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The Relativistic Harmonic Oscillator and the Generalization of Lewis' Invariant

Daniel Reinhart
University of Central Florida

THE RELATIVISTIC HARMONIC OSCILLATOR AND THE GENERALIZATION OF LEWIS' INVARIANT

by

DANIEL REINHART
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ABSTRACT

In this thesis, we determine an asymptotic solution for the one dimensional relativistic harmonic oscillator using multiple scale analysis and relate the resulting invariant to Lewis' invariant. We then generalize the equations leading to Lewis' invariant so they are relativistically correct. Next we attempt to find an asymptotic solution for the general equations by making simplifying assumptions on the parameter characterizing the adiabatic nature of the system. The first term in the series for Lewis' invariant corresponds to the adiabatic invariant for systems whose frequency varies slowly. For the relativistic case we find a new conserved quantity and seek to explore its interpretation.

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CHAPTER 1: INTRODUCTION

For the non-relativistic harmonic oscillator with slowly varying frequency, Lewis [1] determined an invariant that was the usual adiabatic invariant to first order. Eliezer and Gray [2] then investigated an alternate derivation of Lewis' invariant that used the two dimensional harmonic oscillator. This derivation looked at the conserved angular momentum for the system, and it was noted that Lewis' invariant coincides with half of the square of the angular momentum when the angular momentum is taken to have unit value. Shivamoggi and Muilenburg [3] then showed that Lewis' invariant arises in the form of the amplitudes of the various asymptotic solutions.

In this thesis, we develop a generalized version of Lewis' invariant for the relativistic harmonic oscillator in much the same way. We determine an asymptotic solution for the relativistic case, and find an invariant arises of the same form as the non-relativistic case. While a series doesn't appear to be feasible to develop, we find that in the non-relativistic limit it coincides with Lewis' invariant.

We then generalize the procedure of Eliezer and Gray in order to develop an exact expression of the invariant for the relativistic harmonic oscillator. This expression is then used to determine a series for the invariant.

This development would be applicable to any one dimensional relativistic system with a potential expandable about a minimum. In the literature review we also show that for a particle in a slowly varying magnetic field the equations of motion can be reduced to those of the harmonic oscillator (Chandrasekhar [4]).

CHAPTER 2: LITERATURE REVIEW

Landau and Lifshitz Vol. 1: The Adiabatic Invariant

For a closed Hamiltonian system undergoing periodic motion, the energy of the system is an integral of motion and provides a method of reducing the number of parameters parameters in the system by one in favor of the energy. In a system whose frequency is slowly varying (as compared to the period of motion), the energy is no longer a conserved quantity (Landau and Lifshitz [5]). For the sake of argument, suppose that the slowly varying parameter is $\omega = \omega(\tilde{t})$ where $\tilde{t} = \epsilon t$. We then have (Landau and Lifshitz [5]):

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{d\omega(\tilde{t})}{dt}\frac{\partial H}{\partial \omega} = \epsilon\omega'(\tilde{t})\frac{\partial H}{\partial \omega}. \tag{2.1}$$

Hence, the rate at which energy decreases will be small (Landau and Lifshitz [5]). This term can be averaged, where we take ω' to be constant over the average. This gives:

$$\overline{\frac{\partial H}{\partial t}} = \epsilon\omega'\overline{\frac{\partial H}{\partial \omega}}. \tag{2.2}$$

Next note (Landau and Lifshitz [5]):

$$T = \int_0^T dt = \oint \frac{dq}{\dot{q}} = \oint \frac{dq}{\frac{\partial H}{\partial p}}. \tag{2.3}$$

Where we have used Hamilton's equations of motion. Putting all of this into (2.2) and performing the time average:

$$\overline{\frac{dE}{dt}} = \epsilon\omega' \frac{1}{T}\int_0^T \frac{\partial H}{\partial \omega} dt = \epsilon\omega' \frac{\oint \frac{\partial H}{\partial \omega}\frac{dq}{\frac{\partial H}{\partial p}}}{\oint \frac{dq}{\frac{\partial H}{\partial p}}}, \tag{2.4}$$

where the path of integration is along any path where ω is constant, and therefore the Hamiltonian has the constant value of the energy (Landau and Lifshitz [5]). Using this fact:

$$\frac{dH}{d\omega} = \frac{\partial H}{\partial \omega} + \frac{\partial p}{\partial \omega}\frac{\partial H}{\partial p} = 0, \tag{2.5}$$

or (Landau and Lifshitz [5]):

$$\frac{\frac{\partial H}{\partial \omega}}{\frac{\partial H}{\partial p}} = -\frac{\partial p}{\partial \omega}. \tag{2.6}$$

Taking all of this together, and inserting into (2.4), we finally arrive at (Landau and Lifshitz [5]):

$$\frac{dI}{dt} = \frac{1}{2\pi}\frac{d}{dt}\oint p \cdot dq = \frac{1}{2\pi}\frac{d}{dt}\oint \left(\frac{\partial p}{\partial E}\overline{\frac{\partial E}{\partial t}} + \frac{\partial p}{\partial \omega}\frac{\partial \omega}{\partial t}\right) dq = 0, \tag{2.7}$$

where I is the adiabatic invariant. We therefore have that this quantity is conserved to the order of approximation we are working to. For the harmonic oscillator Hamiltonian, we obtain (Landau and Lifshitz [5])

$$I = \frac{E}{\omega}. \tag{2.8}$$

H.R. Lewis

This paper introduces the exact expression for Lewis' invariant when the frequency is slowly varying. The exact expression is given by (Lewis [1]):

$$I = \frac{1}{2}\left(\frac{x^2}{\rho^2} + (\rho p - \dot{\rho} x)^2\right), \tag{2.9}$$

where ρ satisfies the equation:

$$\ddot{\rho} + \omega^2 \rho - \frac{1}{\rho^3} = 0 \tag{2.10}$$

We will see in the next section one way of deriving this equation by examining the two dimensional oscillator.

Eliezer and Gray

In this paper, Lewis' invariant is presented as an integral of motion arising from a two dimensional oscillator. Using the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\dot{\rho}^2 + \dot{\theta}^2 \rho^2) - \frac{1}{2}\omega(t)^2 \rho^2. \tag{2.11}$$

We have that θ is cyclic and the conjugate momentum is conserved:

$$h = \dot{\theta}\rho^2 = const. \tag{2.12}$$

Next, the equations of motion for ρ gives (Eliezer and Gray ([2]):

$$\ddot{\rho} - \rho\dot{\theta}^2 + \omega^2\rho = 0. \tag{2.13}$$

Using (2.12), we can eliminate $\dot{\theta}$ obtaining (Eliezer and Gray[2]):

$$\ddot{\rho} + \omega^2\rho = \frac{h^2}{\rho^3}. \tag{2.14}$$

This is the same equation, (2.10), that appears for the coordinate in Lewis' invariant when $h = 1$ (Eliezer and Gray [2]). The invariant that arises from this consideration is given by:

$$I = \frac{1}{2}\left((\rho p - \dot{\rho}x)^2 + \frac{h^2x^2}{\rho^2}\right). \tag{2.15}$$

This implies that Lewis' invariant equivalent to half the angular momentum of a planar oscillator when the angular momentum is taken to have unit value.

Shivamoggi and Muilenburg

In this paper, Shivamoggi and Muilenburg explored the asymptotic solution for the adiabatic harmonic oscillator. While determining the solution for the equation of motion, the quantity (Shivamoggi and Muilenburg [3]):

$$A_0^2\omega = C^2 \tag{2.16}$$

was a constant of the motion (here A_0 is the slowly varying time dependent amplitude for the first order solution (Shivamoggi and Muilenburg[3]). They were able to obtain a series representation for C^2 by finding the higher order solutions. The series obtained was (Shivamoggi and Muilenbug

[3])

$$C^2 = \frac{p^2 + \omega^2 x^2}{\omega} + \epsilon \frac{\omega'}{\omega^2} px + \epsilon^2 \left(\frac{\omega'}{4\omega^3} x^2 + \left(\frac{3\omega'^2}{8\omega^5} - \frac{\omega''}{4\omega^4} \right) (\omega^2 x^2 - p^2) \right) + O(\epsilon^3). \qquad (2.17)$$

Next this series was compared to the expression obtained by Lewis ([1]) and the two were found to be proportional.

Shivamoggi: Method of Multiple Scales

In the next section, we rely heavily on the method of multiple scales in order to determine the asymptotic solution for our problem. This method is useful when dealing with a system that has time scales of different order (Shivamoggi [6]. For instance, Shivamoggi and Muilenburg ([3]) applied the method to the adiabatic harmonic oscillator with two time scales: a slow time scale which characterized the slowly varying nature of the frequency and a fast time scale, determined as a power series in ϵ. This method will take an ordinary differential equation and turn it into a partial differential equation (Shivamoggi [6]). Much like seperation of variables, this method typically splits our solutions into separate parts. In this case though, the components are functions of the different time scales rather than the spatial variables (as in the separation of variables).

For a charged particle moving in a weakly time dependent magnetic field, the magnetic moment:

$$\mu = \frac{w_\perp}{B} \tag{2.18}$$

is an adiabatic invariant for the system (Chandrasekhar [4]). Here, $w_\perp$ is the kinetic energy associated with the velocity of the particle that is perpendicular to the magnetic field B.

The equations of motion associate with this system are given by (Chandrasekhar [4]):

$$\ddot{\mathbf{r}} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right). \tag{2.19}$$

Fixing the direction of **B** we have:

$$\mathbf{B} = B\hat{\mathbf{z}}. \tag{2.20}$$

Using Maxwell's equations for a source free region, and taking the velocity along the magnetic field lines to be zero, (Chandrasekhar [4]):

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{2.21}$$

and setting:

$$\omega(t) = \frac{eB(t)}{mc} \tag{2.22}$$

we arrive at the pair of coupled equations (Chandrasekhar [4])

$$\ddot{x} - \omega(t)\dot{y} - \frac{1}{2}y\dot{\omega} = 0 \tag{2.23}$$

$$\ddot{y} + \omega(t)\dot{x} + \frac{1}{2}x\dot{\omega} = 0. \tag{2.24}$$

Upon the substitution: $\xi = x + iy$, this can be recast(Chandrasekhar [4]):

$$\ddot{\xi} + i\omega(t)\dot{\xi} + \frac{1}{2}i\dot{\omega}\xi = 0. \tag{2.25}$$

If the middle term could be eliminated, then we would have the oscillator equation. We make one last change of variables via a Liouville transformation (Chandrasekhar [4]):

$$\xi = ze^{-\frac{i}{2}\int_0^t \omega(t')dt'}. \tag{2.26}$$

Under this transformation, (2.25) becomes:

$$\ddot{z} + \frac{1}{4}\omega^2 z = 0. \tag{2.27}$$

This shows that the coordinate z follows the equation for the time dependent oscillator.

CHAPTER 3: ASYMPTOTIC SOLUTION FOR THE RELATIVISTIC HARMONIC OSCILLATOR

For an oscillator system whose speeds reach a significant percentage of the speed of light and whose frequency is slowly changing, Newton's laws break down and we must turn to the relativistic version:

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\right)+\omega^2(\tilde{t})x=0, \tag{3.1}$$

where $\tilde{t}=\epsilon t$ We will also take the following initial conditions for convenience:

$$x(0)=0, \quad \dot{x}(0)=\epsilon v_0. \tag{3.2}$$

The Lorentz factor in our equation of motion can be Taylor expanded and differentiated term by term yielding:

$$\ddot{x}+\frac{3}{2c^2}\ddot{x}\dot{x}^2+\frac{15}{8c^4}\ddot{x}\dot{x}^4+\omega^2(\tilde{t})x+O\left(\frac{\dot{x}^5}{c^5}\right)=0. \tag{3.3}$$

As per the method of multiple scales (Shivamoggi and Muilenburg [3]), we take a fast varying time scale in the form:

$$\hat{t}=\int_0^t f(\epsilon t')dt', \tag{3.4}$$

where f is to be determined.

Following the chain rule we find (Shivamoggi [6]):

$$\frac{d}{dt} \to f(\tilde{t})\frac{\partial}{\partial \hat{t}} + \epsilon \frac{\partial}{\partial \tilde{t}} \tag{3.5}$$

$$\frac{d^2}{dt^2} \to f^2(\tilde{t})\frac{\partial^2}{\partial \hat{t}^2} + 2\epsilon f(\tilde{t})\frac{\partial^2}{\partial \hat{t}\partial \tilde{t}} + \epsilon f'(\tilde{t})\frac{\partial}{\partial \hat{t}} + \epsilon^2 \frac{\partial^2}{\partial \tilde{t}^2}. \tag{3.6}$$

We also take a power series for f (Shivamoggi and Muilenburg [3]):

$$f(\tilde{t}) = \omega(\tilde{t}) + \epsilon f_1(\tilde{t}) + \epsilon^2 f_2(\tilde{t}) + ..., \tag{3.7}$$

and an asymptotic expansion for the solution:

$$x(\hat{t}, \tilde{t}; \epsilon) \to \epsilon x_1(\hat{t}, \tilde{t}) + \epsilon^2 x_2(\hat{t}, \tilde{t}) + ... \tag{3.8}$$

We use a series of this form, with the first term being $O(\epsilon)$, in order to avoid non-linearities that would arise from the Lorentz factor series.

Combining equations $(3.5) - (3.8)$, we obtain to first order in ϵ:

$$O(\epsilon) : \omega^2 \frac{\partial^2 x_1}{\partial \hat{t}^2} + \omega^2 x_1 = 0, \quad x_1(0) = 0, \;\; \frac{\partial x_1}{\partial \hat{t}}(t = 0) = \frac{v_0}{\omega(0)}. \tag{3.9}$$

This can be solved, yielding:

$$x_1(\hat{t}, \tilde{t}; \epsilon) = A_1(\tilde{t}) Sin(\hat{t}), \tag{3.10}$$

where A_1 is to be determined.

Next to order ϵ^2 we find:

$$O(\epsilon^2) : \frac{\partial^2 x_2}{\partial \hat{t}^2} + x_2 = -\frac{\omega'}{\omega^2}\frac{\partial x_1}{\partial \hat{t}} + \frac{2f_1}{\omega}x_1 - \frac{2}{\omega}\frac{\partial^2 x_1}{\partial \hat{t}\partial \tilde{t}}, \qquad x_2(0) = 0, \quad \frac{\partial x_2}{\partial \hat{t}}(t=0) = 0. \tag{3.11}$$

Upon using (3.10) we find:

$$\frac{\partial^2 x_2}{\partial \hat{t}^2} + x_2 = \frac{2A_1(\tilde{t})f_1(\tilde{t})}{\omega(\tilde{t})}Sin(\hat{t}) - \left(\frac{2A_1'(\tilde{t})}{\omega(\tilde{t})} + \frac{A_1(\tilde{t})\omega'(\tilde{t})}{\omega^2(\tilde{t})}\right)Cos(\hat{t}). \tag{3.12}$$

Both of these terms are secular so the coefficients must be zero. Applying these conditions we find:

$$f_1 = 0, \quad 2A_1(\tilde{t})A_1'(\tilde{t})\omega(\tilde{t}) + A_1^2(\tilde{t})\omega'(\tilde{t}) = 0. \tag{3.13}$$

Following Shivamoggi and Muilenburg [3], the second term can be rewritten as:

$$\frac{d}{d\tilde{t}}(A_1^2\omega) = 0 \to A_1^2\omega = C^2. \tag{3.14}$$

i.e. the quantity $A_1^2\omega$ is invariant in time. For the non-relativistic case, this term was found to be directly proportional to Lewis' invariant (Shivamoggi and Muilenburg [3]). Ideally, we should be able to express this constant in terms of our original solutions as in the non-relativistic case.

Applying (3.13), (3.11) and (3.12) becomes:

$$\frac{\partial^2 x_2}{\partial \hat{t}^2} + x_2 = 0, \quad \frac{\partial x_2}{\partial \hat{t}}(t=0) = 0. \tag{3.15}$$

Solving we find:

$$x_2 = 0. \tag{3.16}$$

So far, the solutions have not reflected any of the relativistic corrections, which is expected since the first term where the correction appears is $O(\epsilon^3)$. Continuing to that term:

$$\begin{aligned} O(\epsilon^3): \frac{\partial^2 x_3}{\partial \hat{t}^2} + x_3 = -\frac{\omega'(\tilde{t})}{\omega^2(\tilde{t})}\frac{\partial x_2}{\partial \hat{t}} - 2\frac{f_2(\tilde{t})}{\omega(\tilde{t})}\frac{\partial^2 x_1}{\partial \hat{t}^2} - \\ \frac{2}{\omega(\tilde{t})}\frac{\partial^2 x_2}{\partial \tilde{t}\partial \hat{t}} - \frac{1}{\omega^2(\tilde{t})}\frac{\partial^2 x_1}{\partial \tilde{t}^2} - \frac{3}{2c^2}\omega^2(\tilde{t})(\frac{\partial x_1}{\partial \hat{t}})^2\frac{\partial^2 x_1}{\partial \hat{t}^2}. \end{aligned} \tag{3.17}$$

Applying (3.10), (3.13), and (3.16), equation (3.17) becomes

$$\frac{\partial^2 x_3}{\partial \hat{t}^2} + x_3 = \Big(\frac{2A_1(\tilde{t})f_2(\tilde{t})}{\omega(\tilde{t})} - \frac{A_1''(\tilde{t})}{\omega^2(\tilde{t})} + \frac{3A_1^3(\tilde{t})\omega^2(\tilde{t})}{8c^2}\Big)Sin(\hat{t}) + \frac{3A_1^3(\tilde{t})\omega^2(\tilde{t})}{8c^2}Sin(3\hat{t}). \tag{3.18}$$

The first term here is secular, so we set the coefficient to zero and find:

$$f_2(\tilde{t}) = \frac{-3A_1^3(\tilde{t})\omega^4(\tilde{t}) + 8c^2 A_1''(\tilde{t})}{16c^2 A_1(\tilde{t})\omega(\tilde{t})} = \frac{3(\omega')^2}{8\omega^3} - \frac{\omega''}{4\omega^2} - \frac{3\omega^3 A_1^2}{16c^2}. \tag{3.19}$$

The second term here is not secular, and will lead to the appearance of higher harmonics in the solution.

Using (3.19), (3.18) becomes:

$$\frac{\partial^2 x_3}{\partial \hat{t}^2} + x_3 = \frac{3A_1^3(\tilde{t})\omega^2(\tilde{t})}{8c^2} Sin(3\hat{t}), \qquad x_3(0) = 0. \tag{3.20}$$

This equation has solution:

$$x_3(\hat{t}, \tilde{t}; \epsilon) = A_3(\tilde{t}) Sin(\hat{t}) - \frac{3A_1^3\omega^2}{64c^2} Sin(3\hat{t}). \tag{3.21}$$

We have therefore determined the solution up to order ϵ^3:

$$x(\hat{t}, \tilde{t}; \epsilon) = \epsilon A_1 Sin(\tilde{t}) + \epsilon^3 \left(A_3 Sin(\hat{t}) - \frac{3A_1^3\omega^2}{64c^2} Sin(3\hat{t}) \right). \tag{3.22}$$

This differs from the relativistic case in the $1/c^2$ term that appears. Ideally, the invariant would be calculated from this as follows. Noting $p = \dot{x}$ and using equation (3.5):

$$\begin{aligned} p = f(\tilde{t}) \left(\epsilon A_1 Cos(\hat{t}) + \epsilon^3 \left(A_3 Sin(\hat{t}) - \frac{9A_1^3\omega^2}{64c^2} Cos(3\hat{t}) \right) \right) + \\ \epsilon \left(\epsilon A_1' Sin(\hat{t}) + \epsilon^3 \left(A_3' Sin(\hat{t}) - \left(\frac{9A_1^2\omega^2}{64c^2} + \frac{6A_1^3\omega\omega'}{64c^2} \right) Sin(3\hat{t}) \right) \right). \end{aligned} \tag{3.23}$$

From here, we would like to find an expression for A_1 involving the variables x and p. This is greatly complicated by the appearance of the higher order harmonics. If the higher order harmonics are ignored, we can recover the solution that Shivamoggi and Muilenburg ([3]) obtained. Thus we are able to say that in the non-relativistic limit, our invariant will reduce to Lewis' invariant.

CHAPTER 4: DIRECT CALCULATION OF LEWIS' INVARIANT

We will now generalize Lewis' invariant (Lewis [1]) for the relativistic harmonic oscillator. We begin with the relativistic Lagrangian for a planar oscillator, using polar coordinates (Eliezer and Gray [2], Goldstein [7]):

$$\mathcal{L} = -c^2\sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2}{c^2}} - \frac{1}{2}\omega^2(t)r^2. \tag{4.1}$$

Immediately we see that θ is a cyclic variable and consequently obtain:

$$h = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \gamma r^2\dot{\theta}. \tag{4.2}$$

Here, h is the angular momentum of the system (constant) and

$$\gamma = \frac{1}{\sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2}{c^2}}}. \tag{4.3}$$

We follow Eliezer and Gray [2], using (4.2) to solve for $\dot{\theta}$ and we find:

$$\dot{\theta} = \frac{h/r^2}{\gamma_r\sqrt{1 + \frac{h^2}{r^2c^2}}}, \tag{4.4}$$

where:

$$\gamma_r = \frac{1}{\sqrt{1 - \frac{\dot{r}^2}{c^2}}}. \tag{4.5}$$

Returning to (4.1), we can obtain the equations of motion:

$$\frac{d}{dt}(\gamma\dot{r}) - r\dot{\theta}^2\gamma + \omega^2 r = 0. \tag{4.6}$$

Using (4.2), (4.4) and (4.5), (4.6) can be rewritten as:

$$\frac{d}{dt}\left(\dot{r}\sqrt{\frac{1+\frac{h^2}{r^2c^2}}{1-\frac{\dot{r}^2}{c^2}}}\right) - \frac{h^2}{r^3}\sqrt{\frac{1-\frac{\dot{r}^2}{c^2}}{1+\frac{h^2}{r^2c^2}}} + \omega^2 r = 0. \tag{4.7}$$

Derivation of Lewis' Invariant

The invariant studied by Lewis [1] was for the non-relativistic harmonic oscillator. To derive the invariant for the relativistic case we follow Eliezer and Gray [2], and begin by noting:

$$x = rCos(\theta). \tag{4.8}$$

Differentiating this with respect to time and solving for $Sin(\theta)$ we find:

$$Sin(\theta) = -\frac{1}{\dot{\theta}}\frac{\dot{x}r - x\dot{r}}{r^2} = -\frac{\dot{x}r - x\dot{r}}{h}\gamma_r\sqrt{1+\frac{h^2}{r^2c^2}}, \tag{4.9}$$

where we have used equation (4.4). Using the fact that $Sin^2(t) + Cos^2(t) = 1$:

$$\frac{(\dot{x}r - x\dot{r})^2}{h^2}\frac{1+\frac{h^2}{r^2c^2}}{1-\frac{\dot{r}^2}{c^2}} + \frac{x^2}{r^2} = 1. \tag{4.10}$$

According to Lewis [1] and Eliezer and Gray [2], Lewis' invariant is exactly half the left hand side when $h = 1$:

$$2I = (\dot{x}r - x\dot{r})^2 \frac{1 + \frac{1}{r^2c^2}}{1 - \frac{\dot{r}^2}{c^2}} + \frac{x^2}{r^2}. \tag{4.11}$$

General Solution

Since we will be interested in the case of slowly varying frequencies, we have:

$$\omega \to \omega(\tilde{t}), \quad r \to r(\tilde{t}). \tag{4.12}$$

With this, (4.7) becomes:

$$\epsilon \frac{d}{d\tilde{t}} \left(\epsilon r' \sqrt{\frac{1 + \frac{h^2}{r^2c^2}}{1 - \epsilon^2 \frac{(r')^2}{c^2}}} \right) - \frac{h^2}{r^3} \sqrt{\frac{1 - \epsilon^2 \frac{(r')^2}{c^2}}{1 + \frac{h^2}{r^2c^2}}} + \omega^2 r = 0. \tag{4.13}$$

We will look for an asymptotic solution of the form:

$$r(\tilde{t}) \to r_0(\tilde{t}) + \epsilon r_1(\tilde{t}) + \epsilon^2 r_2(\tilde{t}) + ... \tag{4.14}$$

This is inserted into (4.13), and we obtain to zeroth order in epsilon:

$$O(1) : -\frac{h^2}{r_0^3} \sqrt{\frac{c^2 r_0^2}{h^2 + c^2 r_0^2}} + r_0 \omega^2 = 0. \tag{4.15}$$

This equation yields an eighth degree (exactly solvable) polynomial in r_0:

$$r_0^8 + \frac{h^2}{c^2} r_0^6 - \frac{h^4}{\omega^4} = 0, \tag{4.16}$$

where only one solution reduces to the correct non-relativistic case (Shivamoggi and Muilenburg [3]) when it is applied ($\omega \ll c^2$):

$$\begin{aligned} r_0 = \Bigg(&- \frac{1}{4c^2} - \frac{1}{2}\sqrt{\frac{1}{4c^4} - \frac{4\sqrt[3]{\frac{2}{3}}c^{4/3}}{\omega^2\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}} + \frac{\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}}{\sqrt[3]{2}3^{2/3}c^{4/3}\omega^2}} + \\ &\frac{1}{2}\Bigg(\frac{1}{2c^4} + \frac{4\sqrt[3]{\frac{2}{3}}c^{4/3}}{\omega^2\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}} - \frac{\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}}{\sqrt[3]{2}3^{2/3}c^{4/3}\omega^2} + \\ &\frac{1}{4c^6\sqrt{\frac{1}{4c^4} - \frac{4\sqrt[3]{\frac{2}{3}}c^{4/3}}{\omega^2\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}} + \frac{\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}}{\sqrt[3]{2}3^{2/3}c^{4/3}\omega^2}}}\Bigg)^{1/2}\Bigg)^{1/2}, \end{aligned} \tag{4.17}$$

where we have set $h = 1$ to simplify the solution. A figure of this solution is included in the appendix.

To next order we obtain:

$$r_1 = 0. \tag{4.18}$$

Using the assumptions above, namely (4.18) and (4.12), we find that:

$$2I = p^2 r_0^2 + \frac{x^2}{r_0^2} + \frac{p^2}{c^2} + \epsilon \left(\frac{p^2 \left(r_0'\right)^2}{c^4} + \frac{p^2 r_0^2 \left(r_0'\right)^2}{c^2} - \frac{2pxr_0'}{c^2 r_0} + 2p^2.r_0 r_1 - 2pr_0 x r_0' - \frac{2r_1 x^2}{r_0^3} \right) + O(\epsilon^2) \quad (4.19)$$

The equations from here, while algebraically are simpler, result in solutions whose complexity is incredibly high. We will use what we have obtained so far to determine the invariant to zeroth order (because of the large nature of the expression, a figure of the solution is included in appendix A):

$$\begin{aligned} 2I = \frac{p^2}{4c^2} \Bigg(& -2c^2 \sqrt{\frac{1}{4c^4} - \frac{4\sqrt[3]{\frac{2}{3}}c^{4/3}}{\omega^2 \sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}} + \frac{\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}}{\sqrt[3]{2}3^{2/3}c^{4/3}\omega^2}} + \\ & 2c^2 \left(\frac{1}{2c^4} + \frac{4\sqrt[3]{\frac{2}{3}}c^{4/3}}{\omega^2 \sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}} - \frac{\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}}{\sqrt[3]{2}3^{2/3}c^{4/3}\omega^2} + \right. \\ & \left. \frac{1}{4c^6 \sqrt{\frac{1}{4c^4} - \frac{4\sqrt[3]{\frac{2}{3}}c^{4/3}}{\omega^2 \sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}} + \frac{\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}}{\sqrt[3]{2}3^{2/3}c^{4/3}\omega^2}}} \right)^{1/2} + 3 \Bigg) - \\ 4c^2x^2 \Bigg/ \Bigg(& 2c^2 \sqrt{\frac{1}{4c^4} - \frac{4\sqrt[3]{\frac{2}{3}}c^{4/3}}{\omega^2 \sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}} + \frac{\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}}{\sqrt[3]{2}3^{2/3}c^{4/3}\omega^2}} - \\ & 2c^2 \left(\frac{1}{2c^4} + \frac{4\sqrt[3]{\frac{2}{3}}c^{4/3}}{\omega^2 \sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}} - \frac{\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}}{\sqrt[3]{2}3^{2/3}c^{4/3}\omega^2} + \right. \\ & \left. \frac{1}{4c^6 \sqrt{\frac{1}{4c^4} - \frac{4\sqrt[3]{\frac{2}{3}}c^{4/3}}{\omega^2 \sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}} + \frac{\sqrt[3]{\sqrt{768c^8+81\omega^4}-9\omega^2}}{\sqrt[3]{2}3^{2/3}c^{4/3}\omega^2}}} \right)^{1/2} + 1 \Bigg) + O(\epsilon). \quad (4.20) \end{aligned}$$

This expression is cumbersome and incredibly difficult to work with. It is also difficult to check the non-relativistic limit on, as the small dimensionless quantity in this case would need to be p^2/c^2 and ω/c^2. It has the quality of involving only $x, p,$ and ω, but does not compare to the non-relativistic quantity easily. In the next two sections, we will look at schemes for obtaining expressions that are easier to work with.

First Relativistic Corrections

In this section, we will approximate the roots to the equations determining the functions r_i. Returning to equation (4.16), we set:

$$\delta = \frac{h\omega}{c^2}. \tag{4.21}$$

Here, δ will be our relativistic parameter to check against the non-relativistic case. We then have

$$r_0^8 + \frac{h}{\omega}\delta r_0^6 - \frac{h^4}{\omega^4} = 0. \tag{4.22}$$

This can be solved perturbatively using the series

$$r_0 = \sqrt{\frac{h}{\omega}} + \delta r_{01} + \delta^2 r_{02} + ... \tag{4.23}$$

Inserting this into (4.22), we find:

$$\frac{\left(h^4 + 8h^3 r_{01}\sqrt{h\omega}\right)}{\omega^4}\delta + \left(\frac{28h^3 r_{01}^2}{\omega^3} + \left(\frac{h}{\omega}\right)^{7/2}(6r_{01} + 8r_{02})\right)\delta^2 + O\left(\delta^3\right) = 0. \tag{4.24}$$

We equate each term to zero, and we find:

$$r_{01} = -\frac{1}{8}\sqrt{\frac{h}{\omega}}, \quad r_{02} = \frac{5}{128}\sqrt{\frac{h}{\omega}}. \tag{4.25}$$

We thus have:

$$r_0 = \sqrt{\frac{h}{\omega}} - \frac{1}{8}\sqrt{\frac{h}{\omega}}\delta + \frac{5}{128}\sqrt{\frac{h}{\omega}}\delta^2 + O\left(\delta^3\right). \tag{4.26}$$

We now move forward and determine the corrections to the adiabatic invariant (4.19) by inserting (4.26) and using (4.21), we can write:

$O(1)$:

$$2I = \left(\frac{hp^2}{\omega(t)} + \frac{x^2}{h}\right) + \delta\left(\frac{x^2\omega}{4h} - \frac{hp^2}{4} + \frac{p^2}{h\omega}\right) + \delta^2\left(\frac{3hp^2}{32\omega} - \frac{x^2\omega}{32h}\right) + O(\delta^3). \tag{4.27}$$

This form for the first term is much more manageable, and is easier to check the limits on. In the limit of small speeds $h\omega \propto v^2$, the instantaneous velocity, so $\delta \ll 1$. However, the drawback of this formulation is that we have two independent scales for the system, that which characterizes the adiabatic nature, ϵ, and the relativistic parameter of the system δ.

The case of $\epsilon^2 = \frac{v_0^2}{c^2}$

In the previous section, we saw how the solutions for the asymptotic series became hard to work with, even at first order. One simplifying assumption that can be made that yields equations that are more manageable is that the parameter characterizing the slow variation of our system is equal to the quantity v_0/c, where v_0 is taken to be the initial speed of the system. This will allow us to develop a series that is simpler, and more recognizable when compared to the non-relativistic case. We rewrite equation (4.13)

$$\epsilon \frac{d}{d\tilde{t}} \left(\epsilon r' \sqrt{\frac{1+\epsilon^2 \frac{h^2}{r^2 v_0^2}}{1-\epsilon^4 \frac{(r')^2}{v_0^2}}} \right) - \frac{h^2}{r^3} \sqrt{\frac{1-\epsilon^4 \frac{(r')^2}{v_0^2}}{1+\epsilon^2 \frac{h^2}{r^2 v_0^2}}} + \omega^2 r = 0. \tag{4.28}$$

We again look for solutions of the form (4.14), and find:

$$O(1) : r_0 \omega^2 - \frac{h^2}{r_0^3} = 0, \tag{4.29}$$

$$O(\epsilon) : \frac{3h^2 r_1}{r_0^4} + r_1 \omega^2 = 0, \tag{4.30}$$

$$O(\epsilon^2) : \frac{h^4}{2 r_0^5 v_0^2} - \frac{6h^2 r_1^2}{r_0^5} + \frac{3h^2 r_2}{r_0^4} + r_0'' + r_2 \omega^2 = 0. \tag{4.31}$$

These can be solved successively, yielding:

$$r_0(\tilde{t}) = \sqrt{\frac{h}{\omega}}, \tag{4.32}$$

$$r_1(\tilde{t}) = 0, \tag{4.33}$$

$$r_2(\tilde{t}) = \frac{\sqrt{h}\omega''}{8\omega^{7/2}} - \frac{3\sqrt{h}\,(\omega')^2}{16\omega^{9/2}} - \frac{h^{3/2}\sqrt{\omega}}{8v_0^2}. \tag{4.34}$$

This can be inserted into (4.14) and we find:

$$r(\tilde{t}) = \sqrt{\frac{h}{\omega}} + \epsilon^2 \left(\frac{\sqrt{h}\omega''}{8\omega^{7/2}} - \frac{3\sqrt{h}(\omega')^2}{16\omega^{9/2}} - \frac{h^{3/2}\sqrt{\omega}}{8v_0^2} \right). \tag{4.35}$$

We can now use this in equation (4.11) if we set $h = 1$. From this we obtain:

$$\begin{aligned} 2I = \frac{p^2 + x^2\omega^2}{\omega} + \epsilon \frac{px\omega'}{\omega^2} + \\ \epsilon^2 \left(\frac{x^2\omega^2 + 3p^2}{4v_0^2} + \frac{(\omega')^2}{4\omega^3} x^2 + \left(\frac{3(\omega')^2}{8\omega^5} - \frac{\omega''}{4\omega^4} \right) (\omega^2 x^2 - p^2) \right) + O(\epsilon^3). \end{aligned} \tag{4.36}$$

This expression, when compared to the non-relativistic case, is more recognizable. The first two terms are identical to the non relativistic case, and we see the addition of two new terms which are $O(\epsilon^2)$. We can also recover the non-relativistic case exactly, by letting $\epsilon^2 p^2/v_0^2$ and $\epsilon^2 x^2 \omega^2/v_0^2$ both go to zero.

CHAPTER 5: CONCLUSION

In this thesis, we explored the equations of motion for the relativistic harmonic oscillator. While computing an asymptotic series for the solution, a constant of motion arose. For the non-relativistic case, this constant was proportional to Lewis' invariant. Following the asymptotic solution, we were able to determine that equation (3.14) reduced to Lewis' invariant. The attempts at writing the invariant, with higher order terms, became intractable due to the appearance of the higher order harmonic.

Knowing that the constant that arose was proportional to Lewis' invariant for the non-relativistic case, we investigated the equation for the relativistic case, obtained from the Euler-Lagrange equation and a trigonometric identity. These equations gave an asymptotic solution which exhibited relativistic terms at every order in ϵ. This differed from the solution derived in the previous case, since relativistic terms were kept to $O(\epsilon^3)$ and higher. Equation (4.19) was not investigated beyond zeroth order, as the terms became too cumbersome to work with.

While investigating the solution to equation (4.7), it became apparent that the solutions would simplify greatly if the parameter ϵ, which describes the slowly varying nature of the frequency, were equal to v_0/c. This allowed us to obtain simpler equations for the asymptotic solutions which agreed with the non-relativistic case.

APPENDIX: IMAGES OF SELECTED SOLUTIONS

$$\left\{r0[t] \to \sqrt{\left(-\frac{1}{4\,c^2}-\frac{1}{2}\sqrt{\left(\frac{1}{4\,c^4}-\frac{4\left(\frac{2}{3}\right)^{1/3}c^2}{\left(-9\,c^2\,\omega^8+\sqrt{3}\,\sqrt{256\,c^{12}\,\omega^{12}+27\,c^4\,\omega^{16}}\right)^{1/3}}+\frac{\left(-9\,c^2\,\omega^8+\sqrt{3}\,\sqrt{256\,c^{12}\,\omega^{12}+27\,c^4\,\omega^{16}}\right)^{1/3}}{2^{1/3}\times 3^{2/3}\,c^2\,\omega^4}\right)}+\frac{1}{2}\sqrt{\left(\frac{1}{2\,c^4}+\frac{4\left(\frac{2}{3}\right)^{1/3}c^2}{\left(-9\,c^2\,\omega^8+\sqrt{3}\,\sqrt{256\,c^{12}\,\omega^{12}+27\,c^4\,\omega^{16}}\right)^{1/3}}-\frac{\left(-9\,c^2\,\omega^8+\sqrt{3}\,\sqrt{256\,c^{12}\,\omega^{12}+27\,c^4\,\omega^{16}}\right)^{1/3}}{2^{1/3}\times 3^{2/3}\,c^2\,\omega^4}+\frac{1}{4\,c^6\sqrt{\frac{1}{4\,c^4}-\frac{4\left(\frac{2}{3}\right)^{1/3}c^2}{\left(-9\,c^2\,\omega^8+\sqrt{3}\,\sqrt{256\,c^{12}\,\omega^{12}+27\,c^4\,\omega^{16}}\right)^{1/3}}+\frac{\left(-9\,c^2\,\omega^8+\sqrt{3}\,\sqrt{256\,c^{12}\,\omega^{12}+27\,c^4\,\omega^{16}}\right)^{1/3}}{2^{1/3}\times 3^{2/3}\,c^2\,\omega^4}}}\right)}\right)}\right\}\right\}$$

Figure 5.1: Root of the eighth degree polynomial for r_0

$$
\begin{aligned}
&= \Bigg(\frac{1}{4\,c^2}\Bigg(3 - 2\,c^2\,\sqrt{\frac{1}{4\,c^4} - \frac{4\left(\frac{2}{3}\right)^{1/3} c^{4/3}}{\omega^2\left(-9\,\omega^2 + \sqrt{768\,c^8 + 81\,\omega^4}\right)^{1/3}} + \frac{\left(-9\,\omega^2 + \sqrt{768\,c^8 + 81\,\omega^4}\right)^{1/3}}{2^{1/3}\times 3^{2/3}\,c^{4/3}\,\omega^2}} + \\
&\quad 2\,c^2\,\sqrt{\left(\frac{1}{2\,c^4} + \frac{4\left(\frac{2}{3}\right)^{1/3} c^{4/3}}{\omega^2\left(-9\,\omega^2 + \sqrt{768\,c^8 + 81\,\omega^4}\right)^{1/3}} - \frac{\left(-9\,\omega^2 + \sqrt{768\,c^8 + 81\,\omega^4}\right)^{1/3}}{2^{1/3}\times 3^{2/3}\,c^{4/3}\,\omega^2} + \frac{1}{4\,c^6\,\sqrt{\frac{1}{4\,c^4} - \frac{4\left(\frac{2}{3}\right)^{1/3} c^{4/3}}{\omega^2\left(-9\,\omega^2+\sqrt{768\,c^8+81\,\omega^4}\right)^{1/3}} + \frac{\left(-9\,\omega^2+\sqrt{768\,c^8+81\,\omega^4}\right)^{1/3}}{2^{1/3}\times 3^{2/3}\,c^{4/3}\,\omega^2}}}\right)}\Bigg)\,p[t]^2 - \\
&\quad \left(4\,c^2\,x[t]^2\right)\Bigg/\Bigg(1 + 2\,c^2\,\sqrt{\frac{1}{4\,c^4} - \frac{4\left(\frac{2}{3}\right)^{1/3} c^{4/3}}{\omega^2\left(-9\,\omega^2 + \sqrt{768\,c^8 + 81\,\omega^4}\right)^{1/3}} + \frac{\left(-9\,\omega^2 + \sqrt{768\,c^8 + 81\,\omega^4}\right)^{1/3}}{2^{1/3}\times 3^{2/3}\,c^{4/3}\,\omega^2}} - \\
&\quad 2\,c^2\,\sqrt{\left(\frac{1}{2\,c^4} + \frac{4\left(\frac{2}{3}\right)^{1/3} c^{4/3}}{\omega^2\left(-9\,\omega^2 + \sqrt{768\,c^8 + 81\,\omega^4}\right)^{1/3}} - \frac{\left(-9\,\omega^2 + \sqrt{768\,c^8 + 81\,\omega^4}\right)^{1/3}}{2^{1/3}\times 3^{2/3}\,c^{4/3}\,\omega^2} + \frac{1}{4\,c^6\,\sqrt{\frac{1}{4\,c^4} - \frac{4\left(\frac{2}{3}\right)^{1/3} c^{4/3}}{\omega^2\left(-9\,\omega^2+\sqrt{768\,c^8+81\,\omega^4}\right)^{1/3}} + \frac{\left(-9\,\omega^2+\sqrt{768\,c^8+81\,\omega^4}\right)^{1/3}}{2^{1/3}\times 3^{2/3}\,c^{4/3}\,\omega^2}}}\right)}\Bigg)\Bigg) + O[\epsilon]^1
\end{aligned}
$$

Figure 5.2: Twice the invariant to first order

LIST OF REFERENCES

[1] H. R. Lewis, *Journal of Mathematical Physics*, **9**, 1976 (1968).

[2] C. J. Eliezer and A. Gray: *SIAM Journal on Applied Mathematics*, **30**, 3, (1976).

[3] B. Shivamoggi, and L. Muilenburg: *Physics Letters A*, **154**, 1, (1991).

[4] S. Chadrasekhar: *Plasma Physics*, The University of Chicago Press, (1960).

[5] L. Landau and E. Lifshitz: *Mechanics*, Butterworth Heinemann, (1976).

[6] B. Shivamoggi: *Perturbation Methods for Differential Equations*, Birkhäuser, (2003).

[7] H. Goldstein, C. Poole, and J. Safko: *Classical Mechanics*, Addison Wesley, (2000).