_{n=1}^{\infty}.$$

Then we have

$$\sum_{n=1}^{\infty}\left(c_n - \langle f, S^{-1}f_n\rangle\right) f_n = 0.$$

Thus $\{c_n - \langle f, S^{-1}f_n\rangle\}_{n=1}^{\infty} \in \text{null}(T^*) = [\text{range}(T)]^{\perp}$ and $\{\langle f, S^{-1}f_n\rangle\}_{n=1}^{\infty} = \{\langle S^{-1}f, f_n\rangle\}_{n=1}^{\infty} \in$ range(T). We obtain 1.3. $\square$

Note that 1.3 also says that the frame coefficients are the unique coefficients minimizing the ℓ^2-norm.

Definition 1.1.6. Let $\{f_n\}_{n=1}^{\infty}$ be a frame for $\mathcal{H}$. If $\{g_n\}_{n=1}^{\infty}$ is another frame for $\mathcal{H}$ such that for all $f \in \mathcal{H}$

$$f = \sum_{n=1}^{\infty}\langle f, g_n\rangle f_n$$

then we say that $\{g_n\}_{n=1}^{\infty}$ is a dual frame of $\{f_n\}_{n=1}^{\infty}$.

It turns out that if $\{g_n\}_{n=1}^{\infty}$ is a dual frame for $\{f_n\}_{n=1}^{\infty}$ then $\{f_n\}_{n=1}^{\infty}$ is a dual frame for $\{g_n\}_{n=1}^{\infty}$. Equation (1.2) says that $\{f_n\}_{n=1}^{\infty}$ and $\{S^{-1}f_n\}_{n=1}^{\infty}$ are dual frames. If $\{f_n\}_{n=1}^{\infty}$ is an overcomplete frame then it has many duals. The proofs of the following results can be found in [Chris03].

Proposition 1.1.7. *If $\{f_n\}_{n=1}^{\infty}$ is an overcomplete frame. Then there exists frames $\{g_n\}_{n=1}^{\infty} \neq \{S^{-1}f_n\}_{n=1}^{\infty}$ such that*

$$f = \sum_{n=1}^{\infty}\langle f, g_n\rangle f_n, \forall h \in \mathcal{H}.$$

Proposition 1.1.8. *If $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are frames for $\mathcal{H}$ then the following are equivalent:*

i. $f = \sum_{n=1}^{\infty}\langle f, g_n\rangle f_n, \forall f \in \mathcal{H}.$

ii. $f = \sum_{n=1}^{\infty} \langle f, f_n \rangle g_n, \forall f \in \mathcal{H}$.

The next theorem, also in [Chris03], provides a characterization of all dual frames for a given frame.

Theorem 1.1.9. *Let $\{f_n\}_{n=1}^{\infty}$ be a frame for $\mathcal{H}$. Then $\{g_n\}_{n=1}^{\infty}$ is a dual frame for $\{f_n\}_{n=1}^{\infty}$ if and only if*

$$\{g_n\}_{n=1}^{\infty} = \left\{ S^{-1} f_n + h_n - \sum_{j=1}^{\infty} \langle S^{-1} f_n, f_j \rangle h_j \right\}_{n=1}^{\infty}$$

where $\{h_n\}_{n=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$.

The following theorem states that when an orthogonal projection is applied to a frame we still get a frame.

Theorem 1.1.10 (**Compression Property** [HKLW07])**.** *Let $\mathcal{H}$ be a Hilbert space and let $\{f_n\}_{n=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame bounds A and B. If P is an orthogonal projection of $\mathcal{H}$ onto a closed subspace M, then $\{Pf_n\}_{n=1}^{\infty}$ is a frame for M with frame bounds A and B. In particular, if $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ then $\{Pf_n\}_{n=1}^{\infty}$ is a Parseval frame for M.*

Proof. Pick $f \in M$. Then $Pf = f$ and we have

$$\langle f, f_n \rangle = \langle Pf, f_n \rangle = \langle f, Pf_n \rangle.$$

Therefore

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, Pf_n\rangle|^2 \leq B\|f\|^2 \quad \text{for all} f \in M.$$

Thus $\{Pf_n\}_{n=1}^{\infty}$ is a frame for M. Moreover, if $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ then we can take $A = B = 1$ and it follows that $\{Pf_n\}_{n=1}^{\infty}$ is a Parseval frame for M. $\square$

Theorem 1.1.11 (Dilation Property [HKLW07]**).** *Let $\mathcal{H}$ be a Hilbert space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of vectors in $\mathcal{H}$*

i. If $\{f_n\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a basis $\{g_n\}_{n=1}^{\infty}$ for $\mathcal{K}$ such that $Pg_n = f_n$. Here P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

ii. If $\{f_n\}_{n=1}^{\infty}$ is a Parseval frame for $\mathcal{H}$ then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{u_n\}_{n=1}^{\infty}$ for $\mathcal{K}$ such that $Pu_n = f_n$. Here again P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

Proof. The proofs of both parts are very similar so we only prove item ii. Let T be the analysis operator for $\{f_n\}_{n=1}^{\infty}$. Recall from Theorem 1.1.3 that range(T) is a closed subspace of $\ell^2(\mathbb{N})$. Hence the orthogonal projection Q of $\ell^2(\mathbb{N})$ onto range(T) is defined. Let $Q^{\perp} = I - Q$ be the orthogonal projection onto $[\text{range}(T)]^{\perp}$. The direct sum $H \oplus [\text{range}(T)]^{\perp}$ is a Hilbert space with the inner product given by

$$\langle (f_1, c_1), (f_2, c_2)\rangle = \langle f_1, f_2\rangle_{\mathcal{H}} + \langle c_1, c_2\rangle_{\ell^2(\mathbb{N})}$$

and we can identify $\mathcal{H}$ with $\mathcal{H} \oplus \{0\}$. Let $\{e_n\}_{n=1}^{\infty}$ be the standard orthonormal basis for $\ell^2(\mathbb{N})$ and let $u_n = f_n \oplus Q^{\perp}e_n$. Then $Pu_n = f_n$ where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

Now we show that $\{u_n\}_{n=1}^{\infty}$ is an orhonormal basis for $\mathcal{K}$. Consider the Parseval frame $\{Qe_n\}_{n=1}^{\infty}$ for range(T). Then the restriction of T^* to range(T) is an invertible operator. Denote this restriction by Γ. Since $e_n - Qe_n \in [\text{range}(T)]^{\perp} = \ker(T^*)$ we have $f_n = T^* e_n = T^* Q e_n = \Gamma Q e_n$. Also for every $f \in \mathcal{H}$, we have

$$\begin{aligned}
\|\Gamma^* f\|^2 &= \sum_{n=1}^{\infty} |\langle \Gamma^* f, Qe_n \rangle|^2 \\
&= \sum_{n=1}^{\infty} |\langle f, \Gamma Q e_n \rangle|^2 \\
&= \sum_{n=1}^{\infty} |\langle f, T^* Q e_n \rangle|^2 \\
&= \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \\
&= \|f\|^2
\end{aligned}$$

Thus Γ is a unitary operator. It follows that

$$u_n = f_n \oplus Q^{\perp} e_n = \Gamma Q e_n \oplus Q^{\perp} e_n = U(Qe_n + Q^{\perp} e_n) = U e_n,$$

where $U = \Gamma \oplus I$ is a unitary operator from $\ell^2(\mathbb{N})$ to $\mathcal{K}$. We conclude that $\{u_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{K}$. $\square$

1.2 Frames in Finite-Dimensional Spaces

In the finite-dimensional setting, one can construct frames with infinitely many elements. For example, the sequence

$$\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}\bigcup\left\{\begin{pmatrix}\frac{1}{2^{n/2}}\\0\end{pmatrix}\right\}_{n=1}^{\infty}$$

is a frame for $\mathbb{R}^2$ with upper frame bound $B = 2$ and lower frame bound $A = 1$. But any finite spanning set for a finite-dimensional vector space is a frame for that space. In fact, we have the following theorem.

Theorem 1.2.1. *Let $\mathcal{H}$ be a finite-dimensional vector space and let $\{f_k\}_{k=1}^m$ be a sequence of vectors in $\mathcal{H}$. Then $\{f_k\}_{k=1}^m$ is a frame for $\mathcal{H}$ if and only if $\operatorname{span}\{f_k\}_{k=1}^m = \mathcal{H}$.*

Proof. If $\{f_k\}_{k=1}^m$ does not span $\mathcal{H}$, then we can find a nonzero vector $f \in [\operatorname{span}\{f_k\}_{k=1}^m]^{\perp}$. Thus we have $\sum_{k=1}^m |\langle f, f_k\rangle|^2 = 0$ and so $\{f_k\}_{k=1}^m$ is not a frame for $\mathcal{H}$.

Now suppose that $\{f_k\}_{k=1}^m$ spans $\mathcal{H}$. We may assume that $\mathcal{H}$ is not the trivial vector space and so necessarily not all f_k are zero. By the Cauchy-Schwarz inequality, for any $f \in \mathcal{H}$, we have

$$\sum_{k=1}^m |\langle f, f_k\rangle|^2 \leq \sum_{k=1}^m \|f\|^2\|f_k\|^2 = \left(\sum_{k=1}^m \|f_k\|^2\right)\|f\|^2.$$

Hence we can choose $B = \sum_{k=1}^m \|f_k\|^2 > 0$ as the upper frame bound. We now show the existence of a positive lower frame bound. Consider the continuous mapping $\phi : \mathcal{H} \to \mathbb{R}$, defined for all $f \in \mathcal{H}$, by

$$\phi(f) = \sum_{k=1}^{m} |\langle f, f_k \rangle|^2.$$

Since $\mathcal{H}$ is finite-dimensional, the unit sphere is compact and so ϕ attains a minimum at some unit vector f_0. Call this minimum A. Clearly $A > 0$ and for any $f \in \mathcal{H}$, with $f \neq 0$, we have

$$\sum_{k=1}^{m} |\langle f, f_k \rangle|^2 = \sum_{k=1}^{m} |\langle \frac{f}{\|f\|}, f_k \rangle|^2 \|f\|^2 \geq A\|f\|^2.$$

Thus $\{f_k\}_{k=1}^m$ is a frame for $\mathcal{H}$. □

Theorem 1.2.1 is not true if $\mathcal{H}$ is not finite-dimensional or the sequence is not finite.

Remark 1.2.2. We note that if $\mathcal{H}$ is a finite-dimensional Hilbert space then a sequence $\{x_n\}_{n=1}^\infty$ is a frame for $\mathcal{H}$ if and only if $\{x_n\}_{n=1}^\infty$ contains a basis for $\mathcal{H}$ and $\sum_{n=1}^\infty \|x_n\|^2 < \infty$. To see this, note that if $\{x_n\}_{n=1}^\infty$ contains a basis for $\mathcal{H}$ and $\sum_{n=1}^\infty \|x_n\|^2 < \infty$, then an upper frame bound is obtained as in the proof of Theorem 1.2.1 and a lower frame bound can be obtained by defining the function ϕ for any finite subsequence of $\{x_n\}_{n=1}^\infty$ which contains a basis for $\mathcal{H}$. For the other direction, we first suppose that $\{x_n\}_{n=1}^\infty$ is a Parseval frame for $\mathcal{H}$. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis for $\mathcal{H}$. Then

$$\begin{aligned} m = \sum_{i=1}^{m} \|e_i\|^2 &= \sum_{i=1}^{m} \sum_{n=1}^{\infty} |\langle e_i, x_n \rangle|^2 \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{m} |\langle x_n, e_i \rangle|^2 \\ &= \sum_{n=1}^{\infty} \|x_n\|^2. \end{aligned}$$

If $\{x_n\}_{n=1}^{\infty}$ is not a Parseval frame then $\{S^{-1/2}x_n\}_{n=1}^{\infty}$ is a Parseval frame. Furtheremore, we have

$$\|x_n\|^2 = \|S^{1/2}S^{-1/2}x_n\|^2 \leq \|S^{1/2}\|\|S^{-1/2}x_n\|^2.$$

Since $\sum_{n=1}^{\infty} \|S^{-1/2}x_n\|^2$ is convergent, the desired result follows from the comparison test. In fact, we know that $\sum_{n=1}^{\infty} \|x_n\|^2 \leq m\|S^{-1/2}\|^2$.

Because of Theorem 1.2.1 we only consider frames consisting of finite sequences in the finite-dimensional setting. Theorem 1.1.4 tells us that if $\{f_k\}_{k=1}^{m}$ is a frame for $\mathcal{H}$, then its frame operator S is self-adjoint and positive. As such all the eigenvalues of S are real and positve. We have the following theorem.

Theorem 1.2.3. *Let $\{f_k\}_{k=1}^{m}$ be a frame for and d-dimensional vector space $\mathcal{H}$. Then the following holds:*

i. The optimal lower frame bound is the smallest eigenvalue for S, and the optimal upper frame bound is the largest eigenvalue.

ii. If $\{\lambda_k\}_{k=1}^{d}$ is the set of eigenvalues for S (taking into account the algebraic multiplicity of each eigenvalue), then

$$\sum_{k=1}^{d} \lambda_k = \sum_{k=1}^{m} \|f_k\|^2$$

.

Proof. (i)Since S is self-adjoint, we know that $\mathcal{H}$ has an orthonormal basis $\{u_k\}_{k=1}^{d}$ consisting of eigenvectors for S. Let $\{\lambda_k\}_{k=1}^{d}$ be the corresponding eigenvalues. Then for every $f \in \mathcal{H}$, we have

$$f = \sum_{k=1}^{d} \langle f, u_k \rangle u_k$$

and so

$$Sf = \sum_{k=1}^{d} \langle f, u_k \rangle S u_k = \sum_{k=1}^{d} \lambda_k \langle f, u_k \rangle u_k.$$

This implies that

$$\sum_{k=1}^{m} |\langle f, f_k \rangle|^2 = \langle Sf, f \rangle = \sum_{k=1}^{d} \lambda_k |\langle f, u_k \rangle|^2.$$

From the above, we conclude that

$$\lambda_{\min} \|f\|^2 \leq \sum_{k=1}^{m} |\langle f, f_k \rangle|^2 \leq \lambda_{\max} \|f\|^2.$$

Hence $\lambda_{\min}$ is a lower frame bound and $\lambda_{\max}$ is an upper frame bound. Now if $A > \lambda_{\min}$ and u_j is any eigenvector corresponding to $\lambda_{\min}$ then $\sum_{k=1}^{m} |\langle u_j, f_k \rangle|^2 = \lambda_{\min} \|u_j\|^2 < A \|u_j\|^2$. Hence A cannot be a lower frame bound for $\{f_k\}_{k=1}^{m}$. We conclude that $\lambda_{\min}$ is the optimal lower frame bound. Similarly, $\lambda_{\max}$ is the optimal upper frame bound.

(ii) We have

$$\begin{aligned}
\sum_{k=1}^{d} \lambda_k &= \sum_{k=1}^{d} \lambda_k \|u_k\|^2 \\
&= \sum_{k=1}^{d} \langle S u_k, u_k \rangle \\
&= \sum_{k=1}^{d} \sum_{l=1}^{m} |\langle u_k, f_l \rangle|^2 \\
&= \sum_{l=1}^{m} \sum_{k=1}^{d} |\langle u_k, f_l \rangle|^2 \\
&= \sum_{l=1}^{m} \|f_l\|^2.
\end{aligned}$$

□

1.3 Frames in $\mathbb{F}^d$ and Matrices

In this section we will let $\mathcal{H} = \mathbb{F}^d$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Theorem 1.1.11 takes the following form.

Theorem 1.3.1 (Dilation Property). *Let $\{f_k\}_{k=1}^m$ be a frame for $\mathbb{F}^d$. Then there exist vectors g_k, $1 \leq k \leq m$, in $\mathbb{F}^{m-d}$ such that the vectors $\{\tilde{f}_k\}_{k=1}^m$, where*

$$\tilde{f}_k = \begin{bmatrix} f_k \\ g_k \end{bmatrix},$$

constitute a basis for $\mathbb{F}^m$. If $\{f_k\}_{k=1}^m$ is a Parseval frame then the g_k's can be chosen so that $\{\tilde{f}_k\}_{k=1}^m$ is an orthonormal basis for $\mathbb{F}^m$.

Proof. Consider the matrix

$$F = [f_1 \; f_2 \; \cdots \; f_m] \, .$$

Since $\{f_k\}_{k=1}^m$ spans $\mathbb{C}^d$ we have that rank$(F) = d$. Thus, the rows of F, viewed as vectors in $\mathbb{C}^m$, constitute a linearly independent set. We can therefore extend them to a basis for $\mathbb{C}^m$ by adjoining $m - d$ additional vectors. In doing so, we obtain in an $m \times m$ matrix

$$\tilde{F} = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ g_1 & g_2 & \cdots & g_m \end{bmatrix}.$$

Since rank$(\tilde{F}) = m$, its columns are linearly independent and therefore $\{\tilde{f}_k\}_{k=1}^m$ is a basis for $\mathbb{C}^m$.

Now suppose that $\{f_k\}_{k=1}^m$ is a Parseval frame. Let f^* denote the conjugate transpose $\bar{f}^T$ of f. Then $\langle f, g \rangle = g^* f$. For every f in $\mathbb{C}^m$, we have

$$\begin{aligned} \|f\|^2 &= \sum_{k=1}^m |\langle f, f_k \rangle|^2 \\ &= \sum_{k=1}^m \langle f, f_k \rangle \langle f_k, f \rangle \\ &= \sum_{k=1}^m f^* f_k f_k^* f \\ &= f^* \left(\sum_{k=1}^m f_k f_k^* \right) f. \end{aligned}$$

This holds for every $f \in \mathbb{C}^d$ and it follows that $\sum_{k=1}^m = I_d$ the $d \times d$ identity matrix. If $f_k(j)$ denotes the j^{th} entry of the vector f_k then we obtain that $\sum_{k=1}^m f_k(j)\overline{f_k(l)} = \delta_{jl}$. But $\sum_{k=1}^m f_k(j)\overline{f_k(l)} = \delta_{jl}$ is the inner product between the j^{th} and l^{th} rows of F. So the rows of F

are othormal and so by they Gram-Schmidt algorithm we can extend them to an orthonormal basis for $\mathbb{C}^m$. This concludes the proof. $\square$

Given a frame $\{f_k\}_{k=1}^m$ for $\mathbb{C}^d$, the matrix representation for its analysis operator with respect to the standard orthonormal basis is the $m \times d$ matrix

$$\Theta = \begin{bmatrix} \leftarrow f_1^* \rightarrow \\ \leftarrow f_2^* \rightarrow \\ \vdots \\ \leftarrow f_m^* \rightarrow \end{bmatrix}.$$

Consequently, the matrix representation for its synthesis operator with respect to the standard orthonormal basis is the $d \times m$ matrix

$$\Theta^* = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ f_1 & f_2 & \cdots & f_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Given a matrix F with at least d columns. The following proposition gives a criterion to determine when its column vectors constitute a frame for $\mathbb{F}^d$.

Proposition 1.3.2. *Let $F = [f_1\ f_2 \cdots f_m]$ be an $d \times m$ matrix with column vectors f_k.*

i. $\{f_k\}_{k=1}^m$ *is a frame for* $\mathbb{F}^d$ *if and only if* $\operatorname{rank}(F) = d$.

ii. $\{f_k\}_{k=1}^m$ *is a tight frame for* $\mathbb{F}^d$ *if and only if the rows for* F *are orthogonal and have the same norm. Furthermore,* $\{f_k\}_{k=1}^m$ *is a Parseval frame for* $\mathbb{F}^d$ *if and only if the rows are orthonormal.*

Proof. (i) $\{f_k\}_{k=1}^m$ is a frame for $\mathbb{F}^d$ if and only if $\text{span}(\{f_k\}_{k=1}^m) = \mathbb{F}^m$ and this is true if and only if $\text{rank}(F) = d$.

(ii)$\{f_k\}_{k=1}^m$ is a tight frame with frame bound A if and only if its analysis operator F^* is a scalar multiple of an isometry. That is, if and only if $F^* = AV$ for some isometry V. The columns of V are orthonormal and since the rows of F are obtained by multiplying the columns of V by A we obtain that the rows of F are orthogonal and have the same norm. If $\{f_k\}_{k=1}^m$ is a Parseval frame then $A = 1$ and the result follows immediately from the previous argument. $\square$

1.4 Application of Finite Frames to Signal Processing

One of the most common applications of frames is in signal processing. In this setting we would like to send a signal $f \in \mathcal{H}$, where $\mathcal{H}$ is finite-dimensional, from a transmitter $\mathbf{T}$ to a receiver $\mathbf{R}$. If $\{f_k\}_{k=1}^m$ is a frame for $\mathcal{H}$ we can do this by encoding $\{f_k\}_{k=1}^m$ into both $\mathbf{T}$ and $\mathbf{R}$ and then transmits the coefficients $\{\langle f, f_k\rangle\}_{k=1}^m$ from $\mathbf{T}$ to $\mathbf{R}$. The signal f can then be reconstructed using the formula

$$f = \sum_{k=1}^m \langle f, f_k\rangle S^{-1} f_k.$$

Two complications that may arise during this process are *noises* and *erasures*. Noises occur when at least one of the coefficients is perturbed: $\mathbf{R}$ receives the coefficients $\{\langle f, f_k\rangle + c_k\}_{k=1}^m$. Erasure occurs when at least of the coefficients is deleted. Both complications may take place at the same time but we will look at them separately.

First, note that the overcompleteness of $\{f_k\}_{k=1}^m$ gives it some advantages over a basis. In the case of noises, the reconstructed signal will be given by

$$\sum_{k=1}^{m}(\langle f, f_k\rangle + c_k)S^{-1}f_k = \sum_{k=1}^{m}\langle f, f_k\rangle S^{-1}f_k + \sum_{k=1}^{m} c_k S^{-1} f_k$$
$$= f + S^{-1}\sum_{k=1}^{m} c_k f_k$$

Now if $\{f_k\}_{k=1}^m$ is linearly dependent (overcomplete) there are nonzero coefficients c_k such that $\sum_{k=1}^m c_k f_k = 0$. Thus the noise may not have any effect at all on the reconstruction of the signal. This would never happen with an orthonormal basis.

In the case of erasure, we see that once again overcompleteness is an advantage because the frame $\{f_k\}_{k=1}^m$ may still remain a spanning set even when some of its vectors have been removed. In the best case scenerio we may remove any $m-\dim\mathcal{H}$ elements of the frame and still get a spanning set. For example, this is true when the frame is obtained by compressing a discrete Fourier transform basis. In a slightly more general case, we have the following proposition.

Proposition 1.4.1. *Let $\{f_n\}_{k=1}^m$ for $\mathcal{H}$ with lower frame bound $A > 1$. Suppose that $\|f_k\| = 1$, for $1 \leq k \leq m$. Then for any $J \subset \{1, ..., m\}$ with $|J| < A$, the family $\{f_k\}_{k\notin J}$ is a frame for $\mathcal{H}$ with lower bound $A - |I|$.*

Proof. For $f \in \mathcal{H}$, we have

$$\sum_{k\in J} |\langle f, f_k\rangle| \leq \sum_{k\in J} \|f_k\|^2\|f\|^2 = |J|\|f\|^2.$$

Thus

$$(A - |J|)\|f\|^2 \leq \sum_{k\notin J} |\langle f, f_k\rangle|^2.$$

$\square$

Now going back to signal transmission we see that in order to reconstruct f using the coefficients $\{\langle f, f_k\rangle\}_{k=1}^m$ we need to compute the inverse S^{-1} of the frame operator. This may get costly if the dimension of $\mathcal{H}$ or the condition number of S (the ratio between its largest and smallest eigenvalue) is very large. The next result is a numerical scheme that allows us to get better and better approximation of f without computing S^{-1}.

Proposition 1.4.2 (Frame Algorithm). *Let $\{f_k\}_{k=1}^m$ be a frame for $\mathcal{H}$ with frame bounds A and B. Given $f \in \mathcal{H}$, consider the sequence $\{g_n\}_{n=0}^\infty$ defined recursively by*

$$g_0 = 0, \quad g_n = g_{n-1} + \frac{2}{A+B} S(f - g_{n-1}), n \geq 1.$$

Then $\{g_n\}_{n=0}^\infty$ converges to f and the convergence speed is given by

$$\|f - g_n\| \leq \left(\frac{B-A}{B+A}\right)^n \|f\|.$$

Proof. Let I be the identity operator on $\mathcal{H}$. Then for any $f \in V$, we have

$$\langle (I - \frac{2}{A+B} S) f, f\rangle = \|f\|^2 - \frac{2}{A+B} \sum_{k=1}^m |\langle f, f_k\rangle|^2.$$

By the lower frame bound condition we get

$$\langle (I - \frac{2}{A+B} S) f, f\rangle \leq \|f\|^2 - \frac{2}{A+B}\|f\|^2 = \frac{B-A}{A+B}\|f\|^2.$$

Similarly, by using the upper frame bound condition, we get

$$-\frac{B-A}{B+A} \leq \langle (I - \frac{2}{A+B}S)f, f\rangle.$$

Therefore,

$$\left\| I - \frac{2}{A+B} \right\| \leq \frac{B-A}{B+A}.$$

Now we have

$$\begin{aligned} f - g_n &= f - g_{n-1} - \frac{2}{A+B}S(f - g_{n-1}) \\ &= \left(I - \frac{2}{A+B}S \right)(f - g_{n-1}). \end{aligned}$$

Repeating this argument, we obtain

$$f - g_n = \left(I - \frac{2}{A+B} \right)^n (f - g_0).$$

It follows that

$$
\begin{aligned}
\|f - g_n\| &= \left\| \left(I - \frac{2}{A+B} S \right)^n (f - g_0) \right\| \\
&\leq \left\| I - \frac{2}{A+B} S \right\|^n \|f\| \\
&\leq \left(\frac{B-A}{B+A} \right)^n \|f\|.
\end{aligned}
$$

This completes the proof. □

CHAPTER 2: PHASE RETRIEVAL

Phase retrieval tackles the problem of recovering a signal after loss of phase. One of the earliest instances of the phase problem comes from X-ray crystallography, where the goal is to determine the structure of a crystal based on diffraction images. The structure of the crystal depends on the distribution of the electrons in the crystal and this can be represented by an electron density function. Thus once we know the electron density function we can determnine the structure of the crystal. However, the diffraction image only shows intensity measurements – the number of X-ray photons in a given spot. These can be used to compute the magnitude of the fourier transform of the density function but its phase is lost.

Today, phase retrieval arises in many different settings such as speech recognition, astronomy, and coherent diffractive imaging. Phase retrieval using frames was introduced by Balan, Casazza, and Edidin in 2006 [BCE06].

2.1 Phase-Retrievable Frames

Definition 2.1.1. Let $\mathcal{H}$ be a Hilbert space and let $\{f_n\}_{n=1}^{\infty}$ be a frame for $\mathcal{H}$. We say that $\{f_n\}_{n=1}^{\infty}$ *does phase retrieval* for $\mathcal{H}$ if whenever $h, g \in \mathcal{H}$ and $|\langle h, f_n\rangle| = |\langle g, f_n\rangle|$ for every n then $h = \alpha g$, where $|\alpha| = 1$.

A frame that does phase retrieval for a Hilbert space will often be referred to as a **phase-retrievable frame** or, simply, a **PR-frame**. The numbers $|\langle h, f_n\rangle|$ are called **intensity measurements** (or *measurements*, for short). We provide a quick example.

Example 2.1.2. Let $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $x_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We claim that the frame $\{x_i\}_{i=1}^{3}$ does

phase retrieval for $\mathbb{R}^2$. To see this, suppose we have

$x = \begin{bmatrix} a \\ b \end{bmatrix}, y = \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$ with $|\langle x, x_i \rangle| = |\langle y, x_i \rangle|$ for $1 \leq i \leq 3$. This means that $|a| = |c|, |b| = |d|$ and $|a+b| = |c+d|$. If $a = c$, then $|c+b| = |c+d|$. So either $c+b = c+d$, which implies that $b = d$, or $c+b = -c-d$. This last equality gives us $2c+b = -d$. By the triangle inequality, we obtain

$$|d| \leq |2c| + |b|.$$

Since $|b| = |d|$ we must have $2c = 0$ and so $b = -d$. In either cases, we see that $x = \pm y$.

Now suppose that $a = -c$. Then $|-c+b| = |c+d|$. If $-c+b = c+d$, then $b = 2c+d$ and $|b| \leq |2c| + |d|$. We must have $c = 0$ which gives us $b = d$. If $-c+b = -c-d$ then $b = -d$. Again we see that $x = \pm y$. So $\{x_i\}_{i=1}^3$ does phase retrieval for $\mathbb{R}^2$.

Remark 2.1.3. The theory of phase retrieval in real Hilbert spaces is much simpler than the theory of phase retrieval in complex Hilbert spaces. This is due to the fact that in the real case the phase factors consist only of the two-element set $\{\pm 1\}$ while in the complex case the phase factors consist of the uncountably infinite set $\mathbb{T}$. Because of this, the phase retrievable frames in real Hilbert spaces have been characterized by the complement property (to be defined), which, in the finite-dimensional case, provides a computionally finite method for determining whether or not a real Hilbert space frame does phase retrieval. At the time of this writing there is no computationally finite method to determine if a complex Hilbert space frame does phase retrieval.

2.2 Existence and Characterization of Phase-Retrievable Frames

Example 2.1.2 gives a frame that does phase retrieval for $\mathbb{R}^2$. In this section we will discuss some existence theorems for phase-retrievable frames; we will also present two characterization theorems. The interested reader can find more details in [BCE06], and [BCMN14]. The discussion will be restricted to the finite-dimensional setting.

Definition 2.2.1. A frame $\{f_n\}_{n=1}^m$ for a Hilbert space $\mathcal{H}$ is said to have the *complement property* if for any subset Λ of $\{1, ..., m\}$ we have either $\text{span}\{f_n : n \in \Lambda\} = \mathcal{H}$ or $\text{span}\{f_n : n \notin \Lambda\} = \mathcal{H}$.

The proof of the following theorem can be found in [BCE06].

Theorem 2.2.2. *The complement property is necessary for a frame to be phase-retrievable. It is also sufficient for real Hilbert spaces.*

The following example shows that the complement property is not enough in the complex Hilbert space case.

Example 2.2.3. Let $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $x_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The frame $\{x_i\}_{i=1}^3$ clearly has the complement property but does not do phase retrieval for $\mathbb{C}^2$. To see this consider the two vectors $x = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $y = \begin{bmatrix} i \\ 1 \end{bmatrix} \in \mathbb{C}^2$. Then y cannot be obtained from x by multiplying by a unimodular factor. But a quick computation shows that $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for $1 \leq i \leq 3$. Hence $\{x_i\}_{i=1}^3$ does not do phase retrieval for $\mathbb{C}^2$.

To state the existence theorem we need some preliminary results. The following proposition from [BCE06] is not hard to verify.

Proposition 2.2.4. *Let $\{f_n\}_{n=1}^m$ be a phase-retrievable frame for a Hilbert space $\mathcal{H}$. If T is an invertible operator on $\mathcal{H}$ then $\{Tf_n\}_{n=1}^m$ is a phase-retrievable frame for $\mathcal{H}$.*

Two frames $\{f_n\}_{n=1}^m$ and $\{g_n\}_{n=1}^m$ for $\mathcal{H}$ are said to be **equivalent** if there exists an invertible operator T such that $Tf_n = g_n$, for $1 \leq n \leq m$.

The proof of the following theorem can be found [HKLW07].

Theorem 2.2.5. *Let $\{f_n\}_{n=1}^m$ and $\{g_n\}_{n=1}^m$ be two frames for $\mathcal{H}$ and $\mathcal{K}$, respectively. Then they are equivalent if and only if their analysis operators have the same range.*

We will use the following lemma to help state the existence theorem for phase retrievable frames.

Lemma 2.2.6. *Let $\mathcal{H}$ be an d-dimensional Hilbert space and let W be an d-dimensional subspace of $\mathbb{K}^m$, with $m > d$. Then there exists a frame $\{f_n\}_{n=1}^m$ for $\mathcal{H}$ such that the range of its analysis operator is W.*

Proof. Let P be the orthogonal projection onto W and let $\{e_n\}_{n=1}^m$ be the standard orthonormal basis for $\mathbb{K}^m$. Choose a d-element subset Λ of $\{1, ..., m\}$ such that $\{Pe_n\}_{n\in\Lambda}$ is a basis for W. Pick a basis $\{v_1, ..., v_d\}$ for $\mathcal{H}$ and define an operator

$$B : \mathcal{H} \to \mathbb{K}^m$$

by $B(v_i) = e_{n_i}$, $n_i \in \Lambda$. Since B^* has rank d the sequence $\{B^*Pe_n\}_{n=1}^m$ is a frame for $\mathcal{H}$. Let Γ its analysis operator. Then, for $1 \leq i \leq d$, we have

$$
\begin{aligned}
\Gamma v_i &= \sum_{n=1}^{m} \langle v_i, B^* P e_k \rangle e_n \\
&= \sum_{n=1}^{m} \langle B v_i, P e_n \rangle e_n \\
&= \sum_{n=1}^{m} \langle e_{n_i}, P e_n \rangle e_n \\
&= \sum_{n=1}^{m} \langle P e_{n_i}, e_n \rangle e_n \\
&= P e_{n_i}.
\end{aligned}
$$

Therefore $\Gamma(\mathcal{H}) = W$. □

Now let $\mathcal{H}$ be an d-dimensional Hilbert space and let $\mathcal{F}_{\mathcal{H}}^m$ denote the set of all m-element frames for $\mathcal{H}$. Define an equivalence relation $\sim$ on $\mathcal{F}_{\mathcal{H}}^m$ by using the definition above and let $[\mathcal{F}_{\mathcal{H}}^m] = \mathcal{F}_{\mathcal{H}}^m / \sim$. From Proposition 2.2.4 we see that the set of points in $[\mathcal{F}_{\mathcal{H}}^m]$ corresponding to phase-retrievable frames are precisely the classes where at least one element in the class does phase retrieval. Consider the Grassmannian $\mathbf{Gr}(d, m)$ of $\mathbb{K}^m$. Then by Theorem 2.2.5 and Lemma 2.2.6 there is a one-to-one correspondence between $[\mathcal{F}_{\mathcal{H}}^m]$ and $\mathbf{Gr}(d, m)$. Now recall that $\mathbf{Gr}(d, m)$ is a projective algebraic variety in $\mathbb{KP}^{\binom{m}{d}-1}$; that is, $\mathbf{Gr}(d, m)$ is the common zero set of a collection of homogeneous polynomials in $\binom{m}{d}$ variables, see [SKKT00]. So it inherits the subspace topology from the Zariski topology on $\mathbb{KP}^{\binom{m}{d}-1}$. The correspondence between $[\mathcal{F}_{\mathcal{H}}^m]$ and $\mathbf{Gr}(d, m)$ induces a topology on $[\mathcal{F}_{\mathcal{H}}^m]$: the open sets in $[\mathcal{F}_{\mathcal{H}}^m]$ are inverse images of open sets in $\mathbf{Gr}(d, m)$. The proof of the following theorem can be found in [BCE06]

Theorem 2.2.7. *Let $\mathcal{H}$ be a Hilbert space of dimension d.*

i. If $\mathcal{H}$ is a real Hilbert space and $m \geq 2d - 1$ then the set of points in $[\mathcal{F}_{\mathcal{H}}^m]$ corresponding to

phase-retrievable frames is open dense in $[\mathcal{F}^m_{\mathcal{H}}]$.

ii. *If* $\mathcal{H}$ *is a complex Hilbert space and* $m \geq 4d-2$ *then the set of points in* $[\mathcal{F}^m_{\mathcal{H}}]$ *corresponding to phase-retrievable frames is open dense in* $[\mathcal{F}^m_{\mathcal{H}}]$.

As we saw in example 2.2.3, the complement property does not characterize phase-retrievable frames for complex Hilbert spaces. But it is possible to characterize the complex case by using the PhaseLift operator, even though that method is not finitely computational.

Let $f, g \in \mathcal{H}$. The rank-one operator $f \otimes g$ is defined by $(f \otimes g)k = \langle k, g\rangle f$, for every $k \in \mathcal{H}$. Given a frame $\mathcal{F} = \{f_n\}_{n=1}^m$ for $\mathcal{H}$, consider the family of rank-one Hermitian operators $\{f_n \otimes f_n\}_{n=1}^m$. Identify the operators on $\mathcal{H}$ with the set of matrices $\mathbb{K}^{d\times d}$, endowed with the Hilbert-Schmidt innner product. The mapping $\mathcal{A}_{\mathcal{F}} : \mathbb{K}^{d\times d} \to \mathbb{K}^m$ defined by

$$\mathcal{A}_{\mathcal{F}}(B) = \{\langle B, f_n \otimes f_n\rangle\}_{n=1}^m$$

is called the **PhaseLift operator** (or **super analysis operator**). If $B = x \otimes x$, then we see that $\langle x \otimes x, f_n \otimes f_n\rangle = |\langle x, f_k\rangle|^2$. The proof of the following theorem can be found in [BCMN14].

Theorem 2.2.8. *A frame* $\mathcal{F} = \{f_n\}_{n=1}^m$ *does phase retrieval if and only if the kernel of* $\mathcal{A}_{\mathcal{F}}$ *does not contain any Hermitian matrix of rank 1 or 2. That is,* $\mathcal{F}$ *does phase retrieval if and only if* $\ker(\mathcal{A}_{\mathcal{F}}) \cap \mathcal{S}_2 = \{0\}$, *where* $\mathcal{S}_2$ *is the set of all Hermitian matrices of rank less than or equal to 2.*

2.3 Reconstruction Algorithms and Frames with the Maximal Span Property

In this section we discuss some algorithms that reconstruct a vector up a to unimodular constant, given the absolute values of its frame coefficients. We also discuss conditions under which we can

still perform the reconstruction even after the loss of some of these absolute values. These results can all be found [BBCE09].

Let $\mathcal{F} = \{f_n\}_{n=1}^m$ be a frame for $\mathcal{H}$ and let $S = \{f_n \otimes f_n\}_{n=1}^m$. If span($S$) contains all the rank-one Hermitian operators defined on $\mathcal{H}$ then we say that S has **maximal span**.

Note that if x and y are in $\mathcal{H}$, then $x \otimes x = y \otimes y$ if and only if $x = \alpha y$, where $|\alpha| = 1$. Thus, recovering x up to a unimodular constant is equivalent to recovering the rank-one Hermitian operator $x \otimes x$. But if S has maximal span and we have dual for it, then every rank-one Hermitian operator $x \otimes x$ can be recovered from its inner products with the elements of S. Also since

$$\langle x \otimes x, f_n \otimes f_n \rangle = \mathrm{tr}[(x \otimes x)(f_n \otimes f_n)] = |\langle x, f_n \rangle|^2$$

we see that we can reconstruct x, up to a unimodular constant, from the magnitudes of its frame coefficients.

If $\mathcal{H}$ is a real Hilbert space of dimension d, then the Hermitian operators form a subspace of dimension $d(d+1)/2$. If $\mathcal{H}$ is a complex Hilbert space the Hermitian operators do not form a subspace. Hence in the real case we need $m \geq d(d+1)/2$ for S to have maximal span, while in the complex case we need $m \geq d^2$.

We now give examples of frames whose associated rank-one Hermitian operators have maximal span.

Definition 2.3.1. A tight frame $\mathcal{F} = \{f_n\}_{n=1}^m$ is said to be *2-uniform* if $\|f_n\| = b$ for $1 \leq n \leq m$ and there exists $c > 0$ such that for all $1 \leq j, n \leq m$, $j \neq k$, we have $|\langle f_j, f_n \rangle| = c$.

Example 2.3.2. An example of a 2-uniform tight frame in $\mathbb{R}^2$ is the Mercedes-Benz frame $\{x_1, x_2, x_3\}$

given by

$$\left\{ \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right\}$$

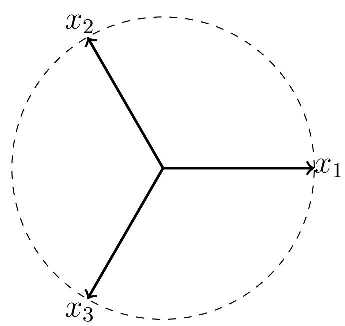

Figure 2.1: The Mercedes Benz frame in $\mathbb{R}^2$.

This is a 2-uniform Parseval frame. We have the following proposition.

Proposition 2.3.3. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a 2-uniform tight frame. Then $S = \{f_k \otimes f_k\}_{k=1}^m$ has maximal span if and only if $m = d(d+1)/2$ in the real case or $m = d^2$ in the complex case.*

Definition 2.3.4. Le $\mathcal{H}$ be a Hilbert space of dimension d. A family of vectors $\{e_k^{(j)}\}$, where $1 \le k \le d$ and $1 \le j \le l$, is said to form l *mutually unbiased bases* for $\mathcal{H}$ if for all j, j' and k, k' we have

$$|\langle e_k^j, e_{k'}^{j'} \rangle| = \delta_{k,k'}\delta_{j,j'} + \frac{1}{\sqrt{d}}(1 - \delta_{j,j'})$$

We have the following proposition due to Delsarte, Goethals, and Seidel.

Proposition 2.3.5. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{F}$ form $[d/2]+1$ mutually unbiased bases in the real case or $d+1$ mutually unbiased bases in the complex case. Then the associated rank-one operators S have maximal span.*

We now present the reconstruction theorem.

Theorem 2.3.6. *Let $\mathcal{H}$ be a d-dimensional Hilbert sapce and let $\mathcal{F} = \{f_n\}_{n=1}^m$ be a m/d-tight frame such that the associated rank-one Hermtian operators S has maximal span. Let M denote the pseudo-inverse of the Grammian G. If $\{R_n\}_{n=1}^m$ is the canonical dual for S then for every $x \in \mathcal{H}$*

$$x \otimes x = \sum_{n=1}^{m} |\langle x, f_n \rangle|^2 R_n.$$

Corollary 2.3.7. *If $\mathcal{F}$ is a 2-uniform m/d-tight frame formed by mutually unbiased bases and S has maximal span then for every $x \in \mathcal{H}$*

$$x \otimes x = \frac{d(d+1)}{m} \sum_{n=1}^{m} |\langle x, f_n \rangle|^2 (f_n \otimes f_n - I/(d+1)).$$

In application we may loose some of the coefficients. The following theorem provide conditions under which we may still be able to perform the reconstruction.

Definition 2.3.8. Let $\{f_n\}_{n=1}^m$ be a frame for $\mathcal{H}$ and let $S = \{f_n \otimes f_n\}_{n=1}^m$ be the associated rank-one Hermitian operators. An erasure of coefficients indexed by $\Lambda \subset \Omega = \{1, ..., m\}$ is *correctible* if span$(\{f_n \otimes f_n\}_{n \in \Omega \setminus \Lambda}) =$ span$(\{f_n \otimes f_n\}_{n \in \Omega})$.

Theorem 2.3.9. *Let $\mathcal{H}$ be a Hilbert space of dimension d. Let $\{e_k^{(j)}\}$, where $1 \leq k \leq d$ and $1 \leq j \leq l$, be a family of l mutually unbiased bases such that $\{e_k^{(j)} \otimes e_k^{(j)}\}$ has maximal span. If*

we lose at most one vector from each family and there is a family with no loss, then the remaining rank-one operators have maximal span.

CHAPTER 3: EXACT PHASE-RETRIEVABLE FRAMES

3.1 Redundancy with Respect to Phase Retrievability

For certain phase-retrievable frames, it is possible to lose some of the frame elements and still obtain a phase-retrievable frame. While that's a desirable property, we may also be interested in minimizing the number of measurements (due to computational cost). In this chapter we introduce a way to measure redundancy with respect to phase retrievability. More precisely, we will study the concept of exact phase-retrievable frames. The results in this chapter are taken from [HJLS].

Theorem 2.2.8 indicates that $\ker(\mathcal{A}_{\mathcal{F}}) \cap \mathcal{S}_2$ is a good candidate to measure the phase-retrievability for a frame $\mathcal{F}$. This motivates the fowllowing definition.

Definition 3.1.1. Given a frame $\mathcal{F} = \{f_n\}_{n=1}^m$ for $\mathcal{H}$. Let k be the smallest integer such that there exists a subset Λ of $\{1, ..., m\}$ with the property that $|\Lambda| = k$ and

$$\ker(\mathcal{A}_{\mathcal{F}_\Lambda}) \cap \mathcal{S}_2 = \ker(\mathcal{A}_{\mathcal{F}}) \cap \mathcal{S}_2.$$

Then we call m/k the *PR-redundancy* of $\mathcal{F}$. A frame is said to have the *exact PR-redundancy property* if its PR-redundancy is 1, and a phase-retrievable frame with the exact PR-redundancy will be called an *exact phase-retrievable frame*.

Given a frame $\mathcal{F} = \{f_n\}_{n=1}^m$ for $\mathcal{H}$. Three conclusions follow immediately from the above definition: (i) There exists a subset Λ of $\{1, ..., m\}$ such that $\mathcal{F}_\Lambda$ is a frame for $\mathcal{H}$ with the exact PR-redundancy property; (ii) $\mathcal{F}$ has the exact PR-redundancy property if and only if for any proper subset Λ of $\{1, ..., m\}$, there exist two vectors $x, y \in \mathcal{H}$ such that $|\langle x, f_j\rangle| = |\langle y, f_j\rangle|$ for every $j \in \Lambda$, but $|\langle x, f_i\rangle| \neq |\langle y, f_i\rangle|$ for some $i \in \Lambda^c$; and (iii) If $\mathcal{F}$ is phase-retrievable, then it is an

exact phase-retrievable frame if and only if $\mathcal{F}_\Lambda$ is no longer phase-retrievable for any proper subset Λ of $\{1, ..., m\}$.

In what follows we always assume that $\mathcal{H} = \mathbb{R}^d$ and use $\mathbb{H}_d$ to denote the space of all the $d \times d$ Hermitian matrices.

Lemma 3.1.2. *If a frame $\mathcal{F} = \{f_n\}_{n=1}^m$ for $\mathbb{R}^d$ has the exact PR-redundancy property, then $\{f_n \otimes f_n\}_{n=1}^m$ is linearly independent (and hence $m \le \dim \mathbb{H}_d = d(d+1)/2$). The converse is false.*

Proof. If $\{f_n \otimes f_n\}_{n=1}^m$ is linearly dependent, then there exists a proper subset Λ of $\{1, ..., m\}$ such that $\operatorname{span}\{f_n \otimes f_n : n \in \Lambda\} = \operatorname{span}\{f_n \otimes f_n : 1 \le n \le m\}$. This implies that $\ker(\mathcal{A}_{\mathcal{F}_\Lambda}) = \ker(\mathcal{A}_{\mathcal{F}})$. Hence $\mathcal{F}$ does not have the exact PR-redundancy property. Therefore $\{f_n \otimes f_n\}_{n=1}^m$ is linearly independent.

Let $d \ge 3$. Then $2d-1 < d(d+1)/2$. Let $\{f_1, ..., f_{2d-1}\}$ be a phase-retrievable frame for $\mathcal{H}$. Then $\{f_n \otimes f_n\}_{n=1}^{2d-1}$ is linearly independent. Since $\dim \mathbb{H}_d = d(d+1)/2$ and $\operatorname{span}\{f \otimes f : f \in \mathcal{H}\} = \mathbb{H}_d$, we can extend $\{f_n \otimes f_n\}_{n=1}^{2d-1}$ to a basis $\{f_n \otimes f_n\}_{n=1}^{d(d+1)/2}$ for $\mathbb{H}_d$. But clearly $\mathcal{F} = \{f_n\}_{n=1}^{d(d+1)/2}$ does not have the exact PR-redundancy. $\square$

Lemma 3.1.2 immediately implies the following length bound for exact phase-retrievable frames.

Corollary 3.1.3. *If $\mathcal{F} = \{f_n\}_{n=1}^m$ is an exact phase-retrievable frame for $\mathbb{R}^d$, then $2d - 1 \le m \le d(d+1)/2$.*

Our main result will show that every m between $2d-1$ and $d(d+1)/2$ is attainable. But first we make the following observation.

For a given frame $\mathcal{F} = \{f_n\}_{n=1}^m$, the **spark** of $\mathcal{F}$ is the cardinality of the smallest linearly dependent subset of the frame. A **full-spark frame** is a frame whose spark is $d+1$; that is, every subcollection

of $\mathcal{F}$ consisting of d vectors is linearly independent. It is known that for each $m \geq d$, the set of full-spark frames of length m is open and dense in the direct sum space $\mathcal{H}^{(m)} := \mathcal{H} \oplus ... \oplus \mathcal{H}$ (m-copies). It is clear that if $m > 2d-1$ and $\mathcal{F} = \{f_n\}_{n=1}^m$ has the full spark, then m can not be an exact phase-retrievable frame. Therefore the set of exact phase-retrievable frames of length m has measure zero, and so the existence proof of exact phase-retrievable frames is quite subtle, as the proof in the next section shows.

3.2 Existence of Exact Phase-Retrievable Frames

In this section we prove the existence theorem for exact phase-retrievable frames of length m with $2d-1 \leq m \leq d(d+1)/2$.

Theorem 3.2.1. *For every integer m with $2d-1 \leq m \leq d(d+1)/2$, there exists an exact phase-retrievable frame of length m.*

Before giving a proof for the above theorem, we introduce some preliminary results. We use the following notations for matrices: $A(I, J)$ is the submatrix of A consisting of the entries with row indices in I and column indices in J. $A(:, J) = A(\{1, \ldots, d\}, J)$ and $A(i, j) = A(\{i\}, \{j\})$

Lemma 3.2.2. *Let $f(x_1, \ldots, x_d)$ be a polynomial and a_i be independent continuous random variables. Then $f(a_1, \ldots, a_d) \neq 0$ almost surely.*

Proof. The conclusion can be proved by induction on d and we omit the details. $\square$

Lemma 3.2.3. *Let A be a $d \times p$ random matrix such that $\operatorname{rank}(A) = r$ almost surely. Let B be a $(d+1) \times (p+1)$ matrix such that $B(1..d, 1..p) = A$ and $B(d+1, p+1)$ is a continuous random variable which is independent of the entries of A. Then we have $\operatorname{rank}(B) \geq r+1$ almost surely.*

Proof. Let Ω be the sample space. Since A has only finitely many submatrices and $\operatorname{rank}(A) = r$ almost surely, there is a partition $\{\Omega_i\}_{i=1}^m$ of Ω such that for each $1 \leq i \leq m$, there is an $r \times r$ submatrix A_i which is of rank r almost surely on Ω_i. Therefore, the submatrix of A consisting of rows and columns in A_i and the $(d+1)$-th row and the $(p+1)$-th column is of rank $r+1$ almost surely on Ω_i by Lemma 3.2.2. This completes the proof. □

The following lemma can be proved similarly.

Lemma 3.2.4. *Let A be a $d \times p$ random matrix such that $\operatorname{rank}(A) = r \leq d-1$ almost surely. Let a be a d-dimensional vector with entries consisting of continuous independent random variables, which are also independent of the entries of A. Then we have $\operatorname{rank}((A\ a)) = r+1$ almost surely.*

We are ready to give a proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. Since every full-spark frame of length $2d-1$ is an exact PR-frame, we only need to prove the theorem for $2d \leq m \leq d(d+1)/2$. First, we show that for $2d \leq m \leq d(d+1)/2$, there exist $d \times m$ matrices A such that

(P1) A contains the $d \times d$ identity matrix as a submatrix;

(P2) the rest $m-d$ columns of A consisting of independent continuous random variables or zeros and each column contains at least one 0 and two non-zero entries;

(P3) there are exactly d non-zero entries in every row of A;

(P4) for each $1 \leq i \leq d$, there exist mutually different indices $j_1, \ldots, j_d$ such that $a_{i,j_l}, a_{l,j_l} \neq 0$;

(P5) columns of A form an exact PR frame with probability 1.

It is obvious that a phase-retrievable frame which satisfies (P3) is exact. Let us explain (P4) in more details.

Fix some i, say, $i = 1$. By (P3), there exist mutually different indices $j_1, \ldots, j_d$ such that $a_{1,j_l} \neq 0$ for $1 \leq l \leq d$. (P4) says that every row contains a non-zero entry in such columns and different rows correspond to different columns.

Consider the following example,

$$A = \begin{pmatrix} 1 & 0 & 0 & a_{1,4} & a_{1,5} & 0 \\ 0 & 1 & 0 & a_{2,4} & 0 & a_{2,3} \\ 0 & 0 & 1 & 0 & a_{3,5} & a_{3,3} \end{pmatrix}, \tag{3.1}$$

where $a_{i,j}$ are independent continuous random variables. For $i = 1$, set $\{j_1, j_2, j_3\} = \{1, 4, 5\}$. Then we have $a_{1,j_l}, a_{l,j_l} \neq 0$ for $1 \leq l \leq 3$.

It is easy to see that A satisfies (P1)$\sim$ (P5). In other words, such matrix exists for $d = 3$.

Now we assume that such matrix A exists for some d and m with $d \geq 3$. Let us consider the case of $d + 1$. We prove the conclusion in the following four steps.

Fix some $0 \leq k \leq d - 2$. By rearranging columns of A, we can assume that $a_{i,d+i} \neq 0$ for $1 \leq i \leq k$ (for $k = 0$, this says nothing).

(I). There is an $(d + 1) \times (m + d + 1)$ matrix satisfying (P1) $\sim$ (P5).

Define the $(d+1) \times (m+d)$ matrix B as follows,

$$B = \begin{pmatrix} & a_{1,m+1} & 0 & & 0 & 0 \\ & 0 & a_{2,m+2} & & 0 & 0 \\ A & 0 & 0 & \cdots & 0 & 0 \\ & & \cdots & & & \\ & 0 & 0 & & a_{d,m+d} & 0 \\ 0 \ldots 0 & a_{d+1,m+1} & a_{d+1,m+2} & & a_{d+1,m+d} & 1 \end{pmatrix}.$$

where all the symbols $a_{i,j}$ are independent continuous random variables. It is easy to see that B meets (P1) $\sim$ (P4). It remains to prove that (P5) holds for B.

Take some $J \subset \{1, \ldots, m+d+1\}$. Set

$$\begin{aligned} J^c &= \{1 \leq j \leq m+d+1 : j \notin J\}, \\ J|_m &= \{j \in J : j \leq m\}, \\ J^c|_m &= \{j \in J^c : j \leq m\}, \end{aligned}$$

Without loss of generality, we assume that $m+d+1 \in J^c$.

Suppose that $\mathrm{rank}(B(:, J^c)) < d+1$ on some sample set Ω' which is of positive probability. Since $m+d+1 \in J^c$, we have $\mathrm{rank}(A(:, J^c|_m)) < d$ a.s. on Ω'. Consequently, $\mathrm{rank}(A(:, J|_m)) = d$ a.s. on Ω'.

On the other hand, Since $m+d+1 \in J^c$, not all of $m+1, \ldots, m+d$ are contained in J^c. Otherwise, $\mathrm{rank}(B(:, J^c)) = d+1$ a.s. on Ω'. Hence there is some $1 \leq i \leq d$ such that $m+i \in J$. By Lemma 3.2.3, $\mathrm{rank}(B(:, J)) = d+1$ a.s. on Ω'.

(II). There is an $(d+1) \times (m+d)$ matrix satisfying (P1) $\sim$ (P5).

Since A satisfies (P2), by rearranging columns of A, we may assume that $A(:, m) = (0, a_{2,m}, \ldots)^t$, where at least two entries are non-zero. Define the $(d+1) \times (m+d)$ matrix B as follows,

$$B = \begin{pmatrix} & 0 & a_{1,m+1} & 0 & & 0 & 0 \\ & a_{2,m} & a_{2,m+1} & 0 & & 0 & 0 \\ \cdots & * & 0 & a_{3,m+2} & \cdots & 0 & 0 \\ & & & \cdots & & & \\ & * & 0 & 0 & & a_{d,m+d-1} & 0 \\ & a_{d+1,m} & a_{d+1,m+1} & a_{d+1,m+2} & & a_{d+1,m+d-1} & 1 \end{pmatrix}.$$

Again, we only need to prove that (P5) holds for B.

As in Step I, we take some $J \subset \{1, \ldots, m+d\}$. We suppose that $m+d \in J^c$ and that $\operatorname{rank}(B(:, J^c)) < d+1$ on some sample set Ω' which is of positive probability. Then we have $\operatorname{rank}(A(:, J|_m)) = d$ a.s. on Ω'.

If there is some $1 \leq i \leq d$ such that $m+i \in J$, then we have $\operatorname{rank}(B(:, J)) = d+1$ a.s. on Ω', thanks to Lemma 3.2.3.

Next we assume that $m+i \in J^c$ for $1 \leq i \leq d$. Since $\operatorname{rank}(B(:, J^c)) < d+1$ a.s. on Ω', for any $j \leq m$ with $A(1, j) \neq 0$, we have $j \in J$, thanks to Lemma 3.2.2. Similarly we get that $m \in J$.

By setting $i = 1$ in (P4), we get mutually different $1 \leq j_1, \ldots, j_d \leq m$ such that $A(1, j_l), A(l, j_l) \neq 0$. Hence $j_1, \ldots, j_d \in J|_m$. Moreover, $\operatorname{rank}(A(:, \{j_1, \ldots, j_d\})) = d$ a.s. on Ω', thanks to Lemma 3.2.2. Note that $m \in J|_m$ and $m \neq j_l$ for $1 \leq l \leq d$. By Lemma 3.2.3, we have

$$\operatorname{rank}(B(:, \{j_1, \ldots, j_d, m\})) = d+1, \qquad \text{a.s. on } \Omega'.$$

Hence

$$\operatorname{rank}(B(:,J)) \geq \operatorname{rank}(B(:,\{j_1,\ldots,j_d,m\})) = d+1, \qquad \text{a.s. on } \Omega'.$$

(III). There is an $(d+1)\times(m+2)$ matrix satisfying (P1) $\sim$ (P5).

By rearranging columns of A, we may assume that

(1) $A(:,\{1,\ldots,d\})$ is the $d \times d$ identity matrix (P1),

(2) $A(d,m)=0$ and there are at least two non-zero entries in the m-th column (P2),

(3) $A(i,m-i), A(d,m-i) \neq 0$ for $1 \leq i \leq d-1$ (P4).

Without loss of generality, we assume that $A(d-1,m)\neq 0$. Define the $(d+1)\times(m+2)$ matrix B as follows,

$$B = \begin{pmatrix} & & * & & * & a_{1,m-1} & * & a_{1,m+1} & 0 \\ & & * & & a_{2,m-2} & * & * & a_{2,m+1} & 0 \\ I_{d\times d} & & * & & * & * & * & a_{3,m+1} & 0 \\ & \cdots & & \cdots & & & & \cdots & \\ & & a_{d-1,m-d+1} & & * & * & * & & 0 \\ & & a_{d,m-d+1} & & a_{d,m-2} & a_{d,m-1} & 0 & a_{d,m+1} & 0 \\ 0\ldots 0 & & a_{d+1,m-d+1} & & a_{d+1,m-2} & a_{d+1,m-1} & a_{d+1,m} & 0 & 1 \end{pmatrix}.$$

Again, we only need to prove that (P5) holds for B.

As in Step I, take some $J \subset \{1,\ldots,m+2\}$ and suppose that $m+2 \in J^c$ and $\operatorname{rank}(B(:,J^c)) < d+1$ on some sample set Ω' which is of positive probability. Then we have $\operatorname{rank}(A(:,J|_m)) = d$ a.s. on Ω'.

There are three cases.

(i). $m+1 \in J^c$

In this case, we conclude that

(a) $\operatorname{rank}(B(1..d, J^c|_m)) \leq d-2$, a.s. on Ω';

(b) there is some $1 \leq j_0 \leq d-1$ such that $m - j_0 \in J$.

In fact, if there is some $\Omega'' \subset \Omega'$ with positive probability such that $\operatorname{rank}(B(1..d, J^c|_m)) = d-1$ a.s. on Ω'', then we see from Lemma 3.2.4 that $\operatorname{rank}(B(1..d, J^c|_m \cup \{m+1\})) = d$ a.s. on Ω''. By Lemma 3.2.3, we get $\operatorname{rank}(B(:, J^c)) = d+1$ a.s. on Ω'', which contradicts with the assumption. This proves (a).

On the other hand, if $m - j \in J^c$ for any $1 \leq j \leq d-1$, then the expansion of the determinant of $B(:, \{m-d+1, m-d+2, \ldots, m-1, m+1, m+2\})$ contains the term $A(d, m+1) \cdot 1 \cdot \prod_{i=1}^{d-1} A(i, m-i)$, which is not zero a.s. By Lemma 3.2.2, $\operatorname{rank}(B(:, J^c)) = d+1$ a.s. on Ω'. Again, we get a contradiction with the assumption. Hence (b) holds.

We see from (a) and (b) that $\operatorname{rank}(B(1..d, J^c|_m \cup \{m-j_0\})) \leq d-1$, a.s. on Ω'. Since A is a PR frame a.s., we have $\operatorname{rank}(B(1..d, J|_m \setminus \{m-j_0\})) = d$ a.s. Now we see from Lemma 3.2.3 that $\operatorname{rank}(B(:, J|_m)) = d+1$ a.s. on Ω'.

(ii). $m+1 \in J$ and $m - j_0 \in J$ for some $0 \leq j_0 \leq d-1$.

Since $\operatorname{rank}(A(:, J|_m)) = d$ a.s. on Ω', by Lemma 3.2.3,

$$\operatorname{rank}(B(\{1, \ldots, d\}, J|_m \cup \{m+1\} \setminus \{m-j_0\})) = d, \qquad \text{a.s. on } \Omega'.$$

Using Lemma 3.2.3 again, we get

$$\operatorname{rank}(B(:, J|_m \cup \{m+1\})) = d+1, \qquad \text{a.s. on } \Omega'.$$

Hence

$$\operatorname{rank}(B(:, J)) = d+1, \qquad \text{a.s. on } \Omega'.$$

(iii). $m+1 \in J$ and $m-j \in J^c$ for any $0 \leq j \leq d-1$.

By (P2), there is some $1 \leq i_0 \leq d-1$ such that $A(i_0, m) \neq 0$. Hence the expansion of the determinant of $B(:, \{m-d+1, m-d+2, \ldots, m, m+2\})$ contains the term $B(d+1, m+2)A(d, m-i_0)A(i_0, m)\prod_{1\leq i\leq d-1, i\neq i_0} A(i, m-i)$, which is not zero a.s. By Lemma 3.2.2, $\operatorname{rank}(B(:, J^c)) = d+1$ a.s. on Ω', which contradicts with the assumption.

(IV). For $2d \leq m \leq d(d+1)/2$, there exist $d \times m$ matrices satisfying (P1) $\sim$ (P5).

Let K_d be the set of all integers k such that there exists an $d \times k$ matrix A satisfying (P1) $\sim$ (P5).

Since $K_3 \supset \{6\}$, we see from the previous arguments that

$$\begin{aligned} K_4 &\supset \{8, 9, 10\}, \\ K_5 &\supset \{10, 11, 12, 13, 14, 15\}. \end{aligned}$$

Hence for $3 \leq d \leq 5$,

$$K_d \supset \{k : 2d \leq k \leq d(d+1)/2\}. \tag{3.2}$$

Now suppose that (3.2) is true for some $d \geq 5$. Since $2d + (d+1) \leq d(d+1)/2 + 2$ for $d \geq 5$, we

have

$$
\begin{aligned}
&\{k+2:\ 2d \le k \le d(d+1)/2\} \cup \{k+d:\ 2d \le k \le d(d+1)/2\} \\
&\qquad \cup \{k+d+1:\ 2d \le k \le d(d+1)/2\} \\
= \ &\{k:\ 2(d+1) \le k \le (d+1)(d+2)/2\}.
\end{aligned}
$$

Hence $K_{d+1} \supset \{k:\ 2(d+1) \le k \le (d+1)(d+2)/2\}$. By induction, (3.2) is true for $d \ge 3$.

Finally, since columns of a randomly generated $d \times (2d-1)$ matrix form an exact PR frame almost surely, we get the conclusion as desired. □

The following are some explicit examples for $d = 5$ and $10 \le m \le 15$. In each case, column vectors of A form an exact PR frame. Moreover, such matrices correspond to exact PR frames almost surely if the non-zero entries are replaced with independent continuous random variables.

$(d, m) = (5, 10)$:

$$
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 6 & 4 & 2 & 11 & 0 \\
0 & 1 & 0 & 0 & 0 & 13 & 10 & 8 & 0 & 3 \\
0 & 0 & 1 & 0 & 0 & 7 & 7 & 0 & 9 & 8 \\
0 & 0 & 0 & 1 & 0 & 16 & 0 & 8 & 30 & 13 \\
0 & 0 & 0 & 0 & 1 & 0 & 4 & 12 & 14 & 18
\end{pmatrix}.
$$

$(d,m) = (5,11)$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 5 & 0 & 3 & 35 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 & 18 & 0 & 14 & 27 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 23 & 5 & 0 & 1 & 14 \\ 0 & 0 & 0 & 1 & 0 & 0 & 8 & 0 & 14 & 7 & 14 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 30 & 3 & 14 \end{pmatrix}.$$

$(d,m) = (5,12)$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 7 & 0 & 10 & 10 & 11 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 4 & 0 & 7 & 16 & 0 & 15 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 16 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 23 & 3 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 12 & 2 & 11 & 0 & 2 \end{pmatrix}.$$

$(d,m) = (5,13)$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 6 & 0 & 4 & 12 & 16 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 6 & 0 & 8 & 5 & 0 & 0 & 15 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 9 & 5 & 0 & 0 & 11 & 12 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 16 & 0 & 6 & 1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 7 & 6 & 0 & 10 & 0 & 9 \end{pmatrix}.$$

$(d,m) = (5,14)$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 11 & 0 & 20 & 0 & 16 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 5 & 0 & 0 & 1 & 16 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 3 & 6 & 0 & 0 & 0 & 13 & 8 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 17 & 0 & 0 & 8 & 8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 3 \end{pmatrix}.$$

$(d,m) = (5,15)$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 12 & 0 & 4 & 0 & 7 & 0 & 13 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 17 & 0 & 0 & 3 & 0 & 10 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 8 & 0 & 0 & 0 & 0 & 12 & 17 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 15 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 13 & 0 & 18 \end{pmatrix}.$$

Remark 3.2.5. In the real case, the minimal length for phase retrievable frames is $2d-1$. But in the complex case, the minimal length varies depending on the demension. For example, it is known that the minimum length satisfies the following inequality [HMW13]

$$m \geq 4d - 2 - 2b(d) + \begin{cases} 2 & \text{if } d \text{ is odd and } b = 3 \bmod 4 \\ 1 & \text{if } n \text{ is odd and } b = 2 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

If $d = 2^k + 1$, then the minimal length is $4d - 4$. Thus determining whether we can have exact phase retrievable frames of any length between the minimum and maximum is a tricky problem.

On a positive note, for any d the d^2-element frames $\{e_1, ..., e_d\} \cup \{e_i + e_j + \sqrt{-1}e_j\}_{1<i<j\leq d}$ are exact.

CHAPTER 4: MAXIMAL PHASE-RETRIEVABLE SUBSPACES

4.1 Phase-Retrievable Subspaces

Frames that do not do phase retrieval can be very useful in applications. For a non-phase-retrievable frame $\mathcal{F}$, researchers have been interested in identifying the subsets of the signal space for which $\mathcal{F}$ can do phase-retrieval. A typical example is the subset of sparse signals (see [EHM16] and [WX14]). To gain a better understanding about frame phase-retrievability, we will look at the problem of identifying the largest subspaces M such that $\mathcal{F}$ does the phase-retrieval for all the signals in M. The results in the first two sections of this chapter are taken from [HJLS].

Definition 4.1.1. Let $\mathcal{F} = \{f_n\}_{n=1}^m$ be a frame for $\mathcal{H}$ and M is a subspace of $\mathcal{H}$. We say that M is a *phase-retrievable subspace* with respect to $\mathcal{F}$ if $\{P_M f_n\}_{n=1}^m$ is a phase-retrievable frame for M, where P_M is the orthogonal projection from $\mathcal{H}$ onto M. A phase-retrievable subspace M is called *maximal* if it is not a proper subspace of any other phase-retrievable subspaces with respect to $\mathcal{F}$.

We will use the abbreviation "$\mathcal{F}$-PR subspace " to denote a phase-retrievable subspace with respect to $\mathcal{F}$. Given a frame $\mathcal{F}$, there are some natural questions that come up about phase-retrievable subspaces. For example: what are possible dimensions k such that there exists a k-dimensional maximal $\mathcal{F}$-PR subspace? What is the largest (or the smallest) dimension for all the maximal $\mathcal{F}$-PR subspaces? Can we characterize all the maximal phase-retrievable subspaces? We will explore the answers to these questions in Section 3 and Section 4.

As a motivating example, we will show that if $\mathcal{F} = \{f_i\}_{n=1}^d$ is a basis for $\mathcal{H}$, then there exists a k-dimensional maximal $\mathcal{F}$-PR subspace if and only if $1 \leq k \leq [(d+1)/2]$. For any general frame $\mathcal{F}$, we will identify the largest k such that there exists a k-dimensional maximal $\mathcal{F}$-PR subspace. This leads to a generalization of the complement property characterization for real phase-retrievable

frames. In the case that $\mathcal{F} = \{f_n\}_{n=1}^d$ is a basis, we show that if M is a $\mathcal{F}$-PR subspace, then the support supp(x) (with respect to the dual basis) of every nonzero vector x in M has the cardinality greater than or equal to k. Moreover, we will prove that for any given vector x with $|\text{supp}(x)| = k$, there exists a k-dimensional maximal $\mathcal{F}$-PR subspace M containing x. This support condition is also necessary in the case that $k < [(d+1)/2]$, i.e, in this case we have that a k-dimensional $\mathcal{F}$-PR subspace M is maximal if and only if there exists an nonzero vector x in M whose support has the cardinality k.

The following simple property will be needed.

Lemma 4.1.2. *Suppose that $\mathcal{H}$ is the direct sum of two subspaces X and Y. If $\mathcal{F}_1$ is a frame for X with the exact PR-redundancy property and $\mathcal{F}_2$ is a frame for Y with the exact PR-redundancy property, then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a frame for $\mathcal{H}$ with the exact PR-redundancy property.*

Proof. By passing to a similar frame we can assume that $Y = X^\perp$. Clearly $\mathcal{F}$ is a frame for $\mathcal{H}$. Now assume that a vector f is removed from $\mathcal{F}_1$. Since $\mathcal{F}_1$ is a frame for X with the exact PR-redundancy property, there exists some nonzero operator $A = u \otimes u - v \otimes v$ with $u, v \in X$ such that $A \in \ker(\mathcal{A}_{\mathcal{F}_1 \setminus \{f\}})$ and $A \notin \ker(\mathcal{A}_{\mathcal{F}_1})$. Since $Y \perp X$, we also have $A \in \ker(\mathcal{A}_{\mathcal{F}_2})$. This implies that $A \in \ker(\mathcal{A}_{\mathcal{F} \setminus \{f\}})$ and $A \notin \ker(\mathcal{A}_{\mathcal{F}})$. The same argument works if we remove one element from $\mathcal{F}_2$. Thus $\mathcal{F}$ has the exact PR-redundancy property.

□

We first prove the following special case.

Theorem 4.1.3. *Let $\mathcal{F} = \{f_n\}_{n=1}^d$ be a basis for $\mathcal{H}$. Then there exists a k-dimensional maximal $\mathcal{F}$-PR subspace if and only if $1 \le k \le [(d+1)/2]$.*

Proof. Suppose that M is a k-dimensional $\mathcal{F}$-PR subspace. Then we have that $d \ge 2k - 1$ and

hence $k \leq (d+1)/2$. For the other direction, note that for each invertible operator T on $\mathcal{H}$, M is a maximal $\mathcal{F}$-PR subspace if and only if $(T^t)^{-1}M$ is a maximal $T\mathcal{F}$-PR subspace. So it suffices to show that for each k-dimensional subspace M with $1 \leq k \leq [(d+1)/2]$ there exists a basis $\{u_n\}_{n=1}^d$ such that M is an maximal PR subspace with respect to $\{u_n\}_{n=1}^d$.

Let $\{\varphi_j\}_{j=1}^{2k-1} \subset M$ be a PR-frame for M. Without losing the generality we can assume that $\{\varphi_1, ..., \varphi_k\}$ is an orthonormal basis for M. Extend it to an orthonormal basis $\{e_n\}_{n=1}^d$ for $\mathcal{H}$, where $e_n = \varphi_n$ for $n = 1, ..., k$. Define u_n by

$$u_n = e_n \ \ (n = 1, .., k, 2k, ..., d) \ \text{ and } u_n = e_n + \varphi_n \ \ (n = k+1, ..., 2k-1).$$

Let P_M be the orthogonal projection onto M. Clearly we have

$$\{P_M u_n\}_{n=1}^d = \{\varphi_1, ..., \varphi_{2k-1}, 0, ..., 0\},$$

and hence $\{u_n\}_{n=1}^d$ is a phase-retrievable for M. It is also easy to verify that $\{u_n\}_{n=1}^d$ is a basis for $\mathcal{H}$. Now we show that M is an maximal PR subspace with respect to $\{u_1, ..., u_d\}$. Let $\tilde{M} = \operatorname{span}\{M, u\}$ with $u = \sum_{j=k+1}^d a_j e_j$ in $M^\perp$ and $||u|| = 1$. Then $P_{\tilde{M}} u_n = e_n$ for $1 \leq n \leq k$, $P_{\tilde{M}} u_n = \varphi_n + a_n u$ for $k+1 \leq n \leq 2k-1$ and $P_{\tilde{M}} u_n = a_n u$ for $n \geq 2k-1$. If $a_n = 0$ for $n = 2k, ..., d$, then $\{P_{\tilde{M}} u_n\}_{n=1}^d$ is not phase-retrievable for $\tilde{M}$ since it only contains at most $2k-1$ nonzero elements. If $a_{n_0} \neq 0$ for some $n_0 \geq 2k$, then clearly $\{P_{\tilde{M}} u_n\}_{n=1}^d$ is phase-retrievable for $\tilde{M}$ if and only if $\{P_{\tilde{M}} u_n\}_{n=1}^{2k-1} \cup \{a_{n_0} u\}$ is phase-retrievable for $\tilde{M}$. Thus $\tilde{M}$ is not a PR subspace with respect to $\{u_1, ..., u_d\}$ since we need at least $2k+1$ number of elements in a phase-retrievable frame for the $(k+1)$-dimensional space $\tilde{M}$. $\square$

We now consider the general frame case. Let $\mathcal{F}$ be a frame for $\mathcal{H}$. For each subset Λ of $\{1, ..., m\}$,

let

$$d_\Lambda = \max\{\dim \mathrm{span}\,(\mathcal{F}_\Lambda), \dim \mathrm{span}\,(\mathcal{F}_{\Lambda^c})\}.$$

Define

$$d(\mathcal{F}) = \min\{d_\Lambda : \Lambda \subset \{1, ..., m\}\}.$$

Theorem 4.1.4. *Let $\mathcal{F}$ be a frame for $\mathcal{H}$. Then k is the largest integer such that there exists a k-dimensional maximal $\mathcal{F}$-PR subspace if and only if $k = d(\mathcal{F})$.*

Clearly, $d(\mathcal{F}) = d$ if and only if $\mathcal{F}$ has the complement property. Thus the above theorem is a natural generalization of Theorem 2.2.2 . We need following lemma for the proof of Theorem 4.1.4.

Lemma 4.1.5. *Let $Tx = \sum_{n=1}^{k} \langle x, x_n\rangle x_n$ be a rank-k operator and M be a subspace of $\mathcal{H}$ such that* $\dim TM = k$*, then* $\dim P(M) = k$*, where P is the orthogonal projection onto* $\mathrm{span}\,\{x_1, ..., x_k\}$.

Proof. Since $\langle x, x_n\rangle = \langle Px, x_n\rangle$, we get that $\mathrm{range}(T|_M) = \mathrm{range}(T|_{PM})$. Thus $\dim P(M) \geq k$ and hence $\dim P(M) = k$. $\square$

Proof of Theorem 4.1.4. Clearly we only need to prove that if $d(\mathcal{F}) = k$, then there exists a k-dimensional $\mathcal{F}$-PR subspace and every $(k+1)$-dimensional subspace is not phase-retrievable with respect to $\mathcal{F}$.

Suppose that M is a $(k+1)$-dimensional subspace of $\mathcal{H}$ and it is also phase-retrievable with respect to $\mathcal{F}$. Then, by Theorem 2.2.2, we get that $d(P\mathcal{F}) = k+1$, and hence $d(\mathcal{F}) \geq d(P\mathcal{F}) \geq k+1$, which leads to a contradiction. Therefore every $(k+1)$-dimensional subspace is not phase-retrievable with respect to $\mathcal{F}$.

Next we show that there exists a k-dimensional $\mathcal{F}$-PR subspace. Let Ω be a subset of $\{1, ..., m\}$

such that $\dim \mathcal{H}_\Omega \geq k$, where $\mathcal{H}_\Omega = \operatorname{span} \mathcal{F}_\Omega$. For $X = (x_1, ..., x_k) \in \mathcal{H}^{(k)} := \mathcal{H} \oplus ... \oplus \mathcal{H}$, define $T_X(z) = \sum_{n=1}^{k} \langle z, x_n \rangle x_n$.

Consider the following set

$$S_\Omega = \{(x_1, ..., x_k) \in \mathcal{H}^{(k)} : \dim T_X(\mathcal{H}_\Omega) = k\}.$$

Since $\dim \operatorname{span} \mathcal{F}_\Omega \geq k$, we get that there exists a linearly independent set $(f_{i_1}, ..., f_{i_k})$ in $\mathcal{F}_\Omega$. This implies that $(f_{i_1}, ..., f_{i_k}) \in S_\Omega$ and hence S_Ω is not empty.

Moreover, since $\dim T_X(\mathcal{H}_\Omega) = k$ if and only if there exists an $k \times k$ submatrix of the $d \times |\Omega|$ matrix $[T_X f_\omega]$ whose determinant is a nonzero polynomial of the input variables $x_1, ..., x_k$, we obtain that S_Ω is open dense in $\mathcal{H}^{(k)}$.

Now for each subset Λ in $\{1, ..., m\}$. Let $\Omega_\Lambda = \Lambda$ if $d_\Lambda = \dim \operatorname{span} (\mathcal{F}_\Lambda)$, and otherwise $\Omega_\Lambda = \Lambda^c$. Thus we have $\dim \operatorname{span} \mathcal{F}_{\Omega_\Lambda} \geq k$ for every subset Λ. Since each S_{Ω_Λ} is open dense in $\mathcal{H}^{(k)}$, we get that

$$S := \bigcap_{\Lambda \subset \{1, ..., m\}} S_{\Omega_\Lambda}$$

is open dense in $\mathcal{H}^{(k)}$. Let $X = (x_1, ..., x_k) \in S$ and $M = \operatorname{span} \{x, ..., x_k\}$. Then by Lemma 4.1.5 we obtain that $\dim P(\mathcal{H}_{\Omega_\Lambda}) = k$. This implies that either $\dim \operatorname{span} P\mathcal{F}_\Lambda = k$ or $\dim \operatorname{span} P\mathcal{F}_{\Lambda^c} = k$ for each subset Λ. Hence $\{Pf_j\}_{j=1}^m$ is a frame for M that has the complement property, which implies by Theorem 2.2.2 that M is a k-dimensional $\mathcal{F}$-PR subspace. $\square$

From the proof of Theorem 4.1.4, we also have the following:

Corollary 4.1.6. *Let $\mathcal{F}$ be a frame for $\mathcal{H}$. Then for almost all the vectors $(x_1, ...x_\ell)$ in $\mathcal{H}^{(\ell)}$ (here $\ell \leq d(\mathcal{F})$), the subspace* $\operatorname{span} \{x_1, ..., x_\ell\}$ *is phase-retrievable with respect to $\mathcal{F}$. More precisely,*

for each $\ell \leq d(\mathcal{F})$, the following set

$$\{(x_1, ...x_\ell) \in \mathcal{H}^{(\ell)} : \text{ span}\{x_1, ..., x_\ell\} \textit{ is phase retrievable with respect to } \mathcal{F}\}$$

is open dense in $\mathcal{H}^{(\ell)}$.

The following lemma follows immediately from the definitions, and it tells us that it is enough to focus on maximal phase-retrievable subspaces for frames that have the exact PR-redundancy property.

Lemma 4.1.7. *Let $\mathcal{F} = \{f_n\}_{n=1}^m$ be a frame for $\mathcal{H}$, and $\Lambda \subset \{1, ..., m\}$. If $\ker(\mathcal{A}_{\mathcal{F}_\Lambda}) \cap \mathcal{S}_2 = \ker(\mathcal{A}_{\mathcal{F}}) \cap \mathcal{S}_2$, then M is a $\mathcal{F}$-PR subspace if and only if it is a $\mathcal{F}_\Lambda$-PR subspace. Consequently, M is an maximal $\mathcal{F}$-PR subspace if and only if it is an maximal $\mathcal{F}_\Lambda$-PR subspace.*

Now we would like to know what are the possible values of $d(\mathcal{F})$. Since every frame contains a basis, we get by Proposition 4.1.3 that $d(\mathcal{F}) \geq [\frac{d+1}{2}]$. The following theorem tells us that for every k between $[(d+1)/2]$ and d, there is a frame $\mathcal{F}$ with the exact PR-redundancy property such that $k = d(\mathcal{F})$.

Theorem 4.1.8. *Let $\mathcal{H} = \mathbb{R}^d$ and k be an integer such that $d \geq k \geq [\frac{d+1}{2}]$. Then for each m between $2k-1$ and $k(k+1)/2 + (d-k)(d-k+1)/2$, there exists a frame $\mathcal{F}$ of length m such that it has the exact PR-redundancy property and $d(\mathcal{F}) = k$, i.e., k is the largest integer such that there exists a k-dimensional maximal $\mathcal{F}$-PR subspace.*

Before giving the proof we remark that while the proof of the this theorem uses Theorem 3.2.1, it is also a generalization of Theorem 3.2.1 since it clearly recovers Theorem 3.2.1 if we let $d = k$.

Proof. Let M be a k-dimensional subspace of $\mathcal{H}$.

We divide the proof into two cases.

Case (i). Assume that $2k-1 \leq m \leq k(k+1)/2$.

By Theorem 3.2.1, there exists an exact PR-frame $\mathcal{G} = \{g_n\}_{n=1}^m$ for M. Without losing the generality we can also assume that $\{g_1, ..., g_k\}$ is an orthonormal basis for M. Extend it to an orthonormal basis $\{e_n\}_{n=1}^d$ with $e_1 = g_1, ..., e_k = g_k$. Let

$$\mathcal{F} = \{f_n\}_{n=1}^m = \{e_1, ..., e_k, g_{k+1} + e_{k+1}, ..., g_d + e_d, g_{d+1}...., g_m\}.$$

Then $\mathcal{F}$ is a frame for $\mathcal{H}$. Consider the subset $\Lambda = \{1, ..., k, d+1, ..., m\}$ of $\{1, ..., m\}$. We have $\dim \operatorname{span} \mathcal{F}_\Lambda = \dim M = k$, and $\dim \operatorname{span} \mathcal{F}_{\Lambda^c} \leq d-k$. Note that from $k \geq [\frac{d+1}{2}]$ we get that $d-k \leq k$. Thus we have $d(\mathcal{F}) \leq \max\{d-k, k\} = k$. On the other hand, it is easy to prove that $d(\mathcal{F}) \geq d(P_M\mathcal{F}) = d(\mathcal{G}) = k$, where P_M is the orthogonal projection onto M. Therefore we have $d(\mathcal{F}) = k$.

Now we show that $\mathcal{F}$ has the exact PR-redundancy property. If fact, if Λ is a proper subset of $\{1, ..., m\}$, then $P_M\mathcal{F}_\Lambda$ is not a PR frame for M since $P_M\mathcal{F} = \mathcal{G}$ is an exact PR-frame for M. Therefore, there exists x and y in M such that $|\langle x, P_M f_n\rangle| = |\langle y, P_M f_n\rangle|$ for all $n \in \Lambda$ and $A = x \otimes x - y \otimes y \neq 0$. Since $P_M\mathcal{F}$ is a PR-frame for M, we obtain that $|\langle x, P_M f_n\rangle| \neq |\langle y, P_M f_n\rangle|$ for some $n \in \Lambda^c$. Note that $|\langle z, f_n\rangle = \langle z, P_M f_n\rangle$ for every $z \in M$. Therefore, we have that $A \in \ker(\mathcal{A}_{\mathcal{F}_\Lambda}) \cap S_2$ but $A \notin \ker(\mathcal{A}_\mathcal{F}) \cap S_2$, and hence $\ker(\mathcal{A}_{\mathcal{F}_\Lambda}) \cap S_2 \neq \ker(\mathcal{A}_\mathcal{F}) \cap S_2$ for any proper subset Λ. So $\mathcal{F}$ has the exact PR-redundancy property.

Case (ii): Assume that $k(k+1)/2 < m \leq k(k+1)/2 + (d-k)(d-k+1)/2$.

Since $k \geq [(d+1)/2] \geq d/2$, it is easy to verify that

$$k(k+1)/2 \geq (2k-1) + 2(d-k) - 1 = 2d - 2.$$

Then we can write $m = m_1 + m_2$ such that

$$2k - 1 \leq m_1 \leq k(k+1)/2 \text{ and } 2(d-k) - 1 \leq m_2 \leq (d-k)(d-k+1)/2.$$

By Theorem 3.2.1, there exist an exact PR-frame $\mathcal{F}_1$ of length m_1 for M and an exact PR-frame $\mathcal{F}_2$ of length m_2 for the $M^\perp$. By Lemma 4.1.2, we know that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a frame of length m with the exact PR-redundancy property. Clearly $d(\mathcal{F}) \leq k$ since

$$\max\{\dim \operatorname{span} \mathcal{F}_1, \dim \operatorname{span} \mathcal{F}_2\} = k.$$

On the other hand, since $\mathcal{F}$ has a k-dimensional PR-subspace M, we get from Theorem 4.1.4 that $d(\mathcal{F}) \geq k$. Thus we have $d(\mathcal{F}) = k$. $\square$

The following example shows that $k(k+1)/2 + (d-k)(d-k+1)/2$ is not necessarily the upper bound of m such that there exists a frame $\mathcal{F}$ of length m with the exact PR-redundancy property and $d(\mathcal{F}) = k$.

Example 4.1.9. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathbb{R}^3$. Consider the following frame

$$\mathcal{F} = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_2 + e_3\}.$$

Then $k = d(\mathcal{F}) = 2$ and $5 > k(k+1)/2 + (3-k)(3-k+1)/2 = 4$. We can check that $\mathcal{F}$ has the exact PR-redundancy property. Let $\mathcal{G}$ be the frame after removing an element f from $\mathcal{F}$. Based on the following five cases, we can easily construct $A = x \otimes x - y \otimes y$ such that $A \neq 0$, $A \in \ker(\mathcal{A}_\mathcal{G})$ but $A \notin \ker(\mathcal{A}_\mathcal{F})$:

(i) $f = e_1$: Let $x = 2e_1 = e_2$ and $y = 4e_1 - e_2$.

(ii) $f = e_2$: Let $x = 2e_2 + e_1$ and $y = 4e_2 - e_1$.

(iii) $f = e_3$: Let $x = e_1 + e_3$ and $y = e_1 - 3e_3$.

(iv) $f = e_1 + e_2$: Let $x = e_1 + (e_2 + e_3)$ and $y = e_1 - (e_2 + e_3)$

(v) $f = e_1 + e_2 + e_3$: Let $x = e_1 + e_3$ and $y = e_1 - e_3$.

Theorem 4.1.10. *Let $\mathcal{H} = \mathbb{R}^d$. Suppose that a frame $\mathcal{F}$ of length m has the exact PR-redundancy property and $d(\mathcal{F}) < d$. Then $m < d(d+1)/2$.*

Proof. Since $\mathcal{F}$ has the exact PR-redundancy, we get that $m \leq d(d+1)/2$. If $m = d(d+1)/2$, then, by Lemma 3.1.2, $\{f_n \otimes f_n\}_{n=1}^m$ is linearly independent and hence a basis for $\mathbb{H}_d$. This implies that $\mathcal{F}$ is phase-retrievable and so $d(\mathcal{F}) = d$. This contradiction shows that $m < d(d+1)/2$. $\square$

4.2 Maximal Phase-Retrievable Subspaces

Given a basis $\mathcal{F} = \{f_1, ..., f_d\}$. We would like to have a better understanding about the maximal phase-retrievable subspaces with respect to $\mathcal{F}$. We will first focus on orthonormal bases and then use the similarity to pass to general bases.

Now we assume that $\mathcal{E} = \{e_1, ..., e_d\}$ is an orthonormal basis for $\mathbb{R}^d$. By Proposition 4.1.3, we know that there exists a k-dimensional maximal $\mathcal{E}$-PR subspace for ever integer k with $1 \leq k \leq [\frac{d+1}{2}]$. What more can be said about these k-dimensional maximal $\mathcal{E}$-PR subspaces? We explore this question by establishing a connection with the support property of the vectors in maximal PR-subspaces. Recall that for a vector $x = \sum_{n=1}^d \alpha_n e_n \in \mathbb{R}^d$, the support of x is defined by $\text{supp}_{\mathcal{E}}(x) := \{n \,|\alpha_n \neq 0\}$. We will also use $\text{supp}(x)$ to denote $\text{supp}_{\mathcal{E}}(x)$ if $\mathcal{E}$ is well understood in the statements, and use $|\Lambda|$ to denote the cardinality of any set Λ.

Proposition 4.2.1. *Suppose that M is a k-dimensional $\mathcal{E} - PR$ subspace. Then for any nonzero vector $x \in M$, we have $|supp(x)| \geq k$.*

Proof. Assume to the contrary that there exists a nonzero $x \in M$ with $|\text{supp}(x)| = j < k$. We may assume that $\|x\| = 1$ and that $\text{supp}(x) = \{1, 2, ..., j\}$. Pick vectors $y_1, ..., y_{k-1}$ in M such that the set $\{x, y_1, ..., y_{k-1}\}$ is an orthonormal basis for M. Then we have,

$$\begin{aligned}
P_M(e_1) &= \langle e_1, x\rangle x + \langle e_1, y_1\rangle y_1 + \cdots + \langle e_1, y_{k-1}\rangle y_{k-1} \\
P_M(e_2) &= \langle e_2, x\rangle x + \langle e_2, y_1\rangle y_1 + \cdots + \langle e_2, y_{k-1}\rangle y_{k-1} \\
&\vdots \\
P_M(e_j) &= \langle e_j, x\rangle x + \langle e_j, y_1\rangle y_1 + \cdots + \langle e_j, y_{k-1}\rangle y_{k-1} \\
P_M(e_{j+1}) &= \langle e_{j+1}, y_1\rangle y_1 + \cdots + \langle e_{j+1}, y_{k-1}\rangle y_{k-1} \\
&\vdots \\
P_M(e_d) &= \langle e_d, y_1\rangle y_1 + \cdots + \langle e_d, y_{k-1}\rangle y_{k-1}.
\end{aligned}$$

The partition $\{P_M(e_1), ..., P_M(e_j)\}$ and $\{P_M(e_{j+1}), ..., P_M(e_d)\}$ does not have the complement property since the first set contains less than k elements and the members of the second set are all contained in the $(k-1)$-dimensional subspace $\text{span}\{y_1, ..., y_{k-1}\}$. Thus M is not a $\mathcal{E} - PR$ subspace, which leads to a contradiction. $\square$

Corollary 4.2.2. *If M is a k-dimensional $\mathcal{E} - PR$ subspace and there exists $x \in M$ such that $|\text{supp}(x)| = k$, then M is maximal.*

Proof. If M is not maximal then there exists a $(k+1)$-dimensional $\mathcal{F} - PR$ subspace W which

contains M. Thus we have $x \in W$ with $|\text{supp}(x)| = k$, a contradiction. $\square$

Now suppose that $k \leq [(d+1)/2]$. Let $x \in \mathcal{H}$ be a vector of norm one and $|\text{supp}(x)| = k$. We show that x can be extended to an orthonormal set $\{x, u_1, ..., u_{k-1}\}$ such that $M = \text{span}\{x, u_1, ..., u_{k-1}\}$ is a k-dimensional $\mathcal{E}$-PR subspace.

Theorem 4.2.3. *Let $u_1 \in \mathbb{R}^d$ be a unit vector such that $|supp(u_1)| = k$ and $k \leq [(d+1)/2]$. Then u_1 can be extended to an orthonormal set $\{u_1, ..., u_k\}$ such that $M = \text{span}\{u_1, ..., u_k\}$ is a k-dimensional maximal $\mathcal{E}$-PR subspace.*

Proof. We can assume that $\{e_1, ..., e_d\}$ is the standard orthonormal basis for $\mathbb{R}^d$ and $u_1 = \sum_{n=1}^{k} \alpha_n e_n$ such that $\alpha_n \neq 0$ for every $1 \leq n \leq k$.

It is easy to observe the following fact: Let $m : 1 \leq m \leq k$. Suppose that $\{u_1, ..., u_m\}$ is an orthonormal set extension of u_1 and

$$A(u_1, ..., u_m) = [u_1, ..., u_m].$$

is the matrix consisting of column vectors $u_1, ..., u_m$. Also let $A_\Lambda(u_1, ..., u_m)$ be the matrix consisting of the row vectors of $A(u_1, ..., u_m)$ corresponding to an index set Λ. If $A_\Lambda(u_1, ..., u_m)$ is invertible for every subset Λ of $\{1, ..., d\}$ of cardinality m with the property that $\Lambda \cap \{1, ..., k\} \neq \emptyset$, then the row vectors of $A(u_1, ..., u_m)$ form a frame for $\mathbb{R}^m$ that has the complement property.

Now we use the induction to show that such an matrix $A(u_1, ..., u_m)$ exists for every $m \in \{1, ..., k\}$. Clearly, the $d \times 1$ matrix $A(u_1)$ satisfies the requirement. Now assume that such an $d \times m$ matrix $A(u_1, ..., u_m)$ has been constructed and $m < k$. We want to prove that there exists a unit vector $u_{m+1} \perp u_i (1 \leq i \leq m)$ such that $A(u_1, ..., u_m, u_{m+1})$ has the required property.

Let $U = \text{span}\{u_1, ..., u_m\}^\perp$, and let Λ be a subset of $\{1, ..., d\}$ such that $|\Lambda| = m + 1$ and $\Lambda \cap$

$\{1, ..., k\} \neq \emptyset$. Define

$$\Omega_\Lambda = \{u \in U : A_\Lambda(u_1, ..., u_m, u) \text{ is invertible}\}.$$

We claim that Ω_Λ is an open dense subset of U.

Using the fact that the set of invertible matrices form an open set in the space of all matrices, it is clear that Ω_Λ is open in U.

Now we show that $\Omega_\Lambda \neq \emptyset$. Let Λ' be a subset of Λ with cardinality m and $\Lambda' \cap \{1, ..., k\} \neq \emptyset$. Then, by our induction assumption, we have that $A_{\Lambda'}(u_1, ..., u_m)$ is invertible, which implies that the m column vectors of $A_\Lambda(u_1, ..., u_m)$ form a linearly independent set in the $m+1$ dimensional space $\mathbb{R}^\Lambda = \Pi_{i\in\Lambda}\mathbb{R}$. Let $z \in \mathbb{R}^{m+1}$ be a nonzero vector such that it is orthogonal to all the column vectors of $A_\Lambda(u_1, ..., u_m)$. Define $u = (u_1, ..., u_d)^T \in \mathbb{R}^d$ by letting $u_i = z_i$ for $i \in \Lambda$, and 0 otherwise. Then $u \in U$ and hence $u \in \Omega_\Lambda$. Therefore we get that $\Omega_\Lambda \neq \emptyset$.

For the density of Ω_U, let $y \in U$ be an arbitrary vector and pick a vector $u \in \Omega_\Lambda$. Consider the vector $u_t = tu + (1-t)y \in U$ for $t \in R$. Since $A_\Lambda(u_1, ..., u_m, u)$ is invertible, we have that $det(A_\Lambda(u_1, ..., u_m, u_t))$ is a nonzero polynomial of t, and hence it is finitely many zeros. This implies that there exists a sequence $\{t_j\}$ such that $u_{t_j} \in \Omega_\Lambda$ and $\lim_{j\to\infty} t_j = 0$. Hence $u_{t_j} \to y$ and therefore Ω_U is dense in U.

By the Baire Category theorem we obtain that the intersection Ω of all such Ω_Λ is open dense in Y. Pick any $u_{m+1} \in \Omega$, then $A(u_1, ..., u_m, u_{m+1})$ has the required property. This completes the induction proof for the existence of such an matrix $A = [u_1, ..., u_k]$, where $\{u_1, ..., u_k\}$ is an orthonormal set extending the given vector u_1.

Write $u_j = (a_{1j}, a_{2j}, ..., a_{dj})^T$ for $1 \leq j \leq k$. Let $M = \text{span}\{u_1, ..., u_k\}$ and P be the orthogonal

projection onto M. Then

$$Pe_i = \sum_{j=1}^{k} < e_i, u_j > u_j = \sum_{j=1}^{d} a_{ij} u_j$$

for all $1 \leq i \leq d$. For every subset Λ of $\{1, ..., n\}$, since $\{u_1, ..., u_k\}$ is an orthonordal set, we have that $\{Pe_j : j \in \Lambda\}$ are linearly independent if and only if A_Λ is invertible. Thus, $\{Pe_i\}_{i=1}^{d}$ has the complement property since the set of row vectors of A has the complement property. $\square$

Remark 4.2.4. Note that from the proof of the above theorem it is easy to see that the existence of such a matrix $A(u_1, ..., u_k)$ does not require the condition $k \leq [(d+1)/2]$. However, the complement property of the row vectors for $\mathbb{R}^k$ does require this condition.

We already knew that if M is a k-dimensional PR-subspace with respect to an orthonormal basis $\mathcal{E}$, then the condition $\min\{|\text{supp}(x)| : 0 \neq x \in M\} = k$ is sufficient for M to be maximal. The following example show that this condition is not necessary in general. However, we will prove in Theorem 4.2.6 that it is indeed also necessary if $k < [\frac{d+1}{2}]$.

Example 4.2.5. There exists a 2-dimensional maximal PR-subspace M in $\mathbb{R}^4$ such that $|\text{supp}(x)| = 3$ for every nonzero $x \in M$. Indeed, let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal basis for $\mathbb{R}^4$ and be $M = \text{span}\,\{e_1 + e_2 + e_3, e_1 - e_2 + e_4\}$. Then it can be easily verified that M is a PR-subspace and $|\text{supp}(x)| = 3$ for every nonzero $x \in M$. It is clear that M is maximal since there is no 3-dimensional PR-subspace with respect to $\{e_1, e_2, e_3, e_4\}$ in $\mathbb{R}^4$.

Theorem 4.2.6. *Assume that $M = \text{span}\,\{u_1, ..., u_k\}$ is a k-dimensional maximal PR-subspace with respect to $\{e_1, ..., e_d\}$ and $k < [\frac{d+1}{2}]$. Then $\min\{|\text{supp}(x)| : 0 \neq x \in M\} = k$.*

Proof. By Proposition 4.2.1, it suffices to show there is an nonzero vector $x \in M$ such that $|\text{supp}(x)| \leq k$.

Let $\{u_1, ..., u_k\}$ be an orthonormal basis for M. We adopt the notation used in the proof of Theorem 4.2.3: For every subset Λ of $\{1, ..., d\}$, let $A_\Lambda(u_1, ..., u_k)$ be the matrix consisting of row vectors

of $[u_1, ..., u_k]$ corresponding to the row index set Λ. It is obvious that if there is a subset Λ with $|\Lambda| = d - k$ such that $\text{rank} A_\Lambda(u_1, ..., u_k) < k$, then there is a nonzero vector $x \in M$ such that $\text{supp}(x) \subseteq \Lambda^c$ and hence $|\text{supp}(x)| \leq k$. We will prove that such a subset Λ exists.

Assume, to the contrary, that $\text{rank} A_\Lambda(u_1, ..., u_k) = k$ for any subset Λ with $|\Lambda| = d - k$. Thus we have $\text{rank} A_\Lambda(u_1, ..., u_k) = k$ for any subset Λ with $|\Lambda| \geq d - k$.

For each subset Λ, since $k < [\frac{d+1}{2}]$, we only have three possible cases:

(i) $|\Lambda| \geq d - k$ and $|\Lambda^c| < d - k$.

(ii) $|\Lambda^c| \geq d - k$ and $|\Lambda| < d - k$.

(iii) $|\Lambda| < d - k$ and $|\Lambda^c| < d - k$.

Note that case (iii) implies that $|\Lambda| > k$ and $|\Lambda^c| > k$. Now we assign each Λ to a subset $S(\Lambda)$ by the following rule: Set $S(\Lambda)$ to be Λ or Λ^c depending case (i) or case (ii). Suppose that Λ satisfies (iii). Since the row vectors of $[u_1, ..., u_k]$ has the complement property, we have that either $\text{rank} A_\Lambda(u_1, ..., u_k) = k$ or $\text{rank} A_{\Lambda^c}(u_1, ..., u_k) = k$. In this case we set $S(\Lambda) = \Lambda$ if $\text{rank} A_\Lambda(u_1, ..., u_k) = k$, and otherwise set $S(\Lambda) = A^c$. Let

$$\mathcal{S} = \big\{S(\Lambda) : \Lambda \subseteq \{1, ..., d\}\big\}.$$

Then for each Λ we have either $S(\Lambda) = \Lambda$ or $S(\Lambda) = \Lambda^c$, $\text{rank} A_{S(\Lambda)}(u_1, ..., u_k) = k$ and $|S(\Lambda)| \geq k + 1$.

Let $U = \text{span}\{u_1, ..., u_k\}^\perp$ and

$$\Omega_\Lambda = \{u \in U : \text{rank} A_{S(\Lambda)}(u_1, ...u_k, u) = k + 1\}.$$

Then by the exact same argument as in the proof of Theorem 4.2.3, we get that Ω_Λ is open dense in U. The Baire-Category theorem implies that there exists unit vector $u_{k+1} \in U$ such that $\text{rank} A_{S(\Lambda)}(u_1, ...u_k, u_{k+1}) = k+1$ for every subset $\Lambda \subseteq \{1, ..., d\}$. This shows that the row vectors of the matrix $[u_1, ..., u_k, u_{k+1}]$ has the complementary property, and hence span$\{u_1, .., u_k, u_{k+1}\}$ is a PR-subspace with respect to the orthonormal basis $\{e_1, ..., e_d\}$, which contradicts the maximality of M. $\square$

Example 4.2.7. Let $\mathcal{F} = \{e_1, ..., e_d\}$ be an orthonormal basis for $\mathbb{R}^d$. Then $M = \text{span}\,\{x\}$ is a one-dimensional maximal $\mathcal{F}$-PR subspace if and only if $|\text{supp}(x)| = 1$.

Example 4.2.8. Let $x \in \mathbb{R}^n$ be a unit vector such that $|\text{supp}(x)| = 2$ and M be a 2-dimensional subspace containing x. Then M is maximal $\mathcal{F}$-PR subspace if and only if there exists an orthonormal basis $\{x, y\}$ for M such that $y = y_1 + y_2$ with $0 \neq y_1 \in \text{span}\,\{e_n : n \in \text{supp}(x)\}$ and $0 \neq y_2 \in \text{span}\,\{e_n : n \notin \text{supp}(x)\}$. Indeed, by Corollary 4.2.2, it suffices to show that M is a $\mathcal{F} - PR$ subspace. We may assume that supp$(x) = \{1, 2\}$. Then we have

$$\begin{aligned}
P_M(e_1) &= \langle e_1, x\rangle x + \langle e_1, y_1\rangle y \\
P_M(e_2) &= \langle e_2, x\rangle x + \langle e_2, y_1\rangle y \\
P_M(e_3) &= \langle e_3, y_2\rangle y \\
&\vdots \\
P_M(e_d) &= \langle e_d, y_2\rangle y \\
&.
\end{aligned}$$

Then it is easy to check that $\{P_M e_i\}$ has the complement property if and only if $\{P_M e_1, P_M e_2\}$ are linearly independent, and $\langle e_i, y_2\rangle \neq 0$ for some $3 \leq i \leq n$. This is in turn equivalent to the

conditions that $y_1 \neq 0$ and $y_2 \neq 0$.

Finally, let examine the general basis case. Let $\mathcal{F} = \{f_1, ..., f_d\}$ be a basis for $\mathbb{R}^d$, and $\mathcal{F}^* = \{f_1^*, ..., f_d^*\}$ be its dual basis. Let T be the invertible matrix such that $f_n = Te_n$ for all n, where $\mathcal{E} = \{e_1, ..., e_d\}$ be the standard orthonormal basis for $\mathbb{R}^d$. We observe the following facts:

(i) M is a PR-subspace with respect to $\mathcal{F}$ if and only if $T^t M$ is a PR-subspace with respect to $\mathcal{E}$.

(ii) The dual basis of $\mathcal{F}$ is $\mathcal{F}^* = \{(T^{-1})^t T^{-1} e_d\}_{n=1}^d$, i.e., $f_n^* = (T^{-1})^t T^{-1} e_n$.

(iii) The coordinate vector of x with respect to the basis $\mathcal{F}^*$ is the same as the coordinate vector of $T^t x$ with respect to the basis $\mathcal{E}$.

Based on the above observations we summarize the main results of this section in the following theorem:

Theorem 4.2.9. *Let $\mathcal{F} = \{f_1, ..., f_d\}$ be a basis for $\mathbb{R}^d$, and $\mathcal{F}^* = \{f_1^*, ..., f_d^*\}$ be its dual basis. Then we have*

(i) If M is a k-dimensional PR-subspace with respect to $\mathcal{F}$, then $|\mathrm{supp}_{\mathcal{F}^*}(x)| \geq k$ for any nonzero vector $x \in M$. Consequently, M is maximal if there exists a vector $x \in M$ such that $|\mathrm{supp}_{\mathcal{F}^*}(x)| = k$.

(ii) For any vector $x \in \mathbb{R}^d$ such that $|\mathrm{supp}_{\mathcal{F}^*}(x)| = k$, there exists a k-dimensional maximal PR-subspace M with respect to $\mathcal{F}$ such that $x \in M$.

(iii) If $k < [(d+1)/2]$ and M is a k-dimensional PR-subspace with respect to $\mathcal{F}$, then M is maximal if and only if there exists a vector $x \in M$ such that $|\mathrm{supp}_{\mathcal{F}^*}(x)| = k$.

4.3 Characterization via the PhaseLift Operator

In this section we provide a characterization of phase-retrievable subspaces using the PhaseLift operator. Recall that if $T \in \mathbb{R}^{d\times d}$ then a subspace M of $\mathbb{R}^d$ is said to be T-invariant if $T(M) \subseteq M$. Let $\mathsf{S} = \{T \in \mathbb{R}^{d\times d} : T^t = T, \text{rank}(T) = 1 \text{ or } 2\}$. If $T \in \mathsf{S}$ and rank(T)=1, then $T = \lambda\,(x \otimes x)$ for some nonzero scalar λ and some unit vector $x \in \mathbb{R}^d$. If rank(T)=2, then by the spectral decomposition theorem there exists orthonormal vectors $x_1, x_2 \in \mathbb{R}^d$, and nonzero scalars λ_1, λ_2 such that $T = \lambda_1\,(x_1 \otimes x_1) + \lambda_2\,(x_2 \otimes x_2)$. In the following preliminary lemmas, we fix a k-dimensional subspace M of $\mathbb{R}^d$.

Lemma 4.3.1. *Assume that $M \not\subseteq \ker(T)$. If $T \in \mathsf{S}$ and $T = \lambda\,(x \otimes x)$, then M is T-invariant if and only if $x \in M$.*

Proof. If $x \in M$ then clearly M is T-invariant. Now suppose that M is T-invariant. Let T_M denote the restriction of T to M. Then T_M is a self-adjoint operator and so there exists an orthonormal basis $\{u_1, ..., u_k\}$ for M consisting of eigenvectors of T_M. T_M has exactly one nonzero eigenvalue so we may assume that $\lambda \neq 0$ is the eigenvalue corresponding to u_1. Thus we have

$$T_M(u_1) = \lambda\,(x \otimes x)\,(u_1) = \lambda\langle u_1, x\rangle x = \lambda u_1.$$

Note that $\langle u_1, x\rangle \neq 0$. Hence $x = \frac{1}{\langle u_1, x\rangle} u_1 \in M$ and the claim is proved. $\square$

Note: We could also argue that T has rank one and both x and u_1 are eigenvectors of T corresponding to the eigenvalue λ.

Lemma 4.3.2. *Assume that there exist linearly independent vectors $x, y \in M$ such that $x, y \notin \ker(T)$. If $T \in \mathsf{S}$ and $T = \lambda_1\,(x_1 \otimes x_1) + \lambda_2\,(x_2 \otimes x_2)$, then M is T-invariant if and only if*

$x_1, x_2 \in M$.

Proof. M is clearly T-invariant if $x_1, x_2 \in M$. So suppose that M is T-invariant. Let $\{u_1, ..., u_k\}$ be an orthonormal basis for M consisting of eigenvectors of T_M. Let u_i be eigenvectors corresponding to λ_i, for $i = 1, 2$. Then we have

$$
\begin{aligned}
T_M(u_1) &= \lambda_1 (x_1 \otimes x_1)(u_1) + \lambda_2 (x_2 \otimes x_2)(u_1) \\
&= \lambda_1 \langle u_1, x_1 \rangle x_1 + \lambda_2 \langle u_1, x_2 \rangle x_2 \\
&= \lambda_1 u_1
\end{aligned}
$$

and

$$
\begin{aligned}
T_M(u_2) &= \lambda_1 (x_1 \otimes x_1)(u_2) + \lambda_2 (x_2 \otimes x_2)(u_2) \\
&= \lambda_1 \langle u_2, x_1 \rangle x_1 + \lambda_2 \langle u_2, x_2 \rangle x_2 \\
&= \lambda_2 u_2
\end{aligned}
$$

From the above we obtain,

$$
\begin{cases}
\lambda_1 \langle u_2, x_1 \rangle u_1 & = \lambda_1 \langle u_2, x_1 \rangle \langle u_1, x_1 \rangle x_1 + \lambda_2 \langle u_2, x_1 \rangle \langle u_1, x_2 \rangle x_2 \\
-\lambda_2 \langle u_1, x_1 \rangle u_2 & = -\lambda_1 \langle u_2, x_1 \rangle \langle u_1, x_1 \rangle x_1 - \lambda_2 \langle u_2, x_2 \rangle \langle u_1, x_1 \rangle x_2
\end{cases}
$$

This implies that

$$\lambda_2[\langle u_2, x_1\rangle\langle u_1, x_2\rangle - \langle u_2, x_2\rangle\langle u_1, x_1\rangle]x_2 = \lambda_1\langle u_2, x_1\rangle u_1 - \lambda_2\langle u_1, x_1\rangle u_2$$

Since u_1 and u_2 are linearly independent we have that $\langle u_2, x_1\rangle\langle u_1, x_2\rangle - \langle u_2, x_2\rangle\langle u_1, x_1\rangle \neq 0$. Thus we have,

$$x_2 = \frac{\lambda_1\langle u_2, x_1\rangle}{A}u_1 - \frac{\lambda_2\langle u_1, x_1\rangle}{A}u_2$$

where $A = \lambda_2[\langle u_2, x_1\rangle\langle u_1, x_2\rangle - \langle u_2, x_2\rangle\langle u_1, x_1\rangle]$. So $x_2 \in M$.

Similarly,

$$x_1 = \frac{\lambda_1\langle u_2, x_2\rangle}{B}u_1 - \frac{\lambda_2\langle u_1, x_2\rangle}{B}u_2$$

where $B = \lambda_1[\langle u_2, x_2\rangle\langle u_1, x_1\rangle - \langle u_2, x_1\rangle\langle u_1, x_2\rangle]$. This proves the claim. $\square$

Before stating the main result, we make the following conventions. For a fixed subspace M of $\mathbb{R}^d$, let $\mathsf{S}_M = \{T \in \mathsf{S} : M$ is T-invariant and M has rank(T) l.i. vectors not in ker$(T)\}$. Define $\mathcal{A}_M : \mathbb{R}^{d\times d} \to \mathbb{R}^d$ by $\mathcal{A}_M(L) = \{\langle L, (P_M e_i) \otimes (P_M e_i)\rangle_{HS}\}_{i=1}^d$, for $L \in \mathbb{R}^{d\times d}$. We have the following result.

Theorem 4.3.3. *Let M be a k-dimensional subspace of $\mathbb{R}^d$. Then M is $\mathcal{F}$-PR if and only if* $\ker(\mathcal{A}_M) \cap \mathsf{S}_M = \emptyset$.

CHAPTER 5: PHASE-RETRIEVABLE OPERATOR-VALUED FRAMES

In this chapter, we consider the phase-retrieval problem for more general system of measurements. More precisely, let $\mathcal{H}$ be a Hilbert (signal) space over $\mathbb{F}$ ($= \mathbb{C}$ or $\mathbb{R}$) and let $F_j : \mathcal{H} \to \mathbb{F}$ ($j \in \mathbb{J}$) be a system of measurement functions that satisfy the condition $F_j(\lambda x) = F_j(x)$ for all $x \in \mathcal{H}$ and every unimodular scalar $\lambda \in \mathbb{F}$. We say that $\{F_j\}_{j\in\mathbb{J}}$ does **phase-retrieval** (or, is a **phase-retrievable system**) if the measurements $\{F_j(x) : j \in \mathbb{J}\}$ uniquely determines $x \otimes x$ for every $x \in \mathcal{H}$.

In the previous chapters, we considered the case where the measurement functions were given by $F_j(x) = |\langle x, x_j\rangle|^2$ and this led to the recently well-studied problems for vector valued phase-retrievable frames $\{x_j\}_{j\in\mathbb{J}}$. Vector-valued phase-retrievable frames can also be viewed as a special case of a phase-retrievable problem from positive (or more generally self-adjoint) operator induced quadratic measurements. This chapter examines several aspects of general phase-retrievable operator-valued frames. The elementary (both in terms of statements and proofs) characterizations of such frames will be presented in connection with several applications to group representation frames, point-wisely tight frames, and almost point-wise phase-retrievable. The results presented in the following three sections have been published by the author as part of [HJ19].

5.1 Phase-Retrievable Operator-Valued Frames

Notation: $\mathcal{A}$ denotes a von Neumann algebra acting on $\mathcal{H}$ and $\mathcal{S}(\mathcal{A})$ denotes the (quantum) state space of $\mathcal{A}$; that is, the set of all normal positive linear functionals f such that $f(I) = 1$.

Definition 5.1.1. Let $\{T_j\}_{j\in\mathbb{J}}$ be an operator system in $\mathcal{A}$. We say that $\{T_j\}_{j\in\mathbb{J}}$ is

(i) an *operator-valued frame* for $\mathcal{H}$ if $\sum_{j\in\mathbb{J}} T_j^*T_j$ is bounded and invertible;

(ii) *phase-retrievable from quadratic measurement* (QM-phase-retrievable for short) if $\{\langle x, T_j x\rangle\}_{j\in\mathbb{J}}$ uniquely determines $x \otimes x$ for every $x \in \mathcal{H}$;

(iii) *state-retrievable* if $\{\rho(T_j)\}_{j\in\mathbb{J}}$ uniquely determines ρ for every $\rho \in \mathcal{S}(\mathcal{A})$.

Recall that a (vector-valued) frame for a Hilbert space $\mathcal{H}$ is a sequence $\{x_j\}_{j\in\mathbb{J}}$ such that

$$AI \leq \sum_{j\in\mathbb{J}} x_j \otimes x_j \leq BI$$

for some constants $A, B > 0$. Therefore if we let $T_j = x_j \otimes x_j$ then $\{T_j\}_{j\in\mathbb{J}}$ is a QM-phase-retrievable operator-valued frame if and only if $\{x_j\}_{j\in\mathbb{B}}$ is a phase-retrievable frame. The following is an easy consequence from the definition.

Proposition 5.1.2. *Let $\{T_j\}_{j\in\mathbb{J}}$ be an operator family in $\mathcal{A}$ such that $\sum_{j\in\mathbb{J}} T_j^* T_j$ is bounded.*

(i) If $\{T_j\}_{j\in\mathbb{J}}$ is state-retrievable , then it is QM-phase-retrievable.

(ii) If $\dim\mathcal{H} < \infty$, then the QM-phase-retrievability of $\{T_j\}_{j\in\mathbb{J}}$ implies that it is an operator-valued frame

Proof. (i) follows from the fact that $\rho_x(T) = \langle Tx, x\rangle$ defines a normal positive linear functional on $\mathcal{A}$ for every vector $x \in H$. For (ii), assume that $\{T_j\}$ is not an operator-valued frame. Then there exists an nonzero vector $x \in H$ such that $\sum_{j\in\mathbb{J}} T_j^* T_j x = 0$, which implies that $T_j x = 0$ for every $j \in \mathbb{J}$. Hence we get $\langle x, T_j x\rangle = 0$ for each j and therefore $\{T_j\}_{j\in\mathbb{J}}$ is not QM-phase-retrievable. □

Remark 5.1.3. Even in the finite-dimensional case, it is easy to construct examples showing that the converses of the statements in the above proposition are false.

It is easy to prove that the complement property is equivalent to the condition that

$$\overline{\text{span}}\{\langle x, x_j\rangle x_j : j \in \mathbb{J}\} = \mathcal{H}$$

for every nonzero vector $x \in \mathcal{H}$. The following slightly more general statement (for non-self-adjoint operators $T_j = x_j \otimes y_j$) remains to be true.

Proposition 5.1.4. *Let $T_j = x_j \otimes y_j$ with $x_j, y_j \in \mathcal{H}$ ($j \in \mathbb{J}$). Then the following are equivalent:*

(i) $\overline{\text{span}}\{T_j x : j \in \mathbb{J}\} = \mathcal{H}$ for every nonzero vector $x \in \mathcal{H}$;

(ii) For any $\Omega \subseteq \mathbb{J}$ we have either $\overline{\text{span}}\{x_j\}_{j\in\Omega} = \mathcal{H}$ or $\overline{\text{span}}\{y_j\}_{j\in\Omega^c} = \mathcal{H}$.

Proof. (i) $\Rightarrow$ (ii): Suppose that (ii) is false. Then there exists $\Omega \subseteq \mathbb{J}$ such that

$$\overline{\text{span}}\{x_j : j \in \Omega\} \neq \mathcal{H} \text{ and } \overline{\text{span}}\{y_j : j \in \Omega^c\} \neq \mathcal{H}.$$

Pick a nonzero vector $x \in \mathcal{H}$ such that $x \perp y_j$ for all $j \in \Omega^c$. Then we get

$$\overline{\text{span}}\{T_j x : j \in \mathbb{J}\} = \overline{\text{span}}\{\langle x, y_j\rangle x_j : j \in \Omega\} \neq \mathcal{H}.$$

Thus (i) implies (ii).

(ii) $\Rightarrow$ (i): Let $x \in \mathcal{H}$ be a nonzero vector. Consider $\Omega = \{j \in \mathbb{J} : \langle x, y_j\rangle = 0\}$. Then clearly $\overline{\text{span}}\{y_j : j \in \Omega\} \neq \mathcal{H}$. Thus we must have that $\overline{\text{span}}\{x_j : j \in \Omega^c\} = \mathcal{H}$. Note that $\langle x, y_j\rangle \neq 0$ for evry $j \in \Omega^c$. Thus we get

$$\overline{\text{span}}\{T_j x : j \in \mathbb{J}\} = \overline{\text{span}}\{\langle x, y_j\rangle x_j : j \in \mathbb{J}\} = \overline{\text{span}}\{x_j : j \in \Omega^c\} = \mathcal{H}.$$

$\square$

Condition (i) (and its variations) of Proposition 5.1.4 can be stated for any operator-valued frames and in fact it has been used in the characterization of phase-retrievable frames of self-adjoint operators (c.f. [CEHV15], [WX17]). Here we generalize some of those to arbitrary operator-valued frames.

Lemma 5.1.5. *An operator-valued frame $\{T_j\}_{j\in\mathbb{J}}$ is not QM-phase-retrievable if and only if there exist nonzero vectors x, y such that $x \notin i\mathbb{R}y$ and*

$$\langle x, T_j y\rangle + \overline{\langle x, T_j^* y\rangle} = 0$$

for $j \in \mathbb{J}$.

(Note that in the real Hilbert space case, the condition $x \notin i\mathbb{R}y$ is automatically satisfied)

Proof. For $x, y \in \mathcal{H}$, write $u = x + y$ and $v = x - y$. Then we have that $\langle u, T_j u\rangle = \langle v, T_j v\rangle$ if and only if $\langle x, T_j y\rangle + \overline{\langle x, T_j^* y\rangle} = 0$.

Now first assume that that $\{T_j\}_{j\in\mathbb{J}}$ is not QM-phase-retrievable. Then there exists u, v such that $u \otimes u \neq v \otimes v$ but $\langle u, T_j u\rangle = \langle v, T_j v\rangle$ for every j. Let $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Then $u = x + y$ and $v = x - y$, and $\langle x, T_j y\rangle + \overline{\langle x, T_j^* y\rangle} = 0$. Clearly x and y are nonzero. If $x = iay$ for some $a \in \mathbb{R}$ in the complex Hilbert space case, then we have $u = (1 + ia)y$ and $v = (1 - ia)y$, which implies that $u \otimes u = v \otimes v$. Thus we also have $x \notin i\mathbb{R}y$.

Conversely, assume that $\langle x, T_j y\rangle + \overline{\langle x, T_j^* y\rangle} = 0$ for some nonzero vectors x and y with $x \notin i\mathbb{R}y$. Then $\langle u, T_j u\rangle = \langle v, T_j v\rangle$ with $u = x+y$ and $v = x-y$. If $u\otimes u = v\otimes v$, then u and v are linearly dependent and $||u|| = ||v||$, which implies that x and y are linearly dependent. Write $x = cy$ with $c \neq 0$. On the other hand, from $||u|| = ||v||$, we also get that $c = 0$ in the real Hilbert space case and $c = ia$ for some $a \in \mathbb{R}$ in the complex Hilbert space case, both lead to contradictions. Thus

we obtain that $u \otimes u \neq v \otimes v$ and so $\{T_j\}_{j\in\mathbb{J}}$ is not QM-phase-retrievable. $\square$

5.2 Characterizations

Proposition 5.2.1. *Let $\{T_j\}_{j\in\mathbb{J}}$ be an operator-valued frame for a* **real** *Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) $\{T_j\}_{j\in\mathbb{J}}$ is QM-phase-retrievable.

(ii) $\overline{\mathrm{span}}\{(T_j + T_j^*)x\}_{j\in\mathbb{J}} = \mathcal{H}$ for every nonzero vector $x \in \mathcal{H}$.

In the finite-dimensional and $|\mathbb{J}| < \infty$ case, this condition is also equivalent to:

(iii) $\sum_{j\in\mathbb{J}}(T_j + T_j^*)(x \otimes x)(T_j + T_j^*)$ is invertible for every nonzero vector $x \in \mathcal{H}$.

Proof. Clearly (ii) and (iii) are equivalent in the finite-dimensional case. Write $S_j = T_j + T_j^*$. Then $\langle x, S_j y\rangle + \langle x, S_j^* y\rangle = 2(\langle x, T_j y\rangle + \langle x, T_j^* y\rangle)$. Thus, by Lemma 5.1.5, we have that $\{T_j\}$ is QM-phase-retrievable if and only if $\{T_j + T_j^*\}$ is QM-phase-retrievable. Therefore we can assume that $T_j^* = T_j$. By Lemma 5.1.5 we have that $\{T_j\}$ is not QM-phase-retrievable if and only if there exist nonzero vectors x, y such that $\langle y, T_j x\rangle = 0$. Thus we establish the equivalence of (i) and (ii). $\square$

Remark 5.2.2. If we do not assume that T_j is self-adjoint, then the condition that span$\{T_j x\} = \mathcal{H}$ for every nonzero vector $x \in \mathcal{H}$ is neither necessary nor sufficient for the QM-phase-retrievabilty of $\{T_j\}$. For example, in $\mathbb{R}^2$, let $T_1 = e_1 \otimes e_1$, $T_2 = e_2 \otimes e_2$ and $T_3 = e_1 \otimes e_2$. Then it is easy to verify that span$\{(T_j + T_j^*)x\}_{j=1}^3 = \mathbb{R}^2$ for every nonzero vector $x \in \mathbb{R}^2$. Thus, by Proposition 5.2.1, $\{T_j\}_{j=1}^3$ is QM-phase-retrievable. However, we also have span$\{T_j x\}_{j=1}^3 = \mathbb{R}e_1$ for every nonzero vector x.

Conversely, if we let $T_1 = e_1 \otimes e_1$, $T_2 = e_2 \otimes e_2$ and $T_3 = e_1 \otimes e_2 - e_2 \otimes e_1$. Then $\sum_{j=1}^{3} T_j(x \otimes x)T_j^*$ is invertible for every $x \neq 0$. However, by using the fact that $T_3 + T_3^* = 0$, we get that span$\{(T_j + T_j^*)x\}_{j=1}^{3} \neq \mathbb{R}^2$ for $x = e_1$, which shows by Proposition 5.2.1 that $\{T_j\}$ is not QM-phase-retrievable.

For complex Hilbert spaces, without losing the generality, we will work on the concrete space $\mathcal{H} = \ell_2^d(\mathbb{C})$, where $\ell_2^d(\mathbb{C}) = \mathbb{C}^d$ when d is finite and $\ell_2^d(\mathbb{C}) = \ell_2(\mathbb{C})$ is the space of square-summable sequences when $d = \infty$. Similarly $\ell_2^d(\mathbb{R})$ represents the real Hilbert space. For every vector $x \in \ell_2^d(\mathbb{C})$, we write $x = Re(x) + iIm(x)$ with $Re(x), Im(x) \in \ell_2^d(\mathbb{R})$. A closed subspace of a Hilbert space W is said to have co-dimension one if $\dim W^{\perp} = 1$.

Proposition 5.2.3. *An operator-valued frame for $\ell_2^d(\mathbb{C})$ is QM-phase-retrievable if and only if for every nonzero vector $x \in \ell_2^d(\mathbb{C})$ the subspace*

$$W_x := \overline{\text{span}}\{Re((T_j + T_j^*)x) \oplus Im((T_j + T_j^*)x), Im((T_j^* - T_j)x) \oplus Re((T_j - T_j^*)x) : j \in \mathbb{J}\}$$

has co-dimension one in $\ell_2^d(\mathbb{R}) \oplus \ell_2^d(\mathbb{R})$.

Proof. A simple calculation shows that

$$\begin{aligned} \langle y, T_j x\rangle + \overline{\langle y, T_j^* x\rangle} &= \langle Re(y) \oplus Im(y), Re((T_j + T_j^*)x) \oplus Im((T_j + T_j^*)x)\rangle \\ &\quad + i\langle Re(y) \oplus Im(y), Im((T_j^* - T_j)x) \oplus Re((T_j - T_j^*)x)\rangle. \end{aligned}$$

Thus $\langle y, T_j x\rangle + \overline{\langle y, T_j^* x\rangle} = 0$ for all j if and only if $Re(y) \oplus Im(y)$ is orthogonal to W_x.

If $y = iax$ for some nonzero vector x and nonzero scalar $a \in \mathbb{R}$, then $\langle y, T_j x\rangle + \overline{\langle y, T_j^* x\rangle} = 0$. Note that $Re(y) \oplus Im(y) = a((-Im(x)) \oplus Re(x))$. Thus span$\{(-Im(x)) \oplus Re(x)\}$ is a one-dimensional subspace of the $W_x^{\perp}$. Thus $\dim W_x^{\perp} \geq 1$.

On the other hand, we also have $\dim W_x^\perp > 1$ if and only if there exist vectors $u, v \in \ell_2^d(\mathbb{R})$ such that $u \oplus v \notin$ span$\{(-Im(x)) \oplus Re(x)\}$ and $u \oplus v \in W_x^\perp$, which, in turn, is equivalent to the condition that there exists $y = u + iv \notin i\mathbb{R}x$ such that $\langle y, T_j x\rangle + \overline{\langle y, T_j^* x\rangle} = 0$ for every $j \in \mathbb{J}$. Therefore we get that for a nonzero vector x, $\dim W_x^\perp = 1$ if and only if there exists no nonzero vector y such that $\langle y, T_j x\rangle + \overline{\langle y, T_j^* x\rangle} = 0$ for every $j \in \mathbb{J}$. Hence, Lemma 5.1.5 implies that $\{T_j\}_{j\in\mathbb{J}}$ is QM-phase-retrievable if and only if the following W_x has co-dimension one for every nonzero vector $x \in \ell_2^d(\mathbb{C})$. $\square$

Remark 5.2.4. Although we have assumed that $\{T_j\}_{j\in\mathbb{J}}$ is a finite or countable set, from their proofs it is clear that all the above results remain to be true even when the index set $\mathbb{J}$ is not countable (in this case we drop off the requirement that $\{T_j\}_{j\in\mathbb{J}}$ is an operator-valued frame).

In the case that $\mathcal{H}$ is finite-dimensional and each T_j is self-adjoint we get the following consequence that was due to Wang and Xu [WX17] (also see the work of P. Casazza and his collaborators for the case where each T_j is an orthogonal projection).

Corollary 5.2.5. *Let $\mathcal{H} = \mathbb{C}^d$ and assume that each $T_j \in B(\mathcal{H})$ is self-adjoint. Then $\{T_j\}$ is QM-phase-retrievable if and only if*

$$\dim \operatorname{span}\{Re(T_j x) \oplus Im(T_j x) : j \in \mathbb{J}\} = 2d - 1$$

holds for every nonzero $x \in \mathbb{C}^d$. In particular $|\mathbb{J}| \geq 2d - 1$ if $\{T_j\}_{j\in\mathbb{J}}$ does QM-phase-retrieval.

Remark 5.2.6. Note that for a finite sequence $\{z_j\}_{j=1}^k$, span$\{z_j\}$ is m-dimensional if and only if rank$(\sum_{j=1}^k z_j \otimes z_j) = m$. Therefore for self-adjoint matrices $T_j \in M_d(\mathbb{C})$, we have that $\{T_j\}_{j=1}^k$ is QM-phase-retrievable if and only if

$$\text{rank} \begin{bmatrix} \sum_{j=1}^k Re(T_j x) \otimes Re(T_j x) & \sum_{j=1}^k Re(T_j x) \otimes Im(T_j x) \\ \sum_{j=1}^k Im(T_j x) \otimes Re(T_j x) & \sum_{j=1}^k Im(T_j x) \otimes Im(T_j x) \end{bmatrix} = 2d - 1$$

for every nonzero vector $x \in \mathbb{C}^d$.

The following corollary is a simple consequence of the characterizations of QM-phase-retrievable operator-valued frames.

Corollary 5.2.7. *Let $\{T_j\}_{j\in\mathbb{J}}$ be an operator-valued frame on a Hilbert space H. Then QM-phase-retrievability of $\{T_j + T_j^*\}$ implies the QM-phase-retrievability of $\{T_j\}$. The converse is also true for real Hilbert case but false for the complex case.*

Proof. The real Hilbert space case follows from the proof of Proposition 5.2.1, and the complex Hilbert space case follows from Proposition 5.2.3. For a counterexample of the converse, let $H = \mathbb{C}^2$, $T_1 = i(e_1 \otimes e_1), T_2 = i(e_2 \otimes e_2), T_3 = e_1 \otimes e_2$ and $T_4 = e_2 \otimes e_1$. Then span$\{T_j\}_{j=1}^4 = M_{2\times 2}(\mathbb{C})$. Thus $\{T_j\}_{j=1}^4$ is QM-phase-retrievable. However, by Proposition 5.2.3 , we get that $\{T_j + T_j^*\}_{j=1}^4 = \{0, 0, T_4, T_3\}$ is not QM-phase-retrievable. $\square$

5.3 Point-Wise Tight Phase-Retrievable Operator-Valued Frames

For an operator-valued frames $\{T_j\}_{j\in\mathbb{J}}$, we establish in this section some connections among the phase-retrievability of the operator-valued frames, the (almost everywhere) point-wise phase-retrievability, and point-wise tight frame property for $\{T_j\}$. We will assume that $\mathcal{H} = \mathbb{R}^d$ or $\mathbb{C}^d$ is finite-dimensional and $\mathbb{J} = \{1, ..., N\}$ is finite.

For the real Hilbert space $\mathbb{R}^d$ case, it is easy to prove that $\{T_jx\}_{j=}^N$ is phase-retrievable for some x if and only if $\{T_jx\}_{j=}^N$ is phase-retrievable for any generic vector x. Moreover, if $T_j = x_j \otimes x_j$, then $\{T_j\}_{j=1}^N$ is QM-phase-retrievable if and only if and if $\{T_jx\}_{j=}^N$ is phase-retrievable for some $x \in \mathcal{H}$. However, this is no longer true in general as demonstrated by the following example.

Example 5.3.1. Again we use the example of Z. Xu [Xu18] which is a QM-phase-retrievable operator-valued frame with six Hermitian operators $\{T_j\}_{j=1}^6$ for $\mathbb{R}^4$. Clearly $\{T_j\xi\}_{j=}^6$ is not phase retrievable for any $\xi \in \mathbb{R}^4$ since it requires at least 7 vectors for a vector-valued frame to be phase-retrievable for $\mathbb{R}^4$.

Conversely, let $\mathcal{H} = \mathbb{R}^2$ and $\{e_1, e_2\}$ be its standard orthonormal basis. Define $T_1 = e_1 \otimes e_1 + e_2 \otimes e_2$, $T_2 = e_1 \otimes e_1 + 2e_2 \otimes e_2$ and $T_3 = e_1 \otimes e_1 + 3e_2 \otimes e_2$. Then $\{T_1, T_2, T_3\}$ is an operator-valued frame for $\mathbb{R}^2$. For $x = e_1$ we have span$\{T_1x, T_2x, T_3x\} = \mathbb{R}e_1 \neq \mathbb{R}^2$. Thus, by Proposition 5.2.1, $\{T_1, T_2, T_3\}$ is not QM-phase-retrievable. However, for $x = e_1 + e_2$ we have $\{T_1x, T_2x, T_3x\} = \{e_1 + e_2, e_1 + 2e_2, e_1 + 3e_2\}$, which is clearly is phase-retrievable for $\mathbb{R}^2$.

If an operator-valued frame $\{T_j\}_{j\in\mathbb{J}}$ has the property that $\{T_jx\}_{j\in\mathbb{J}}$ is phase-retrievable for some $x \in \mathcal{H}$ (and hence for almost all $x \in \mathcal{H}$), then we say that $\{T_j\}_{j\in\mathbb{J}}$ is **almost everywhere point-wise phase-retrievable**. The above example naturally leads to the the following question: Can we characterize all the operator-valued frames$\{T_j\}_{j\in\mathbb{J}}$ such that $\{T_j\}_{j\in\mathbb{J}}$ is QM-phase-retrievable if and only if $\{T_j\}_{j\in\mathbb{J}}$ is almost everywhere point-wise phase-retrievable?

The following was recently proved by Y. Wang and Z. Xu.

Theorem 5.3.2 ([WX17], Theorem 4.1)**.** *Let $N \geq 2d - 1$ Then a generic operator-valued frame $A = (A_1, ..., A_N)$ of Hermitian matrices has the phase retrieval property for $\mathbb{R}^d$.*

By Corollary 5.2.7 and the above theorem we immediately get

Corollary 5.3.3. *Let $N \geq 2d - 1$ Then a generic operator-valued frame $A = (A_1, ..., A_N)$ has the phase retrieval property for $\mathbb{R}^d$.*

Next we prove that the same result holds for almost everywhere point-wise phase-retrievable operator-valued frames.

Theorem 5.3.4. *Assume that $N \geq 2d-1$. Let $\mathcal{P}$ the set of all n-tuples $(A_1, ..., A_N)$, where $A_j \in M_d(\mathbb{R})$, such that $\{A_j x\}_{j=1}^N$ is phase-retrievable for some $x \in \mathbb{R}^d$. Then $\mathcal{P}$ is open dense in the direct sum space $M_d(\mathbb{R}) \oplus ... \oplus M_d(\mathbb{R})$ (N-copies).*

Proof. Write $A = (A_1, ..., A_N)$. Let $\{x_j\}_{j=1}^N$ be a phase-retrievable frame for $\mathbb{R}^d$ such that $x_j \neq 0$ for each j. Set $A_j = x_j \otimes x_j$, and pick $x \in \mathbb{R}^d$ such that $\langle x, x_j \rangle \neq 0$ for every j. Then clearly $\{A_j x\}$ is phase-retrievable and hence $\mathcal{P}$ is nonempty.

Now let $A = (A_1, ..., A_N) \in \mathcal{P}$ and $x \in \mathbb{R}^d$ be such that $\{A_j x\}$ is phase-retrievable. We clearly can assume that $||x|| = 1$. Since the set of all the phase-retrievable vector-valued frames of length N is open in $\mathbb{R}^d \oplus ... \oplus \mathbb{R}^d$, there exists $\delta > 0$ such that $\{y_j\}_{j=1}^N$ is phase-retrievable whenever $\sum_{j=1}^N ||A_j x - y_j||^2 < \delta$. This implies that if $\sum_{j=1}^N ||A_j - B_j||^2 < \delta$, then $\{B_j x\}_{j=1}^N$ is phase-retrievable and consequently $B = (B_1, ..., B_N) \in \mathcal{P}$. Thus $\mathcal{P}$ is open.

For density, let $B = (B_1, ..., B_N) \in M_d(\mathbb{R}) \oplus ... \oplus M_d(\mathbb{R})$ be an arbitrary element and let $A = (A_1, ..., A_N) \in \mathcal{P}$ be a fixed element with $\{A_j x\}$ being phase-retrievable for some $x \in \mathbb{R}^d$. Consider $C(t) = tA + (1-t)B$. We show that $C(t)$ is in $\mathcal{P}$ for all but finitely many number of $t's$, which will imply that B is a limit point of $\mathcal{P}$. Since $\{A_j x\}$ is phase-retrievable, we have that either $\text{span}\{A_j x : j \in \Lambda\} = \mathbb{R}^d$ or $\text{span}\{A_j x : j \in \Lambda^c\} = \mathbb{R}^d$ for every $\Lambda \subseteq \{1, ..., N\}$. Thus we can associate every Λ with a set $\Phi(\Lambda)$ of cardinality d such that it is either a subset of Λ or a subset of Λ^c and $\det[A_j x]_{j \in \Phi(\Lambda)} \neq 0$. Define

$$f_\Lambda(t) = \det[(tA_j + (1-t)B_j)x]_{j \in \Phi(\Lambda)}.$$

Then these are nonzero polynomials since $f_\Lambda(1) \neq 0$ for every Λ. By the complement property for phase-retrievable frames, we clearly have that $\{tA_j + (1-t)B_j)x\}_{j=1}^N$ is phase retrievable if $f_\Lambda(t) \neq 0$ for every Λ. Since the union of the zero sets of f_Λ is finite, we conclude that $C(t)$ is in

$\mathcal{P}$ for all but finitely many number of $t's$. $\square$

We have seen that the characterizations of QM-phase-retrievable frames are much more simpler for frames of self-adjoint operators. However this might be too restrictive since, there are many useful and interesting examples (e.g. frames of unitary operators) that do not fall into this category. In what follows, we will call an operator family $\mathcal{S}$ a **self-adjoint family** if $T \in \mathcal{S}$ implies $T^* \in \mathcal{S}$.

Lemma 5.3.5. *Let $\{T_j\}_{j=1}^N$ be a self-adjoint family. If $\{T_j\}_{j=1}^N$ is QM-phase-retrievable, then for every nonzero vector $x \in \mathcal{H}$ we have that $\sum_{j=1}^N T_j(x \otimes x)T_j^*$ is invertible.*

Proof. We only need to prove for the complex Hilbert space case. Assume that $\sum_{j=1}^N T_j(x \otimes x)T_j^*$ is not invertible for some $x \neq 0$. Then there exists $y \neq 0$ such that $\langle y, T_j x\rangle = 0$ for every j. If $y = iax$ for some $0 \neq a \in \mathbb{R}$, then we get $\langle x, T_j^* x\rangle = 0$, which implies that $\{T_j\}$ is not QM-phase-retrievable. So we have that $y \notin i\mathbb{R}x$. Since $\mathcal{S}$ is self-adjoint we obtain that that $\langle y, T_j^* x\rangle = 0$ for every j. Thus $\langle y, T_j x\rangle + \overline{\langle y, T_j^* x\rangle} = 0$, which implies by Lemma 5.1.5 that $\{T_j\}$ is not QM-phase-retrievable. This contradiction shows that $\sum_{j=1}^N T_j(x \otimes x)T_j^*$ must be invertible for every $x \neq 0$. $\square$

Remark 5.3.6. The converse of the above lemma is not true. For the complex case, let $\{x_j\}_{j=1}^N$ be a frame for $\mathbb{C}^d$ such that it has the complement property but not phase-retrievable (existence of such a frame is guaranteed for complex Hilbert spaces). Let $T_j = x_j \otimes x_j$. Clearly $\{T_j\}_{j=1}^N$ is a self-adjoint family, $\sum_{j=1}^N T_j(x \otimes x)T_j^*$ is invertible for every nonzero vector x and $\{T_j\}$ is not QM-phase-retrievable. However, this phenomenon can not happen for some well-structured operator-valued frames. Here we examine the example of projective unitary group representation frames. For a counterexample for real space case, we use the modified example of Remark 5.2.2: Conversely, if we let $T_1 = e_1 \otimes e_1$, $T_2 = e_2 \otimes e_2$, $T_3 = e_1 \otimes e_2 - e_2 \otimes e_1$ and $T_4 T_3^*$. Then $\sum_{j=1}^4 T_j(x \otimes x)T_j^*$ is invertible for every $x \neq 0$. However, by using the fact that $T_3 + T_3^* = 0 = T_4 + T_4^*$, we get

that span$\{(T_j + T_j^*)x\}_{j=1}^4 \neq \mathbb{R}^2$ for $x = e_1$, which shows by Proposition 5.2.1 that the self-adjoint family $\{T_j\}_{j=1}^4$ is not QM-phase-retrievable.

We would like to see how much of the previous proposition can be generalized to more general operator-valued frames. For this purpose we introduce:

Definition 5.3.7. We say an operator family $\{T_j\}_{j\in\mathbb{J}}$ is *point-wisely tight* if for every $x \neq 0$, $\{T_j x\}_{j\in\mathbb{J}}$ is a tight frame for $\mathcal{H}$, i.e., $\sum_{j\in\mathbb{J}} T_j x \otimes T_j x = \lambda_x I$ for some $\lambda_x > 0$.

Theorem 5.3.8. *Let $\{T_j\}_{j\in\mathbb{J}}$ be an operator family for a complex Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) There exists a positive invertible operator B such that $\{T_j B\}_{j\in\mathbb{J}}$ is a Parserval frame for $B(\mathcal{H})$.

(ii) $\{T_j\}_{j\in\mathbb{J}}$ is point-wisely tight.

(iii) For every $x, y \in \mathcal{H}$, there exists $\lambda_{x,y}$ such that

$$\sum_{j\in\mathbb{J}} T_j(x\otimes y)T_j^* = \lambda_{x,y} I,$$

and $\lambda_{x,x} > 0$ when $x \neq 0$.

Proof. $(i) \Rightarrow (ii)$: Suppose that there exists a positive invertible operator B such that $\{T_j B\}$ is a Parserval frame for $B(\mathcal{H})$. Then for any $T \in B(\mathcal{H})$ we have

$$\sum_{j\in\mathbb{J}} \langle T, T_j B\rangle T_j B = T.$$

Note that

$$\langle TB^{-1}, T_j B\rangle = Tr(TB^{-1}(T_j B)^*) = Tr(TT_j^*) = \langle T, T_j\rangle.$$

Thus, by replacing T with TB^{-1}, we get $\sum_{j\in\mathbb{J}}\langle T, T_j\rangle T_j B = TB^{-1}$ and hence we have $\sum_{j\in\mathbb{J}}\langle T, T_j\rangle T_j = T(B^{-1})^2$.

Let $S = (B^{-1})^2$. Now for a fixed $x \neq 0$, define

$$A = \sum_{j\in\mathbb{J}} T_j x \otimes T_j x.$$

Then for any $z, y \in H$ we have that

$$\begin{aligned}
\langle Az, y\rangle &= \sum_{j\in\mathbb{J}}\langle z, T_j x\rangle \cdot \langle T_j x, y\rangle = \sum_{j\in\mathbb{J}}\langle z\otimes x, T_j\rangle\langle T_j, y\otimes x\rangle \\
&= \langle\sum_{j\in\mathbb{J}}\langle z\otimes x, T_j\rangle T_j, y\otimes x\rangle \\
&= \langle (z\otimes x)S, y\otimes x\rangle = \langle x, Sx\rangle\langle z, y\rangle.
\end{aligned}$$

This implies that $Az = \langle x, Sx\rangle z$ for any $z \in \mathcal{H}$, and hence $A = \langle x, Sx\rangle I$. Therefore $\{T_j\}_{j\in\mathbb{J}}$ is point-wisely tight.

$(ii) \Rightarrow (iii)$: Assume that $\sum_{j\in\mathbb{J}} T_j x \otimes T_j x = \lambda_x I$ for each x with $\lambda_x > 0$ when $x \neq 0$. Now fix any $x, y \in H$. We have

$$\begin{aligned}
\lambda_{x+y} I &= \sum_{j\in\mathbb{J}} T_j(x+y)\otimes T_j(x+y) \\
&= (\lambda_x + \lambda_y)I + \sum_{j\in\mathbb{J}} T_j y\otimes T_j x + \sum_{j\in\mathbb{J}} T_j x \otimes T_j y,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{x+iy} I &= \sum_{j\in\mathbb{J}} T_j(x+iy)\otimes T_j(x+iy) \\
&= (\lambda_x + \lambda_y)I + i\sum_{j\in\mathbb{J}} T_j y\otimes T_j x - i\sum_{j\in\mathbb{J}} T_j x \otimes T_j y,
\end{aligned}$$

This implies that

$$\sum_{j\in\mathbb{J}} T_j x \otimes T_j y = \frac{1}{2}[\lambda_{x+y} + i\lambda_{x+iy} - (1+i)(\lambda_x + \lambda_y)]I.$$

Thus we get (iii) by setting $\lambda_{x,y} = \frac{1}{2}[\lambda_{x+y} + i\lambda_{x+iy} - (1+i)(\lambda_x + \lambda_y)]$.

$(iii) \Rightarrow (i)$: Assume that

$$\sum_{j\in\mathbb{J}} T_j(x\otimes y)T_j^* = \lambda_{x,y} I.$$

and $\lambda_{x,x} > 0$ when $x \neq 0$. Then clearly $\lambda_{x,y}$ defines a positive bilinear form on $\mathcal{H}$, and hence there exists a positive invertible operator S such that $\lambda_{x,y} = \langle Sx, y\rangle$ for all $x, y \in \mathcal{H}$. Let $B = S^{-1/2}$. A simple calculation shows that (iii) implies

$$\sum_{j\in\mathbb{J}} \langle u\otimes v, T_j B\rangle T_j B = u \otimes v$$

for all $u, v \in \mathcal{H}$. Since span$\{u\otimes v : u, v \in H\} = B(\mathcal{H})$, we get that $\{T_j B\}_{j\in\mathbb{J}}$ is a Parserval frame for $B(\mathcal{H})$. □

From the proof of Theorem 5.3.8, we have that $\lambda_{x,y} = \langle B^{-1/2}x, y\rangle$ when $\{T_j B\}_{j\in\mathbb{J}}$ is a Parserval frame for $B(\mathcal{H})$. Thus we obtain the following consequence:

Corollary 5.3.9. *Let $\{T_j\}_{j\in\mathbb{J}}$ be an operator family for a complex Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) $\{T_j\}_{j\in\mathbb{J}}$ is a tight frame for $B(\mathcal{H})$.

(ii) There exists $\lambda > 0$ such that $\sum_{j\in\mathbb{J}} T_j(x\otimes x)T_j^* = \lambda\langle x, x\rangle I$.

(iii) There exists $\lambda > 0$ such that $\sum_{j\in\mathbb{J}} T_j(x\otimes y)T_j^* = \lambda\langle x, y\rangle I$.

Using the fact that every operator is a linear combination of rank-one operators, we also have:

Corollary 5.3.10. *An operator family $\{T_j\}_{j\in\mathbb{J}}$ is a point-wisely tight operator system for a complex Hilbert space $\mathcal{H}$ if and only if there exists a positive operator B such that*

$$\sum_{j\in\mathbb{J}} T_j A T_j^* = tr(AB)I$$

for every $A \in B(\mathcal{H})$.

Remark 5.3.11. In the real Hilbert space case, $\lambda_{x,y}$ in (iii) is positive symmetric. So it is obvious from the proof of Theorem 5.3.8 that we still have the equivalence between (i) and (iii).

CHAPTER 6: PHASE-RETRIEVABLE PROJECTIVE REPRESENTATION FRAMES

We know that when a frame has the maximal span property, we can perform linear phase-less reconstruction. In this chapter we prove that every irreducible projective representation of a finite abelian group admits a frame with maximal span property. This generalizes results obtained by Bojarovska and Flinth for Gabor frames [BojFli16]. The results presented here have been published by the author in [LJBCH19].

Definition 6.0.1. Let G be a group and let $U(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$. We say that the mapping $\pi : G \to U(\mathcal{H})$ is a *projective unitary representation* if there exist a function $\mu : G \times G \to \mathbb{T}$ such that

$$\pi(g)\pi(h) = \mu(g,h)\pi(gh) \quad \text{for all } g, h \in G.$$

μ is called a *multiplier* of π and π is said to be a μ-*projective* unitary representation. π is called *irreducible* if span$\{\pi(g) : g \in G\} = B(\mathcal{H})$, the set of all operators on $\mathcal{H}$.

Definition 6.0.2. Let π be a projective group representation. A vector $x \in \mathcal{H}$ is called a π-*maximal spanning vector* if $\{\pi(g)x\}_{g\in G}$ has the maximal span property.

Let $\mathcal{M}_\pi$ denote the set of all π-maximal spanning frame vectors. We have the following result.

Proposition 6.0.3. *If $\mathcal{M}_\pi$ is not empty, then it is an open dense subset of $\mathcal{H}$.*

Proof. We can assume that $\mathcal{H} = \mathbb{F}^d$. Let $\mathbb{H}_d$ be the space spanned by the all the Hermitian operators and let $\{B_1, ..., B_k\}$ be a basis for $\mathbb{H}_d$ (note that $k = d(d+1)/2$ if $\mathbb{F} = \mathbb{R}$ and $k = d^2$ if $\mathbb{F} = \mathbb{C}$).

For each $g \in G$ and $x = (x_1, ...x_d)$, write

$$\pi(g)x \otimes \pi(g)x = \sum_{i=1}^{k} c_{i,g}(x) B_i.$$

The $c_{i,g}(x)$'s are polynomials in the entries of x when $\mathbb{F} = \mathbb{R}$ and they are polynomials in the real and imaginary parts of the entries of x when $\mathbb{F} = \mathbb{C}$. Let $C(x) = [c_{i,g}]_{k \times |G|}$ and let P_Λ be the determinant of the submatrix of $C(x)$ indexed by $\Lambda \subset G$, where $|\Lambda| = k$. Each P_Λ is a polynomial in the entries of x or a polynomial in the real and imaginary parts of x. Since $x \in \mathcal{M}_\pi$ if and only if rank$(C(x)) = k$, we have that

$$\mathcal{M}_\pi = \mathcal{H} \setminus \cap_{\Lambda \subset G, |\Lambda| = k} Z(P_\Lambda),$$

where $Z(P) = \{x \in \mathcal{H} : P(x) = 0\}$.

Since $\mathcal{M}_\pi$ is nonempty, there exist $x \in \mathcal{H}$ and $\Lambda \subset G$ such that $P_\Lambda(x) \neq 0$. Since P_Λ is a nonzero polynomial, $\mathcal{H} \setminus \mathcal{M}_\pi$ is open dense in $\mathcal{H}$. $\square$

The main theorem of this chapter is the following

Theorem 6.0.4. *Suppose that π is a μ-projective unitary representation for a finite Abelian group G on an d-dimensional complex Hilbert space $\mathcal{H}$. If π is irreducible, then π admits a frame vector with the maximal span property. Moreover, $\{\pi(g)x\}_{g \in G}$ has the maximal span property if and only if $\langle \pi(g)x, x \rangle \neq 0$ for every $g \in G$.*

To prove the theorem we need a couple of lemmas and the following notation. For $x \in \mathcal{H}$, the matrix

$$A(x) = [a_{g,h}(x)]_{G\times G}$$

has entries $a_{g,h} = \langle \pi(h)\pi(g)x, \pi(g)x\rangle$.

Lemma 6.0.5. *If there exists $x \in \mathcal{H}$ such that $A(x)$ has rank d^2, then π is irreducible and $\{\pi(g)x\}_{g\in G}$ has the maximal span property.*

Proof. Let $X = \{\pi(g)\}_{g\in G}$ and $Y = \{\pi(g)x \otimes \pi(g)x\}_{g\in G}$ be two sequences in $B(\mathcal{H})$. Note that the mixed Gramian matrix $\Theta_Y\Theta_X^*$ is exactly the matrix $A(x)$ which is assumed to have rank d^2. Thus $\mathrm{rank}(\Theta_Y) \geq d^2$ and $\mathrm{rank}(\Theta_X) \geq d^2$. Since we also have $\mathrm{rank}(\Theta_Y) \leq d^2$ and $\mathrm{rank}(\Theta_X) \leq d^2$, we get that $\mathrm{rank}(\Theta_Y) = d^2 = \mathrm{rank}(\Theta_X)$, which implies that π is irreducible and $\{\pi(g)x\}_{g\in G}$ has the maximal span property. □

Lemma 6.0.6. *Suppose that π is a μ-projective unitary representation for a finite group G on an d-dimensional complex Hilbert space $\mathcal{H}$. Then there exists $x \in \mathcal{H}$ such $\langle \pi(g)x, x\rangle \neq 0$ for all $g \in G$. Moreover, the set of all such vectors x is open and dense in $\mathcal{H}$.*

Proof. We can assume that $\mathcal{H} = \mathbb{C}^d$. By the Baire-Category theorem it is enough to prove that for each $g \in G$, the set $\{x \in \mathbb{C}^d : \langle \pi(g)x, x\rangle \neq 0\}$ is open and dense in $\mathbb{C}^d$. Since $\langle \pi(g)x, x\rangle$ is a quadratic polynomial of x, we only need to point out that this is a nonzero polynomial. Indeed, if $\langle \pi(g)x, x\rangle = 0$ for all $x \in \mathbb{C}^d$, then we have $\pi(g) = 0$, which is a contradiction. □

Lemma 6.0.7. *Suppose that π is a μ-projective unitary representation for an Abelian group G. If there exists $x \in \mathcal{H}$ such that $\{\pi(g)x\}_{g\in G}$ has the maximal spanning property, then $\langle \pi(g)x, x\rangle \neq 0$ for every $g \in G$.*

Proof. Since $\{\pi(g)x\}_{g\in G}$ has the maximal spanning property we have that $\mathrm{span}\{\pi(g)x \otimes \pi(g)x : g \in G\} = B(\mathcal{H})$. So if $\langle \pi(h)x, x\rangle = 0$ for some $h \in G$, then for every $g \in G$ we get

$$\begin{aligned} |\langle \pi(h)\pi(g)x, \pi(g)x\rangle| &= |\langle \pi(g^{-1})\pi(h)\pi(g)x, x\rangle| = |\langle \pi(g^{-1}hg)x, x\rangle| \\ &= |\langle \pi(h)x, x\rangle| = 0. \end{aligned}$$

Thus $\mathrm{tr}(\pi(h)(\pi(g)x \otimes \pi(g)x)) = 0$, and so $\pi(h) = 0$ which leads to a contradiction. □

The proof of the following lemma can be found in [BB72]

Lemma 6.0.8. *Let μ be a multiplier for an abelian group G. Then all the irreducible μ-projective representations have the same representation dimension.*

Let μ be a multiplier for an abelian group G. The *symmetric multiplier matrix* is defined by $C_\mu = [c_{g,h}]$ with $c_{g,h} = \mu(g,h)\overline{\mu(h,g)}$.

Theorem 6.0.9. *Suppose that π is a μ-projective unitary representation for an abelian group G on $\mathcal{H} = \mathbb{C}^d$. Then $\mathrm{rank}(C_\mu) \leq d^2$. Moreover, π is an irreducible μ-representation if and only if $\mathrm{rank}(C_\mu) = d^2$.*

Proof. By Lemma 6.0.6, there exists $\eta \in \mathbb{C}^d$ such that

$\langle \pi(g)\eta, \eta\rangle \neq 0$ for any $g \in G$. Let $\Theta_1 : M_{d\times d}(\mathbb{C}) \mapsto \ell^2(G)$ be the analysis operator for $\{\pi(g)\}_{g\in G}$, and $\Theta_2 : M_{d\times d}(\mathbb{C}) \mapsto \ell^2(G)$ be the analysis operator for $\{\pi(g)\eta \otimes \pi(g)\eta\}_{g\in G}$. Then we have

$$\Theta_2\Theta_1^* = [\langle \pi(g)\pi(h)\eta, \pi(h)\eta\rangle]_{G\times G}.$$

Note that

$$\langle \pi(g)\pi(h)\eta, \pi(h)\eta\rangle = c_{g,h}\langle \pi(g)\eta, \eta\rangle.$$

and $\langle \pi(g)\eta, \eta\rangle \neq 0$ for every $g \in G$. So we get that

$$\mathrm{rank}(C_\mu) = \mathrm{rank}(\Theta_2\Theta_1^*) \leq \mathrm{rank}(\Theta_1) = \dim\left(\mathrm{span}\{\pi(g) : g \in G\}\right) \leq d^2.$$

Now assume that $\mathrm{rank}(C_\mu) = d^2$. Then the above inequality implies that $\dim\left(\mathrm{span}\{\pi(g) : g \in G\}\right) = d^2$, and thus π is irreducible. Conversely, let us assume that π is irreducible. We will prove that $\mathrm{rank}(C_\mu) = d^2$.

We first introduce a couple of notations: Let $\hat{G}$ be the dual group of G, and $\bar{\pi} : g \mapsto \overline{\pi(g)}$, the complex conjugation of $\pi(g)$. Then $\bar{\pi}$ is a projective representation with multiplier $\bar{\mu}$. Consider the group representation $\pi \otimes \bar{\pi} : g \mapsto \pi(g) \otimes \overline{\pi(g)}$. Then it is a projective representation with multiplier $\mu\bar{\mu} = 1$, and so it is a group representation. Hence $\pi \otimes \bar{\pi}$ can be decomposed as the direct sum of one-dimensional group representations of G. Moreover, each one dimensional representation of G appears at most once in the direct sum decomposition of $\pi \otimes \bar{\pi}$. Let $T_\mu = \{\chi \in \hat{G} : \chi \subset \pi \otimes \bar{\pi}\}$. Then T_μ is a subgroup of $\hat{G}$. Define

$$G_\mu = T_\mu^\perp = \{g \in G : \chi(g) = 1, \forall \chi \in T_\mu\}.$$

Note that $|T_\mu| = \dim\ H \times \dim\ H = d^2$. Thus $|G_\mu| = [G : G_\mu] = |T_\mu| = d^2$.

Since G is abelian, it is easy to verify that $c : G \times G \to T$ defined by $c(g,h) = c_{gh} = \mu(g,h)\overline{\mu(h,g)}$ is a bi-homomorphism, i.e., $c(gg', h) = c(g,h)c(g'h)$ and $c(g, hh') = c(g,h)c(g,h')$ for all $g, g', h, h' \in G$. This induces a homomorphism $\lambda_\mu : G \mapsto \hat{G}$. By Proposition 2.4 in [**?**] we know that

$$G_\mu = Ker(\lambda_\mu) = \{g \in G : \lambda_\mu(g) = 1\}.$$

Therefore we get

$$|\lambda_\mu(G)| = [G : Ker(\lambda_\mu)] = d^2.$$

Recall that the characters of G are linearly independent. Since each row of the symmetric multiplier matrix C_μ defines a character of G by $h \mapsto c(g,h)$, we get that the rank of C_μ is equal to the number of different characters appeared in the rows of C_μ. By the definition of λ_μ, we know that this number is exactly the cardinality of the image of λ_μ. This implies that $\mathrm{rank}(C_\mu) = |\lambda_\mu(G)| = d^2$ as claimed. □

Corollary 6.0.10. *Let μ be a multiplier of an abelian group G and $d^2 = \mathrm{rank}(C_\mu)$. Then every d-dimensional μ-projective representation π of G is irreducible.*

Proof. Let σ be an irreducible subrepresentation of π on a n-dimensional π-invariant subspace. Then, by Theorem 6.0.9, the representation dimension of σ is equal to $\mathrm{rank}(C_\mu) = n^2$. This implies that $n = d$ and thus $\sigma = \pi$. Therefore π is irreducible. □

We now prove the main theorem

Proof of Theorem 6.0.4:

Assume that π is an irreducible μ-projective representation of G on $H = \mathbb{C}^d$. By Lemma 6.0.7 we know that if $\{\pi(g)x\}_{g\in G}$ has the maximal span property, then $\langle \pi(g)x, x\rangle \neq 0$ for every $g \in G$. Therefore, to complete the proof, it suffices to show that $\{\pi(g)x\}_{g\in G}$ has the maximal span property when $\langle \pi(g)x, x\rangle \neq 0$ for every $g \in G$.

Let Θ_1 and $\Theta_2 : M_{d\times d}(\mathbb{C}) \to \ell^2(G)$ be the analysis operators defined in the proof of Theorem 6.0.9. Then we know that $\mathrm{rank}(\Theta_2\Theta_1^*) = \mathrm{rank}(C_\mu)$. Since π is irreducible, we get that $\mathrm{rank}(\Theta_1^*) = d^2$ and by Theorem 6.0.9 that $\mathrm{rank}(C_\mu) = d^2$. This implies that $\mathrm{rank}(\Theta_2\Theta_1^*) = d^2$ which implies that

$\operatorname{rank}(\Theta_2) = d^2$ since we also have $\operatorname{rank}(\Theta_2) \leq d^2$. Therefore $\{\pi(g)x \otimes \pi(g)x : g \in G\}$ spans $M_{d\times d}(\mathbb{C})$, i.e., $\{\pi(g)x\}_{g\in G}$ has the maximal span property. $\square$

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