2019

Multi-level Optimization with Pricing Strategy under Boundedly Rationality User Equilibrium

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MULTI-LEVEL OPTIMIZATION AND APPLICATIONS WITH NON-TRADITIONAL GAME THEORY

by

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A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Industrial Engineering and Management Systems
in the College of Engineering and Computer Science
at the University of Central Florida
Orlando, Florida

Fall Term
2019

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ABSTRACT

We study multi-level optimization problem on energy system, transportation system and information network. We use the concept of boundedly rational user equilibrium (BRUE) to predict the behaviour of users in systems. By using multi-level optimization method with BRUE, we can help to operate the system work in a more efficient way. Based on the introducing of model with BRUE constraints, it will lead to the uncertainty to the optimization model. We generate the robust optimization as the multi-level optimization model to consider for the pessimistic condition with uncertainty. This dissertation mainly includes four projects. Three of them use the pricing strategy as the first level optimization decision variable. In general, our models' first level's decision variables are the measures that we can control, but the second level's decision variables are users behaviours that can only be restricted within BRUE with uncertainty.

Keywords: Boundedly Rational, Robust Optimization, Non-linear Programming, Linear Programming

ACKNOWLEDGMENTS

First, I would like to present my deepest acknowledgment to my advisor, Dr. Qipeng P. Zheng and co-advisor Dr. Vladimir Boginski for their professional advising, education and kindness. I am also grateful to all of my group colleagues.

I would like also to express my thanks to my committee members: Dr. Waldemar Karwowski and Dr. Jiongmin Yong. for their great efforts and supports in my qualify exam and candidacy exam.

I would also like to thanks to my collaborator Dr. Eduardo Pasiliao and Dr. Lihui Bai.

Finally, I would like to thanks to my family, My father Gang Yun, my mother Yurong Qiao for their encourages and supports.

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CHAPTER 1: INTRODUCTION

Mathematical optimization is a widely used method in multiple areas. The general formulation of optimization problem is shown as follows,

(GO):

$$\min_{\mathbf{x}} \quad f(\mathbf{x}) \tag{1.1}$$

$$\text{s.t.} \quad g(\mathbf{x}) \leq 0, \tag{1.2}$$

Where $\mathbf{x}$ is the decision variable in $\mathbf{R}^n$. $f(\mathbf{x})$ is the relative objective function respect to $\mathbf{x}$. $g(\mathbf{x})$ is the constraints with m dimension.

In this proposal, we also use the robust optimization model, which is a special case of the general optimization problem. It has two levels for the optimization problem. The general formulation of robust optimization is shown as follows,

(RO-GO):

$$\max_{\beta \in \mathbf{B}} \min_{\mathbf{x}} \quad f(\beta, \mathbf{x}) \tag{1.3}$$

$$\text{s.t.} \quad g(\beta, \mathbf{x}) \leq 0, \tag{1.4}$$

$$\mathbf{x} \in \mathbf{X}, \tag{1.5}$$

For this two-level optimize model. β is the first level decision variable. $\mathbf{B}$ is the feasible region for β. $\mathbf{X}$ is the feasible region for $\mathbf{x}$.

The application fields of mathematical optimization include but not limit to the energy system, transportation system and information network system. Four projects in this proposal are also in these three areas.

The method robust optimization can be traced back to 1950s to the decision theory which use for worst case analysis under uncertainty. Over the years, especially the recent two decades after the work of Ben-Tal [11, 12], it is widely applied in the areas includes operations research, statistics, control theory, finance, logistics and computer science.

Stochastic optimization is also another method to deal with the problem with uncertainty. But it can only be used when the probability of each scenario is known. But under many cases in the real world we can not know these probability. Then the study of robust optimization becomes extremely important.

In this dissertation, the concept of boundedly rational user equilibrium (BRUE) is introduced to estimate the users' behaviour. BRUE model is proposed by Simon in year 1957 [77, 80, 81, 79], which means for one individual, when the difference of the utilities of different options that the individual can choose are below a level, this individual will regard the utilities of such different options as the same. He or she may choose any options within that level as his or her final decisions. The following is a mathematical constraint for BURE. We suppose one user in the system have multiple choices, for choice i it has utility $U(i)$. With out loss of generality, we can set the choice i^* has the optimal utility value. Then BRUE tells us that for any choice i has the following property will be deemed as the possible future choice for this user.

$$U(i) \geq \rho * U(i^*). \tag{1.6}$$

Where ρ is called the bounded rationality coefficient. And we must have $\rho \leq 1$ because of the optimality of the choice i^*.

Here in our model of the transportation system, the utility includes the travel time and the surplus price. The concept of boundedly rational can be used in many fields, such as the energy system [96], psychology [44], military [69], transportation [58, 24, 56]and so on [70, 58, 36, 27].

The structure of this dissertation is that the first chapter is the introduction part of the whole dissertation. The second to five chapter is the work for four projects. And the six chapter is the conclusion.

The first project studies a new time-of-day pricing framework to reduce the Peak-To-Average ratio in residential electricity usage while considering consumers' boundedly rational behaviors. Instead of always choosing the optimal electricity consumption profiles as described in traditional game theory models, consumers tends to simply pick solutions that are acceptable in terms of cost or preference in reality. To address this, this paper proposes a Boundedly Rational User Equilibrium (BRUE) to model residential electricity consumption in smart grid with advanced metering infrastructure. Upon the BRUE models, this paper studies two pricing strategies, i.e., optimistic and robust, to minimize the total system cost, via bilevel optimization models. In order to address the computational difficulty caused by the nonconexity of the lower level problem, this paper studies three cutting plane methods, i.e., direct cutting-plane method, penalty-based cutting-plane method and Lagrangian-dual-based cutting plane method. Due to the property of hidden convexity, the Lagrangian dual method outperforms the other two methods. Numerical experiments show that by introducing the time-of-day pricing, it can decrease the total cost of the system. The results also suggest that the more users with flexible preferred time windows for electricity usage, the lower total cost the system can achieve by pricing.

The second project gives a model for the static transportation path based problem. We suppose

that all the users in the system will obey the bounded rational principle. In real world instances, people will feel just fine even if they do not reach the best utility they can achieve but only attain a certain level. We propose totally four conditions for our static models with two of them having the pricing strategy. By using the pricing strategy, the total time cost of the system can be reduced. And we also have the robust optimization model by using the pricing strategy, we solve it by using the column and constraint generation method. For transportation path based problem, it is also a large scale problem because the number of pathes have the exponentially relation with the number of acres in the system.

The third project talks about the social media platforms, which have become very popular for people to share information and make new friends. By expanding their own connections to new users in the social network, commercial users can greatly increase their influences leading to much higher profits. In order to optimize a information provider's network connections, we establish a mathematical model to simulate behaviours of other users within the information provider's network. The behaviours include the information repost as well as following/unfollowing other users. We apply the linear threshold propagation model to determine the action of repost. In addition, the action of following or unfollowing other users is restricted by boundedly rational user equilibrium (BRUE). The topology of the network can change depending on the information provider's plan of posting information. The connections for the information provider, therefore, may change as well. We establish a three-level optimization model for the information provider. The first level is to maximize the information provider's connections. The second level is to simulate users' behaviours under BRUE. The third level is to maximize the other users' utility that need to be used in the second level. We solve this problem by using exact algorithms for a small-scale synthetic network. For a large-scale problem, we tackle it by using the heuristic large neighbourhood search algorithms. In this paper, we discuss possible reasons why the BRUE model may be a more accurate simulation of users' actions compared to game theory. We compare results from the BRUE

model to game theory, and find that the BRUE model performs significantly better than game theory.

The fourth project is research about blockchain technology used in energy transaction. BlockChain technology guarantees the safety of transactions between two users who do not know each other without any central institute. We apply this characteristic of blockchain to power system. It can help prosumers within the power networks transact electricity and money. The users who have redundant amount of power can sell it to other users who need power with the price lower than the power company. It is double win for both these two users. This paper establish a mathematical game theory model for user's decision to buy or sell power in the system. We can see what is the influence to the price of the central company by introducing the blockchain to the power transaction system. In addition, we generate the simulation with hyperledge to see its indluence to the price. We use the KKT condition algorithm to solve the multiple level game theory model. We find the price of the power can decrease dramatically by applying the transaction among prosumers unless the amounts of generation power from prosumers are much more less than their demands.

CHAPTER 2: ROBUST PRICING AND BOUNDEDLY RATIONAL USEREQUILIBRIUM FOR MARKET STUDIES OF RESIDENTIALELECTRICITY CONSUMERS IN SMART GRID

NOMENCLATURE

A. Sets, Indices, Parameters and Variables for the Equilibrium Model

A Set of appliances indexed by a

I Set of users indexed by i

i^- Denote all other users in I except user i

T Set of time periods indexed by t

$T^0_{i,a}$ The unacceptable time periods for user i to use appliance a

$T^1_{i,a}$ The preferred time periods for user i to use appliance a

B. Parameters

$D_{i,a}$ The total daily demand of user i on appliance a

$E_{i,a}$ The maximum electricity that can be consumed by user i on appliance a in one time period

π_i The momentary value of the time-of-use convenience for user i

$c_0,\ c$ Electricity price coefficients in the cost function

C. Variables

$x_{i,a}^t$ Electricity consumption of user i on appliance a at time t

$\mathbf{x}_i$ The electricity usage profile of user i, i.e., a vector of $x_{i,a}^t$ for all appliances and time periods

l_t, l_i^t Total electricity loads/consumptions at time t for all users and user i, respectively

$p_{i,a}$ Electricity consumption of user i on appliance a during preferred times

β_t Surplus price for time period t

μ_i Lagrangian dual multiplier for user i

p_i Penalty coefficient for user i

s_i Auxiliary variable for user i when using the penalty cutting plane method

D. Functions

$f(\cdot)$ Unit electricity cost as of a function of total load

$u_i(\cdot)$ Utility function for user i

Introduction

With increasing world population and rapid development and use of new electrical appliances, residential power consumption has significantly increased in the past decades. According to U.S. Energy Information Administration (EIA), the residential consumption accounts for about 37% of the total electricity end use in the past decade (2008-2017), rising from around around 33%

in the 1980s and 25% in the 1950s. Because the introduction of electrical vehicles (EV) [20] is shifting from gas consumption toward electricity consumption, it is estimated that mass adoption of EVs will double the residential electricity consumption. In the foreseeable future the percentage of residential use in total electricity end use will continue to increase. Therefore, it is imperative to achieve high efficiency in residential electricity consumption for a sustainable energy system. Peak-to-Average (PTA) ratio of electricity demands indirectly reflects the system redundancy and additional cost for system stability due to the fact electrical power being a instantly perishable commodity. Hence it is considered as an important index for power systems' efficiency. Demand Side Management (DSM) by leveraging time-of-day electricity prices, has long been proposed since 1980s to reduce PTA ratio in the commercial or industrial use. The introduction of Advanced Metering Infrastructure (AMI) in smart grid has enable DSM in the residential sector by sending price signals to residential consumers in real time. Various pricing schemes [59] such as time-of-use, critical peak pricing, day ahead pricing [55, 42, 47] and real-time pricing [72, 8] have been adopted in practice (e.g., ComEd in Chicago area) and all have achieve some level of successes in peak load reduction.

In the literature, many have built mathematical models to study the optimal electricity consumption profiles and the optimal design of pricing schemes for DSM [17]. Since DSM is centered around altering consumer behaviors via financial incentives, many have devoted to develop game theoretical models to describe electricity consumption profiles of residential users. For example, [3] proposed a system optimum model and a Nash Equilibrium model considering both electricity costs and convenience of electricity consumption profiles. Similar works based on Nash Equilibrium also appeared in [71, 29]. [23] studied the use of energy management controller for electric vehicles, and conclude that the game-theory-based controller on the New European Driving Cycle (NEDC) works better than the existing baseline controller. Further, [100] proposed an integer linear programming models based on game theory for optimal scheduling the use of power-shiftable

appliances. In addition to modeling user behaviors and optimal scheduling, many have also studied the pricing problems. For example, [37] showed that a proper pricing strategy is important to minimize the total cost and proposes optimal pricing policies under certain conditions. [72] demonstrated that pricing strategy has big influence in the electricity system and proposed strategies that can effectively shift users' consumption from peak to off-peak time. In addition, similar work also appeared in [73, 18]. At the same time, many have combined the use of game theory and pricing model to determine the optimal pricing strategies in smart grids. For example, [92] integrated distributed generations to reduce the energy losses. They achieve this in two ways: making a game theory-based loss reduction allocation and making a load feedback control with price elasticity. [97], on the other hand, combined game theory and pricing strategies using virtual machines (VMs) placement. They propose new algorithms to solve the problem of dynamic placement of VMs for energy consumptions' optimization.

This paper considers a mathematical model framework to reflect the reality that not every user or even no user seeks to perfectly minimize their electricity bill (e.g., [44]) or maximize their personal utility (including electricity cost, comfort and convenience). This is because humans have limited cognitive ability to solve optimization problem in practice and that usage of some home appliances can be flexible so long as certain threshold or range is maintained. In other words, most users are satisfied or have no incentive to change their consumption profiles so long as the total utility (including electricity cost, comfort and convenience) of their consumption profiles reaches a certain threshold. In economic literature, this phenomenon is referred to as "bounded rationality." In contrast to Nash equilibrium [64] where individual players optimize their own problem until no unilateral change of strategies occurs and the system reaches an equilibrium, [78] defines "boundedly rational user equilibrium" (BRUE) as the system reaches an equilibrium when no unilateral change is needed when individual players accept the utility to be at least at a certain percentage of the maximum value of their optimal utility. The notion of BRUE has been widely

used in modeling users behaviors. For example, [70] use the BRUE concept in the bank system, while [58] use the boundedly rationality in their transportation planning models. In addition, [56] used the BRUE concept in the static traffic assignment problem, and they solved the resulting mathematical program with equilibrium constraints (MPEC) by using the penalty method. [36] also use the BRUE in a dynamic traffic assignment problem. Examples of BRUE related works in other fields include [44] in psychology, [26] in industrial organization, and [96] in energy systems. To our best knowledge, this paper is the first to use the BRUE framework to study residential electricity consumption and related pricing strategies in DSM in a smart grid. Under the BRUE framework, this paper studies four core problems, extending from the system optimum model, which aims to minimize the total generation cost while satisfying all shift-able demands. The four core problems are actually categorized into two groups. The first group includes two models built to explore the best and the worst possible performances of the BRUE conditions in terms of total generation cost. The second group contains two models (with pessimistic v.s. optimistic viewpoints) that aim to provide pricing strategies via bi-level optimization.

All four core problems are very difficult to solve due to nonlinearity and nonconvexity. Of particular, the proposed bi-level pricing models are even more challenging. However, we show that the special structure of the lower-level electricity consumption game allows the bi-level model to satisfy the "hidden convexity" property first introduced by [13] in 1996. By exploiting the hidden convexity of the BRUE models, it is guaranteed that the proposed Lagrangian dual cutting plane method will produce optimal solution with zero duality gap. They especially focused on the nonconvex, quadratically constrained problem. In a more recent study, [10] found that under some conditions, the nonconvex quadratic problem is equivalent to a convex problem. Since then, [53] and [95] analyze the conditions of hidden convex in more general cases beyond the quadratic constrained problems. [25] uses the relaxation techniques to transform the nonconvex to an approximately hidden convex problem. [15] discusses the hidden convex property under the

condition with positive eigenfunctions. [63] applied the hidden convexity to communication problems. To our best knowledge, this paper is the first to exploit hidden convexity to solve hard BRUE related pricing models in a smart grid.

The contributions of this paper are summarized as follows. First, we propose mathematical models under the principle of bounded rationality user equilibrium in residential electricity consumption games compared to the most existing works in the literature that are under the Nash equilibrium principle. The BRUE models are more realistic in that they adequately acknowledge that electricity consumers do not necessarily optimize their energy consumption in real life. Second, we consider four cases best-performance and worst-performance system optimal models with BRUE constraints, and pessimistic and optimistic pricing models with BRUE constraints. Furthermore, we show that by introducing a carefully chosen time-varying surcharge, electricity users will change their energy consumption behavior ultimately leading to higher system efficiency (i.e., lower peak-to-average ratio). Third, we show that even though BRUE pricing models are non-convex, the Lagrangian method still satisfies the property of strong duality due to its hidden convexity. Finally, we conduct extensive sensitivity analysis to provide managerial insights for stakeholders of the DSM program in a smart grid.

Mathematical Models

Two Energy Consumption Models

We introduce two basic energy consumption models, i.e., the system optimal and user equilibrium models in energy consumption. The definition of the sets, indices, parameters and variables are listed at the beginning of this chapter. As in [3], we considers a local residential electrical power system with n users and a set of appliances A for each user. In the system optimal model (SO),

the energy company want to minimize their total electricity cost based on the certain customer demand. In this model, $t \in T = \{1, 2, \cdots, 24\}$ has 24 time periods in a daily cycle.

$$
\begin{aligned}
\text{(SO):} \quad \min \quad & \sum_{t \in T} f\left(\sum_{i \in I} \sum_{a \in A} x_{i,a}^{t}\right) \cdot \sum_{i \in I, a \in A} x_{i,a}^{t} && && (2.1a)\\
\text{s.t.} \quad & \sum_{t \in T} x_{i,a}^{t} = D_{i,a}, && \forall\, i \in I,\ a \in A, && (2.1b)\\
& x_{i,a}^{t} \leq E_{i,a}, && \forall\, i \in I,\ a \in A,\ t \in T, && (2.1c)\\
& x_{i,a}^{t} = 0, && \forall\, i \in I,\ a \in A,\ t \in T_{i,a}^{0}, && (2.1d)\\
& x_{i,a}^{t} \geq 0, && \forall\, i \in I,\ a \in A,\ t \in T, && (2.1e)
\end{aligned}
$$

where $f(l_t)$ is the utility cost function, which is a monotone increasing function of the total electricity consumption l_t at time t. In this paper, we let $f(l_t) = p \cdot l_t + q$. p and q are constant value here. In the SO model, we suppose the central power company can control all users consume behavior to let the system work in the best way. The constraints here means the power supply meet the customer demands.

On the other hand, unlike the central controller's system optimal model, the user equilibrium model assumes each user optimizes her/his own objective which is a combination of electricity cost and self convenience based on electricity consumption profile, i.e., when and how much the consumer uses his/her appliances. Hence each user i maximizes the following payoff or utility:

$$
U_i = -\left[\sum_{t \in T} f(l_t) \cdot l_i^t\right] + u_i(\mathbf{x}_i) \tag{2.2}
$$

where l_i^t is the total electricity consumption by user i at time t, and $\mathbf{x}_i$ is the electricity usage profile of user i, a vector of $x_{i,a}^t$ for all appliances and time periods. In this payoff U_i, the first term is the total power cost, which is actually the disutility for user i. The second term defines the convenience utility by user i, it means when a user use the appliance within his/her desired time period, it has

positive convenience utility. We define $u_i(\mathbf{x}_i) = \sum_{a\in A}\sum_{t\in T}\pi^t_{i,a}v^t_{i,a}(x^t_{i,a})$, where $\pi^t_{i,a}$ and $v^t_{i,a}$ are the parameter of convenience. Hence, in the equilibrium model, each user needs to solve the following UO_i model:

$$
\begin{aligned}
(\mathrm{UO}_i):\quad \max\quad & U_i = -\left[\sum_{t\in T} f\left(\sum_{a\in A} x^t_{i,a} + \sum_{j\in I\setminus\{i\},a\in A} x^t_{j,a}\right)\cdot\left(\sum_{a\in A} x^t_{i,a}\right)\right] + u_i(\mathbf{x}_i) && (2.3a)\\
\text{s.t.}\quad & \sum_{t\in T} x^t_{i,a} = D_{i,a}, && \forall\, a\in A, && (2.3b)\\
& x^t_{i,a} \le E_{i,a}, && \forall\, a\in A,\ t\in T, && (2.3c)\\
& x^t_{i,a} = 0, && \forall\, a\in A,\ t\in T^0_{i,a}, && (2.3d)\\
& x^t_{i,a} \ge 0, && \forall\, a\in A,\ t\in T. && (2.3e)
\end{aligned}
$$

The UO$_i$ model means each user want to maximize their unilities. Thus, the objective (2.3a) for user i is to minimize the total disutility, but each users decision also depends on other user's decision. It is a game theory model.

Finally, in an n-user system where each user solves their own UO$_i$ model, we define the user equilibrium for the energy consumption game as follows. In game theory model, each user does not have the incentive to change his/her decision, it means the equilibrium will have the following inequality.

$$
(\mathrm{UE})\quad U_i(\mathbf{x}^*_i;\mathbf{x}^*_{i-}) \ge U_i(\mathbf{x}_i;\mathbf{x}^*_{i-}), \forall \mathbf{x}_i \in X_i,\ \forall i = 1,\cdots,n. \tag{2.4}
$$

Two Boundedly Rational User Equilibrium Models

While the UE model (2.4) represents the decentralized behavior for all energy users instead of a centralized control scheme by the utility firm as in the (SO) model, still one drawback of the UE is that in practice no user has either the desire or cognitive ability to *optimize* a utility function. Based

on Simon's notion of **bounded rationality**, people are assumed to be happy if they can reach some level of utility without maximizing his/her utility. Hence, the boundedly rational user equilibrium (BRUE) is defined by a set of constraints compared to simultaneous optimization problems for equilibrium with perfect rationality. Assume user i has a satisfaction level $\xi_i \in [0,1]$, i.e., he/she is happy if his/her utility is within the range $[\xi_i R_i, R_i]$. Here, R_i is the upper bound of user i's utility and can be computed by assuming zero consumptions from others, i.e., $R_i = U_i(\mathbf{x}_i^*; \mathbf{0})$. Hence, any $\mathbf{x} \in F$ satisfies the BRUE condition if the following holds:

$$-\left[\sum_{t\in T} f\left(\sum_{i\in I, a\in A} x_{i,a}^t\right)\cdot\left(\sum_{a\in A} x_{i,a}^t\right)\right] + u_i\left(\mathbf{x}_i\right) \geq \xi_i R_i, \quad \forall\, i \in I. \tag{2.5}$$

The BRUE constraint (2.5) enforces a lower bound on the utility of user i. Alternatively, the above BRUE condition can be rewritten in terms of disutility as the following:

$$\left[\sum_{t\in T} f\left(\sum_{i\in I, a\in A} x_{i,a}^t\right)\cdot\left(\sum_{a\in A} x_{i,a}^t\right)\right] - u_i\left(\mathbf{x}_i\right) \leq \rho_i W_i, \quad \forall\, i \in I. \tag{2.6}$$

where $\rho_i \geq 1$ is a scalar and W_i is the minimum disutility for user i assuming no other users exist.

Because essentially bounded rationality (BR) is represented by a lower (or upper) bound constraint on individual's utility (or disutility), it gives rise to subsequent optimization models with such BR constraint. Below we introduce two modeles representing the best and worst BRUE conditions, respectively.

To formulate the best performance BRUE conditions, we aim to minimize the total system generation cost while respecting the BR constraint.

$$\text{(B-BRUE):} \quad \min \quad \sum_{t\in T} f\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^t\right)\cdot \sum_{i\in I, a\in A} x_{i,a}^t$$

$$\text{s.t.} \quad \left[\sum_{t\in T} f\left(\sum_{i\in I, a\in A} x_{i,a}^t\right)\cdot\left(\sum_{a\in A} x_{i,a}^t\right)\right] - u_i(\mathbf{x}_i) \leq \rho_i W_i, \quad \forall\, i\in I \tag{2.7a}$$

$$\mathbf{x}\in F, \tag{2.7b}$$

The above (B-BRUE) is a non-convex quadratic problem where both the objective function and the constraints (2.7a) are quadratic.

On the other hand, the worst performance BRUE conditions are found by maximizing the total system genration cost with the BR constraints.

$$\text{(W-BRUE):} \quad \max \quad \sum_{t\in T} f\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^t\right)\cdot \sum_{i\in I, a\in A} x_{i,a}^t$$

$$\text{s.t.} \quad (2.7\text{a}), (2.7\text{b})$$

Pricing Strategies in Boundedly Rational User Equilibrium for Electricity Consumer Market

Building on the BRUE energy consumption models the previous section, we now consider the pricing strategies to be employed by the utility firm under the two BRUE energy consumption behavior scenarios. Because under BRUE, consumer behaviors are within a given range, the performance of the system also falls into a range. Hence it is necessary to discuss pessimistic and optimistic pricing strategies acknowledging varying system performances under BRUE.

The optimistic pricing model below determines an optimal pricing scheme (or surcharge on top of the generation cost), β^t, so that the resulting total system cost is minimum given any users' behaviors under BRUE falling within their own rational bounds.

$$\text{(O-P):} \quad \min_{\beta^t\in B, \forall t\in T} \quad \min \quad \sum_{t\in T} f(l_t)\cdot l_t \tag{2.9a}$$

$$\text{s.t.} \quad \left[\sum_{t\in T}\left[\beta^t + f\left(l_i^t + l_{i-}^t\right)\right]\cdot l_i^t\right] - u_i\left(\mathbf{x}_i\right) \le \rho_i W_i, \qquad \forall\, i \in I \tag{2.9b}$$

$$l_i^t = \sum_{a\in A} x_{i,a}^t, \qquad \forall\, t\in T, i\in I \tag{2.9c}$$

$$\mathbf{x} \in F, \tag{2.9d}$$

where β^t is the surcharge price at time t for all users, and β is the vector composed by all β^t and B is the set of β^t.

The (O-P) problem is a bi-level optimization problem. The upper level minimizes the total system cost for the utility firm to select an optima price strategy β, and the lower level minimizes the same objective for users to select an optimal consumption profile $\mathbf{x}_i$. As will be discussed in Section 2, the solution method for solving the (O-P) is similar to that for solving (B-BRUE) as the two levels of minimization in (O-P) can be merged into a single level objective.

Similarly the pessimistic/robust pricing strategy determines a proper pricing strategies so that the resulting maximum system cost due to varying consumer behaviors under BRUE can be minimized. In other words, we assess the highest possible system cost given users' rational bounds of their satisfactory utilities, and then minimize this worst case system cost. The pricing problem can be formulated as a two-stage robust optimization problem as follows:

$$\text{(PR-P):} \quad \min_{\beta^t\in B, \forall t\in T} \quad \max \quad \sum_{t\in T} f\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^t\right)\cdot\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^t\right) \tag{2.10a}$$

$$\text{s.t.} \quad \sum_{t\in T}\left(\beta^t + f\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^t\right)\right)\cdot\left(\sum_{a\in A} x_{i,a}^t\right) - u_i\left(\mathbf{x}_i\right) \le \rho_i W_i, \forall\, i\in I \tag{2.10b}$$

$$\mathbf{x} \in F \tag{2.10c}$$

The (PR-P) model is a non-linear and non-convex problem due to the nonlinearity of the objective function and the BRUE constraint. This section investigates conditions under which the so-called "hidden convexity" [13] holds for the (PR-P) problem and the next section presents its solution algorithm by exploiting the hidden convexity property.

Hidden Convexity

"Hidden convexity" [13] refers to a non-convex optimization problem for which the Lagrangian dual has zero duality gap. Hence, computationally it can be treated as a convex optimization problem. Below we present three main results. First, the (W-BRUE) problem has the hidden convexity property when there is $n = 1$ user in the system. Second, in general though there is no guarantee that the hidden convexity holds for a system with $n > 1$. Third, when $n > 1$, under certain conditions with respect to the BRUE constraint in the (W-BRUE) problem, strong duality can still satisfy with and without the pricing strategy.

Lemma 1. *Recall the following (W-BRUE) problem.*

$$\begin{aligned}
\textit{(W-BRUE):} \quad \min \quad & \sum_{t\in T} f\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^t\right)\cdot \sum_{i\in I,a\in A} x_{i,a}^t \\
\textit{s.t.} \quad & \left[\sum_{t\in T} f\left(\sum_{i\in I,a\in A} x_{i,a}^t\right)\cdot\left(\sum_{a\in A} x_{i,a}^t\right)\right] - u_i(\mathbf{x}_i) \leq \rho_i W_i, \quad \forall\, i\in I \quad (2.7a) \\
& \mathbf{x}\in F,
\end{aligned}$$

Let $g_i(x) = \left[\sum_{t\in T} f\left(\sum_{i\in I,a\in A} x_{i,a}^t\right)\cdot\left(\sum_{a\in A} x_{i,a}^t\right)\right] - u_i(\mathbf{x}_i) - \rho_i W_i$ *and* λ_i *be the associated Lagrangian multiplier for the ith constraint in (2.7a). Suppose* x^* *and* λ^* *are the optimal solutions to the primal and Langrangian dual problems, respectively, then (W-BRUE) has strong duality if the*

following holds: $g_i(x^*) \leq 0, u_i^* \cdot g_i(x^*) = 0, \quad \forall i \in I$.

Proof. Let Z^* and Z_D^* be, respectively, the optimal objective values for the (W-BRUE) and its Lagrangian dual after relaxing constraint $2.7a$ with Lagrangian multiplier λ. Let

$$h(x) = \sum_{t \in T} f\left(\sum_{i \in I}\sum_{a \in A} x_{i,a}^t\right) \cdot \sum_{i \in I, a \in A} x_{i,a}^t$$

be the objective function. From weak duality, $Z_D^* \geq Z^*$. Further, if $g_i(x^*) \leq 0, \forall i \in I$, then x^* is a feasible solution to (W-BRUE) and thus $Z^* \geq h(x^*)$. Hence, $Z_D^* = h(x^*) - \sum_{i \in I} u_i^* \cdot g_i(x^*) = h(x^*) \leq Z^*$. Therefore, $Z^* = Z_D^*$, i.e., the Lagrangian dual method has the strong duality. □

Lemma 2. [39]

Let A and B be two real symmetric matrices. If there exist α, $\beta \in \mathbb{R}$ *such that* $\alpha A + \beta B > 0$, *then there exists a nonsingular matrix* $C \in \mathbb{R}^{n \times n}$ *such that both* $C^T AC$ *and* $C^T BC$ *are diagonal.*

Lemma 3. [13]

Consider the following nonlinear program:

$$\begin{aligned} \text{(BT):} \qquad \min \qquad & 1/2x^T Ax + c^T x \\ \text{s.t.} \qquad & 1/2x^T Bx \leq d; \\ & 1/2x^T Gx + h^T x + k \leq 0; \end{aligned}$$

where $A, B, G \in \mathbb{R}^{n \times n}$ *are symmetric, and* $c, d, k, h \in \mathbb{R}^n$. *The following holds:*

(1) If one of the matrices A, B, G *is a zero matrix and the other two are simultaneous diagonalizable, then separability is obtained.*

(2) Let $\nu_1^* \geq 0$ *and* $\nu_2^* \geq 0$ *be the KKT multipliers of the two constraints of (BT), respectively, corresponding to an optimal solution* x^*. *Then, either* $\nu_1^* = 0$ *or* $\nu_2^* = 0$ *holds.*

We rewrite the W-BRUE model as follows:

$$
\begin{aligned}
\text{(W-BRUE-Q):} \quad \min \quad & p \cdot x^T A x + q_1^T x; \\
\text{s.t.} \quad & x^T B^k x + d_1^T x - d^k \leq 0, \quad k = 1, 2, \cdots, n \\
& H^T x + m \leq 0, \\
& x \in \mathbb{R}^{nT}, q_1 = [q, q, \cdots, q] \\
& A,\ B^k \in \mathbb{R}^{nT \times nT}, d^k \in \mathbb{R}, m \in \mathbb{R}
\end{aligned}
$$

Where p and q are relative value in function $f(l_t) = p \cdot l_t + q$. $d_1^T x = q\left[\sum_{t \in T}\left(\sum_{a \in A} x_{i,a}^t\right)\right] - u_i(\mathbf{x}_i)$, it has a linear relation with x. $d^k = \rho_k W_k$. We can have linear transformation for the variable x to let d_1=0. n is the number of users, T is the number of time periods and A and B^k are symmetric whose $(i, j)^{th}$ elements are defined below.

$$
a_{ij} = \begin{cases} 1, & \forall (i,j) \in A_1 \\ 0, & else \end{cases} \quad , \quad b_{ij}^l = \begin{cases} 1, & \forall (i,j) \in B_1^l \\ 1/2, & \forall (i,j) \in B_2^l \cup B_3^l \\ 0, & else \end{cases}
$$

$$
\begin{gathered}
A_1 = \Big\{(i,j) : n(t-1)+1 \leq i \leq nt, n(t-1)+1 \leq j \leq nt, \forall t = \{1, 2, \cdots, T\}\Big\}. \\
B_1^l = \Big\{(i,j) : i = j = n(t-1)+l, \forall t = \{1, 2, \cdots, T\}\Big\}. \\
B_2^l = \Big\{(i,j) : i \neq j, i = n(t-1)+l, n(t-1)+1 \leq j \leq n(t-1)+T, \forall t = \{1, 2, \cdots, T\}\Big\}. \\
B_3^l = \Big\{(i,j) : i \neq j, j = n(t-1)+l, n(t-1)+1 \leq i \leq n(t-1)+T, \forall t = \{1, 2, \cdots, T\}\Big\}.
\end{gathered}
$$

To illustrate, when $n = 2$ the matrices A and B^1 are as follows.

$$
\mathbf{A} = \begin{pmatrix}
1 & 1 & 0 & 0 & \dots & 0 & 0 \\
1 & 1 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 1 & 1 & \dots & 0 & 0 \\
0 & 0 & 1 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 1 & 1 \\
0 & 0 & 0 & 0 & \dots & 1 & 1
\end{pmatrix}, \quad
\mathbf{B^1} = \begin{pmatrix}
1 & 1/2 & 0 & 0 & \dots & 0 & 0 \\
1/2 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 1 & 1/2 & \dots & 0 & 0 \\
0 & 0 & 1/2 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 1 & 1/2 \\
0 & 0 & 0 & 0 & \dots & 1/2 & 0
\end{pmatrix}
$$

If there is more than one person in the system, the BRUE constraints for the model (W-BRUE) are nonconvex constraint. Because with the definition of convex feasible region in such form, we need the matrix B^k to be an positive semi definite (PSD) matrix. But actually in our case if we choose $x = [1, -2, 0, 0, \cdots, 0]$, then now $x^T B^1 x = -1 < 0$, and now $x \neq 0$, so B^1 is not a PSD matrix, then the feasible region for such constraint is not a convex set.

Theorem 1. *Consider the (W-BRUE) problem and its Lagrangian dual after relaxing constraint (2.7a).*

(1) If $n = 1$, then strong duality holds for the (W-BRUE) and the Lagrangian dual.

(2) If $n > 1$ and at least $n - 1$ constraints in (2.7a) are non-binding at the optimal solution, then strong duality holds for the (W-BRUE) and the Lagrangian dual.

Proof. First consider the special case where $n = 1$. In this case, A and B are both an identity matrix. According to Lemma 2, there exists a nonsingular matrix C such that both $C^T AC$ and $C^T BC$ are diagonal. We call C as the simultaneous diagonalization matric for A and B. Subsequently, applying Lemma 3's result (1), one concludes that the W-BRUE problem with $n = 1$ is convex.

On the other hand, if $n > 1$ and at least $n-1$ (2.7a) constraints are non-binding at the optimal solution, then there is only one constraint that is active in the system. Hence, the problem also reduces to a convex problem. $\square$

Note that Theorem 1 suggests that when the value of *rho* in constraint (2.7a) is chosen such that conditions (1) or (2) in the theorem holds, then strong duality holds between W-BRUE and its Lagrangian dual. Below is a counterexample when neither (1) nor (2) is satisfied, then there may be duality gap between W-BRUE and its dual.

Finally we present some strong duality results for the pessimistic pricing problem (PR-P) in (2.10a)-(2.10c). Let Z^* and β^* be the optimal objective value and the associated optimal pricing, respectively. Further, let $Z^*_{\beta'}$ denote the optimal objective value for the inner maximum problem when β is fixed at $\beta^{'}$. When relaxing constraint (2.10b) for the Lagrangian dual problem (PR-PD) (see details in section 2), we use Z^*_D to denote the optimal objective value for the dual problem (PR-PD), and β^*_D to denote the corresponding optimal value for β for (PR-PD). Finally, let $Z^*_{D,\ \beta'}$ be the optimal objective value for problem (PR-PD) when β is fixed at $\beta^{'}$.

Theorem 2. *The following holds for the pessimistic/robust pricing model (PR-P).*
(1): Z^* = Z^*_D if and only if $Z^*_{\beta^*}$ = $Z^*_{D,\ \beta^*}$, when it has the unique solution.
(2): Strong duality holds for (PR-P) and its Lagrangian dual when relaxing constraint (2.10b) if and only if it has the strong duality to fix $\beta = \beta^*$.

Proof. For part (1), firstly, if $Z^* = Z^*_D$ is satisfied, then we show that $Z^*_{\beta^*} = Z^*_{D,\ \beta^*}$. This is because $Z^* = Z^*_{\beta^*}, Z^*_D = Z^*_{D,\beta^*_D}$ From the definition of β^*, one has $Z^*_{\beta^*} \leq Z^*_{\beta^*_D}$ Further, by weak duality, $Z^*_{\beta^*_D} \leq Z^*_{D,\beta^*_D}$. Hence $Z^* = Z^*_{\beta^*} \leq Z^*_{\beta^*_D} \leq Z^*_{D,\beta^*_D} = Z^*_D = Z^*$. Therefore, all inequalities hold as equality, i.e., $Z^*_{\beta^*_D} = Z^*_{\beta^*} = Z^*_{D,\ \beta^*_D}$. Because it has the solution, one obtains $\beta^*_D = \beta^*$, thus $Z^*_{\beta^*} = Z^*_{D,\ \beta^*}$.

Secondly, we show that if $Z^*_{\beta^*} = Z^*_{D,\ \beta^*}$ then $Z^* = Z^*_D$. This is because $Z^* = Z^*_{\beta^*} \leq Z^*_{\beta^*_D} \leq Z^*_{D,\ \beta^*_D} =$

$Z_D^* \leq Z_{D,\ \beta^*}^* = Z_{\beta^*}^*$. Similar to the above, all inequalities must hold as equalities, therefore $Z^* = Z_D^*$.

For part (2), this is a direct result of part (1). □

Algorithms for BRUE and Pricing Models

We have four BRUE related models. (B-BRUE) is the Best Performance of the BRUE Conditions, whereas (W-BRUE) is the Worst Performance of the BRUE Conditions. When pricing is considered, (O-P) solves for the optimistic Pricing Strategies whereas (PR-P) determins the pessimistic/Robust Pricing Strategies. For these four models, we apply three calculation methods for different models. First method is using solver BARON [87] directly. BARON can be used tor solve (B-BRUE), (W-BRUE) and (O-P). Second method is that we develop a penalty cutting plane method for each model and then apply BARON to solve the sub-problems. Third method is applying the lagrangian dual cutting plane method for each model and then solve sub-problems with BARON.

Cutting Plane with Penalty Method

In model (B-BRUE), (W-BRUE) and (O-P), we can use BARON to solve the problem directly. However, since (PR-P) is a robust optimization, BARON cannot be used directly to solve it. In this section, we first propose a penalty cutting plan method for solving the (RP-P) and then propose a Lagrangian dual cutting plane method, and finally compare the efficiency of the two methods.

In devising the penalty cutting plane, we penalize all BRUE constraints in (PR-P) into the objective

function, and therefore obtain the following problem (PR-PP).

$$\text{PR-PP:} \quad \min_{\beta^t \in B, \forall t \in T} \quad \max_{\mathbf{x} \in F} \quad P(\mathbf{x};\beta), \tag{2.12a}$$

where $P(\mathbf{x};\beta)$ is new penalty objective function. More specifically,

$$\begin{aligned} P(\mathbf{x};\beta) \;=\; & \sum_{t\in T} f\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^{t}\right)\cdot\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^{t}\right) \\ & -\sum_{i\in I} p_i \left[\rho_i W_i + \sum_{t\in T}\left(\beta_t + f\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^{t}\right)\right)\cdot\left(\sum_{a\in A} x_{i,a}^{t}\right) - u_i(\mathbf{x}_i) + s_i\right]^2, \end{aligned}$$

where p_i is the penalty value for the corresponding constraint (2.10b), $u = [u_i,\ \forall i \in I]^{\mathsf{T}}$ is as before the utility coefficient, and $s_i \geq 0$ is the slack variable for the constraint (2.10b).

Hence the model of the cutting plane method can be formulated as follows.

$$\begin{aligned} \text{(PR-MPP):} \quad & \min_{\beta\in B,\ \pi} \quad \pi & (2.13a)\\ & \text{s.t.} \quad \pi \geq a_k\beta + b_k\beta^2 + c_k, \quad \forall k = 0,1,2,\cdots,l & (2.13b) \end{aligned}$$

a_k, b_k, c_k are the corresponding coefficient for β, β^2 and constant value in $P(\mathbf{x};\beta)$ when $\mathbf{x} = \mathbf{x}_k$, and π is the cutting plane value. Below is the outline for the resulting penalty cutting plane method.

Remark. Under optimal pricing strategies, the optimal system cost falls into the range of $[Z_{\text{O-P}}, Z_{\text{PR-P}}]$ due to user's behavior under BRUE.

Algorithm 1 ALG – PR – PP

1: Set k=0, $\beta_0 = 0$, UB = M, LB = -M.
2: Solve $\max\{P(\mathbf{x};\beta_k)|\mathbf{x}_i \in X_i, \forall i\}$ by BARON. We can get $\mathbf{x}_k$ and the upper bound of PR-PP, denote as UB_k.
3: If $UB_k < UB$, $UB = UB_k$.
4: Get a_k, b_k, c_k by using $\mathbf{x}_k$.
5: Add the cutting plane (2.13b) by using the relative a_k,b_k,c_k.
6: Solve PR-MPP by BARON and get β_{k+1} . And π_{k+1} is the lower bound of PR-PP, denote as LB.
7: If UB $>$ LB, $k = k+1$, go to Step 2. Otherwise, stop.

Lagrangian Dual Cutting Plane Method

We recognize that the second stage in (PR-P) is a maximization problem for a given β. This is indeed the same problem as (W-BRUE). Therefore, hidden convexity holds if we relax constraint (2.10b), and this has motivated us to study the Lagrangian dual cutting plan method as an alternative to the penalty cutting plan method.

After obtaining the Lagrangian dual for the inner maximization problem of the (PR-P) by relaxing constraint (2.10b), the new equivalent problem (PR-PD) is as follows.

$$\text{(PR-PD):} \quad \min_{\beta^t \in B,\ \mu \geq 0} \quad \max_{\mathbf{x} \in F} \quad L(\mathbf{x};\beta,\mu),$$

where $L(\mathbf{x};\beta,\mu)$ is the new objective function. More specifically,

$$\begin{aligned} L(\mathbf{x};\beta,\mu) \quad = \quad & \sum_{t\in T} f\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^t\right)\cdot\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^t\right) \\ & -\sum_{i\in I}\mu_i\left[\rho_i W_i + \sum_{t\in T}\left(\beta^t + f\left(\sum_{i\in I}\sum_{a\in A} x_{i,a}^t\right)\right)\cdot\left(\sum_{a\in A} x_{i,a}^t\right) - u_i\left(\mathbf{x}_i\right)\right], \end{aligned}$$

where μ_i is the lagrange dual variable corresponding to constraint (2.10b) and $\mu = [\mu_i,\ \forall i]^\mathsf{T}$.

Hence the Lagrangian dual cutting plane model can be formulated as follows.

$$\text{(PR-MP):} \quad \min_{\beta \in B,\ \mu \geq 0,\ \pi} \quad \pi$$
$$\pi \geq a_k \cdot \mu + b_k \cdot \beta + c_k \cdot \mu \cdot \beta + d_k, \quad \forall k = 0, 1, 2, \cdots, l,$$

where as previously, a_k, b_k, c_k and d_k are corresponding coefficient for μ,β, $\mu \cdot \beta$ and the constant in $L(\mathbf{x};\beta,\mu)$ when $\mathbf{x} = \mathbf{x}_k$. Also similar to previously, π is the cutting plane value. Below is the outline of the Lagrangian dual cutting plane method.

Algorithm 2 $\mathsf{ALG-PR-PD}$

1: Set k=0, $\beta_0 = 0$, $\mu_0 = 0$, UB = M, LB = -M.
2: Solve $\max\{L(\mathbf{x};\beta_k,\mu_k)|\mathbf{x}_i \in X_i, \forall i\}$ by BARON. We can get $\mathbf{x}_k$ and the upper bound of PR-PD, denote as UB_k.
3: If $UB_k < UB$, $UB = UB_k$.
4: Get a_k, b_k, c_k and d_k by using $\mathbf{x}_k$.
5: Add the cutting plane (2.14) by using the relative a_k, b_k, c_k and d_k.
6: Solve PR-MP by BARON and get β_{k+1}, μ_{k+1}. And π_{k+1} is the lower bound of PR-PD, denote as LB.
7: If UB >LB, $k = k+1$, go to Step 2. Otherwise, stop.

Computational Experiments and Results

In this section, we compute the result for all of the four former cases. We solved it in Matlab using the solver BARON. All of the calculations were run on a Intel(R) Core(TM)2 Duo CPU serve with 4GB of memory in 64-bit Operating System.

We define the preferred usage utility function as below,

$$u_i(\mathbf{x}_i) = \pi_i \sum_{a \in A} \left(\frac{p_{i,a}}{D_{i,a}} \right) \tag{2.14}$$

where π_i is the corresponding utility coefficient for user i and $p_{i,a} = \sum_{t \in T^1_{i,a}} x^t_{i,a}$ is the total amount for the user i to use appliance a in the preferred time period. We define the unit cost function as $f_t(l_t) = c_0 + c_1 l_t$, here f_t means the total load at the time period t.

Results under Different Data Sets

We will use differ data set to calculate for our four different models, and will analysis the calculation time and accuracy by using three algorithms mentions before.

A Simple Example With Four Time Period

We consider a quick example with two users, two appliances and four time periods. We use $[\alpha, \beta]$ to demonstrate $T - T^0_{i,a}$, which means the time period that available for the user i with appliance a. Use $[\alpha_p, \beta_p]$ to demonstrate $T^1_{i,a}$, which means the preferred time periods for the user i with appliance a. In this case, we have $1 \leq \alpha \leq \alpha_p \leq \beta_p \leq \beta \leq 4$. The relative parameters are shown in Table 2.1. And we set $c_0 = 10$, $c_1 = 3$, which is used in the unit price function $f_t(l_t) = c_0 + c_1 l_t$ in example 1. Set $\rho_1 = 1.214$,$\rho_2 = 1.054$, $\pi_1 = \pi_2 = 100$, penalty value $p_1 = p_2 = 100$, set B is $-1 \leq \sum_t \beta^t \leq 1$, $-0.5 \leq \beta^t \leq 0.5$.

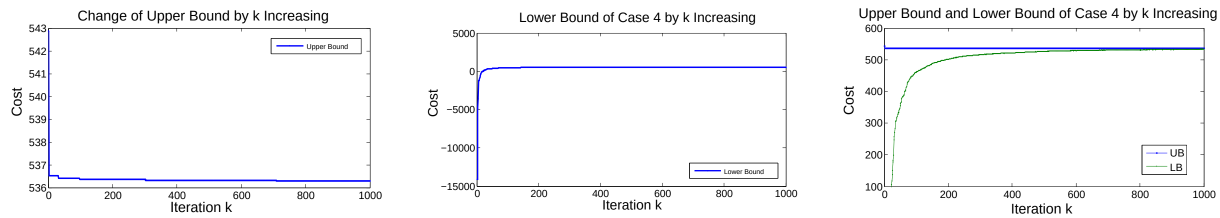
Table 2.1: Parameters of Simple Example

$User$	$Appliance$	$D_{i,a}$	$E_{i,a}$	α	β	α_p	β_p
1	1	7	3	1	3	1	1
1	2	3	1	2	4	4	4
2	1	6	4	2	4	3	3
2	2	4	3	1	3	2	2

We solve this problem with solver BARON directly and penalty method, the results are shown in Table 2.2 and Figure 2.1(a), 2.1(b), 2.1(c). w_1 is the minimum optimal cost for the user 1, if we suppose user 2 do not need to obey the BRUE. It has the similar definition for w_2.

Table 2.2: Results of Simple Example

$\mathbf{w_1 = 164.81}$ $\mathbf{w_2 = 123.30}$	**(B-BRUE)**	**(W-BRUE)**	**(O-P)**	**(PR-P)**
Cost	513.2	542.9	510.2	LB: 533.5 UB: 536.3
Solve Method	BARON	BARON	BARON	Penalty
Iterations	N/A	N/A	N/A	1000
Time Cost	0.79s	0.7s	1.12s	8 days



(a) Upper Bound (b) Lower Bound (c) Upper Bound and Lower Bound

Figure 2.1: Upper Bound and Lower Bound of Model (PR-P) by k Increasing

Table 2.2 shows that the result for four cases is different and the cost of (B-BRUE) = 513.2 is greater than the cost of (O-P) = 510.2, the cost of (W-BRUE) = 542.9 is greater than the cost of (PR-P) = $[533.5, 536.3]$. The values within the brackets are relatively the lower bound and upper bound of model (PR-P). This is reasonable because the optimal solution of (B-BRUE) must be a solution of (O-P) = 510.2, it just means $\beta^t = 0, \forall t \in T$ in (O-P). And the same reason with (W-BRUE) and (PR-P).

The calculation time cost for the first three cases are just seconds, but the calculation time for the (PR-P) is about 8 days, and it also has the gap of 2.8. This is slow first because we use the cutting plane method and the iterations are 1000. Second reason is that we also use the penalty method, then in this case we introduce new slack variables $s_i, \forall i \in I$ and make a square for the BRUE constraints, now the objective function become more complex.

Figure 2.1(a) shows that upper bound is decreasing when k increasing, it is because we use the judgement (if $UB(k) < UB, UB(k) = UB$). Figure 2.1(b) shows that lower bound is increasing when k increasing, it is because for a new iteration, we add a new constraint to the former iteration, and it is a minimum problem, so the result for the Lower Bound is increasing. Figure 2.1(c) shows that when k increasing, the upper bound and Lower Bound go to a convergency value.

Results for Instances with 24 Time Periods

From the result of four time periods case study. We know that the calculation time for the model (PR-P) which is a max-min problem with the penalty and cutting plane method is 8 days. And the gap is still 2.8 between the upper bound and lower bound from the cutting plane method. Then we use the lagrangian cutting plane method to solve our problem. In this example, the data are based on daily energy consumptions of three appliances: dishwasher, vehicle and air conditioner [1]. The unit cost here $f_t(l_t) = c_0 + c_1 l_t, c_0 = 7.43$ cents, $c_1 = 1.55$ cents per KWh. $\pi_1 = \pi_2 = 100, \rho_1 = \rho_2 = 1.7$. The upper bound of the lagrangian dual is $u_0 = 100$. set B is $-1 \leq \sum_t \beta^t \leq 1$, $-0.1 \leq \beta^t \leq 0.1$. The other parameters are based on the Table 2.3.

Table 2.3: Parameters of 24 Hours Example

$User$	$Appliance$	$D_{i,a}$	$E_{i,a}$	α	β	α_p	β_p
1	1	6.0753	1.1703	1	24	3	10
1	2	13.0305	3.2684	1	24	4	9
1	3	18.5805	2.3439	1	24	5	15
2	1	6.0753	1.1703	1	24	3	13
2	2	13.0305	3.2684	1	24	4	12
2	3	18.5805	2.3439	1	24	5	18

First under the current parameters, we can calculate w_1,w_2. Then we calculate the four cases by different methods. The results are shown in Table 2.4.

Table 2.4: The Results of 24 Hours Example

$\mathbf{w_1 = 150.2}$ $\mathbf{w_2 = 120.0}$		**Solve Method**		
		Baron	**Penalty**	**Lagrange Dual**
(B-BRUE)	Lower Bound	N/A	-	936.677
	Upper Bound	936.675	-	936.704
	Time Spend	1 day	-	2.5 hours
	Iteration	N/A	-	11
(W-BRUE)	Lower Bound	N/A	-	987.678
	Upper Bound	987.696	-	987.697
	Time Spend	1 day	-	2.5 hours
	Iteration	N/A	-	10
(O-P)	Lower Bound	N/A	-	933.985
	Upper Bound	933.991	-	934.047
	Time Spend	1 day	-	2.5 hours
	Iteration	N/A	-	11
(PR-P)	Lower Bound	N/A	853.471	972.371
	Upper Bound	N/A	978.151	973.030
	Time Spend	N/A	7 days	3 hours
	Iteration	N/A	100	12

'N/A': There has no meaning for such condition.

'-': Not calculate the result under that case.

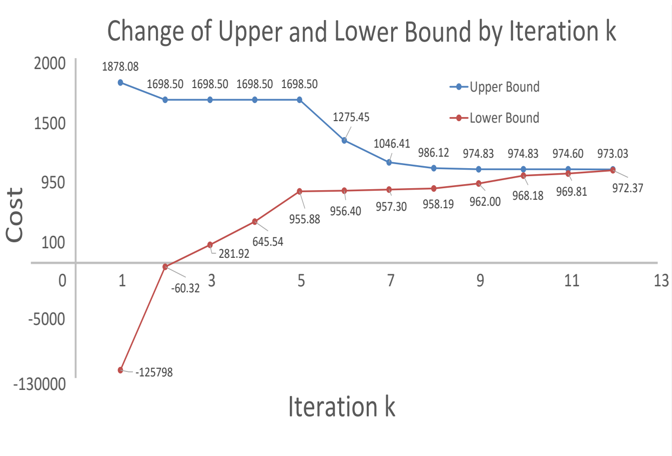


Figure 2.2: Upper Bound and Lower Bound by Iteration k in 24 Hours Model

From the Table 2.4, we can know that the calculation speed of penalty method is much slower than the lagrangian method, using the penalty method, we get the result with a gap of $978.151 - 853.471 = 124.68$ after 100 iterations and the time cost is about 7 days for model (PR-P). But using the lagrangian method, we get the result with a gap of $973.030 - 972.371 = 0.659$ only in 12 iterations and the time cost is about 3 hours for model (PR-P). We found that the lagrangian method can get more accuracy result compare to the penalty method and also spend less time to calculate. Figure 2.2 gives out the upper bound and lower bound by increasing the iteration k using algorithm 2 $\mathsf{ALG-PR-PD}$.

We also found that the time to solve model (PR-P) and model (W-BRUE) is almost the same with lagrangian method, and the numbers of iterations to be converged is also the same(for (W-BRUE) is 10, for model (PR-P) is 12). Without using the lagrangian method, the model (PR-P) is a min-max problem and (W-BRUE) is just a max problem. But using the lagrangian method, calculation speed of them become similar. It is because we introduce a lagrangian dual μ to the system, and in model (PR-P), we can mix the master problem variable unit price β and μ together, so we can

solve the master problem just like the (W-BRUE). Then we can know that the lagrangian method is an efficient method for the min-max or max-min problem if there is no lagrangian gap exists.

Figure 2.3 give us the result for the residual values for each BRUE constraint for different iteration k. From the results we can know that the tendency of the residual is becoming 0 when the iteration is increasing. And from the theorem 1 and 2 we can know that now our lagrangian dual method has the strong duality.

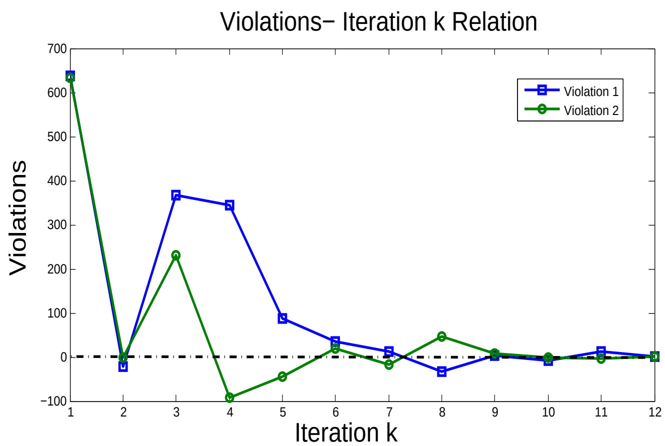


Figure 2.3: Residual For BRUE Constraints with Tow Users

The Results in 24 *Hours for Multiple Users*

In the real world, we can divided the people in different group of people with different rationality coefficient ρ. In table 2.5, we give the results for 4 users and 10 users system with the lagrange dual method. For the 4 users system, we set two users with $\rho = 1.7$ and other two users with $\rho = 1.8$. For the 10 users system. we set the ten users in five groups with the different ρ, and relatively $\rho = 1.7, 1.8, 1.9, 2.0, 2.1$.

Table 2.5: The Results of 24 Hours for Multiple Users Example

		(B-BRUE)	(W-BRUE)	(O-P)	(PR-P)
4 users	Lower Bound	2613.7	2756.6	2604.6	2714.5
	Upper Bound	2623.4	2766.2	2614.4	2724.2
	Error(%)	0.37	0.35	0.38	0.36
	Time Spend	8 hours	8 hours	8 hours	9 hours
	Iteration	23	23	24	29
10 users	Lower Bound	12099	12618	12053	12428
	Upper Bound	12297	12812	12248	12626
	Error(%)	1.6	1.5	1.6	1.6
	Time Spend	1 day	1 day	1 day	30 hours
	Iteration	73	76	74	94

We also calculation for the residual value for each BRUE constraints when the iteration k increasing in 10 users system. If we set

$$Violation_k = max\{(g_i(x_k^*), 0) | i = 1, 2, \cdots, 10\}, \quad \forall k$$

Then we have the figure 2.4, Which shows that the convergency solution almost to be zero, that means now it has the strong duality.

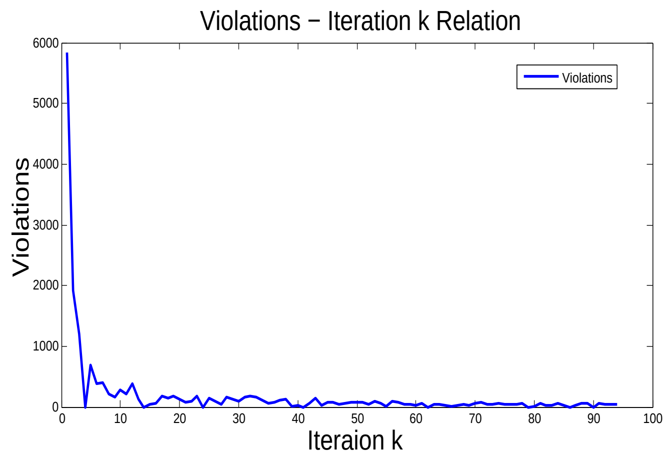


Figure 2.4: Residual For BRUE Constraints with 10 Users

Sensitivity Analysis

As we know, if the coefficient parameter's of human beings was changed, the final result will also change. We want to know the relations for the influence by changing the coefficient parameter's of human such as the boundedly rationality coefficient ρ, preference coefficient π and the customer's demand D. Here we give some theorems about the sensitivity analysis. And we will test them in the computer results part.

Impacts of π and ρ to the System

As we know, π is the utility coefficient for the user and ρ is the Boundedly Rational coefficient for the user. These two parameters are determined by the user, it changes with different groups of users. So we want to discuss how the different values of π and ρ influence the system and total cost. The results for the influence of π are shown in Figure 2.5(a). The results for the influence of ρ are shown in Figure 2.5(b). The followings are some remarks for the influence of π and ρ to the

energy system.

Remark. When the utility coefficient π is increasing, the minimum optimal objective value is also increasing.

This is because when π is increasing, actually it means the users have more incentive to use the energy in their prefer using time period because the utilities for them to use the energy in such time period are higher. Then this will lead to the peak of the energy consumption is higher, so the minimum optimal objective value is also higher.

Remark. The influence by introducing the pricing strategy is better when the utility coefficient π is greater in the minimization problem of BRUE model.

As we discussed before, when π is smaller, the users prefer to use the energy in average for each time period. So the pricing strategy does not give much influence to the user's behaviour. But when π is larger, the user prefer to use the energy in their prefer time period, at this time, we can using pricing strategy to encourage some users to move their time to use the energy to their unlike time period. So the effect for the pricing strategy is better when π is larger.

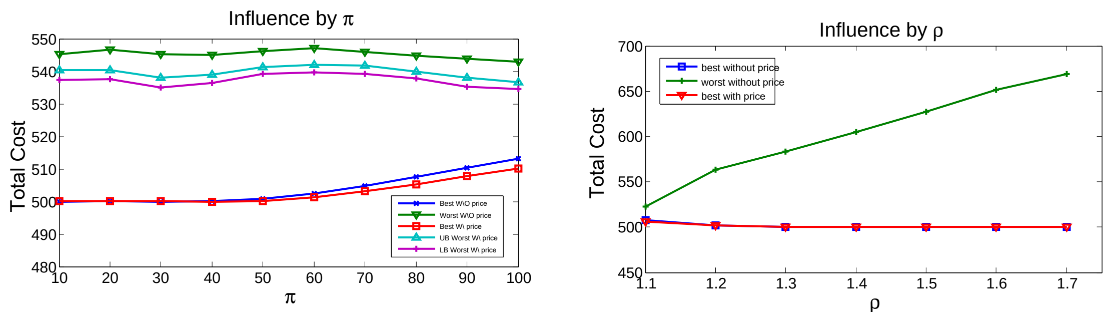
Remark. When the BRUE coefficient ρ is increasing, the minimum optimal objective value is decreasing together with the maximum optimal objective value is increasing.

When ρ is increasing, it will lead that the feasible region for the BRUE constraint is increasing. And the objective function and other constraints are keep the same. So for minimization problem, the optimal value become smaller. And for maximization problem, the optimal value become greater.

Remark. The influence by introducing the pricing strategy is better when the BRUE coefficient ρ is smaller in the minimization problem of BRUE model.

When ρ is greater enough, it means actually we do not have the BRUE constraint, now we will

have no influence by using the pricing strategy. But when ρ become smaller, the uncertainty set for the users behaviour that restricted by the BRUE constraint also becomes smaller. This will lead to the peak hours energy consumption become higher. Now the pricing strategy can make flatten for the energy consumption curve.



(a) Influence by π

(b) Influence by ρ

Figure 2.5: Sensitivity Analysis

Figure 2.5(a) shows Remark 2 and Remark 2, when π is increasing, the cost is also increasing. And it also shows that for the maximization problem, it does not have this property. And the gap of use or not use the pricing strategy is increasing when π is increasing.

Figure 2.5(b) shows Remark 2 and Remark 2. When ρ is increasing the optimal cost of the minimum problem is decreasing, and for the maximum problem the optimal cost is increasing. And the gap of use or not use the pricing strategy is decreasing when ρ is increasing.

The relative Difference of the Total Cost With or Without β

From the above we know that, if we use the β to the unit price, the total cost can decrease no matter in the best condition or the worst condition. And we also want to know that how much

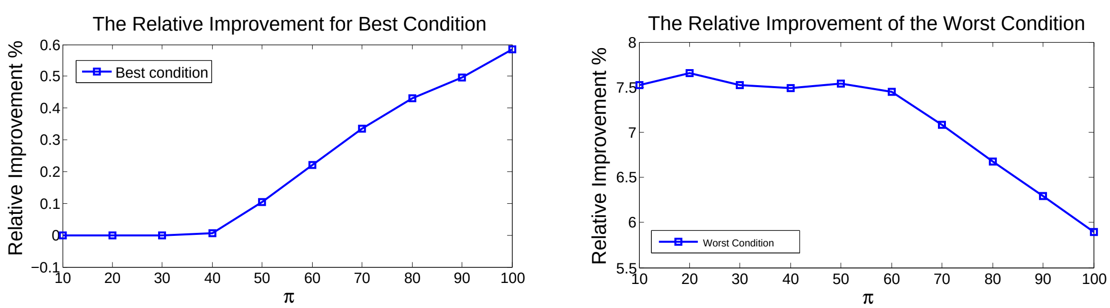
improvement by introducing β, we want to calculate the relative improvement in both the best and worst condition. We use the formulation that. The relative improvement of best condition improvement

$$\mathbf{IB} = \frac{\textbf{(Total cost of best without } \beta \textbf{- Total cost of best with } \beta\textbf{)}}{\textbf{Total cost of best without } \beta} \cdot 100\%$$

The relative improvement of best condition improvement

$$\mathbf{IW} = \frac{\textbf{(Total cost of worst without } \beta \textbf{- Total cost of worst with } \beta\textbf{)}}{\textbf{Total cost of worst without } \beta} \cdot 100\%$$

Figure 2.6(a), 2.6(b) shows the relative improvements. We can know from the result that when π increasing, the relative improvement to introduce β is also increasing. This is because when π increasing, the w is decreasing, then$(\rho - 1)w$ is decreasing, it lead to the impact of β is increasing. When $\pi = 100$, the we can improve the system in about 0.6% at best condition and about 5.7% at the worst condition.



(a) The Relative Improvement of the Best Condition (b) The Relative Improvement of the Worst Condition

Figure 2.6: The Relative Improvement of the Best and Worst Condition by Using Pricing Strategy

The Influence of the Change for the Customer Demand

In this section, we make changes to the customer demand for different users. For some theoretical analysis, we list them in following. It is the theory for the special case with two users and two appliances. We compare their influence to the total cost in minimization case and maximization case under different customer demand. In numerical result, we make changes to the demand by the following rules, $D_2^{'} = D_2 - \sigma$ or $D_4^{'} = D_4 - \sigma$, here D_2 is $D_{1,2}$ and D_4 is $D_{2,2}$ in table 2.1. The results are shown in the following figures. Figure 2.7 and 2.8 shows the results for the new w_1/w_2 and the relative minimization/maximization cost when we change the customer demand. We can know that in our example, the change of D_2 has more influence to the system compare than the change of D_4 no matter in the minimum or the maximum cases. And the relation of w_1/w_2 and the relative minimization/maximization cost have linear relation respect to the change of demand with sensitivity analysis.

In the case with two users and two appliances model, the definition of the variables and the preference time period are shown in Table 2.6. We will name this model as (Sim) and we have our model as

Table 2.6: The Definition of Variables for the Simple Example Used for Sensitivity Analysis

Variables	T_1	T_2	Preference
Person 1	x_{11}	x_{12}	T_1
Person 2	x_{21}	x_{22}	T_2

$$\text{(SW1):}\quad w_1 = min \quad [c_1 \cdot (x_{11}+x_{21})+c_0]\cdot x_{11} + [c_1 \cdot (x_{12}+x_{22})+c_0]\cdot x_{12} - \pi_1 \cdot x_{11}$$

$$\text{s.t.}\quad x_{11}+x_{12} = D_1;$$

$$x_{21}+x_{22} = D_2;$$

$$x_{11}, x_{12}, x_{21}, x_{22} \geq 0;$$

$$\text{(SW2):}\quad w_2 = min \quad [c_1 \cdot (x_{11}+x_{21})+c_0]\cdot x_{21} + [c_1 \cdot (x_{12}+x_{22})+c_0]\cdot x_{22} - \pi_1 \cdot x_{22}$$

$$\text{s.t.}\quad x_{11}+x_{12} = D_1;$$

$$x_{21}+x_{22} = D_2;$$

$$x_{11}, x_{12}, x_{21}, x_{22} \geq 0;$$

$$\text{(Sim):}\quad min/max \quad (c_1 \cdot (x_{11}+x_{21})+c_0)\cdot (x_{11}+x_{21}) + (c_1 \cdot (x_{12}+x_{22})+c_0)\cdot (x_{12}+x_{22})$$

$$\text{s.t.}\quad x_{11}+x_{12} = D_1;$$

$$x_{21}+x_{22} = D_2;$$

$$(c_1 \cdot (x_{11}+x_{21})+c_0)\cdot x_{11} + (c_1 \cdot (x_{12}+x_{22})+c_0)\cdot x_{12} - \pi_1 x_{11} \leq \rho w_1;$$

$$(c_1 \cdot (x_{11}+x_{21})+c_0)\cdot x_{21} + (c_1 \cdot (x_{12}+x_{22})+c_0)\cdot x_{22} - \pi_2 x_{22} \leq \rho w_2;$$

$$x_{11}, x_{12}, x_{21}, x_{22} \geq 0;$$

Where c_1, c_0 is the coefficient for the unit energy price, and π_1, π_2 is the relative preference coefficient for the users, D_1, D_2 is the relative customer demand.

Theorem 3. *It has more influence to the optimal value of the problem (Sim) when we change D_1 compare to we change D_2, when any one of the following two conditions is achieved.*

Condition 1, $2D_1 \geq D_2 + \pi_1/c_1$, $2D_2 \leq D_1 + \pi_2/c_1$, $2D_2 \leq (\pi_2 - c_0)/c_1$.

Condition 2, $2D_1 \leq D_2 + \pi_1/c_1$, $2D_1 \geq (\pi_1 - c_0)/c_1$, $2D_2 \leq D_1 + \pi_2/c_1$, $2D_2 \leq (\pi_2 - c_0)/c_1$.

Proof. First we make the definition that $(x_{11}^1, x_{12}^1, x_{21}^1, x_{22}^1)$ is the optimal solution to calculate w_1, and $(x_{11}^2, x_{12}^2, x_{21}^2, x_{22}^2)$ is the optimal solution to calculate w_2, and

$$\begin{aligned} f_1 &= (c_1 \cdot (x_{11} + x_{21}) + c_0) \cdot x_{11} + (c_1 \cdot (x_{12} + x_{22}) + c_0) \cdot x_{12} - \pi_1 \cdot x_{11}, \\ f_2 &= (c_1 \cdot (x_{11} + x_{21}) + c_0) \cdot x_{21} + (c_1 \cdot (x_{12} + x_{22}) + c_0) \cdot x_{22} - \pi_2 \cdot x_{22}, \end{aligned}$$

We have that $x_{11}^1 \geq x_{12}^1$, because if not we can choose $(\widetilde{x_{11}^1}, \widetilde{x_{12}^1}, \widetilde{x_{21}^1}, \widetilde{x_{22}^1}) = (x_{12}^1, x_{11}^1, x_{22}^1, x_{21}^1)$ which is also feasible to the problem $SW1$, but the value of the objective function is obviously less than the optimal value with $(x_{11}^1, x_{12}^1, x_{21}^1, x_{22}^1)$, that is contradict to it is the optimal solution to problem $SW1$. Now obviously we have (x_{12}^1, x_{22}^1)=$(0, D_2)$, this is because first the feasible set for the person 1 and person 2 is separated. Second, the coefficient for x_{12} is greater or equal to the coefficient for x_{22}. Now if we let $x_{12} = D_1 - x_{11}$, then

$$f_1(x_{11}) = c_1 \cdot (x_{11} \cdot x_{11} + (D_1 - x_{11} + D_2) \cdot (D_1 - x_{11})) + c_0 \cdot D_1 - \pi_1 \cdot x_{11}$$

Then

$$d(f_1(x_{11}))/d(x_{11}) = 4c_1 x_{11} - (2c_1 D_1 + c_1 D_2 + \pi_1)$$

$$d^2(f_1(x_{11}))/d(x_{11})^2 = 4c_1 > 0.$$

So if we let $d(f_1(x_{11}))/d(x_{11}) = 0$ and the solution (x_{11}^1, x_{12}^1)=$(D_1/2 + (D_2 + \pi_1/c_1)/4, D_1/2 - (D_2 + \pi_1/c_1)/4)$for this is also feasible for the problem $SW1$, then this solution must be the optimal solution for $SW1$, if the solution not optimal for problem $SW1$, then (x_{11}^1, x_{12}^1)=$(D_1, 0)$. The

feasibility need that $2D_1 \geq D_2 + \pi_1/c_1$, so we can get the result that

$$w_1 = \begin{cases} c_1 D_1^2 + c_0 D_1 - \pi_1 D_1, & 2D_1 \leq D_2 + \pi_1/c_1 \\ c_1 \cdot ((D_1/2 + (D_2 + \pi_1/c_1)/4)^2 + \\ (D_1/2 + (D_2 + \pi_1/c_1)/4 + D_2) \cdot (D_1/2 + (D_2 + \pi_1/c_1)/4)) \\ +c_0 D_1 - \pi_1 (D_1/2 + (D_2 + \pi_1/c_1)/4), & 2D_1 \geq D_2 + \pi_1/c_1 \end{cases}$$

And similarly for the problem $SW2$, we can get the result

$$w_2 = \begin{cases} c_1 D_2^2 + c_0 D_2 - \pi_2 D_2, & 2D_2 \leq D_1 + \pi_2/c_1 \\ c_1 \cdot ((D_2/2 + (D_1 + \pi_2/c_1)/4)^2 + \\ (D_2/2 + (D_1 + \pi_2/c_1)/4 + D_1) \cdot (D_2/2 + (D_1 + \pi_2/c_1)/4)) \\ +c_0 D_2 - \pi_2 (D_2/2 + (D_1 + \pi_2/c_1)/4), & 2D_2 \geq D_1 + \pi_2/c_1 \end{cases}$$

For condition 1,

$$\begin{aligned} d(w_1)/d(D_1) &= c_1(4D_1 + 2D_2 - 2\pi_1/c_1)/4 + c_0, \\ d(w_1)/d(D_2) &= c_1(2D_1 - D_2 - \pi_1/c_1)/4, \\ d(w_2)/d(D_1) &= 0, \\ d(w_2)/d(D_2) &= 2c_1 D_2 + c_0 - \pi_2, \end{aligned}$$

And

$$\begin{aligned} d(w_1)/d(D_1) - d(w_1)/d(D_2) &= c_1(2D_1 + 3D_2 - \pi_1/c_1)/4 + c_0 \geq 0, \\ d(w_2)/d(D_1) - d(w_2)/d(D_2) &= -(2c_1 D_2 + c_0 - \pi_2) \geq 0, \end{aligned}$$

For condition 2,

$$
\begin{aligned}
d(w_1)/d(D_1) &= 2c_1D_1 + c_0 - \pi_1, \\
d(w_1)/d(D_2) &= 0, \\
d(w_2)/d(D_1) &= 0, \\
d(w_2)/d(D_2) &= 2c_1D_2 + c_0 - \pi_2,
\end{aligned}
$$

And

$$
\begin{aligned}
d(w_1)/d(D_1) - d(w_1)/d(D_2) &= 2c_1D_1 + c_0 - \pi_1 \geq 0, \\
d(w_2)/d(D_1) - d(w_2)/d(D_2) &= -(2c_1D_2 + c_0 - \pi_2) \geq 0,
\end{aligned}
$$

So either the condition 1 or condition 2 happens, we will have that the change for w_1, w_2 by changing D_1 is larger than by changing D_2. So the influence is greater to the optimal value of the problem (Sim) when we change D_1 compare to we change D_2.

□

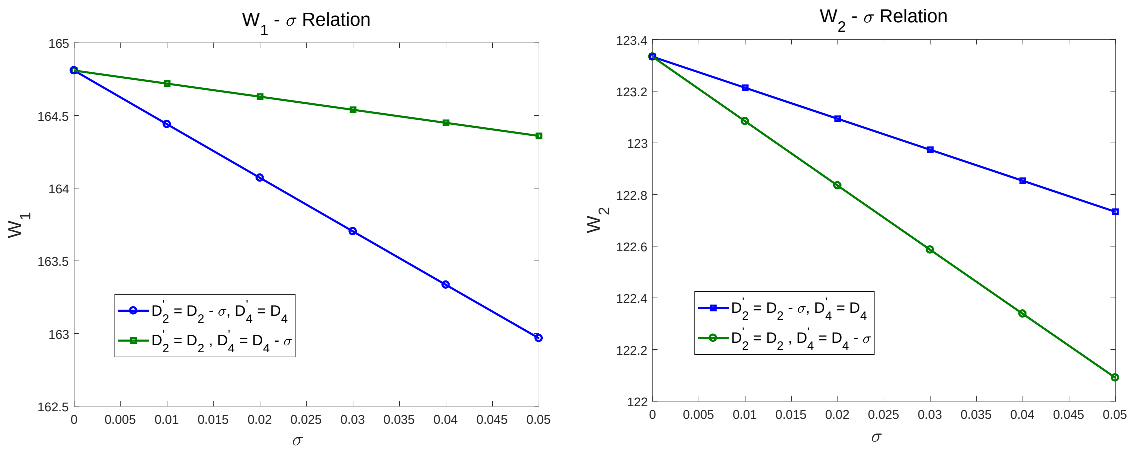


Figure 2.7: The Influence of Optimal Utility W by Change of Demand

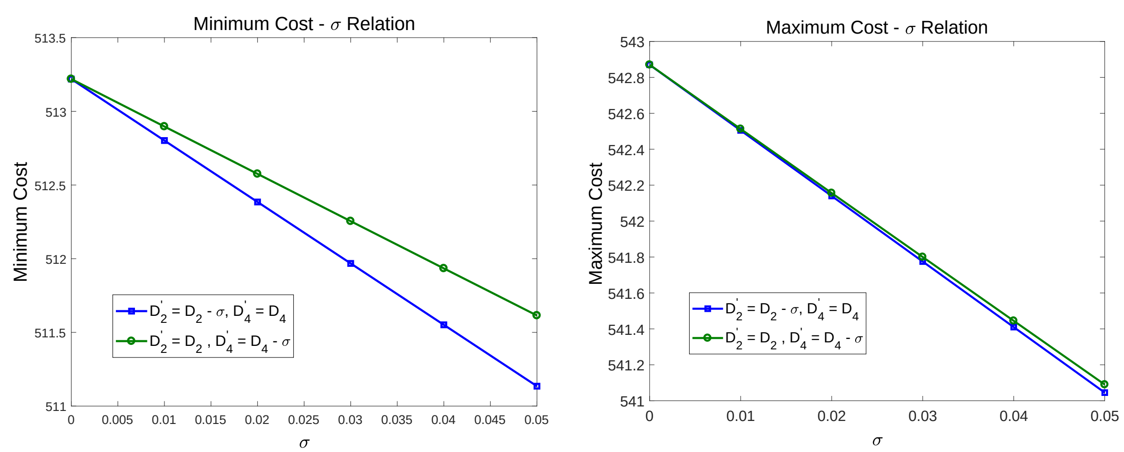
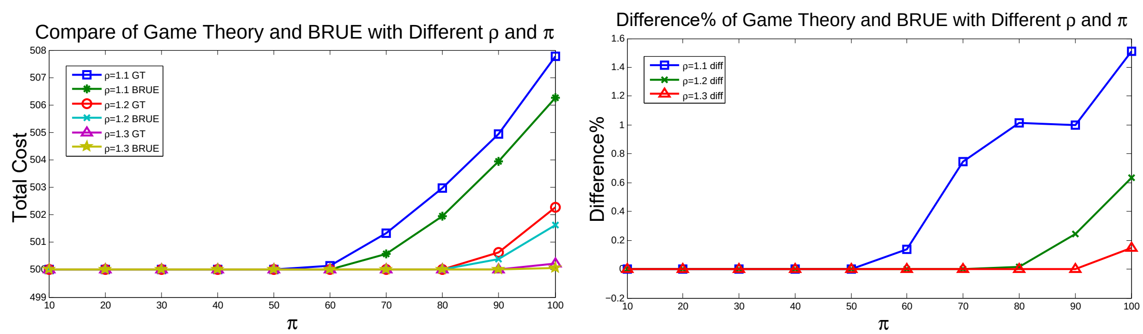


Figure 2.8: The Influence of Total Cost by Change of Demand

Compare of the Game Theory Model and BRUE with Pricing Strategy

In this section, we will compare the result of different pricing strategy that relatively determined by the game theory model and the BRUE model. We will first calculate the optimal price under

the game theory model. Then we use this price to our BRUE model and get the result under such pricing strategy. And we can get what is the error that we will get if we use the game theory model to determine the pricing strategy.



(a) Compare of Game Theory (GT) and BRUE with Different ρ and π

(b) Difference(%) of Game Theory and BRUE with Different ρ and π

Figure 2.9: The Difference of BRUE and Game Theory Model

Figure 2.9(a) gives the difference of the cost when we use the price that get from the game theory model to our BRUE model for different ρ and different π. We can know that all of the difference is positive.

CHAPTER 3: ROBUST OPTIMIZATION WITH SURPLUS PRICE TRANSPORTATION UNDER BOUNDEDLY RATIONALITY USER EQUILIBRIUM

NOMENCLATURE

A. Sets, Indices, Parameters and Variables for the Static Model

I, J Set of the node in the network, indexed by i, j

(I, J) Set of arcs, indexed by (i, j)

O, D Set of original and destination, indexed by o, d

(O, D) Set of arcs, indexed by (od)

W_{OD} Possible pairs of routes for (od), indexed by w_{od}

B. Parameters

D_{od} Demand for (od)

$\alpha_{ij;w_{od}}$ Binary value equals to 1 when the path w_{od} go through arc (i, j). Else equals to 0.

$b_{i,j}$, $k_{i,j}$ Relative coefficient of arc (i, j) for the Time function.

C. Variables

$x_{\rho,(od);w_{od}}$ Flows go into the system with rationality ρ on (od) on the path w_{od}

f_{ij} Flows on arc (i,j)

Binary β_{ij} Whether or not to change the direction of the road of (i,j)

U_{od} Minimum utility for (od)

D. Functions

$T(\cdot)$ Time cost function

Introduction

In recent world, the transportation methods are very important to our lives. We need transport the food, clothes, daily necessities and so on everyday all over the world. Also we have a lot of different kinds of methods to transport them, such as through the air, rail, road, water or tube. In this paper, we will emphasis on the road transportation. In 2010, there were over 1 billion automobiles in the world [93]. Road transport undertakes the most part of the transportation. We know that under some conditions, the road will become crowed and the travelling speed will reduce. Such as when the evacuation happens, the flows on the road will influence the evacuation speed dramatically. There were some research in this field before [76, 75, 94, 84]. But only few researchers [56] consider the boundedly rational to the problem and models. We will solve our problem under the boundedly rational user equilibrium (BRUE) by introducing the pricing strategy to our models. Different with Lou's work [56] with arc based flows, our model is path based flows, which is the actually conditions for the BRUE model.

BRUE model is proposed by Simon in year 1957 [77, 80, 81, 79], which means for one individual, when the difference of the utilities of different options that the individual can choose are below

a level, this individual will regard the utilities of such different options as the same. He or she may choose any options within that level as his or her final decisions. Here in our model of the transportation system, the utility includes the travel time and the surplus price. The concept of boundedly rational can be used in many fields, such as the energy system [96], psychology [44], military [69], transportation [58, 24, 56]and so on [70, 58, 36, 27]. When a individual in the transportation system, the individual will have a bunch of choices for the path go from the original place to the destination. Different pathes will have different utilities. And the individual may choose any path that fulfill the BRUE constraints. It means that for the pathes that do not fulfill the BRUE constraints, the flows will be zero. Actually the difference between BRUE and game theory is just the tolerance level. In the well known game theory model, the tolerance level can only be zero. But in the BRUE model, the tolerance can be greater or equal to zero. So game theory is just a special case of BRUE. Some former research was also done under the game theory [3] to the energy system. And also some work was done under game theory to the transportation system [32, 9, 61]. The transportation problem is a classical illustration for the game theory model.

As discussed above, the flows maybe only aggregate in some pathes. This will lead the time cost for the total system large. We introduced the pricing strategy to disperse the flows. The pricing strategy is a well known method [49, 51] to make the system work better. Especially in the transportation system [54, 60], by introducing the price to the arcs in the network, we can control the behaviour of the individuals under the BRUE conditions by controlling the price that we can determine for each arc. But after introducing the price to the system, our model will be a two level optimization model. The outer is to make minimization for the price. And the inner level is to make maximization under the worst case and minimization under the best case for the individual behaviour. When we want to consider the worst case, actually our model become a robust optimization model, the price become the robust price. We gave the algorithms [99] how to solve the

robust problem.

In our models, we proposed a path flow based system. It means that our main decision variables are the flows for each path. As we know, the number of pathes have the exponential relations with the number of arcs in the system. So the number of pathes are actually a large value even if the network itself is not so large. It will be hard for us to solve the problem directly, then we use the column generation and branch and price method to solve this problem. The column generation [6, 22, 91] is a well known method to solve the big problem. And in our problem, originally the BRUE constraints are complementary nonlinear constraints, we make transform for them and make them to the mixed integer linear problem(MILP). Because of the integer introduced to our model, we also use the branch and price method [74, 5] to solve our problem.

And in our models, we also use a method to find the kth shortest path in a known network. we found some former researchers who also get some results for this. The first article for this problem was done by Hoffman and Pavley [38] in 1959. Yen [98] did this in year 1971, the author gave us a method that with the computational linear time relations to the number of k. Am et al.'s work [88] gave us a method to find out all of the pathes by order between two nodes in the network, and apparently it can also find out the kth shortest path in the system. By using Eppstein's method [28] we can also find out the kth shortest path in time $O(m+nlogn+k)$, where m is the number edges in the network and n is the number of nodes in the network. Aljazzar and Leue's work [1] did this by using the heuristic method to solve this problem. In their methods, we do not need to store the whole graph in the main memory, just part of the graph need to be generated. So their advantage is that they do not need too much memory to calculate and also less time for calculation. But their disadvantage is their method is heuristic method. Also we can find some other researcher did the work in this direction [2, 35, 16, 40, 41]. Because of the work [62], in general, people can only make choice from 7-2 to 7+2. Thus, we only need to consider the value of k from 5 to 9.

We proposed a model for the static transportation path flow based problem when the evacuation happens. We supposed that all the users in the system will obey the BRUE constraint. We totally consider about four cases 1, Best case without the price. 2, Worst case without the price. 3, Best case with the price. 4, Worst case with the price. For the fist two cases, we want to find if we do not introduce the price to our system, how best and how worst the total system can be under BRUE. For the last two cases, we tried to find out the situations when we introduce the pricing strategy to the system. In our model, first we changed the originally nonlinear constraints to the MILP constraints in order to use the cplex to solve. But now the objective function is still a quadratic form. The cplex still can not be used directly, then we make the linear transform as the first level's algorithm. After this our model become the robust MILP optimization problem. Then we use the algorithm 4 for the robust part. This algorithm is our second level algorithm. After this, we change our problem to a MILP problem with huge number of variables and huge number of constraints. We use our algorithm 5 as the third level algorithm to find and check the convergency for the columns and then use the algorithm 6 and algorithm 7 to find and check the convergency for the constraints.

Overall, our original problem is a nonlinear problem with huge variables and huge constraints. After our method, we first succussed change our problem to MILP robust optimization problem. And then use the linearization and other methods to decompose our problem to just a normal MILP problem which can be solved with cpelx. Our model is a path flow based model, which is the actually model for the BRUE problem. Lou's work [56] is arc flow based model, in their model, the feasible region is smaller than the actually feasible region. So their optimal solutions are actually an upper bound for the best cases and lower bound for the worst cases. But our model can get the real optimal solution. So by using our model and method, we can get the optimal solutions for the static transportation problem.

As a reminder, section 3 is our static model section, it includes five subsections. The first four subsections are the relative subsections for the four cases. The fifth subsection is to compare our model with Lou's model [56]. Section 3 is the theorem, which will be used in the algorithm. Section 3 is our algorithm section, we totally have four algorithms, they are relatively to the four level iterations. Section 3 is the experimental results for four node example, nine node example and sioux network. The final section is the conclusion and future work.

BRUE For the Static Network Models

We proposed totally four conditions for our static models. In first two conditions, we do not use the pricing strategy to optimize our system. In the last two conditions, we illustrated the pricing strategy to see how the system will work. And for the worst case by using pricing strategy, we will have a robust optimization model.

Case 1, For the best condition without the surplus price β.

In this case we will make the model for the best condition without using the price to system. The model is in follows,

(B-BRUE):

$$\min \sum_{\rho} \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{(i,j)\in(I,J)} (\alpha_{ij;w_{od}} * \gamma_{ij} * x_{\rho,(od);w_{od}}) \tag{3.1a}$$

$$\text{s.t.} \sum_{w_{od}\in W_{OD}} x_{\rho,(od);w_{od}} = D_{\rho,od}, \quad \forall \rho, (od) \in (OD) \tag{3.1b}$$

$$\gamma_{ij} = c_{ij} + k_{ij} * f_{ij}^{p}, \quad \forall (i,j) \in (I,J) \tag{3.1c}$$

$$f_{ij} = \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{\rho} (\alpha_{ij;w_{od}} * x_{\rho,(od);w_{od}}), \quad \forall (i,j) \in (I,J) \tag{3.1d}$$

$$x_{\rho,(od);w_{od}} * (\sum_{(i,j)\in(I,J)} (\alpha_{ij;w_{od}} * \gamma_{ij}) - U_{od} - \rho) \leq 0, \quad \forall \rho, (od) \in (OD), w_{od} \in W_{OD} \tag{3.1e}$$

$$\sum_{(i,j)\in(I,J)} (\alpha_{ij;w_{od}} * \gamma_{ij}) - U_{od} \geq 0, \quad \forall (od) \in (OD), w_{od} \in W_{OD} \tag{3.1f}$$

$$x_{\rho,(od);w_{od}} \geq 0, \quad \forall \rho, (od) \in (OD), w_{od} \in W_{OD} \tag{3.1g}$$

(3.1a) is the objective function contains the summation of all arcs' flows times the time cost on that arc. (3.1b)(3.1c)(3.1d) are the relative constraints for the demand, time cost function and the time cost for arc (ij). (3.1e) and (3.1f) are the BRUE constraint. (3.1g) is the constraint to let all of the flows to be nonnegative.

Case 2, For the worst condition without the surplus price β.

In this case we will make the model for the worst condition without using the price to system. The model is in follows,

(W-BRUE):

$$\max \sum_{\rho} \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{(i,j)\in(I,J)} (\alpha_{ij;w_{od}} * \gamma_{ij} * x_{\rho,(od);w_{od}}) \tag{3.2a}$$

$$\text{s.t.} \sum_{w_{od}\in W_{OD}} x_{\rho,(od);w_{od}} = D_{\rho,od}, \quad \forall \rho, (od) \in (OD) \tag{3.2b}$$

$$\gamma_{ij} = c_{ij} + k_{ij} * f_{ij}^{p}, \quad \forall (i,j) \in (I,J) \tag{3.2c}$$

$$f_{ij} = \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{\rho} (\alpha_{ij;w_{od}} * x_{\rho,(od);w_{od}}), \quad \forall (i,j) \in (I,J) \tag{3.2d}$$

$$x_{\rho,(od);w_{od}} \leq M * z_{\rho,(od);w_{od}} \tag{3.2e}$$

$$x_{\rho,(od);w_{od}} \geq -M * (1 - z_{\rho,(od);w_{od}}) + \delta \tag{3.2f}$$

$$\sum_{(i,j)\in(I,J)}(\alpha_{ij;w_{od}}*\gamma_{ij})-U_{od}-\rho\leq M*(1-z_{\rho,(od);w_{od}}),\forall\rho,(od)\in(OD),w_{od}\in W_{OD}$$

(3.2g)

$$\sum_{(i,j)\in(I,J)}(\alpha_{ij;w_{od}}*\gamma_{ij})-U_{od}\geq 0,\quad\forall(od)\in(OD),w_{od}\in W_{OD}\tag{3.2h}$$

$$x_{\rho,(od);w_{od}}\geq 0,\quad\forall\rho,(od)\in(OD),w_{od}\in W_{OD}\tag{3.2i}$$

We can know that actually in these two models, the only difference should be just the first one is to make minimization of the objective function and the second one is to make maximization of the objective function. But we make another form for the second case. In the first case, our model for the BRUE constraint is nonlinear constraint. But in the second case, we change our model to a linear model by illustrating the big M method. But actually they show the same constraints. (3.2e)(3.2f)(3.2g)(3.2h) are the constraints to fulfill this. Here δ is a small positive value.

Case 3, For the best condition with the surplus price β.

In this case we will make the model for the best condition by using the price to system. The model is in follows,

(BP-BRUE):

$$\min_{\beta}\min_{x,z}\sum_{\rho}\sum_{(od)\in(OD)}\sum_{w_{od}\in W_{OD}}\sum_{(i,j)\in(I,J)}(\alpha_{ij;w_{od}}*\gamma_{ij}*x_{\rho,(od);w_{od}})\tag{3.3a}$$

$$\text{s.t.}\sum_{w_{od}\in W_{OD}}x_{\rho,(od);w_{od}}=D_{\rho,od},\quad\forall\rho,(od)\in(OD)\tag{3.3b}$$

$$\gamma_{ij}=c_{ij}+k_{ij}*f_{ij}^{p},\quad\forall(i,j)\in(I,J)\tag{3.3c}$$

$$f_{ij}=\sum_{(od)\in(OD)}\sum_{w_{od}\in W_{OD}}\sum_{\rho}(\alpha_{ij;w_{od}}*x_{\rho,(od);w_{od}}),\quad\forall(i,j)\in(I,J)\tag{3.3d}$$

$$x_{\rho,(od);w_{od}} \leq M * z_{\rho,(od);w_{od}} \tag{3.3e}$$

$$x_{\rho,(od);w_{od}} \geq -M * (1 - z_{\rho,(od);w_{od}}) + \delta \tag{3.3f}$$

$$\sum_{(i,j)\in(I,J)} (\alpha_{ij;w_{od}} * \gamma_{ij}) + \beta - U_{od} - \rho \leq M * (1 - z_{\rho,(od);w_{od}}), \forall \rho, (od) \in (OD), w_{od} \in W_{OD} \tag{3.3g}$$

$$\sum_{(i,j)\in(I,J)} (\alpha_{ij;w_{od}} * \gamma_{ij}) + \beta - U_{od} \geq 0, \quad \forall (od) \in (OD), w_{od} \in W_{OD} \tag{3.3h}$$

$$x_{\rho,(od);w_{od}} \geq 0, \quad \forall \rho, (od) \in (OD), w_{od} \in W_{OD} \tag{3.3i}$$

This is add the price to the system, we add the minimization of $beta$ to the objective function, and also add the price β to the relative constraint. (3.3g)(3.3h) are the constraints have relation with $beta$.

Case 4, For the worst condition with the surplus price β.

In this case we will make the model for the worst condition by using the price to system. The model is in follows, it is a robust optimization model.

(WP-BRUE):

$$\min_{\beta} \max_{x,z} \sum_{\rho} \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{(i,j)\in(I,J)} (\alpha_{ij;w_{od}} * \gamma_{ij} * x_{\rho,(od);w_{od}}) \tag{3.4a}$$

$$\text{s.t.} \sum_{w_{od}\in W_{OD}} x_{\rho,(od);w_{od}} = D_{\rho,od}, \quad \forall \rho, (od) \in (OD) \tag{3.4b}$$

$$\gamma_{ij} = c_{ij} + k_{ij} * f_{ij}^{p}, \quad \forall (i,j) \in (I,J) \tag{3.4c}$$

$$f_{ij} = \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{\rho} (\alpha_{ij;w_{od}} * x_{\rho,(od);w_{od}}), \quad \forall (i,j) \in (I,J) \tag{3.4d}$$

$$x_{\rho,(od);w_{od}} \leq M * z_{\rho,(od);w_{od}} \tag{3.4e}$$

$$x_{\rho,(od);w_{od}} \geq -M * (1 - z_{\rho,(od);w_{od}}) + \delta \tag{3.4f}$$

$$\sum_{(i,j)\in(I,J)} (\alpha_{ij;w_{od}} * \gamma_{ij}) + \beta - U_{od} - \rho \leq M * (1 - z_{\rho,(od);w_{od}}), \forall \rho, (od) \in (OD), w_{od} \in W_{OD} \tag{3.4g}$$

$$\sum_{(i,j)\in(I,J)} (\alpha_{ij;w_{od}} * \gamma_{ij}) + \beta - U_{od} \geq 0, \quad \forall (od) \in (OD), w_{od} \in W_{OD} \tag{3.4h}$$

$$x_{\rho,(od);w_{od}} \geq 0, \quad \forall \rho, (od) \in (OD), w_{od} \in W_{OD} \tag{3.4i}$$

This is a robust optimization model, which we need to first make the minimization for the price β and then make the maximization for the path flows x.

As we given before the flows on arc (i,j) are $f_{ij} = \sum_{\rho} \sum_{od\in OD} \sum_{w_{od}\in W_{OD}} \alpha_{ij;w_{od}} * x_{\rho,(od);w_{od}}$. The time function for the arc (i,j) is $T(ij) = b_{ij} + k_{ij} * f_{ij}$. And $T(w_{od}) = \sum_{(ij)\in(IJ)} \alpha_{ij;w_{od}} * T(ij)$, $T(w_{od})$ is the time cost for the path w_{od}. Now the total system time cost function is $f(x) = \sum_{(ij)\in(IJ)} T(ij) * f_{ij}$.

We can get the result

Gradient of $f(x)$:

$$\nabla_{x_i} f(x) = 2 * T(w_{od}) - \sum_{(ij)\in(IJ)} \alpha_{ij;w_{od}} * b_{ij} \tag{3.5}$$

To be simplify, we suppose in our system we only have one group of people who have the same boundedly rational coefficient ρ and one (OD) pair. There are totally n possible pathes in the

system. The time cost for each path denoted as $\{T(1), T(2), \cdots, T(k), T(k+1), \cdots, T(n)\}$. Only the first k path has flows. Now the relative formulations becomes

Simplify formulations :

$$f_{ij} = \sum_{i=1}^{n} \alpha_{ij;i} * x_i \tag{3.6a}$$

$$T(ij) = b_{ij} + k_{ij} * f_{ij} \tag{3.6b}$$

$$T(i) = \sum_{(ij)\in(IJ)} \alpha_{ij;i} * T(ij) \tag{3.6c}$$

$$f(x) = \sum_{(ij)\in(IJ)} \left[\left(\sum_{i=1}^{n} \alpha_{ij;i} * x_i \right) \cdot \left(b_{ij} + k_{ij} \cdot \sum_{i=1}^{n} \alpha_{ij;i} * x_i \right) \right] \tag{3.6d}$$

$$\nabla_{x_i} f(x) = 2 * T(i) - \sum_{(ij)\in(IJ)} \alpha_{ij;i} \cdot b_{ij} \tag{3.6e}$$

In order to use the algorithms in the following, we want to first show the theorem 1, 2, 3

Theorem 1. The optimal solution for the maximization problem must have the property that

$$\max_{1\le i,j\le k} \{|T(i) - T(j)|\} = \rho$$

For the non-trivial conditions. **Proof.**

First, because of the constraints for the BRUE restrictions we must have

$$\max_{1\le i,j\le k} \{|T(i) - T(j)|\} \le \rho$$

Second, suppose if

$$\max_{1 \leq i,j \leq k} \{|T(i) - T(j)|\} < \rho$$

Then Without loss of generality, we can suppose

$$T(1) - \frac{1}{2} \sum_{(ij) \in (IJ)} \alpha_{ij;1} \cdot b_{ij} \leq T(2) - \frac{1}{2} \sum_{(ij) \in (IJ)} \alpha_{ij;2} \cdot b_{ij} \leq \cdots \leq T(k) - \frac{1}{2} \sum_{(ij) \in (IJ)} \alpha_{ij;k} \cdot b_{ij}$$

For a $\sigma > 0$, we can let

$$\begin{aligned} x^{'}(k) &= x(k) + \sigma \geq 0, \\ x^{'}(1) &= x(1) - \sigma \geq 0, \\ x^{'}(i) &= x(i) \geq 0, \quad \forall i = 2, 3, \cdots, k-1, k+1, \cdots, n \end{aligned}$$

We denote

$$\triangle T(i) = T^{'}(i) - T(i)$$

as the difference of the time cost for the path i when we change the solution from x to $x^{'}$. Then we can know that

$$\triangle T(i) = \sum_{(ij) \in (IJ)} k_{(ij)} \cdot \alpha_{ij;i} \cdot \alpha_{ij;1} \cdot (-\sigma) + \sum_{(ij) \in (IJ)} k_{(ij)} \cdot \alpha_{ij;i} \cdot \alpha_{ij;k} \cdot \sigma = K(i) \cdot \sigma$$

Where $K(i)$ is the relative constant coefficient for the path i.

$$K(i) = \sum_{(ij) \in (IJ)} k_{(ij)} \cdot \alpha_{ij;i} \cdot (\alpha_{ij;k} - \alpha_{ij;1}) \cdot \sigma$$

We can know that $\triangle T(1) \leq 0, \triangle T(k) \geq 0$, this is because

$$\begin{aligned}
\alpha_{ij;1} \cdot (\alpha_{ij;k} - \alpha_{ij;1}) &= \alpha_{ij;1} \cdot (\alpha_{ij;k} - 1) \leq 0 \\
\alpha_{ij;k} \cdot (\alpha_{ij;k} - \alpha_{ij;1}) &= \alpha_{ij;k} \cdot (1 - \alpha_{ij;1}) \geq 0 \\
k_{(ij)} \geq 0 \quad &, \quad \sigma > 0
\end{aligned}$$

Then

$$\begin{aligned}
\triangle T(1) &= K(1)\sigma \leq 0 \\
\triangle T(k) &= K(k)\sigma \geq 0
\end{aligned}$$

We know that

$$\left|T^{'}(i) - T^{'}(j)\right| = |T(i) + \triangle T(i) - T(j) - \triangle T(j)| \leq |T(i) - T(j)| + |\triangle T(i) - \triangle T(j)|$$

Then

$$\begin{aligned}
\max_{1 \leq i,j \leq k}\{\left|T^{'}(i) - T^{'}(j)\right|\} &\leq \max_{1 \leq i,j \leq k}\{|T(i) - T(j)| + |\triangle T(i) - \triangle T(j)|\} \\
&\leq \max_{1 \leq i,j \leq k}\{|T(i) - T(j)|\} + \max_{1 \leq i,j \leq k}\{|\triangle T(i) - \triangle T(j)|\}
\end{aligned}$$

And because

$$|\triangle T(i) - \triangle T(j)| = |(K(i) - K(j))| \cdot \sigma$$

So

$$\max_{1 \leq i,j \leq k}\{|\triangle T(i) - \triangle T(j)|\} = \{\max_{1 \leq i,j \leq k}\{|(K(i) - K(j))|\}\} \cdot \sigma = K\sigma$$

Where

$$K = \max_{1 \le i,j \le k} \{|(K(i) - K(j))|\}$$

is a constant value, as we suppose before,

$$\max_{1 \le i,j \le k} \{|T(i) - T(j)|\} < \rho$$

So we can always find small enough σ to let

$$\begin{aligned} \max_{1 \le i,j \le k} \{\left|T^{'}(i) - T^{'}(j)\right|\} &< \rho + K\sigma \\ \max_{1 \le i,j \le k} \{\left|T^{'}(i) - T^{'}(j)\right|\} &\le \rho \end{aligned}$$

It means that the current solution $x^{'}$ are also a feasible solution to the primal BRUE problem. And as we know, if now the solution x is the optimal solution for the maximization problem. We can get the optimal value as

$$\begin{aligned} f(x^{'}) &= f(x) + \nabla f(x) \cdot (x^{'} - x) \\ &= f(x) + 2\left(T(k) - \frac{1}{2} \sum_{(ij) \in (IJ)} \alpha_{ij;k} \cdot b_{ij}\right)\sigma - 2\left(T(1) - \frac{1}{2} \sum_{(ij) \in (IJ)} \alpha_{ij;1} \cdot b_{ij}\right)\sigma \\ &> f(x) \end{aligned}$$

It is contradict to the conclusion x is the optimal solution for the maximization problem. So the former suppose

$$\max_{1 \le i,j \le k} \{|T(i) - T(j)|\} < \rho$$

is not right. So we can just have

$$\max_{1 \le i,j \le k} \{|T(i) - T(j)|\} = \rho$$

Proved

Theorem 2. For minimization problem, if

$$\max_{1\le m,l\le n}\left|\sum_{(ij)\in(IJ)}\left(\alpha_{ij;m}-\alpha_{ij;l}\right)\cdot b_{ij}\right|<2\rho \tag{3.7a}$$

then we can always find an optimal solution has the following property

$$T(m)-\frac{1}{2}\sum_{(ij)\in(IJ)}\alpha_{ij;m}\cdot b_{ij}=\overline{T},\quad \forall\{m|x_m>0\} \tag{3.7b}$$

$$\max_{1\le i,j\le k}\{|T(i)-T(j)|\}<\rho \tag{3.7c}$$

$$T(l)-\frac{1}{2}\sum_{(ij)\in(IJ)}\alpha_{ij;l}\cdot b_{ij}\ge\overline{T},\quad \forall\{l|x_l=0\} \tag{3.7d}$$

Else if

$$\max_{1\le m,l\le n}\left|\sum_{(ij)\in(IJ)}\left(\alpha_{ij;m}-\alpha_{ij;l}\right)\cdot b_{ij}\right|\ge 2\rho \tag{3.7e}$$

then

$$\max_{\{m,l|x_m,x_l>0\}}|T(m)-T(l)|=\rho \tag{3.7f}$$

Proof.

If the formulation (3.7a) happens, suppose the condition (3.7b) is not right. And now the optimal solution is x. If now

$$\max_{\{m,l|x_m,x_l>0\}}|T(m)-T(l)|<\rho$$

Then similar as the prove in Theorem 1, without loss of generality, we can find $\{p,q|x_p,x_q>0\}$ to

let

$$T(p) - \frac{1}{2} \sum_{(ij) \in (IJ)} \alpha_{ij;p} \cdot b_{ij} > T(q) - \frac{1}{2} \sum_{(ij) \in (IJ)} \alpha_{ij;q} \cdot b_{ij}$$

and to let

$$\begin{aligned}
x^{'}(p) &= x(p) - \sigma \geq 0, \\
x^{'}(q) &= x(q) + \sigma \geq 0, \\
x^{'}(i) &= x(i) \geq 0, \quad \forall i = 1, 2, 3, \cdots, n, and\ i \neq p, q
\end{aligned}$$

with

$$\triangle T(p) \leq 0, \triangle T(q) \geq 0$$

From the prove of Theorem 1, now the solution $x^{'}$ is also a feasible solution for the minimization problem.

The new objective value is

$$\begin{aligned}
f(x^{'}) &= f(x) + \nabla f(x) \cdot (x^{'} - x) \\
&= f(x) + 2\left(T(p) - \frac{1}{2} \sum_{(ij) \in (IJ)} \alpha_{ij;p} \cdot b_{ij} \right)(-\sigma) + 2\left(T(q) - \frac{1}{2} \sum_{(ij) \in (IJ)} \alpha_{ij;q} \cdot b_{ij} \right)\sigma \\
&< f(x)
\end{aligned}$$

Contradict to the conclusion x is the optimal solution for the minimization problem.

Else if

$$\max_{\{m,l | x_m, x_l > 0\}} |T(m) - T(l)| = \rho$$

We denote the pair as (p,q). Without loss of generality, we can suppose

$$T(p) - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;p}\cdot b_{ij} > T(q) - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;q}\cdot b_{ij} \tag{3.8}$$

We also make the same change for the variable x. Then if $T(p) \geq T(q)$, we have

$$T(p) - T(q) = \rho, \triangle T(p) \leq 0, \triangle T(q) \geq 0$$

so now $T^{'}(p) - T'(q) \leq \rho$.

Else if $T(p) \leq T(q)$, then we can know that $T(q) - T(p) = \rho$, from condition (3.8), we can get

$$\rho = T(q) - T(p) < \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;q}\cdot b_{ij} - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;p}\cdot b_{ij}$$

contradict to our primal suppose formulation (3.7a). So this condition can not be happen. $x^{'}$ is feasible for the problem. Then similar as before, we can know that x is not the optimal solution which is contradict to the former suppose. So in any cases we discussed, we must have the condition (3.7b).

Suppose condition (3.7c) is not right, then

$$\max_{1\leq i,j\leq k}\{|T(i) - T(j)|\} = \rho$$

To let

$$\max_{1\leq i,j\leq k}\{|T(i) - T(j)|\} = |T(p) - T(q)|$$

Because of (3.7b), we will have

$$\left|\sum_{(ij)\in(IJ)} \left(\alpha_{ij;p} - \alpha_{ij;q}\right)\cdot b_{ij}\right| = 2\rho$$

Which is contradict with (3.7a). So (3.7c) is proved.

Suppose the condition (3.7d) is not right. Then for some $\{l|x_l = 0\}$,

$$T(l) - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;l}\cdot b_{ij} < \overline{T} = T(m) - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;m}\cdot b_{ij} \tag{3.9}$$

Where m the index that to let $x_m > 0$. We can find small enough σ to get the following transforms

$$\begin{aligned} x^{'}(l) &= \sigma > 0, \\ x^{'}(m) &= x(m) - \sigma \geq 0, \\ x^{'}(i) &= x(i) \geq 0, \quad \forall i = 1,2,3,\cdots,n, and\ i \neq l,m \end{aligned}$$

If $T(l) \leq T(m)$, obviously, we have the solution $x^{'}$ is also feasible. Else if $T(l) \geq T(m)$ From (3.9) we can get

$$T(l) - T(m) < \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;l}\cdot b_{ij} - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;m}\cdot b_{ij} < \rho$$

And because of (3.7c) and the prove of Theorem 1, we know that now $x^{'}$ is feasible. And obviously

$$f(x^{'}) < f(x)$$

It is contradict to x is the optimal solution. So the condition (3.7d) is right.

The prove of (3.7f) is similar with the prove of Theorem 1. We do not list the details here.

Theorem 3. For the minimization problem, when the ρ is increasing, the objective function value will be strictly decreasing for $\rho \leq \overline{\rho}$, and when $\rho > \overline{\rho}$, the objective function value will keep the

same. Where

$$\overline{\rho} = \frac{1}{2} \max_{1 \le m,l \le n} \left| \sum_{(ij)\in(IJ)} \left(\alpha_{ij;m} - \alpha_{ij;l}\right) \cdot b_{ij} \right| \tag{3.10}$$

Proof. When $\rho > \overline{\rho}$, it is just the condition(3.7a), actually the optimal solution has no relation with ρ, in such cases, the BRUE constraints actually are redundant. So even though the value of ρ is still increasing, the objective value will keep the same. When $\rho > \overline{\rho}$, it is just the condition(3.7f), now we can know that when ρ increasing, the feasible region is also strict enlarged, and the enlarged part of the feasible region is also be used, so the objective value is strictly increasing.

Lemma 1.

Counter part for the Difference Between Path Flow And Link-based Flow Distribution

By using the example in Lou's work [56], the data was shown in Table 3.1. The network picture was shown in 3.2.

Figure 3.1: Parameters For the Four Nodes Example

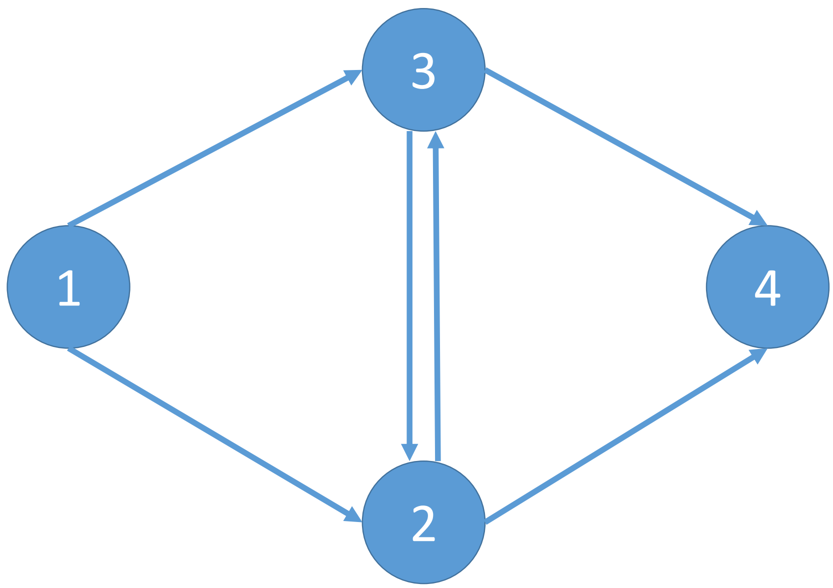


Figure 3.2: Four Nodes Flow Distribution

Link	**Time function($T(ij)$)**	**Path**	**Relative Variables**
(1,3)	$3+f_{13}$	1-3-4	$x1$
(1,2)	$7+f_{12}$	1-3-2-4	$x2$
(2,3)	$0+f_{23}$	1-2-4	$x3$
(2,4)	$5+f_{24}$	1-2-3-4	$x4$
(3,2)	$0+f_{32}$		
(3,4)	$2+f_{34}$		

If we set the flow capacity for each path to be 1. Then we can get the solution for case 1 of this problem as it shown in the table. But if we use the model presented in Lou's work. It will be infeasible for the potential value of the node 3 and 2. It is because if we set the potential of node 3 and 2 as x, y. Then we will have the following constraint.

$$\text{s.t.} 9 - 5 \leq x \tag{3.11a}$$

$$7 - 4 \leq 4 - x \tag{3.11b}$$

$$8 - 6 \leq y \tag{3.11c}$$

$$6 - 5 \leq 4 - y \tag{3.11d}$$

Obviously, this is no feasible solution for this part. So These two model are not equals at least for the minimum problem.

Algorithm

Actually for our robust optimization problem, we can simplify to write our our model as follows,

(WP-BRUE):

$$\min_{\beta} \max_{x,z} f(x,z) = x^T Dx + dx + e \tag{3.12a}$$

$$\text{s.t.} Ax + Bz + C\beta \leq R \tag{3.12b}$$

(WP-BRUE):

$$\min_{\beta} \max_{x,z} f(x,z) = x^T D x + dx + e \tag{3.13a}$$

$$\text{s.t.} Ax + Bz + C\beta \leq R \tag{3.13b}$$

To simplify, actually the problem WP-BRUES, WP-BRUEM has a lot of constraints and variables like the follows.

(BRANCHP):

$$\max_{x,z} f(x,z) = e^T x \tag{3.14a}$$

$$\text{s.t.} A^n x + B^n z \leq R^n \tag{3.14b}$$

We can use our algorithm 3 to first make the linearization of our objective function. Then use the algorithm 4 to make the robust part min-max to a master problem and a sub problem we can solve these two problems separately by using the algorithm 5. Then at last after the four level iterations, we can get our result for our case 4 problem. And for other three cases, actually we can just use part of the algorithms we posted to solve it.

To simplify state the problem for B-BRUE and W-BRUE, we can write the problem as the follows,

(P):

$$\min_{x_{l_n}, z_{l_n}} \sum_{l} \sum_{n} c_{l_n}^T x_{l_n} \tag{3.15a}$$

$$\text{s.t.} \sum_l \sum_n A_{l_n,l_n} x_{l_n} + \sum_l \sum_n B_{l_n,l_n} z_{l_n} \leq b_{l_n}, \forall l_n \tag{3.15b}$$

$$\sum_n E_{l,l_n} x_{l_n} = D_l, \forall l \tag{3.15c}$$

Where $A_{l_n,l_n} \in R^{l_n * l_n}$, because this problem has huge number of l_n, we can not solve it directly, so we want to solve a subproblem of it, we just consider about the first l_k pathes in the system. We call it *SP* problem. And we define that $l_k \in J, l_n/l_k \in J^{'}$, J is the set that $\{x_{l_k} > 0, z_{l_k} = 1 | l_k \in J\}$

(SP):

$$\min_{x_{l_k}, z_{l_k}} \sum_l \sum_k c_{l_k}^T x_{l_k} \tag{3.16a}$$

$$\text{s.t.} \sum_l \sum_k A_{l_k,l_k} x_{l_k} + \sum_l \sum_k B_{l_k,l_k} z_{l_k} \leq b_{l_k}, \forall l_k \tag{3.16b}$$

$$\sum_k E_{l,l_k} x_{l_k} = D_l, \forall l \tag{3.16c}$$

(SPC):

$$\min_{x_{l_k}, z_{l_k}} \sum_l \sum_k c_{l_k}^T x_{l_k} \tag{3.17a}$$

$$\text{s.t.} \sum_l \sum_k A_{l_k,l_k} x_{l_k} + \sum_l \sum_k B_{l_k,l_k} e_{l_k} \leq b_{l_k}, \forall l_k \tag{3.17b}$$

$$\sum_k E_{l,l_k} x_{l_k} = D_l, \forall l \tag{3.17c}$$

Where e_{l_k} is a vector has l_k elements with 1.

As the definition of the set J above, we know that $SP = SPC$. And we also establish two other problems. called SPC_1 and SPC_2.

(SPC_1):

$$\min_{x_{l_k}, z_{l_k}} \sum_l \sum_k c_{l_k}^T x_{l_k} + \sum_l c_{l_{k+1}} x_{l_{k+1}} \tag{3.18a}$$

$$\text{s.t.} \sum_l \sum_k A_{l_k, l_k} x_{l_k} + \sum_l \sum_k B_{l_k, l_k} e_{l_k} +$$

$$\sum_l a_{l_k, l_{k+1}} x_{l_{k+1}} + \sum_l b_{l_k, l_{k+1}} \leq b_{l_k}, \forall l_k \tag{3.18b}$$

$$\sum_k E_{l, l_k} x_{l_k} = D_l, \forall l \tag{3.18c}$$

(SPC_2):

$$\min_{x_{l_k}, z_{l_k}} \sum_l \sum_k c_{l_k}^T x_{l_k} \tag{3.19a}$$

$$\text{s.t.} \sum_l \sum_k A_{l_k, l_k} x_{l_k} + \sum_l \sum_k B_{l_k, l_k} e_{l_k} \leq b_{l_k}, \forall l_k \tag{3.19b}$$

$$\sum_l a_{l_{k+1}, l_k} x_{l_k} + \sum_l b_{l_{k+1}, l_k} e_{l_k} \leq b_{l_{k+1}}, \tag{3.19c}$$

$$\sum_k E_{l, l_k} x_{l_k} = D_l, \forall l \tag{3.19d}$$

$(SPCN_1)$:

$$\min_{x_{l_n}, z_{l_n}} \sum_l \sum_n c_{l_n}^T x_{l_n} \tag{3.20a}$$

$$\text{s.t.} \sum_{l}\sum_{n} A_{l_k,l_n} x_{l_n} + \sum_{l}\sum_{n} B_{l_k,l_n} z_{l_n} \le b_{l_n}, \forall l_k \tag{3.20b}$$

$$\sum_{n} E_{l,l_n} x_{l_n} = D_l, \forall l \tag{3.20c}$$

$(SPCN_2)$:

$$\min_{x_{l_k}, z_{l_k}} \sum_{l}\sum_{k} c_{l_k}^T x_{l_k} \tag{3.21a}$$

$$\text{s.t.} \sum_{l}\sum_{k} A_{l_n,l_k} x_{l_k} + \sum_{l}\sum_{k} B_{l_n,l_k} e_{l_k} \le b_{l_k}, \forall l_n \tag{3.21b}$$

$$\sum_{k} E_{l,l_k} x_{l_k} = D_l, \forall l \tag{3.21c}$$

And we also use the following model to choose the minimum reduce cost of the problem to determine which path we will choose to add the system.

(RC):

$$\min_{\alpha_{ij}} \sum_{(ij)} [(\lambda_{ij} - 1/2c_{ij})\alpha_{ij} - \pi^T \lambda_{ij}\alpha_{ij}] \tag{3.22a}$$

$$\text{s.t.} \sum_{j} \alpha_{ij} = \sum_{j} \alpha_{ji}, \forall i \neq o, i \neq d \tag{3.22b}$$

$$\sum_{j} \alpha_{oj} = 1, \sum_{j} \alpha_{jd} = 1, \tag{3.22c}$$

$$\alpha ij \in Bin, \forall (ij) \tag{3.22d}$$

Where π is the dual value for problem SPC.

And we will also use the algorithm 6 to find out which constraints we need to add to our model

SPC_2 for our system.

Theorem 4. If $SPC_1 = SPC_2$, then $SPC = P$. Where the new series of columns to add is the path with the maximum reduce cost of problem RC. And the new series of constraints to add to the system is the time cost for the path with the minimum time cost except the l_k pathes that already existed after current iteration for OD pair l.

Proof. As we know we will have the following relation $SPC_2 \geq SPC \geq SPC_1$. This is because first the feasible region for SPC is actually a subset with SPC_1 for which to let the variable x_{l_k} to be zero in SPC, so for minimization problem, $SPC \geq SPC_1$. Second, the only difference of the feasible region with SPC and SPC_2 is that SPC_2 has a new serious of constraints. So $SPC_2 \geq SPC$.
Now if $SPC_1 = SPC_2$, it means $SPC_1 = SPC = SPC_2$. The new serious of columns were chose to add from SPC to SPC_1 is from the maximum reduce cost, so if $SPC_1 = SPC$, it means that $SPC = SPCN_1$. And the new serious of constraints is the constraints that violate most for the left pathes. If $SPC_2 = SPC$, it means that the most possibly violation constraints are still within the current feasible region. So now $SPC = SPCN_2$ and then $SPCN_1 = SPC = SPCN_2$. Also as we know $SPCN_1 \leq P \leq SPCN_2$. Then now we will have $SPCN_1 = P = SPC = SPCN_2$. So $SPC = P$. □

As we illustrated above we totally have four levels of optimizations, we find actually we do not need the accuracy too much in the lower levels when the higher levels are just in the beginning several iterations. It means for the first several iterations of the higher levels we just need coarse control by calculating the lower levels. But together with the iterations become larger especially the last several iterations for the outer level, we need fine control by calculating the inner levels, that means we need to set the small tolerance for the check of convergency of the lower iterations.

Algorithm 3 LINEARIZATION

Step 0: Select $\mathbf{x}(1) \in X$ such that

$$A^n x + B^n z + C^n \beta \leq R^n.$$

Set $k = 1$.

Step 1:Solve the following robust mixed linear problem:

$$\begin{aligned} \underline{T(x)} = \min_{\beta} \max_{x,z} \quad & {x^k}^T D^n x^k + d^T x^k (D^n x^k)^T (x - x^k) \\ \text{s.t.} \quad & A^n x + B^n z + C^n \beta \leq R^n \\ & x \geq 0, z \in Bin \end{aligned}$$

and let $x^* = y^k, \beta^k, z^k$ as the optimal solution. $p^k = y^k - x^k$ is the resulting search direction.

Step 2:Convergency Check:
Let $LBD = max\{LBD, \underline{T(x^k)}\}$. If

$$\frac{T(x^k) - LBD}{LBD} < \varepsilon,$$

then stop and β^k, x^k, z^k is the solution. Else

Step 3:Line search.
Find a step length l^k which solves the following problem,

$$\min\{T(x^k + l p^k) | 0 \leq l \leq 1\}.$$

Update $x^{k+1} = x^k + l^k p^k$. $k = k + 1$, and go to step 1

Result

Simple Example

First we solve our models under a simple example, the data is from Lou's article [56]. it just has four nodes and six arcs in the system. The relative figure and data are shown in Figure 3.2 and Table 3.1, and we add an allowance for the surplus price is that the bound for the price is in $[-1, 1]$.

Algorithm 4 Robust Transform

Step 0: Select $\beta(1) \in B$. Set $m = 1.UB = M, LB = -M$

Step 1:Solve WP-BRUES:

(WP-BRUES):

$$\max_{x,z} T(x) = e^T x \tag{3.23a}$$

$$\text{s.t.} A^n x + B^n z + C^n \beta_m \leq R^n \tag{3.23b}$$

We can get x^{m+1},z^{m+1} and π^{m+1}. Set $UB = min\{UB, T(x^{m+1})\}$.

Step 2:Solve WP-BRUEM:

(WP-BRUEM):

$$\min_{\beta} T(\tilde{x}) = e^T \tilde{x} \tag{3.24a}$$

$$\text{s.t.} A^n \tilde{x} + B^n \tilde{z} + C^n \beta \leq R^n \tag{3.24b}$$

$$e^T \tilde{x} \geq e^T \tilde{x}^j, \ \ 1 \leq j \leq m \tag{3.24c}$$

$$A^n \tilde{x}^j + B^n \tilde{z}^j + C^n \beta \leq R^n, \ \ 1 \leq j \leq m \tag{3.24d}$$

$$A^{n^T} \pi^j \geq e^T \tag{3.24e}$$

$$\tilde{x}^j (A^{n^T} \pi^j - e^T) = 0 \tag{3.24f}$$

$$\pi^j (R^n - A^n \tilde{x}^j - B^n \tilde{z}^j - C^n \beta) = 0 \tag{3.24g}$$

We can get β^{m+1} and $\tilde{x}$. Let $LB = T(\tilde{x})$. If

$$\frac{UB - LB}{LB} < \varepsilon,$$

then stop and $\beta^{m+1}, x^{m+1}k, z^{m+1}$ is the solution. Else $m = m+1$, go to step 1.

The results are shown in table 3.1, in this table the meaning of iteration row $i * j * k$ means the relative iterations by using the algorithms 1 is i, algorithm 2 is j, algorithm 3 is k. And $\tilde{i}$ means the approximate times for the iteration of that algorithm, because for different outer iterations, we will

Algorithm 5 Convergency for Big Problem

Step 0: Set $k = 1$.

Step 1:Solve the problem *SPC*, if all of the variable $x_{l_k} > 0$. Go to step 3, else go to step 2.

Step 2:Delete all of the variables with the optimal value of 0 from the set J, then go to step 1.

Step 3:Solve the problem *RC*, insert the new founded path to set J.

Step 4:Solve the problem SPC_1 to get the lower bound of *SPC* as LB, and solve the problem SPC_2 to get the upper bound of *SPC* as UB. If

$$\frac{UB - LB}{LB} < \varepsilon$$

stop,else go to step 1.

Algorithm 6 Add Constraint

In this algorithm we want to find out the shortest path for the *OD* pair l except the known path $l_1, l_2, \cdots, l_k$.

Step 0: Set p=1. Make the order of $l_1, l_2, \cdots, l_k$ from lower to higher. Without loss of generality, we can assume $l_1 \leq l_2 \cdots \leq l_k$. Set the time for the path l_i as T_{l_i} for $i = 1, 2, \cdots, k$.

Step 1: Find the pth shortest path between *OD* pair l. The time for this path is $T_{l'_p}$

Step 2: If $T_{l'_p} = T_{l_p}$, $p = p + 1$. Go to step 1.
Else set $l_{k+1} = l'_p$, stop

have different inner level iterations to be converge.

We can see from the table that by introducing the pricing strategy, we can make the system work better.

We can get the relative improvement by introducing the price for the best and worst case are relatively 0.38% and 0.82%. The improvement is not very large, this is because we mandatory to set the scale of the price within a certain scale.

Algorithm 7 *pth* shortest path

In this algorithm we want to find out the pth shortest path for the OD pair l. And set the rth shortest path as l_r

Step 0: Set $q = 1$.

Step 1: If $q \leq p$, set the time cost of one arc on path l_r to be big M for every $r = 1, 2, \cdots, q-1$. Then calculate the shortest path in the new network. and set the newly time as T_i for the ith combinations. After we calculate all of the possible combinations of different arcs on path$l_r, r = 1, 2, \cdots, q-1$. We can choose the T_i with the minimum value as T_q and set the relative path of it is l_q. $q = q + 1$. Go to step 1.
Else stop.

Table 3.1: The Results of Four Nodes Example

	Case 1	*Case* 2	*Case* 3	*Case* 4
Total cost	49.82	51.31	49.63	50.89
Iteration	1*1*3	4*1*3	1*1*3	5*$\tilde{6}$*$\tilde{3}$
Time spend(s)	0.76	1.93	0.82	$\tilde{6}0$

Check the Theorems

In this example, there are totally four pathes, we set them as path $1, 2, 3, 4,$. We set the D=30. And the relatively$\frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;1} \cdot b_{ij} = 2.5$, $\frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;2} \cdot b_{ij} = 4$, $\frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;3} \cdot b_{ij} = 6$, $\frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;4} \cdot b_{ij} = 4.5$. Then

$$\overline{\rho} = \max_{1\leq m,l\leq n} \left| \sum_{(ij)\in(IJ)} \left(\alpha_{ij;m} - \alpha_{ij;l}\right) \cdot b_{ij} \right| = 3.5$$

The calculation results for different ρ are shown in table 3.2.

Table 3.2: The Results of Four Nodes Example

ρ	**3.3**	**3.4**	**3.5**	**3.6**
x_1	15.8625	15.8375	15.8125	15.8125
x_2	0.125	0.125	0.125	0.125
x_3	14.0125	14.0375	14.0625	14.0625
x_4	0	0	0	0
T_1	36.85	36.8	36.75	36.75
T_2	38.25	38.25	38.25	38.25
T_3	40.15	40.2	40.25	40.25
T_4	38.875	38.875	38.875	38.875
$T_1 - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;1} \cdot b_{ij}$	34.35	34.3	34.25	34.25
$T_2 - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;2} \cdot b_{ij}$	34.25	34.25	34.25	34.25
$T_3 - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;3} \cdot b_{ij}$	34.15	34.2	34.25	34.25
$T_4 - \frac{1}{2}\sum_{(ij)\in(IJ)} \alpha_{ij;4} \cdot b_{ij}$	34.375	34.375	34.375	34.375
$f(x)$	1151.916	1151.908	1151.906	1151.906

All of the data accords with the theorems.

Nine Nodes Example

The data we use is still from the article [56], and the the network is shown in figure 2. We can see from the result that the relative improvement by introducing the price for the best and worst case are relatively 1.82% and 2.01%

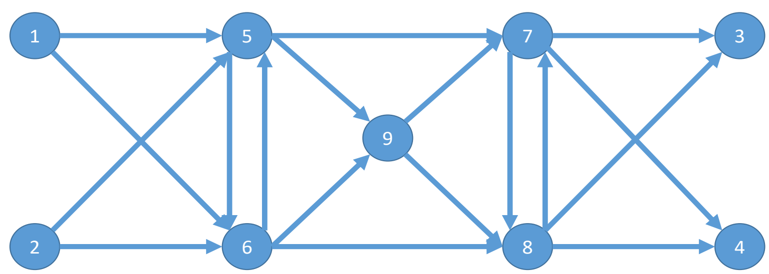


Figure 3.3: The nine nodes network

Table 3.3: The Results of Nine Nodes Example

	Case 1	*Case* 2	*Case* 3	*Case* 4
Total cost	2347.15	2532.68	2305.74	2481.89
Iteration	1*1	7*1	1*1	6*$\tilde{1}$1
Time spend(s)	1.02	5.88	1.32	1$\tilde{8}$0

Sioux Fall System

Table 3.4 is the result with two original places and two destinations, so totally have 4 *OD* pairs. Figure 3.4 gives us the compare for different boundedly rationality coefficient ρ's influence to our system. Such results are got by not using the algorithm 6 and 7. We just still check the path for

the constraints that get from the column generation. But actually such method can not guarantee us this is a convergency result for our initially problem. This just give us a comparison result. And the calculation time for Case 4 are just about 2 hours.

Table 3.4: The Results for Sioux Fall network with 4 OD pairs

Total Cost	*Case* 1	*Case* 2	*Case* 3	*Case* 4
$\rho = 0.05$	2347.66	2446.37	2311.49	2370.36
$\rho = 0.1$	2312.59	2502.38	2297.68	2386.76
$\rho = 0.2$	2298.28	2715.69	2295.46	2462.38

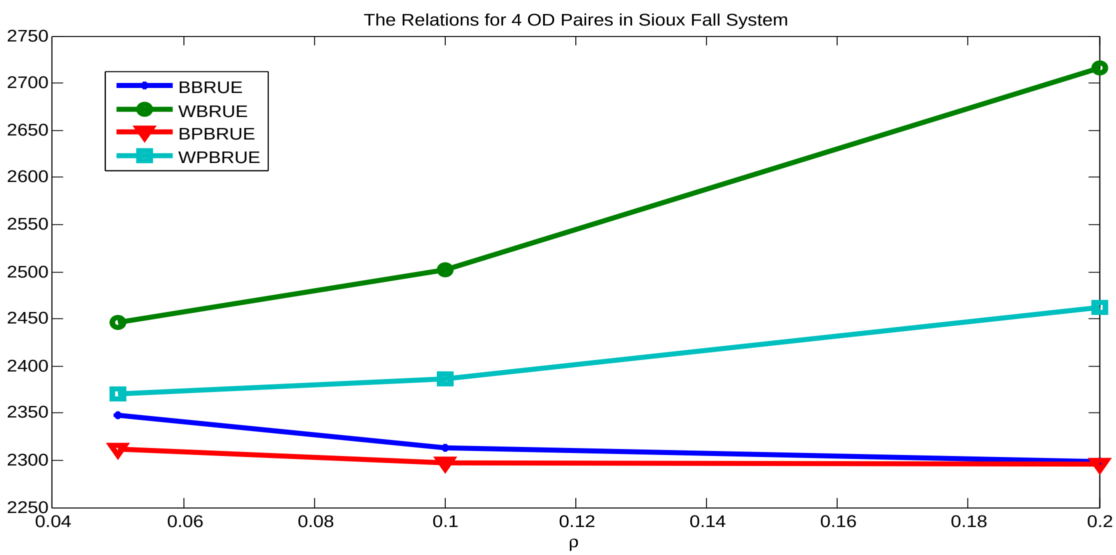


Figure 3.4: The Sioux Fall Network

So we still want to add our algorithm 6 and 7 to our system. These results are shown in table 3.5, in such results we can find that, all of the best cases of our problem we have higher results compare

to the former method, and all of the worst cases of our problem we can get lower results. This is because when we use the former method, actually we just check the rightness for the column generation, so our results actually is the results for SPC_1. Here for the best case, it is the lower bound for the real results, and for worst case it is the upper bound for the real results. The figure 3.5 shows the relations for the new method. And we give an upper bound of the number of pathes for finding the new pathes to add to check the convergency for the constraints as 15. We can find from the two tables that the most relative difference of the two method is 0.4%. It is not a big difference, but still the new method is more close to the real result. But the calculation time for case 4 is about 33 hours. It is more than the former method.

Table 3.5: The Results for Sioux Fall Network With 4 OD Pairs By Add Constraints

Total Cost	*Case* 1	*Case* 2	*Case* 3	*Case* 4
$\rho = 0.05$	2352.45	2444.58	2313.22	2365.18
$\rho = 0.1$	2314.29	2495.26	2304.43	2382.47
$\rho = 0.2$	2306.76	2703.72	2299.28	2453.28

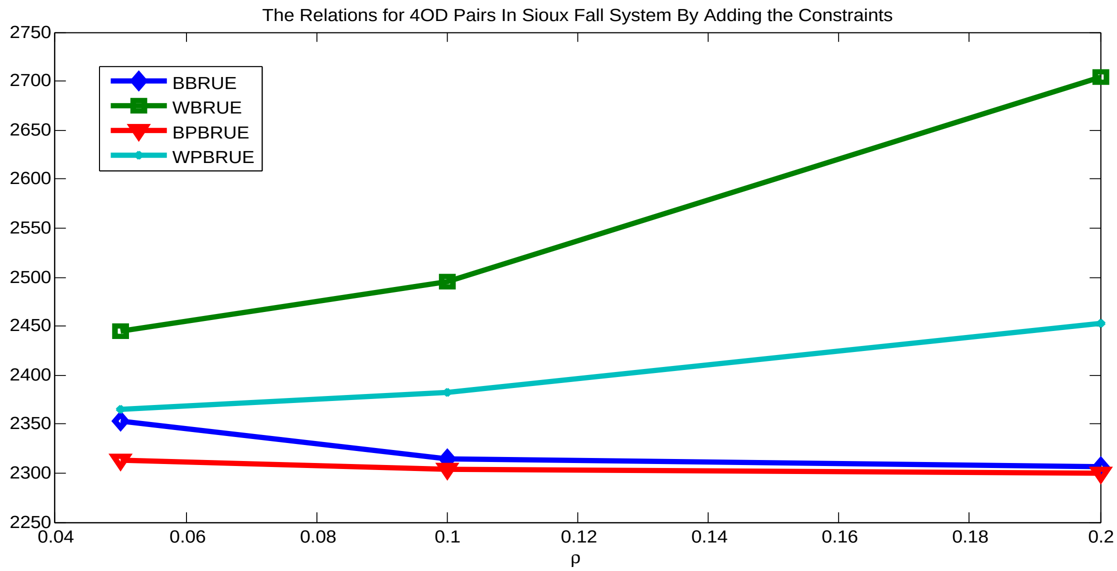


Figure 3.5: The Sioux Fall Network 2

Then we find out that because we have totally four levels of iterations, so actually for the beginning several iterations of the outer levels, actually we do not need to require the high accuracy for the inner level. By introducing this to our algorithms. We change the number of 15 we mentioned above to 5 for the first ten iterations of our Linearization level, and still keep 15 for the last iterations of the first level. Then we get some new results below, we can know from the table 3.6 that the most relative different between these two methods is only 0.04%. But this can save a lot of calculation time for our problem. The calculation time for case 4 is about 21 hours. But how to get the most efficient and accuracy for setting the convergency, it still need us to have further research. Here is just an illustration that we can get the similar results compared by using the same convergency principles.

Table 3.6: The Results for Sioux Fall Network With 4 OD Pairs By Add Constraints With Different Accuracy

Total Cost	*Case* 1	*Case* 2	*Case* 3	*Case* 4
$\rho = 0.05$	2353.16	2444.29	2313.76	2365.88
$\rho = 0.1$	2313.79	2496.32	2304.12	2381.86
$\rho = 0.2$	2306.22	2703.34	2299.76	2453.43

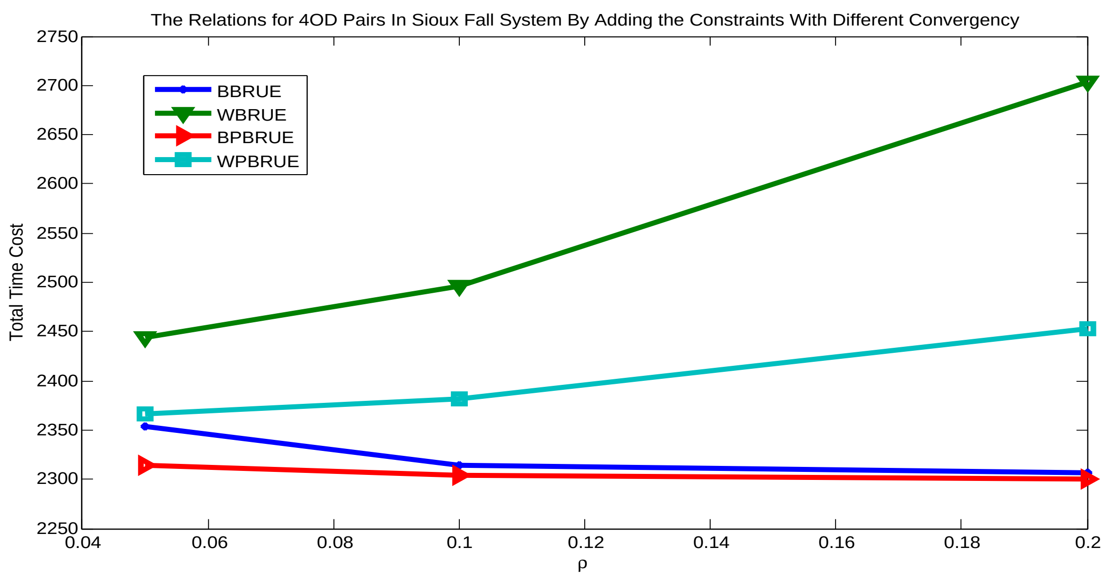


Figure 3.6: The Sioux Fall Network 3

Then this is the results with multiple OD pairs, the pairs were shown in [52]. We use $\rho = 0.1$. And for each level, we set the tolerance to be 2%, then results were shown in table 3.7. We did not use the algorithm for this problem. And the time for calculation is about 10 hours.

Table 3.7: The Results for Sioux Fall network with multiple OD pairs

Total Time Cost	*Case* 1	*Case* 2	*Case* 3	*Case* 4
	33468.1	35776.3	32867.4	34368.7

CHAPTER 4: INFORMATION NETWORK CASCADING ANDNETWORK RE-CONSTRUCTION WITH BOUNDEDLY RATIONALUSER BEHAVIORS

NOMENCLATURE

A. Sets, Indices for the Equilibrium Model

N Set of users index by i,j,k

L Set of information indexed by$l$

i Index of information provider

B. Parameters

p_{jl} Information post plan of user j

d_{ki} The value for the connection benefit to user i if followed by user k

v_{kl} Value of information l to user k

b_{kl} Unit boring value for user k to receive information l for multiple times

T_{jl} Threshold for user j to re-post information l

$\hat{x_{ij}}$ Initial network connection for user i follows to user j

ρ_j BRUE coefficient for user j

c_l Posting cost for information l

B_i Budget for information provider i

C. Variables

p_{jl} Binary variable for user j whether to post or re-post information l

x_{kj} Binary variable for user k whether to follow user j

z_{kl} Number of times for user k to receive information l

g_{kl} Binary variable for user k whether to receive information l

U_k Total information utility for user k

U_{kl} Information utility for user k from information l

D. Function

$F_{kl}(\cdot)$ Relation function for U_{kl} respect to z_{kl} for user k with information l

$G(\cdot)$ Linear threshold function for user's re-post decision p_{jl} respect to information provider's post plan p_{il}

Introduction

With the development of information technology, the social media platform plays a vital role in most people's life. For some commercial users or non-profit organizations, their profits or influences can increase by using the social media system [57, 65]. Especially for some commercial users that highly depend on the social media platform such as News Media, YouTuber or We Media Organization. Lots of people are willing to expand their network connection in social media in order to expand their influences. The number of followers will dramatically influence their profits or influences.

The behaviour of user's connectivity to a information provider is primarily influenced by two aspects [90]: the content of information posted by the information provider[89], and the personality of the user who follows the information provider. In order to expand the connections of the information provider, we need to study for what kind of content to post and also study the human

personality and human behaviours. The increasing rate of the follower will highly depend on what kind of information they post. We call these users need to make choice for information post plans as information provider. In this paper, we will optimize the information provider's information post plan in order to expand the network connections of it. Here, the information post plan means in one time period the information provider need to decide to post and not post what kind of information. The other part need to consider is different people will have different personalities, their interesting items can varies widely. And different people will also have different criteria for connectivity action.

The influence of the information to one user can only be activate if this user can receive the information. We need to consider about the information cascading route. Linear threshold model [33, 19, 14] and independent cascading model [46] are two widely used models for information cascading. in this paper, linear threshold propagation model is used for information cascading. This model was first proposed by Granovetter [34] to describe the people's behaviour. It means when the linear summation of the influence of one user's followees exceed the threshold the this user. Then this user will become active. In our case, it means when some of one user's followees re-post the information and the linear summation of these followees influence exceed the threshold of this user, then this user will also re-post this information.

After information cascades in the network, each user in the network will have a decision whether they have willing to change their followees. Users can get utility from each information, they want to have a network that can get more information they want to get. But they still want to avoid multiple times to receive the same information. because it is actually a redundant for them to get new information. We apply the concept bounded rationality user equilibrium (BRUE) as the decision principle of user's for actions of connectivity. It means one users want to choose connective schedule that can help them to achieve higher utility for different information but not need the maximum utility.

This idea of BRUE originally comes from Simon's Theory [77] in 1957. In this paper, it tells us that human behavior will has its bounded rationality. In 1972, Simon published another paper [79] which gives us an fundamental illustration about the theory of bounded rationality. The rationality of human beings will have the style of their behaviour's utility to achieve a percentage of the ideal goals, within the limits by given conditions and constraints. Simon also continuously worked on bounded rationality [80, 81] to expand the application of the theory.

The concept of bounded rationality can also be used in many fields, such as the energy system [96], psychology [44], military [69], transportation [58, 24, 56]and so on [70, 36, 27]. But as far as we know, fewer people used it on information network system. We suppose that in the information network system users' action of connectivity will obey BRUE. It means one user do not need to get the maximum information utility they can theoretically get. They just need their utilities are greater then a percentage of the maximum utility. They may execute any connection plan's utility that fulfill such criteria.

Some researchers used game theory model in social media network to determine the user's decision [82, 85]. But in the real world, the BRUE model should be more close to the natures compare to game theory model. First, the users in the information networks system will not take care about the little difference for their utility function. Second, in the information network system the utility function is not an exactly function, no one can know the exactly utility function value by using the information network, it is just an appropriate value. Third, based on the reinforcement learning [43], the users in the information networks system will also obey the BRUE principle. Because there should take some time or steps to get to the optimal condition. And before it gets the optimal condition, the topology already changed. So it is actually But in the information networks system, there does not have long enough time to let the equilibrium occurs before the next post come out. Compare to the game theory model, BRUE model should be used for the users' behavior in the information networks system. Some recent research tells us that bounded rationality of

individual users will influence the information network. Kasthurirathna and Piraveenan [45] made the simulation for a number of strategic games. Then they regenerated the network so that the network on average converged towards Nash equilibrium, despite the bounded rationality of nodes. The link between bounded rationality distributions and social structure is important in explaining social phenomena.

We generate a three-level mathematical optimization model. The first level is to optimize the information post plan of information provider in order to maximize its connections. The second level is to optimize the human behaviours of other users under BRUE. It has two formats. In optimistic condition, we maximize human behaviours for the connections of information provider. But in pessimistic condition, we minimize these variables. The reason we have two conditions is because, as we discussed before, by introducing BRUE, users' behaviours will drop in an uncertainty set. We have interesting in how best and how worst this uncertainty will influence the information provider's network connections. That is the reason we study the optimistic and pessimistic conditions for BRUE. The third level is to calculate the maximum information utility for one user can get, which need to be used in second level for BRUE constraints.

We solve a small-scale synthetic network by exact algorithms. But for large-scale network, the calculation time is increasing exponentially. We tackle this problem by using large neighbourhood search (LNS) algorithms. It is a heuristic algorithms [50] used to solve large-scale problem. It is an effect way to find a good solution quickly when the time to find the global optimal solution is too long. The main idea of this method is to block the local optimal solution and then find its neighbourhood to get a new solution. Even though sometimes the second solution is not as better as the first one. But by using such ways, sometimes the new solution can get rid of one local optimal solution. Then we can find another local optimal solution that is better than the first local solution we find.

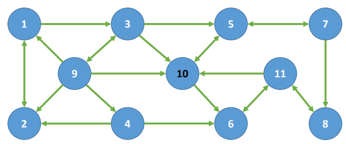
Math Model Formulation

We propose the following model to information network system. Our objective is to maximize our information provider's connections by controlling its information post plan. With different post plan, firstly, the information cascading process will be different. Secondly After information cascades, some users may have a choice to connect to a new followee or disconnect to an exiting followee. Different information post plan of our information provider will lead to different number of its followees. We use the linear threshold principle to determine users' information post behavior. And use BRUE for users to simulation the network reconstruction after information cascading.

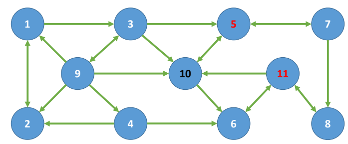
Linear Threshold Model for Information Cascading

We use the linear threshold propagation model to determine whether or not one user decides to re-post the information when this user receive it several times. When the summation of the influence of one user's followers who post the information exceeds this users threshold, it will choose to re-post this information. We will give the detail of the linear threshold constraint inside the model CP_i later. Figure 4.1 gives one example of the procedure for information cascading process and the final connection of the network after the information cascades.

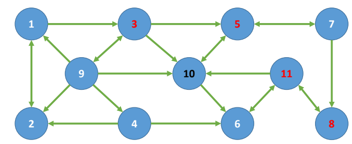
Figure 4.1(a) is the initial network of follower's linking network, the link and arrow between node 3 and node 5 means user 3 follows user 5. Some links between two users have two arrows mean these two users follow with each other. In this example our information provider is node 10 in black. And we just simulate for the cascading of one kind of information. Figure 4.1(b) is the first step cascading, by principle of linear threshold the influence of node 10 is greater than the threshold of node 5 and node 11. Then after node 10 post this information, node 5 and node 11 re-post it. We



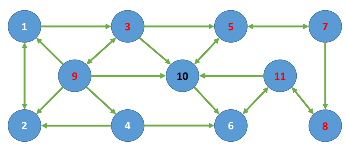
(a) Initial Graph



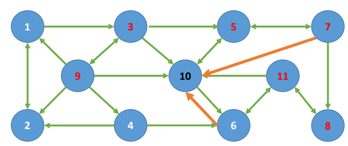
(b) First Step Cascading



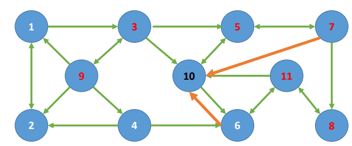
(c) Second Step Cascading



(d) Third Step Cascading



(e) New Followers



(f) New Unfollowers

Figure 4.1: Cascading for the information and the final connection network

mark the re-post node in red color. Figure 4.1(c) is the second step of cascading, we can find that now node 3 and node 8 also re-post this information. Even though node 3 already follow node 10 directly in the initial network, but the influence of node 10 does not beat the threshold of node 3. At that time the influence of 10 to 3 is less than the threshold of node 3. After the first step node 5 also re-post this information and node 3 can receive this information from both node 5 and node 10. And currently the summation of the influence of node 5 and node 10 is over the threshold of node 3, so node 3 re-post this information in the second step. Figure 4.1(d) is the third step of cascading, the similar reason for node 9 and node 7 to re-post the information. And after figure 4.1(d), the information cascading will stop. Because the node 1,2,4,6 will not re-post this information any more. And the network become stable. Figure 4.1(e) adds two new followers to our target node 10 after the cascading based on our BRUE model. Figure 4.1(f) shows that node 9 determine to un-follow node 10 based on the BRUE model. So from the principle of linear threshold, if we know the post plan of the information provider, we can know the information cascading route and procedure. Our next step is to optimize for the action of connection for other users in the system,

which will follow the BRUE constraints.

BRUE Model

BRUE is the math model's equilibrium constraints from the bounded rationality. We suppose one user in the system have multiple choices, for choice i it has utility $U(i)$. With out loss of generality, we can set the choice i^* has the optimal utility value. Then BRUE tells us that for any choice i has the following property will be deemed as the possible choice for this user.

$$U(i) \geq \rho * U(i^*). \tag{4.1}$$

Where ρ is called the bounded rationality coefficient. And we must have $\rho \leq 1$ because of the optimality of the choice i^*. From this constraint, we can know that by introducing BRUE to our math model, we will have an uncertainty feasible region for the users. And when $\rho = 1$, it is the perfect rationality user equilibrium (PRUE). And it is actually the nash equilibrium. Beacuse it means the user can only accept the plan's utility to be the maximum utility, and it has no motivation to move to another paln. Game theory nash equilibrium is a special condition of BRUE when $\rho = 1$.

In this paper, after information cascades, users may change their connections. We assume the way of users' choice to connect or disconnect will obey BRUE constraints, which is also the nature of human beings.

Pessimistic Condition

The following model (CP_i) works for our information provider i to maximize the connections by determine their post plan. It is constructed under the pessimistic condition by introducing the

BRUE constraints. pessimistic condition means under BRUE constraint the behaviour of the other users in the system act in the worst case for our information provider. The other users' choice to connect or dis-connect will lead to the minimization the number of the information provider's connections.The first level is to find the best choice for the information provider to maximize the possible worst case. In this model it also has the third level, it is to maximize for the information utility of each user in the system except the information provider. This information utility need to be used in BRUE constraint. But this value also depends on the variables in the first level.

(CP_i):

$$\max_{p_{il}} \min_{x_{ki}} BP(x_{ki}) = \sum_{k\in N, k\neq i} d_{ki} x_{ki} \tag{4.2a}$$

$$\text{s.t. } U_k = \sum_l U_{kl}, \qquad \forall k \in N, \tag{4.2b}$$

$$U_{kl} = F_{kl}(z_{kl}), \qquad \forall k \in N, \forall l \in L \tag{4.2c}$$

$$z_{kl} = \sum_{j\in N, j\neq k} x_{kj} \cdot p_{jl}, \qquad \forall k \in N, \forall l \in L \tag{4.2d}$$

$$p_{jl} = G(p_{il}), \qquad \forall j \in N, j \neq i, \forall l \in L \tag{4.2e}$$

$$x_{ki} \leq \sum_l \sum_{k'\in N, k'\neq i, k'\neq k} p_{k'l} * x_{k'k} + \hat{x}_{ki}, \qquad \forall k \in N, k \neq i, \forall l \in L \tag{4.2f}$$

$$x_{ij} \in \{0,1\}, \qquad \forall i, j \in N, i \neq j \tag{4.2g}$$

$$U_k \geq U_k^* * \rho_k, \qquad \forall k \in N, k \neq i \tag{4.2h}$$

$$U_k^* = \max_{x'_{kj}} U'_k, \qquad \forall k \in N, k \neq i \tag{4.2i}$$

$$s.t.\ (4.2b') - (4.2g') \tag{4.2j}$$

In model CP_i, the objective function is the total connection benefit the information provider i can

get from the network.

The first level's decision variable p_{il} is a binary variable to indicate whether the information provider i will post information l to the system, this kind of variable can be controlled by the information provider.

In the second level, x_{ij} is a binary variable, when it is equal to 1, it means user i follows user j in the network. else, $x_{ij} = 0$. Constraint (4.2b) gives the total information utility can get for user k equals the summation of the utility from each information in the system. We denote the function as $F_{kl}(\cdot)$. Where

$$F_{kl} = \begin{cases} v_{kl} - (z_{kl} - 1) * b_{kl}, & \forall z_{kl} \geq 1 \\ 0, & z_{kl} = 0 \end{cases}$$

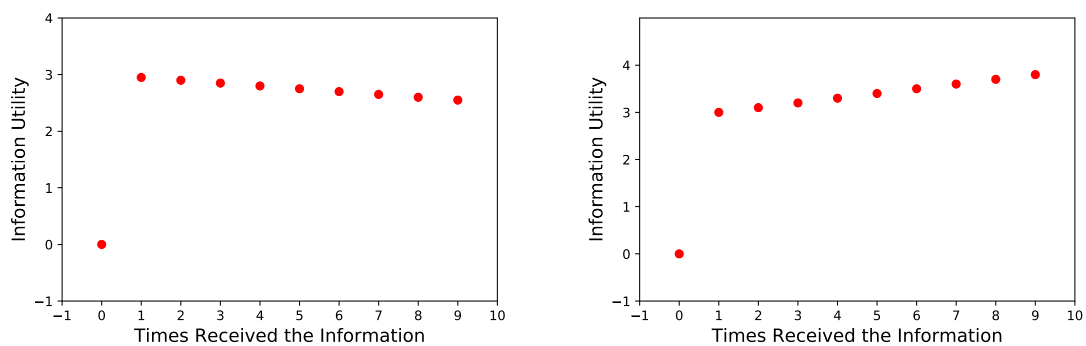
Two examples for the utility from the information respect to the number of times of one user to receive that information is showed in figure 4.2. Figure 4.2(a) has the parameter $b_{kl} > 0$ and figure 4.2(b) has $b_{kl} < 0$. Actually This function can be linearized, it is given later in (LN).

The relation of the variable p_{jl} and variable p_{il} is restricted by linear threshold principle, we denote their relation in function $G(\cdot)$. This function can not simply write out in formula. We just show the mechanism to determine p_{jl} from p_{il} in algorithm 8 ALG-LT, where T_{kl} is the information re-post threshold for user k with information l. We can notice that if we get the value of p_{il}, we can directly get the value of p_{jl} by linear threshold principle.

Constraint (4.2d) gives that for user k, the frequency of information l it gets equals to the number of its followee who re-post information l. It will be a mixed linear constraint by given p_{il}.

Constraint (4.2f) shows that user k can not have the choice to follow the information provider i if it does not follow user i originally and there has no followee of user k repost any information generated by information provider i. $\hat{x}_{ki}$ is the originally connection from user k to user i.

Constraint (4.2h) is the BRUE constraint. It gives out that user k can accept any follow-unfollow plan for which information utility drop within the BRUE gaps. Where ρ_k is the BRUE coefficient for user k. Constraint (4.2i) is the third level problem. It calculates the maximum information utility that user k can get in the system. The constraints of the third level has the same formula compare to the constraint (4.2b)-(4.2g) in the second level. But they did not share the same variable x_{jk} and U_k. We should replace all of the relative variable x_{jk} in the second level with new variable x'_{jk} and U'_k in third level. This is the reason we write the constraints in third level as $(4.2b') - (4.2g')$.



(a) Decreasing utility function (b) Increasing utility function

Figure 4.2: Relation of Information Utility to Times User Receive this information

In addition, we propose another information utility function of all users $k \in N$ for information l. It shows in figure 4.2(b). It means when a user receive the same information multiple, it's utility is increasing. In this case, we consider that if one user receive the information from other user's re-post, it means the other user likes this information. Then this information can have commonality among the user's friend circle. Then the initial value of the information will be larger if this user receive the information more times. For example, if one user has a lot of friends who re-post one information of the result of super bowl, then this user should feel this information will have more utility value to him/her. Because if he/she get this information, it is more easy to chat about it

with his/her friends. After consider this issue and the boring effect for receive same information multiple time, people with personality have strong boring feelings with obey figure 4.2(a). The people with personality have strong feeling for friend circle with obey figure 4.2(b). Actually in this case, we can seem it as this user has negative boring coefficient.

The following is the linearization for function F_{kl},

(LN) F_{kl}:

$$z_{kl} \leq M * g_{kl}, \quad \forall k \in N, \forall l \in L \tag{4.3a}$$

$$g_{kl} \leq z_{kl}, \quad \forall k \in N, \forall l \in L \tag{4.3b}$$

$$U_{kl} \geq v_{kl} - (z_{kl} - 1) * b_{kl} + M(g_{kl} - 1), \quad \forall k \in N, \forall l \in L \tag{4.3c}$$

$$U_{kl} \leq v_{kl} - (z_{kl} - 1) * b_{kl} - M(g_{kl} - 1), \quad \forall k \in N, \forall l \in L \tag{4.3d}$$

$$0 \leq U_{kl} \leq M * g_{kl}, \quad \forall k \in N, \forall l \in L \tag{4.3e}$$

$$g_{kl} \in \{0, 1\}, \quad \forall k \in N, \forall l \in L \tag{4.3f}$$

these constraints give that when $z_{kl} = 0$, we will get $g_{kl} = 0$ and $U_{kl} = 0$. When $z_{kl} > 0$, we will get $g_{kl} = 1$ and $U_{kl} = v_{kl} - (z_{kl} - 1) * b_{kl}$.

Optimistic Condition

Under Optimistic condition, the users in the system will choose the schedule which will be the best case for our information provider i. So the optimistic model (CO_i) will just make change for the objective function in model (CP_i).

(CO_i):

$$\max_{p_{il}} \max_{x_{ki}} \quad BP(x_{ki}) = \sum_{k\in N, k\neq i} d_{ki}x_{ki} \tag{4.4a}$$

$$\text{s.t.} \quad (4.2b) - (4.2j), \tag{4.4b}$$

where all of the constraints will keep the same with model (CP_i).

BRUE Model with Budget Restriction

In this model, we will consider the edit and post budget for different information. And also consider in the objective function for the cost to edit and post the information. Here just write out the model under pessimistic condition.

(CPB_i):

$$\max_{p_{il}\in \mathbf{P}} \min_{x_{ki}} \quad BP(x_{ki}) = \sum_{k\in N, k\neq i} b_{ki}x_{ki} - s_i \tag{4.5a}$$

$$\text{s.t.} \quad s_i = \sum_{l\in L} c_l * p_{il}, \tag{4.5b}$$

$$(4.2b) - (4.2j), \tag{4.5c}$$

$\mathbf{P}$ is set of post plan by consider the budget. that is to say $\mathbf{P} = \{p_{il} | \sum_{l\in L} c_l * p_{il} <= B_i\}$, B_i is the budget for user i to edit and post information. c_l is the cost for information $l.s_i$ is the information edit cost.

Algorithm

The following algorithms is to determine user's re-post decision based on the linear threshold principle and known post plan of information provider i.

Algorithm 8 $\mathsf{ALG-LT}$

---

for $j \in N, j \neq i$ **do**
    for $l \in L$ **do**
        $p_{jl} \leftarrow 0$
    end for
end for
for $l \in L$ **do**
    if $p_{il} = 1$ **then**
        for $j \in N, j \neq i$ **do**
            for $k \in N, k \neq i$ **do**
                if $\sum_{s \in N} \hat{x}_{ks} \cdot p_{sl} \geq T_{kl}$ and $p_{kl} = 0$ **then**
                    $p_{kl} = 1$
                end if
            end for
        end for
    end if
end for

Remark:

The algorithm can guarantee the information cascades by linear threshold. If after one iteration, the information post decision of all users keep the same. Then it means the cascading already finished, there will have no extra cascading. If not, it means at least one user will change their decision, so if we copy the cascading process for $n-1$ times. The cascading must be finished.

In model (CP_i), it is a three-level optimization problem, and in constraint (4.2d) it has the quadratic terms. But we can notice that, if we know the value of variables p_{il} in the first level. The problem will decompose to several one level mixed linear integer program (MILP) problem. We can imagine if we have the value of $\{p_{il}|\forall l \in L\}$, we can get all values of $\{p_{jl}|\forall j \in N, j \neq i, l \in L\}$. Then constraint (4.2d) becomes a linear constraint. And we can also calculate the value for the third level problem. It means we can get the value of $\{U^*(k)|\forall k \in N, k \neq i\}$. Then the total problem was decomposed to (n-1) one level MILP problem in the third level. And (1) MILP problem in the second level. Where n is the number of total users in the system include the information provider.

One possible method to solve this problem is to numerate all possible plans for the first level problem. But we can know that the number of different schedule to post information for user i is $2^{|L|}$, where $|L|$ is the number of information our information provider may post. When $|L|$ is increasing, the number of schedule will increasing exponentially. It is not a good method for the problem with large number of information to decide whether or not to post.

For large-scale problem, we use the idea of large neighbourhood search method for the first level. The detailed algorithm is shown in algorithm 9 ALG-LNS. These two methods can be used for both pessimistic and optimistic condition. The following algorithms just shows the pessimistic condition.

In algorithm 9 ALG-LNS, S is the set for all possible schedule to post information of user i. The number of elements in S is $2^{|L|}$. K is the number of iterations needed for the problem. we can know that the calculation complexity for it is $O(|L| * K)$. Based on different requirements of accuracy, we can use different K. The value $h[s]$ is an indicator whether the plan s already been seem as the location maximum before. ct is the calculation time. TL is the time allowance for calculation.

From the algorithm, we can know that within each iteration k, the post plan p_{il} is actually become a parameter. And based on the linear threshold principle, we can get all of the variables' value p_{jl}

Algorithm 9 ALG − LNS

1: Set $k = 0$, $\{p^k_{il} = 0, |\forall l \in L\}.\{h[s] = 0|\forall s \in S\}$

2: Let $\{p_{il} = p^k_{il}|\forall l \in N\}$, calculate constraint (4.2i,4.2j) in model (CP_i) to get $\{U^*(k)|\forall k \in N, k \neq i\}$, implement this value to BRUE constraint (4.2h) to get objective function value η^k. Set h[p^k_{il}]=1.

3: Set $m = 0$.

4: Let $\{p^{k,m}_{im} = 1 - p^k_{im}\}$, $\{p^{k,m}_{in} = p^k_{in}|\forall n \in L, n \neq m\}$. Let $\{p_{il} = p^{k,m}_{il}|\forall l \in N\}$. Use it to calculate constraint (4.2i,4.2j), use the solution for BRUE constraint (4.2h) to get the objective function value $\eta^{k,m}$.

5: If m $< |L| - 1$. m=m+1, go to step 4. Otherwise, go to step 6.

6: $\eta^{k+1} = \{max_m \eta^{k,m}|s.t.h[p^{k,m}_{il}] = 0\}$, $m^* = \{argmax\, \eta^{k,m}|s.t.h[p^{k,m}_{il}] = 0\}$. Let $\{p^{k+1}_{in} = p^{k,m^*}_{in}|\forall n \in N\}$

7: n= $\sum_{m\in N} h[p^{k,m}_{il}]$.

8: If $n < |L|$ and $k \leq K$ and $ct \leq TL$, k=k+1, go to step 2. Otherwise, stop.

in the math model. So all of the users' post behavior are parameters now. This is the reason we do not need to linearize the function $G(\cdot)$.

Computation Result

Data Set

We use the synthetic network to calculate for the model. We generate the network by different types of information, different number of users in the network, different link density between users and with or without budget as a constraint to evaluate the performance of the system. The following Table 4.1 gives the structure of the network. We totally have 14 different types of data structure to be calculated.

Table 4.1: Data Structure

$Case$	$Node$	$Information$	$Density$	$Budget$
1	20	5	0.3	No
2	20	10	0.3	No
3	20	20	0.3	Yes
4	20	20	0.3	No
5	100	5	0.1	No
6	100	10	0.1	No
7	100	20	0.1	No
8	100	20	0.1	Yes
9	100	20	0.3	Yes
10	100	20	0.5	Yes
11	1000	5	0.1	No
12	1000	10	0.1	No
13	1000	20	0.1	Yes
14	1000	20	0.1	No

The coefficient b_{kl} is the coefficient for user k to get boring about information l. We generate it by using the random function in [0, 2]. The coefficient v_{kl} is the coefficient for user k to get how much value of the information l if just receive it for one time. We generate it randomly in [0, 10]. The coefficient d_{ki} is the importance of user k to user i . We generate it by using the random function in [0, 100]. Threshold value for one user to repost the information is generated in two categories. First category includes the user with high threshold. With probability of 40% in total users. we generate them randomly in [2, HT]. The second category includes the users with low threshold. We

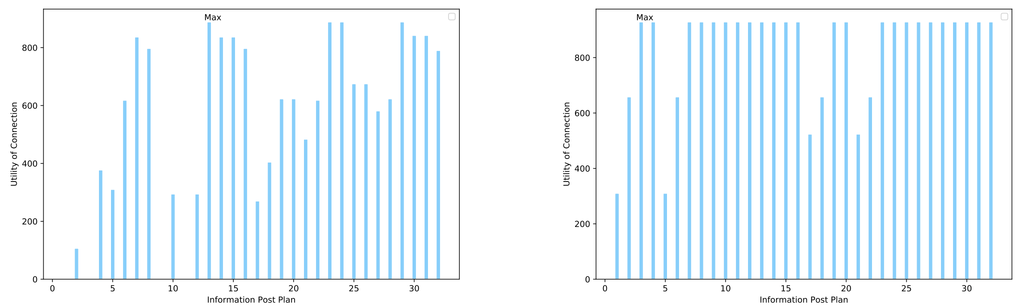
generate them randomly in [2, LT]. Where $HT = 10000 * N * D$ and $LT = N * D$. N is the number of users in the network. D is the density of arcs in the network. The influence between two users is generated randomly between [1, HI], where $HI = N * D/2$. The edit cost for the information$l$ is c_l. It is generated randomly between [0,300]. The total budget for the edit cost is 2000.

Result of Connection Utility for Information provider

Post plan of information provider

Figure 4.3(a) gives out the maximum utilities for different post plan under BRUE coefficient $\rho = 0.8$ by using the data set under case 1 shown in table 4.1. It uses the information utility function in figure 4.2(a). We can find out that maximum plan has the index 13, which means the post plan is $p_{max} = [0, 1, 1, 0, 0]$. Under this case, the maximum utility for post plan is to post information 2,3 and not post information 1,4,5 for our information provider. This is the result for a simple example that can be enumerate all schedules to remove the first level in our model. But in large scale problem, we will use algorithm 9 ALG-LNS to give out the heuristic solutions.

Figure 4.3(b) gives out the result by using the information utility function in figure 4.2(b). We can see that under this case more post plans can get the maximum utility of connection for the information provider. It means the users in the system will easily to choose connecting to others because this user can have more information utility when he or she will get the information more times.



(a) Information provider's utility for Different Post Plan by decreasing information utility (b) Information provider's utility for Different Post Plan by increasing information utility

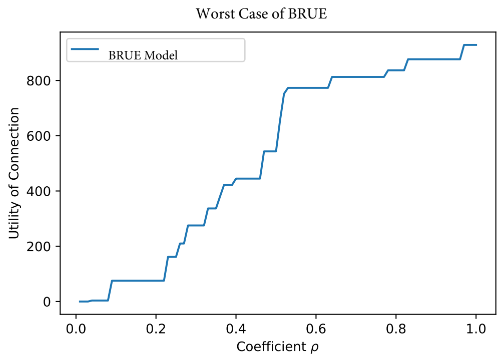
Figure 4.3: Utility for Different Post Plan

Influence of BRUE Coefficient ρ

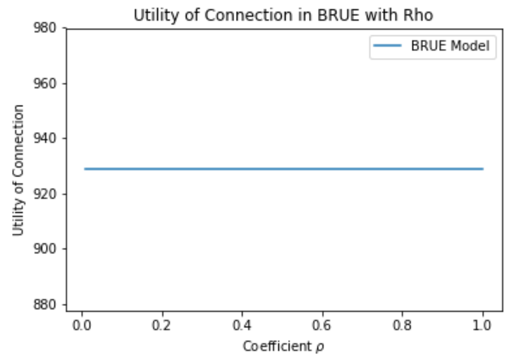
Figure 4.4 shows the result of the influence of BRUE coefficient ρ in worst condition and best condition with case 1 in table 4.1. We can see that under worst condition, when ρ is increasing, the total utility is also increasing. This is because by increasing of ρ, the feasible region of BRUE constraint is decreasing. For worst case, the objective function value is increasing. And for best case is a horizon line, this is because when $\rho = 1$, it already get the maximum utility under this condition. And the values of utility under best case and worst are also equal when $\rho = 1$. Which means under the condition game theory, worst case and best case are the same case.

Compare BRUE and Game Theory

In this section, we calculate the post plan for our information provider if we use the game theory model to forecast the users' behaviour in the network under the worst condition. Then we use this post schedule in our BRUE model, under different BRUE coefficient, we will get different maximum or minimum connection utilities for our information provider. We compare the relative



(a) Influence of BRUE coefficient ρ in worst case

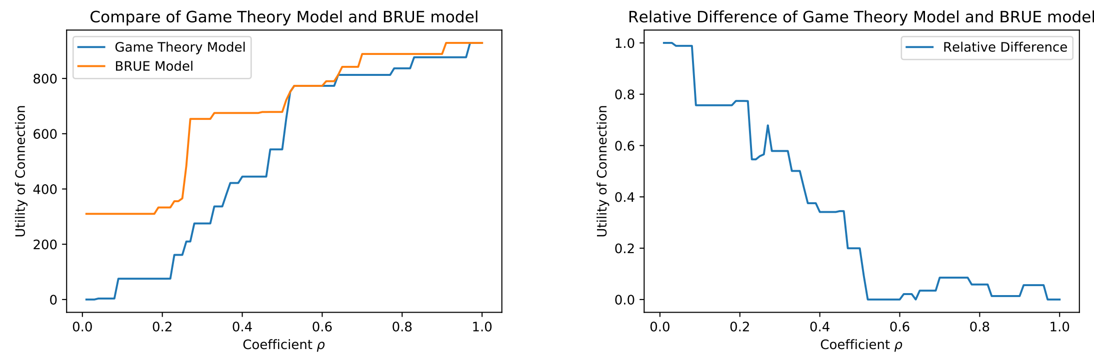


(b) Influence of BRUE coefficient ρ in best case

Figure 4.4: Influence of BRUE coefficient ρ

difference to show how much the system will perform better by using BRUE instead of game theory model under different BRUE coefficients.

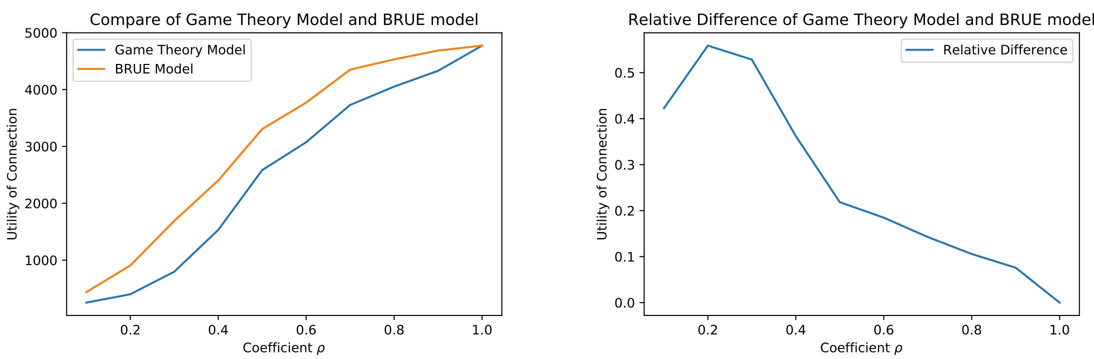
Figure 4.5 is the comparison the connection utility of BRUE and Game Theory Model with data set case 1 in table 4.1 under different BRUE coefficient ρ. We can see that by using the post behaviour got from game theory model and use it in BRUE, the total utility is less or equal to the post behaviour directly got from BRUE model. By decreasing of ρ, the relative difference generally becomes larger.



(a) Compare of Game Theory and BRUE Model

(b) Relative Difference

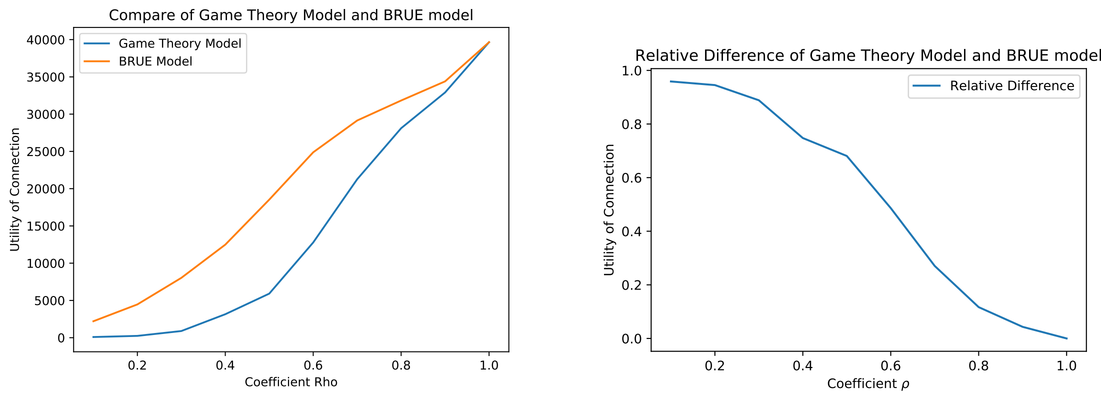
Figure 4.5: Compare of BRUE and Game Theory Model 1



(a) Compare of Game Theory and BRUE Model

(b) Relative Difference

Figure 4.6: Compare of BRUE and Game Theory Model 2



(a) Compare of Game Theory and BRUE Model

(b) Relative Difference

Figure 4.7: Compare of BRUE and Game Theory Model 3

Figure 4.6 and 4.7 are the solutions that is got relatively from case 5 and case 11 in table 4.1. We can see that the larger the number of nodes in the system, the more smooth the curves are.

Result by Using Large Neighbour Search Method

We calculate the result by using the data set in case 7 and case 8. And we also use different initial solution when use large neighbour search.

Result Without Budget and Cost Penalty

Figure 4.8 is the result without budget and penalty in the objective function by using large neighbourhood search method under case 7. The value is the same after about iteration 8 is the same is because for different information post schedule, it may lead to the same connection of the network. Then the result of different iteration in large neighbourhood search just keep the same. This

result is get from the initial solution 0, which means the information provider will not post any information at the first iteration. The calculation time we set is 1 hour. We can also get the result from enumerate method, which is to enumerate all possible post schedule. Compare to the large neighbourhood search method ,it gets the same solution. But the calculation time is about 2 days. It is much longer then LNS.

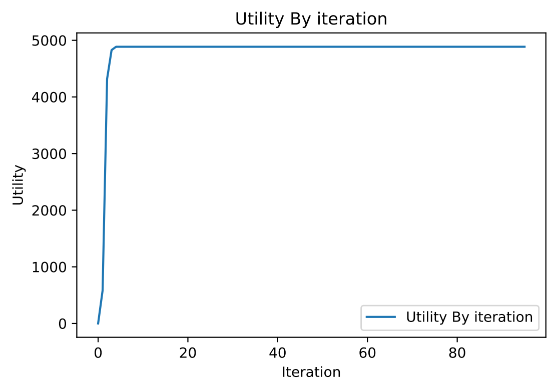


Figure 4.8: Large Neighbour hood Search without budget

Result With Budget and Cost Penalty

Figure 4.9(a) is the result from data case 8. We solve it from the starting point '1', which means the information provider post all information at the first iteration. We can see the optimal solution actually already get at iteration 10. And the optimal solution value is 4798.84. The time cost for the first one is 1 hour. For the second one is just 10 minutes that can get the optimal solution. After we enumerate all possible schedules, we also get the global optimal solution is 4798.84. But the time cost is about 2 days, which is much longer than the time by using large neighbour hood search method.

Figure 4.10 is the result with budget and cost penalty, and the starting point is '0'. It means the

first post plan is to post nothing to the network. We can first know that the total objective value is smaller than the model without budget. It is obviously because in this case actually some post plan is prohibited by the budget.

The starting point from '0' can get the optimal solution more quickly than start from '1'. This is because under this example, the final post schedule does not need to post too much information in order to get the optimal solution.

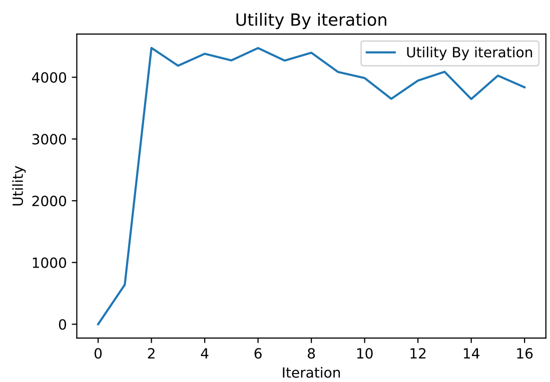
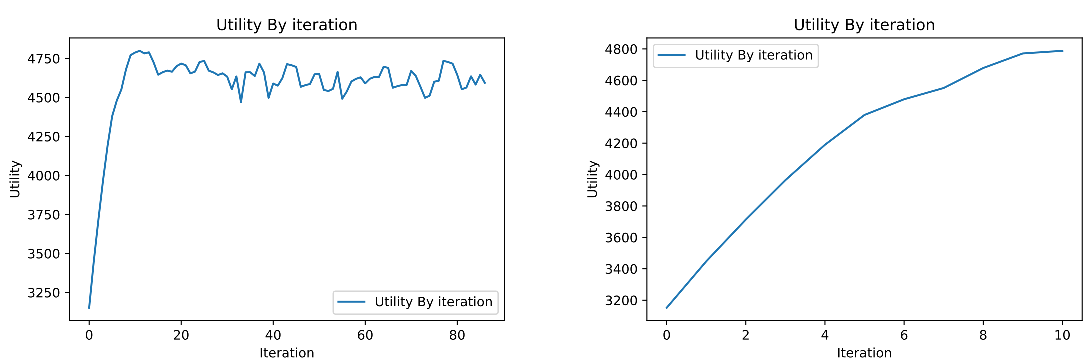


Figure 4.10: Large Neighbourhood Search with budget with Cost Penalty1



(a) Long Time Range

(b) Short Time Range

Figure 4.9: Result of Large Neighbourhood Search without budget with Cost Penalty

CHAPTER 5: RESIDENTS ENERGY TRANSACTION WITHBLOCKCHAIN TECHNOLOGY

Introduction

With the development of solar energy technology, more and more consumers of electricity become prosumers [48]. Here the prosumer means a user use and also generate power for themselves to use. Some prosumers generate too much power in one day and they can not use them all. The idea for them to transact their redundant power to other users to earn money is better than the idea to just release the redundant power. But if two users in the network do not know with each other, they can not establish the sense of trust to make transaction for energy and money directly. Currently they must use a central agency that both of them believe it. But apparently, this agency will charge money for them to provide such services. In this paper, we generate a model for power and money transaction between prosumers without the central agency by using the blockchain technology. Each prosumer in the system will obey a game theory model for them to maximize their profits. After that, we can get the influence to the current power price by introducing the transaction among prosumers.

Blockchain technology [86, 68, 21] is one of the keys to guarantee the safety of the cryptographic currency such as Bitcoin and Ethereum. The blockchain technology has also been widely used recently. One of the most useful aspects of blockchain is that users can transact directly without any central authority with blockchain. Each user can be a witness for the blocks. No one can have fraud in the system unless one user can control more than 50% of the blocks. But that is impossible in the real world. In addition, blockchain can help to protect personal data [101] . Because no one knows the real name of the user of the blockchain.

With the using of blochchain technology, the transaction between two unknown users can happen [7, 67]. Such transaction mainly has two advantages. Firstly, it can reduce the waste of power in total. It is because, without transaction between users, if one user generate much more power than its demand, it can only release the exceed amount. But now it has an option that to put that amount online to sell to other users. Secondly, the price of the power will decrease [30]. The reason is that with the transaction between porsumers, the power company will have a potential competition, and the price set by them should be influenced by the prosumers' selling price, which is apparently lower than the company's price.

This paper studies the influence for the power company's pricing strategy by introducing the transaction. Power company will set their selling price to maximum their profit. If they set the price too high, users will buy the power from other user' who have redundant power. If they set the price too low, they will also have low profits. So they should have a pricing strategy based on the generation amount and demand amount of prosumers in the system. We will use the operations research model to help predict the price. In addition, the decrease of price from the power company should have influence back to the porsumer' buying and selling behavior. Each prosumer will perform in the way to maximize their own profit. But the behavior of one user will influence other users in the system. We conduct a game theory model for prosumers' performance behavior. The game theory model is also used by some previously researchers in power transaction system [31, 66, 83, 4], but they did not use it in the transaction between prosumers.

Mathematical Model

We propose a mathematical optimization model from the aspect of the central energy company to determine their pricing strategy. The price of the central energy company will influence the prosumer's behaviour. In addition, each consumer will obey their game theory model. That means

each prosumer's buying and selling behaviour will maximize their own profits.

It is a three-level optimization model. The first level optimizes for the central company. The second level is the game theory model for each individual prosumers' actually buying and selling behaviour in the system. The third level is the game theory model for each users' ideal selling amount.

We have the following assumptions. 1, The energy selling price p_e is a constant in the math model. But we will use different p_e to calculate the model to give out what is the best pricing strategy for the center company. 2, The energy selling price of each user is a constant small value. We assume they can be fulfilled if this value can cover their transaction costs by using blockchain technology. But for different time periods, this value can be different.

(FL):

$$\max_{b_{iet}} / \min_{b_{iet}} \quad \sum_t (p_e - c) * (\sum_i b_{iet}) \tag{5.1a}$$

$$s.t. \quad (SL_i), \qquad \forall i \tag{5.1b}$$

Model (FL) is the first level optimization model to maximize or minimize the central user's benefit. The reason we need to consider the minimization of the profit is because the behaviour of user b_iet is not controlled by the central company. They just follow their game theory model in constraint (5.1b). The detailed discussion is in model (SL_i). Constraint (5.1b)is defined in the second level.

(SL_i):

$$\max_{b_{iet}, b_{ijt}} \quad \sum_t p_i * (\sum_j b'_{jit}) - \sum_t p_j * (\sum_j b_{ijt}) - \sum_t p_e * b_{iet} \tag{5.2a}$$

$$\text{s.t.} \quad G_{it} - D_{it} + b_{iet} + \sum_j \varepsilon_{ij} * b_{ijt} - \sum_j b'_{jit} + r_{i,t-1} - w_{it} = r_{it}, \qquad \forall t \tag{5.2b}$$

$$b_{ijt} \leq s^*_{jt} - \sum_{i^-} b'_{i^- jt}, \qquad \forall j,t \tag{5.2c}$$

$$b_{iet} \geq 0, b_{ijt} \geq 0, 0 \leq r_{it} \leq CB_i, w_{it} \geq 0, \qquad \forall j,t \tag{5.2d}$$

Model (SL_i) is the game theory model of each prosumer in the system. The objective function the total profit user i get from the network. b_{ijt} is variable that user i can control. But b'_{jit} is the amount determined by user j, it is the relative value of b_{jit} of the equilibrium for user j. But here in (SL_i), it is parameter.

Constraint (5.2b) is the balance equation for user i. The rest of energy left for user i at time t is determined by the receiving amount using amount. The receive amount includes amount of generation itself, amount buy from the power company, amount buy from other users and the rest amount from the previously time period. The using amount has the amount of demand of user i, the selling amount to other users and the release amount. The release amount can be positive only if the energy left for user i at time t exceeds the capacity of its battery.

Constraint (5.2c) illustrates that the limitation amount user i can buy from user j at time t is the amount user j want to sell minus the amount other users buy from user j. Here i^- means the other users in the system except user i. s^*_{jt} is the amount user j want to sell at time t. We can get it from the third level model. But this value will also depend on the equilibrium of the model (SL_i). It is another level of game theory model. In this model, the chance for other users that can successfully get the amount they need from user j is equal. We do not give them an priority. For example, it just have three users in the system, user i, j, k. And user j want to sell amount 10 at time t. But user i want to buy 8 and user k want to buy 7 at time t. Then the final value of the equilibrium for the variable b_{ijt} and b_{kjt} can be any value to let $b_{ijt} + b_{kjt} = 10$. But in real world, the value should be $b_{ijt} = 8, b_{kjt} = 2$ or $b_{ijt} = 3, b_{kjt} = 7$. It depend on the reaction time of user i and user k. We

will show in theorem (2) that the person with the fast reaction can buy the full amount he/she want to buy from the result of the model. The final equilibrium value for b_{ijt} must fulfill the first several L fast reaction users' whole buying amount. Then the rest can be got by the $(L+1)th$ fast reaction user.

Without the priority of different users' reaction time. It may has an uncertainty set for user's behaviours in the equilibrium condition. Different behaviours will lead to different profit for the central company.

(TL_i):

$$s_{it}^{*} = argmax_{s_{it}^{*},b_{ijt}^{''},b_{iet}^{''}} \quad s_{it}^{*} - \sum_{t} p_j * (\sum_{j} b_{ijt}^{''}) - \sum_{t} p_e * b_{iet}^{''} \tag{5.3a}$$

$$s.t. \quad G_{it} - D_{it} + b_{iet}^{''} + \sum_{j} \varepsilon_{ij} * b_{ijt}^{''} - s_{it}^{*} + r_{i,t-1}^{''} - w_{it}^{''} = r_{it}^{''}, \qquad \forall t \tag{5.3b}$$

$$b_{ijt}^{''} \leq s_{jt}^{*} - \sum_{i^{-}} b_{i^{-}jt}^{'}, \qquad \forall j,t \tag{5.3c}$$

$$b_{iet}^{''} \geq 0, b_{ijt}^{''} \geq 0, s_{it}^{*} \geq 0, 0 \leq r_{it}^{''} \leq CB_i, w_{it}^{''} \geq 0, \qquad \forall j,t \tag{5.3d}$$

Model (TL_i) is the Nash Equilibrium model for each user's expected power selling amount. It means at time t user i will post its schedule to sell s_{it}^{*}. But perhaps it can not sell such amount, the real selling amount will also depend on the amount buy from other users. The real selling amount is in model SL_i, it is $\sum_j b_{jit}$. Here the variables $b_{ijt}^{''}$, $b_{iet}^{''}$ are artificial variables that help to get s_{it}^{*}. It is also not the real transaction amount.

Theorem

Theorem 4. *When all of the erosion coefficient $\varepsilon_{ij} = 1$, the energy price of the central power company should only be one of the prosumer's selling price or the upper bound of the their price*

UB_p to maximize their profit.

Here UB_p is defined in the beginning of the article, it is the maximum selling price of the central company bounded by the government principle.

Proof. Without loss of generality, we can assume the selling price of prosumer 1 to n in the system is in a non-decreasing sequence, $p_1 \leq p_2 \leq p_3 \leq \cdots p_{n-1} \leq p_n \leq UB_p$.

By contradiction, p_e is not equal to any of the upper price. Then we must have p_e drops between p_i and p_{i+1} or drops between p_n and UB_p. Where i must be one of $1, 2, \cdots n-1$. It also means $p_i < p_e < p_{i+1}$ or $p_n < p_e < UB_p$. We let $\delta = p_{i+1} - p_e$ or $\delta = UB_p - p_e$.

From the assumption $\delta > 0$. We can always find a $0 < \delta_1 < \delta$. And let $p_e^{'} = p_{i+1} - \delta_1$ or $p_e^{'} = UB_p - \delta_1$. Then we have $p_e^{'} > p_e$. From the model SL_i and TL_i, we can know that if p_e is the price to maximize the central company's profit. Then the current value of the decision variables b_{iet} must also be the solution if we change p_e to $p_e^{'}$. The reason is if $b_{iet} > 0$, for user i it can not find any other prosumers to sell power at time t with price lower than p_e. Then by the definition of $p_e^{'}$. This user also can not find any other prosumers to sell power at time t with price lower than $p_e^{'}$. So the value of decision variable b_{iet} will keep the same after the change of central company's price.

But we will find $\sum_t (p_e - c) * (\sum_i b_{iet}) < \sum_t (p_e^{'} - c) * (\sum_i b_{iet})$. It is contradict to the assumption that p_e is price to maximize the company's profit. □

From theorem 4, we know the central company only need to consider some price that equal to the prosumer's price or UB_p if there has no power loss of transaction between prosumers. It is more easy for us to determine the optimal price for the central company. In the real case, that power loss

of transaction is also very small. So even if we just consider the discrete pricing strategy, we can also find the sub-optimal price from it.

Theorem 5. *If it does have any two prosumers with the same selling price. In model SL_i if the solution $s^*_{jt} > 0$. And among b_{ijt}, if it has l of them is greater than 0 with $l \geq 1$. Then it has at least $l-1$ prosumers in that l prosumers will not buy any power from other prosumers except j or the central company. If $l = 1$, we denote this user as user k, then the model will have the result $\{b_{kj^-t} = 0, \forall j^- \neq j\}$ and $b_{ket} = 0$ except $b_{kjt} = s^*_{jt}$*

It means in the model when a user make a decision to buy power from user j, it will buy the total amount that can fill its willing amount except the rest of user j's selling amount (perhaps already be bought some amounts by other users with quick reaction) is less than its buying amount. In the real world, it also happens like this.

Proof. If $l = 1$, by contradiction, it must have a solution like $b_{kjt} > 0, b_{kj_1t} > 0$ and $b_{kjt} < s^*_{jt}, b_{kj_1t} < s^*_{j_1t}$. Without loss of generality, we have $p_j/\varepsilon_{kj} < p_{j_1}/\varepsilon_{kj_1}$. We can find a small enough δ to let $b^1_{kjt} = b_{kjt} + \varepsilon_{kj_1}\delta > 0$ and $b^1_{kj_1t} = b_{kj_1t} - \varepsilon_{kj}\delta > 0$. Both these two new varialbes are also feasible to model (SL_i). Because the constraints (5.2b), (5.2c)and constraints (5.2d) will still keep its feasibility. But currently, the objective function value is increasing. It is contradict to the assumption $b_{kjt} > 0, b_{kj_1t} > 0$ are the game theory solution.

If $l > 1$, the prove is similar. By contradiction, if it has more than 2 prosumers buy power from other users except j. We just apply these two user with the same method prove in the condition $l = 1$ to get the contradiction. □

Algorithm

In model (FL), it need to obey the constraint defined in model (SL_i). In model (SL_i), one of the variables are got from the model (TL_i). But we can not calculate these three models separately. Because it is all game theory model, other user's decision also influence their decisions with each other. But as we know, we can use KKT condition method for model (SL_i) and model (TL_i). Because all of these two model are game theory models, user will play games to determine their relative decision variables. The KKT model is shown as follows.

$(FL - KKT_{SL_i,TL_i})$:

$$\max/\min \quad \sum_t (p_e - c) * (\sum_i b_{iet}) \tag{5.4a}$$

$$s.t. \quad (SL_i - PF_1): G_{it} - D_{it} + b_{iet} + \sum_j \varepsilon_{ij} * b_{ijt} - \sum_j b_{jit} + r_{i,t-1} - w_{it} = r_{it}, \qquad \forall i,t \tag{5.4b}$$

$$(SL_i - PF_2): b_{ijt} \le s^*_{jt} - \sum_{i^-} b_{i^- jt}, \qquad \forall i,j,t \tag{5.4c}$$

$$(SL_i - PF_3): b_{iet} \ge 0, b_{ijt} \ge 0, 0 \le r_{it} \le CB_i, w_{it} \ge 0, \qquad \forall i,j,t \tag{5.4d}$$

$$\begin{aligned}(SL_i - DF): \triangledown_i [\sum_t p_i * (\sum_j b^{'}_{jit}) - \sum_t p_j * (\sum_j b_{ijt}) - \sum_t p_e * b_{iet} + \nu_{it} \\ * (G_{it} - D_{it} + b_{iet} + \sum_j \varepsilon_{ij} * b_{ijt} - \sum_j b_{jit} + r_{i,t-1} - w_{it} - r_{it}) \\ + \mu_{it} * g_{it}] = 0, \qquad \forall i,j,t\end{aligned} \tag{5.4e}$$

$$(SL_i - CS): \mu_{it} * g_{it} = 0, \qquad \forall i,t \tag{5.4f}$$

$$(TL_i - PF_1): G_{it} - D_{it} + b^{''}_{iet} + \sum_j \varepsilon_{ij} * b^{''}_{ijt} - s^*_{it} + r^{''}_{i,t-1} - w^{''}_{it} = r^{''}_{it}, \qquad \forall i,t \tag{5.4g}$$

$$(TL_i - PF_2): b^{''}_{ijt} \le s^*_{jt} - \sum_{i^-} b^{'}_{i^- jt}, \qquad \forall i,j,t \tag{5.4h}$$

$$(TL_i - PF_3): b^{''}_{iet} \geq 0, b^{''}_{ijt} \geq 0, s^*_{it} \geq 0, 0 \leq r^{''}_{it} \leq CB_i, w^{''}_{it} \geq 0, \qquad \forall i,j,t \quad (5.4i)$$

$$\begin{aligned}(TL_i - DF): \bigtriangledown_i \, [s^*_{it} - \sum_t p_j * (\sum_j b^{''}_{ijt}) - \sum_t p_e * b^{''}_{iet} + \nu^{''}_{it} \\ * (G_{it} - D_{it} + b^{''}_{iet} + \sum_j \varepsilon_{ij} * b^{''}_{ijt} - s^*_{it} + r^{''}_{i,t-1} - w^{''}_{it} - r^{''}_{it}) \\ + \mu^{''}_{it} * g^{''}_{it}] = 0, \end{aligned} \qquad \forall i,j,t \quad (5.4j)$$

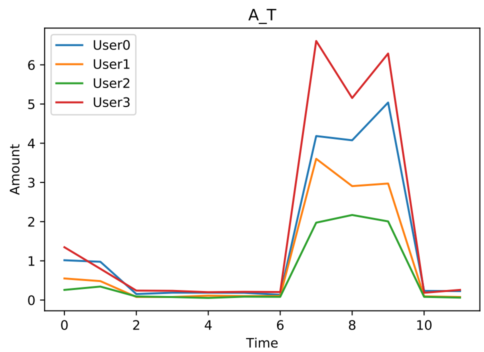
$$(TL_i - CS): \mu^{''}_{it} * g^{''}_{it} = 0, \qquad \forall i,t \quad (5.4k)$$

$$\mu_{it}, \mu^{''}_{it} \geq 0, \qquad \forall i,t \quad (5.4l)$$

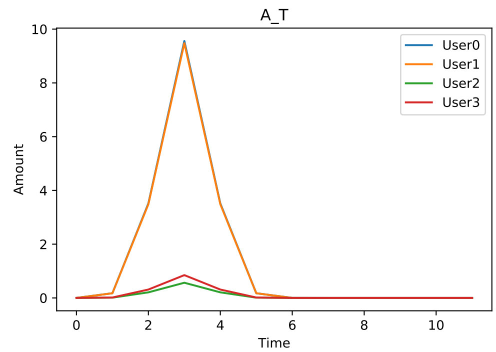
Where ν_{it} and $\nu^{''}_{it}$ are the KKT multiplier for equation constraints (5.2b) and (5.3b). They are free variables. μ_{it} and $\mu^{''}_{it}$ are the KKT multiplier for non-equation constraints (5.2c,5.2d) and (5.3c,5.3d). They are non-negative variables. And g_{it} and $g^{''}_{it}$ are the corresponding inequations as $g_{it} \leq 0$ and $g^{''}_{it} \leq 0$. The detailed formulation are not listed here.

Computational Results

In game theory model, based on different equilibrium of users in the system, we focus on the optimistic and pessimistic conditions to the central energy company.We have the results got from math model by using different customer data.



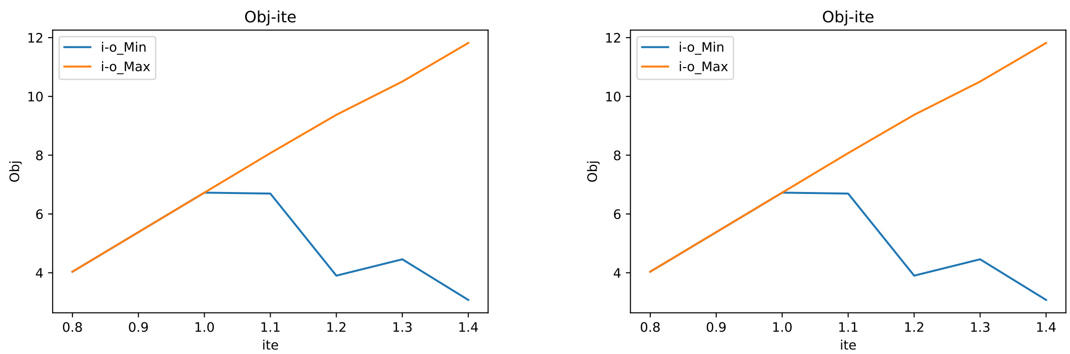
(a) Demand Amount



(b) Generation Amount

Figure 5.1: The Demand and Generation Amount for each user in different time periods

Figure 5.1 is the generation amount and demand amount with in one day's time. It has 12 time periods in one day. It is the data we used in four users system. Here time index 0 is the 6:00 AM in one day. When the index increases 1, it means the hour time increases 2. And we assume that the users' demand have the small peak in the morning time. And high peak is at night. The generation amount is positive from 6:00 AM to 6:00 PM, 12:00 PM has the peak value. It is a simulation for solar energy generation.

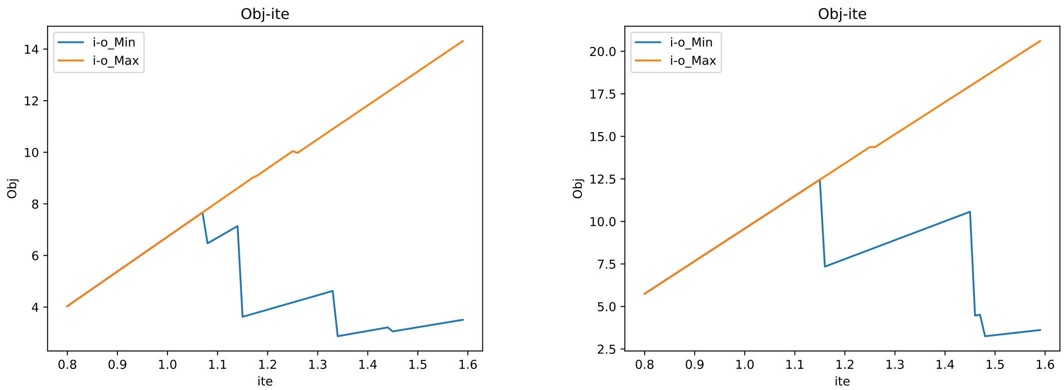


(a) Without Transaction Loss

(b) With Transaction Loss

Figure 5.2: Profit of Central Company in Minimization and Maximization Condition for Different Pricing Strategy

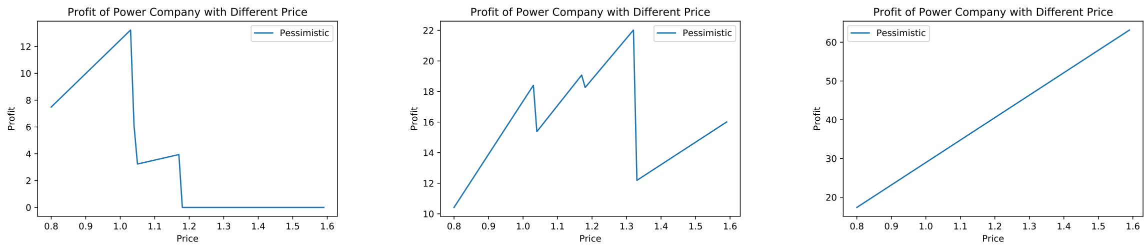
In figure 5.2 we just make iteration for price in a increasing step of 0.1. It is just a first glance of the result. It has four prosumers. Figure 5.2 shows the conditions with $\varepsilon = 1$ and $\varepsilon \neq 1$. The two lines are in pessimistic and optimistic condition for the profit of the power company, we need to consider more about the pessimistic case because we do want even the worst case of profit can also be in a acceptable scale. We can find in pessimistic condition, it has three inflection points. It means the behaviour of prosumer will influence when the price is equal to such three selling price points of the the prosumers. We also find that even ε is close to 1, the influence is not too much.



(a) Without Transaction Loss (b) With Transaction Loss

Figure 5.3: Profit of Central Company in Minimization and Maximization Condition for DifferentPricing Strategy

This is the figure with the price increasing step 0.01. Figure 5.3(a) is the condition with ε=1. We can find it has four inflection points. They are just the four selling price of the prosumers. And we can find in the figure if the price UB_p is not larger enough, then a good pricing strategy for power company should be about 1.05. It can make the total profit most highest. But if UB_p can be a large enough value, then the power company will still have the price UB_p. And figure 5.3(b) is the result with $\varepsilon \neq 1$. We found in this case, it only have three inflection points. We can see the maximum price for power company change to 1.13. This is because we let ε to be the value not very close to 1. And this may influence the decision of prosumers a lot in the system. But this is unusual in the real world, we just want to show if this happens, what is the influence here.



(a) Generation Amount Domination (b) No Domination (c) Demand Amount Domination

Figure 5.4: Influence to Price with Different Generation and Demand Amount

Figure 5.4 is the pessimistic condition for power company with 10 prosumers and 12 time periods. Figure 5.4(a) is the condition with prosumers' total generation amounts are larger than the prosumers' demand amounts. Figure 5.4(b) is the condition with prosumers' total generation amounts are almost equal to the prosumers' demand amounts. Figure 5.4(c) is the condition with prosumers' total generation amounts are much smaller than the prosumers' demand amounts. We can see in figure 5.4(a) after the price increasing to about 1.18. The power company can not get any profit, this is because, the redundant amounts of prosumers with selling price lower than 1.18 is already larger than the needed amount from other users. If the power company set the price larger than 1.18. Then users need power will just buy the power from other prosumers with lower price. Figure 5.4(b) shows the condition that may happen more often in the real life, that means the total generation amount is almost the total demand amount. And the power company will set their price in a moderate level. Here in this example it is about 1.32. In figure 5.4(c), when the generation amounts are more smaller than the demand, the power company will still set their price to UB_P. It means under this condition by introducing transaction among prosumer will not influence the power price of power company.

CHAPTER 6: CONCLUSION

This Dissertation totally four projects, they all have multiple level optimization math model with non-traditional game theory.

In the first project, We consider a problem in smart grid system with users' boundedly rational user equilibrium (BRUE) principle. We introduce pricing strategy to minimize the total energy cost with certain customer demand. In this article, we totally have four cases for our problems by using pricing and BRUE. We consider the optimistic and pessimistic condition with and without pricing. And we also compare the results among these four cases. We solve the problem by using three methods,solver BARON, penalty cutting plane and lagrangian dual cutting plane method.

From the case study, we know first that the system with people have higher BRUE coefficient will lead to more uncertainty to the feasible set. And the difference of the total cost for optimistic and pessimistic condition is larger. Second, by introducing the pricing strategy to our system, the total cost will decrease for both the optimistic and pessimistic condition. Third, if we use Nash Equilibrium instead of using BRUE to determine the optimal price. We will find that such pricing strategy works bad than the strategy directly got from BRUE model. Fourth, the larger value of the appliance preference coefficient π will lead to larger improvement by introducing the pricing strategy. Fifth, for the min-max or max-min problem, the lagrangian dual cutting plane method is an useful method if no lagrangian gap exists. Because the use of it does not increase the complex of our problem and we change the constraint to the objective function.

Future research includes the research for stage pricing systems and the extended using of lagrangian dual cutting plane method to our system and some other problems by using the BRUE principle.

In the second project, From the results of the examples we can get the conclusion that under the BRUE conditions, we can make the total system work better by introducing the surplus price. This is because by changing the value of the surplus price we can get the target to flat the user flows in each arc.

We also found that if we check the convergency for the columns and just check the value of the constraint that generated by the column generation of our system. We can still get a result, but this result is not the whole correct result even though it doesn't have much different with the right one. But by using the new method with algorithm 6 and algorithm 7 we need much more time for the calculation. So we illustration another method that to set the accuracy for the inner level not so much for the beginning several iterations for the outer level. And then we found out that the calculation almost decreased down by half, but the result is almost the same.

We also get some theorems about the characters of our problem under the boundedly rational user equilibrium system. And we checked all of them under the 4 nodes example. Such characters may be used for further work.

In the third project, Based on the BRUE model, we can get the best plan for how to post our information provider's information. By using the best plan, the user can expand its connections. In BRUE model, the smaller BRUE coefficient ρ has, the less connection of our information provider has in pessimistic condition. It means no matter what kind of information the information provider post, it is more easy to lose such users in the network. So the information provider should pay more attention to the users with high BRUE coefficient ρ. BRUE model performs better than the game theory model to maximize the information provider's connection. Especially when BRUE coefficient ρ is smaller, the difference is more larger. Even though the calculation method for game theory is relatively simple because there has no uncertain set. But it is still useful to use BRUE model to simulate users' behaviour especially more users in the system have smaller ρ.

we can also get the conclusion that large neighbourhood search method is a useful algorithms for large-scale problem, we can get the solution in a reasonable time. In addition, the local optimal get from LNS is equal to the global optimal solution in our case study. But the time spend is extremely smaller than the time spend to get the global optimal solution. Another advantage in our model by using LNS is when the variable of first level is fixed. The three-level optimization problem directly decomposed to several one level MILP problem. The starting point by using LNS can also influence the calculation to reach a acceptable local optimal solution. In general, when we have more people with high boring coefficient, it is better to start from the plan that post 0 information. But when we have more people with low boring coefficient or negative coefficient, it is better to start from the plan post all information within the cost budget.

In the fourth project, we establish a multi-level game theory model for power company and prosumers in the power network. We solve it by using the KKT condition and linerization method. In general, by introducing the transaction among prosumer with blockchain technology will decreasing the current power price.

From the side of power company, they only need to consider to set their price equal to the prosumer's selling price or UB_p. They can find a price with maximum profit for them among the price list of prosumer.

From the side of prosumers, if their generation capability is much more higher than their demand, actually they do not need the power company, but it must have a waste for their power. It may just happen to a few users who do not care about the money and the environment. In real world, within the permission of solar energy generation technology, more prosumer will have a generation machine with the power almost equal to their demands. Under this case, using blockchain to make transactions can push the power company to decrease their selling price. But if in the case that prosumers' generation power are much less than their demands. Then the transactions almost have

no influence to the price.

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