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FILTERED BACKPROJECTION INVERSION OF THE CONE BEAM TRANSFORM FOR A GENERAL CLASS OF CURVES*

ALEXANDER KATSEVICH[†] AND MIKHAIL KAPRALOV[†]

Abstract. We extend a cone beam transform inversion formula, proposed earlier for helices by one of the authors, to a general class of curves. The inversion formula remains efficient, because filtering is shift-invariant and is performed along a one-parametric family of lines. The conditions that describe the class are very natural. Curves C are smooth, without self-intersections, have positive curvature and torsion, do not bend too much, and do not admit lines which are tangent to C at one point and intersect C at another point. The notions of PI lines and PI segments are generalized, and their properties are studied. The domain U is found, where PI lines are guaranteed to be unique. Results of numerical experiments demonstrate very good image quality.

Key words. shift-invariant filtering, theoretically exact, PI lines

AMS subject classifications. 44A12, 65R10, 92C55

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1. Introduction. Image reconstruction from projections is important both in pure mathematics (as a problem of integral geometry) and in applications (as a problem of computed tomography (CT)). Cone beam CT is one of the most common medical imaging modalities. Here one recovers a function $f(x)$, $x \in \mathbb{R}^3$, knowing the integrals of f along lines that intersect a curve C . The curve C is usually called a source trajectory. The ever-increasing needs of medical imaging require the development of inversion algorithms for more and more general source trajectories.

A number of theoretically exact algorithms have been proposed in the past several years. They can be classified into three groups: filtered backprojection (FBP) algorithms, slow-FBP algorithms, and backprojection filtration (BPF) algorithms. Slow-FBP and BPF algorithms are quite flexible, allow some transverse data truncation, and can be used for virtually any complete source trajectory [20, 19, 27, 21, 25, 23, 26]. FBP algorithms are less flexible, but they are by far the fastest and have been developed for a range of source trajectories. They include constant pitch helix [9, 12, 13, 15], dynamic pitch helix [7, 6], circle-and-line [11], circle-and-arc [14, 3], circle-and-helix [2], and saddle [22]. A very nice FBP algorithm was recently proposed by Pack and Noo [20]. It applies to almost any reasonable source trajectory. However, it sometimes leads to excessive detector requirements. The problem is that the algorithm is too general and does not take the geometry of the curve into account. Significant progress has also been achieved in the development of quasi-exact algorithms [1, 16].

With one exception, FBP algorithms have been proposed only for certain types of well-defined trajectories: helices, saddles, etc. There is no FBP algorithm developed specifically for a general class of curves. Ideally, such a class would be described only in terms of some basic geometric properties (e.g., smoothness, curvature, etc.) rather than specifying the types of curves (helices, etc.). In this paper we develop a theoretically exact shift-invariant FBP algorithm for a wide class of source trajectories.

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The conditions describing our class are very natural. We consider curves C that are smooth, have no self-intersections, have positive curvature and torsion, do not bend too much, and do not admit lines which are tangent to C at one point and intersect C at another point. Our algorithm applies to any curve with these properties. The inversion algorithm of this paper is a generalization of the formula proposed for constant- and variable-pitch helices in [9, 12, 7].

The importance of our results is twofold. First, the algorithm can be used in a variety of applications. For example, in electron-beam CT/micro-CT there arise source trajectories that can be described as helices with variable radius and pitch [24]. The FBP algorithms of [9, 12, 7] do not work for such curves, but the new algorithm can easily handle them. Second, the results have theoretical value as well. They provide a deeper understanding of the available algorithms, put them into the context of a more general approach, and demonstrate which geometrical properties the curve is required to have for an efficient FBP algorithm to apply.

The paper is organized as follows. In section 2 we define PI lines for general curves, describe precisely the class of curves considered in the paper, and study properties of their PI segments. In section 3 we find the set U where PI lines are guaranteed to be unique. The result is based on the notions of maximal and minimal PI lines. These critical PI lines can be viewed as a generalization of the axial direction for regular helices. Also we find the special planes, such that the stereographic projection of C onto these planes has very useful properties. In section 4 we study more properties of the PI segments of C . Then the inversion formula is given. Finally, the results of numerical experiments are presented in section 5.

2. PI lines and their properties. The objective of this section is to define PI lines for a general class of smooth curves and study their properties. Let C be a smooth curve:

$$(2.1) \quad I := [a, b] \ni s \rightarrow y(s) \in \mathbb{R}^3, \quad |\dot{y}(s)| \neq 0.$$

Here and below, the dot above a variable denotes differentiation with respect to s . Define the functions

$$(2.2) \quad \Phi(s, s_0) := [y(s) - y(s_0), \dot{y}(s), \ddot{y}(s)], \quad Q(s, s_0) := [y(s) - y(s_0), \dot{y}(s_0), \dot{y}(s)],$$

where $[e_1, e_2, e_3] := e_1 \cdot (e_2 \times e_3)$ denotes the scalar triple product of three vectors. If C is a helix, then Φ and Q are precisely the functions that have been introduced under the same names in [7]. Similarly to [7], it turns out later that Φ is intimately related to the convexity of the projection of C onto a detector plane (cf. (4.6) below), and Q is related to the uniqueness of PI lines (cf. Definitions 2.1 and 2.2, (3.6), and the proof of Proposition 3.3). Given any $s_0, s_1 \in I$, $H(s_0, s_1)$ denotes the line segment with the endpoints $y(s_0), y(s_1) \in C$.

DEFINITION 2.1. *Pick two points $y(s_0), y(s_1) \in C$, $s_0 < s_1$. The line segment $H(s_0, s_1)$ is called a PI segment if $Q(s_0, q) \neq 0$ for any $q \in (s_0, s_1)$.*

DEFINITION 2.2. *Pick two points $y(s_0), y(s_1) \in C$, $s_0 < s_1$. The line segment $H(s_0, s_1)$ is called a maximal PI segment if $Q(s_0, s_1) = 0$, but $Q(s_0, q) \neq 0$ for any $q \in (s_0, s_1)$.*

If C is a helix, Definition 2.1 gives the usual PI segments $H(s, q), 0 < q - s < 2\pi$, and Definition 2.2 gives the maximal PI segments $H(s, s + 2\pi)$.

FIG. 1. Critical case.

Next we discuss how a smooth curve bends. Consider two points: $y(s_0), y(s) \in C$. Assume $y(s_0)$ is fixed, and $y(s)$ moves along C . The line segment joining $y(s_0)$ and $y(s)$ rotates about the instantaneous axis $e(s, s_0) = (y(s) - y(s_0)) \times \dot{y}(s) / |(y(s) - y(s_0)) \times \dot{y}(s)|$. The point $y(s)$ rotates also about the instantaneous axis, which is obtained by finding the circle of curvature of C at $y(s)$ (also known as the osculating circle). The corresponding axis of rotation is $\mathbf{b}(s)$, i.e., the binormal vector. If $s \rightarrow s_0$, then $e(s, s_0) \rightarrow \mathbf{b}(s)$. Thus, the difference in directions of the two vectors can measure how much the curve bends between the two points. The maximum possible “bent” occurs when the two axes point in the opposite directions: $e(s, s_0) = -\mathbf{b}(s)$ (see Figure 1).

For the convenience of the reader we recall the definitions of the curvature κ and torsion τ of a smooth curve:

$$(2.3) \quad \kappa(s) := \frac{|\dot{y}(s) \times \ddot{y}(s)|}{|\dot{y}(s)|^3}, \quad \tau(s) := \frac{[\dot{y}(s), \ddot{y}(s), \ddot{\ddot{y}}(s)]}{|\dot{y}(s) \times \ddot{y}(s)|^2}.$$

Now we can formulate the main assumptions on the curve C .

- C1. C is smooth, and the curvature and torsion of C are positive;
- C2. C does not self-intersect within any PI segment (or a maximal PI segment) of C ;
- C3. given any PI segment (or a maximal PI segment) $H(s_0, s)$ of C , there is no line tangent to C at $y(s_1)$ and intersecting C at $y(s_2)$, with $s_1, s_2 \in [s_0, s]$, $s_1 \neq s_2$;
- C4. C does not bend too much; i.e., given any PI segment (or a maximal PI segment) $H(s_0, s)$ of C , one has $e(s_1, s_2) \neq -\mathbf{b}(s_2)$ for any $s_1, s_2 \in [s_0, s]$, $s_1 \neq s_2$.

If a curve satisfies conditions C1–C4, then its PI segments have a number of nice properties.

PROPOSITION 2.3. *Let C be a curve which satisfies conditions C1–C4, and let $H(s_0, s_1)$ be its (possibly maximal) PI segment. Then for any $s, q \in [s_0, s_1]$ one has $\Phi(s, q) > 0$ if $s > q$ and $\Phi(s, q) < 0$ if $s < q$.*

Proof. By shrinking the PI line if necessary, the proposition follows if we show that $\Phi(s, s_0) \neq 0$ for any $s \in (s_0, s_1]$ and $\Phi(s, s_1) \neq 0$ for any $s \in [s_0, s_1)$. We prove only the first statement, because the proof of the second one is analogous.

FIG. 2. Projection of $y(s_0)$ onto the plane through $y(s)$ with normal vector $\dot{y}(s)$.

Let us assume that the parameterization of $y(s)$ is natural, i.e., $|\dot{y}(s)| \equiv 1$. For convenience, recall the Frenet–Serret formulas:

$$(2.4) \quad \begin{bmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

where $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ are the unit tangent and the normal and binormal vectors, respectively, $\kappa(s)$ is the curvature, and $\tau(s)$ is the torsion of the source trajectory (cf. (2.3)). Using (2.4), we get

$$(2.5) \quad \begin{aligned} \Phi(s, s_0) &= [y(s) - y(s_0), \dot{y}(s), \ddot{y}(s)] = \kappa(s)[y(s) - y(s_0), \mathbf{t}(s), \mathbf{n}(s)] \\ &= \kappa(s)\mathbf{b}(s) \cdot (y(s) - y(s_0)). \end{aligned}$$

Since we are interested in the sign of $\Phi(s, s_0)$ and $\kappa(s) > 0$, we determine the sign of

$$(2.6) \quad \begin{aligned} \mathbf{b}(s) \cdot (y(s) - y(s_0)) &= \int_{s_0}^s (\mathbf{b}(t) \cdot (y(t) - y(s_0)))'_t dt \\ &= - \int_{s_0}^s \tau(t)\mathbf{n}(t) \cdot (y(t) - y(s_0))dt. \end{aligned}$$

Let $\mathbf{t}^\perp(s)$ denote the plane passing through $y(s)$ and perpendicular to $\mathbf{t}(s)$. We assume that $\mathbf{n}(s)$ and $\mathbf{b}(s)$ are the coordinate axes on the plane, and $y(s)$ is the origin (see Figure 2).

Let $\Pi_{osc}(s)$ denote the osculating plane of C at $y(s)$. Recall that $\Pi_{osc}(s)$ contains $y(s)$ and is parallel to $\dot{y}(s)$ and $\ddot{y}(s)$. If $y(s_0)$ projects onto the ray $L := y(s) - \lambda\mathbf{n}(s), \lambda > 0$, then $y(s_0)$ belongs to $\Pi_{osc}(s)$. Moreover, the two rotation axes—one determined by rotation of $y(s)$ around $y(s_0)$ and the other, $\mathbf{b}(s)$, determined by rotation of $y(s)$ relative to the intrinsic center of rotation—are parallel and point in opposite directions. This is prohibited by the assumption that the curve does not bend too much (see C4), so $y(s_0)$ never projects onto L .

Let $\hat{y}(s_0)$ denote the projection of $y(s_0)$ onto $\mathbf{t}^\perp(s)$. The Taylor series expansions show that $\tau > 0$ and $\kappa > 0$ imply

$$(2.7) \quad \mathbf{n}(t) \cdot (y(s) - y(s_0)) < 0, \quad \mathbf{b}(s) \cdot (y(s) - y(s_0)) > 0,$$

FIG. 3. Illustration of the containment property: orthogonal projection onto $H^\perp(s_0, s_1)$.

for $s - s_0 > 0$ small enough. Hence, initially $\hat{y}(s_0)$ is located in the third quadrant (see Figure 2). Suppose now s increases. If $\hat{y}(s_0)$ appears in the third quadrant, then $\mathbf{n}(t) \cdot (y(t) - y(s_0)) < 0$. So $\mathbf{b}(s) \cdot (y(s) - y(s_0))$ increases and $\hat{y}(s_0)$ moves down and does not cross the $\mathbf{n}$ -axis. If $\hat{y}(s_0)$ appears in the fourth quadrant, then $\mathbf{n}(t) \cdot (y(t) - y(s_0)) > 0$ and $\mathbf{b}(s) \cdot (y(s) - y(s_0))$ decreases. This implies that in the fourth quadrant $\hat{y}(s_0)$ moves up. However, our assumption precludes $\hat{y}(s_0)$ from crossing L . Consequently, $\hat{y}(s_0)$ never crosses the $\mathbf{n}$ -axis and $\Phi(s, s_0) > 0$ for any $s \in (s_0, s_1]$. $\square$

Let $H(s_0, s_1)$ be a PI segment (possibly maximal) and $C(s_0, s_1)$ the corresponding curve segment. Project $C(s_0, s_1)$, $\dot{y}(s_0)$, and $\dot{y}(s_1)$ orthogonally onto a plane perpendicular to $H(s_0, s_1)$. Such a plane is denoted $H^\perp(s_0, s_1)$. The corresponding projections are denoted $\hat{C}(s_0, s_1)$, $\hat{y}(s_0)$, and $\hat{y}(s_1)$, respectively (see Figure 3). Let O be the projection of $H(s_0, s_1)$. Because of condition C3, the vectors $\hat{y}(s_0)$ and $\hat{y}(s_1)$ determine two rays:

$$(2.8) \quad \begin{aligned} R_+(s_0, s_1) &:= \{x \in H(s_0, s_1)^\perp : x = O + \lambda \hat{y}(s_0), \lambda \geq 0\}, \\ R_-(s_0, s_1) &:= \{x \in H(s_0, s_1)^\perp : x = O + \lambda (-\hat{y}(s_1)), \lambda \geq 0\}. \end{aligned}$$

PROPOSITION 2.4. *Let C be a curve which satisfies conditions C1–C4. If $H(s_0, s_1)$ is a (possibly maximal) PI segment of C , then one has the following:*

- (1) $\hat{C}(s_0, s_1)$ is contained inside the wedge with vertex O and formed by the rays $R_+(s_0, s_1)$ and $R_-(s_0, s_1)$;
- (2) $\hat{C}(s_0, s_1)$ is smooth, and no line through O is tangent to $\hat{C}(s_0, s_1)$ at an interior point;
- (3) if $H(s_0, s_1)$ is not maximal, the angle between $R_+(s_0, s_1)$ and $R_-(s_0, s_1)$ is less than π . If $H(s_0, s_1)$ is maximal, the angle between the rays equals π ;
- (4) no line through O intersects the interior of $\hat{C}(s_0, s_1)$ at more than one point.

The property of C described in statement (1) of the proposition is important for us, so it will be given the name *containment property*. In other words, statement (1) says that PI segments of curves which satisfy conditions C1–C4 have the containment property.

Proof. To show that $\hat{C}(s_0, s_1)$ is contained inside the wedge, we first consider $\hat{C}(s_0, s_1)$, where $s_1 = s_0 + \epsilon$ for some $0 < \epsilon \ll 1$. As is easily seen, containment follows from the two inequalities:

$$(2.9) \quad \begin{aligned} [y(t) - y(s_0), y(s_1) - y(s_0), \dot{y}(s_0)] &> 0 \quad \forall t \in (s_0, s_1), \\ [y(t) - y(s_0), y(s_1) - y(s_0), \dot{y}(s_1)] &> 0 \quad \forall t \in (s_0, s_1). \end{aligned}$$

To prove the first inequality introduce the function

$$(2.10) \quad \Psi(s_1, t) := \left[\frac{y(t) - y(s_0) - \dot{y}(s_0)(t - s_0)}{(t - s_0)^2}, \frac{y(s_1) - y(s_0) - \dot{y}(s_0)(s_1 - s_0)}{(s_1 - s_0)^2}, \dot{y}(s_0) \right].$$

By using the Taylor series expansions we see that $\Psi(s_1, t)$ is smooth and bounded on compact sets. Notice also that

$$(2.11) \quad \Psi(s_1, s_1) = 0, \quad \Psi'_t(s_1, t) < \infty.$$

Hence $\Psi(s_1, t)/(s_1 - t)$ is bounded as well, which implies

$$(2.12) \quad \begin{aligned} & [y(t) - y(s_0), y(s_1) - y(s_0), \dot{y}(s_0)] \\ &= \frac{(t - s_0)^2 (s_1 - s_0)^2 (s_1 - t)}{12} ([\ddot{y}(s_0), \ddot{y}(s_0), \ddot{y}(s_0)] + o(1)) > 0, \end{aligned}$$

where $o(1) \rightarrow 0$ as $s_1 \rightarrow s_0$. We used the argument based upon the function Ψ , because we needed an asymptotic result that holds when $t \rightarrow s_0$ and when $t \rightarrow s_1$. The second inequality in (2.9) can be proven for small $s_1 - s_0 > 0$ in a similar fashion.

Suppose now that $s_1 - s_0$ is not necessarily small. Note that $\hat{C}(s_0, s_1)$ is tangent to the rays $R_+(s_0, s_1)$ and $R_-(s_0, s_1)$ at the point O of order precisely one. Consider, for example, the ray $R_+(s_0, s_1)$. To determine the order of tangency we need to find the asymptotics of the first expression in (2.9) as $t \rightarrow s_0$, with s_0 and s_1 fixed. We have

$$(2.13) \quad \begin{aligned} & [y(t) - y(s_0), y(s_1) - y(s_0), \dot{y}(s_0)] \\ &= [\ddot{y}(s_0), y(s_1) - y(s_0), \dot{y}(s_0)] \frac{(t - s_0)^2}{2} + O((t - s_0)^3) \\ &= -\Phi(s_0, s_1) \frac{(t - s_0)^2}{2} + O((t - s_0)^3). \end{aligned}$$

Similarly,

$$(2.14) \quad [y(t) - y(s_0), y(s_1) - y(s_0), \dot{y}(s_1)] = \Phi(s_1, s_0) \frac{(t - s_1)^2}{2} + O((t - s_1)^3), \quad t \rightarrow s_1.$$

By Proposition 2.3, $\Phi(s_0, s_1) < 0$, $\Phi(s_1, s_0) > 0$, and the desired assertion follows.

Suppose $C(s_0, s_1)$ does not have the containment property. Assume, for example, that the first inequality in (2.9) is violated. A violation of the other inequality can be considered analogously. From (2.13) and Proposition 2.3, the inequality holds for some $t > s_0$, where $t - s_0$ is sufficiently small. Thus there exists $t \in (s_0, s_1)$ such that

$$(2.15) \quad [y(t) - y(s_0), y(s_1) - y(s_0), \dot{y}(s_0)] = 0.$$

Let us show that (2.15) defines t as a function of s_1 . Formally differentiating (2.15) with respect to s_1 gives

$$(2.16) \quad \frac{dt}{ds_1} = - \frac{[y(t) - y(s_0), \dot{y}(s_1), \dot{y}(s_0)]}{[\dot{y}(t), y(s_1) - y(s_0), \dot{y}(s_0)]}.$$

The denominator in (2.16) does not vanish. Otherwise, from the linear independence of $\dot{y}(s_0)$ and $y(s_1) - y(s_0)$ (property C3) and (2.15) we get $Q(t, s_0) = [y(t) -$

$y(s_0), \dot{y}(s_0), \dot{y}(t)] = 0$. Since $H(s_0, s_1)$ is a PI line, this is a contradiction. Thus (2.15) does define t as a function of s_1 , and $C(s_0, s_1)$ does not have the containment property for all s_1 in an open set. Hence we can consider the function $t(s)$ for some $s \leq s_1$ using the fact that $Q(t, s_0) \neq 0$ for $t \in (s_0, s_1)$. As s decreases from s_1 towards s_0 , one of the following must happen:

- (a) $s, t \rightarrow s^* \neq s_0$. Replacing s_1 with s and t with $t(s)$ in (2.15) gives $Q(s^*, s_0) = [y(s^*) - y(s_0), \dot{y}(s_0), \dot{y}(s^*)] = 0$, which contradicts the assumption that $H(s_0, s_1)$ is a PI line.
- (b) $t \rightarrow s_0, s \rightarrow s^* > s_0$. From (2.15), $\Phi(s_0, s^*) = [y(s_0) - y(s^*), \dot{y}(s_0), \ddot{y}(s_0)] = 0$, which contradicts Proposition 2.3.

Note that $s, t \not\rightarrow s_0$ because of (2.12). Thus the containment property is established.

To prove the second statement we argue by contradiction. Suppose there exists $t \in (s_0, s_1)$, where either $\hat{C}(s_0, s_1)$ is nonsmooth or where the line through O and $\hat{y}(t)$ is tangent to $\hat{C}(s_0, s_1)$. Here $\hat{y}(t)$ is the projection of $y(t)$ onto $H^\perp(s_0, s_1)$. In both cases

$$(2.17) \quad [y(s_1) - y(s_0), \dot{y}(t), y(t) - y(s_0)] = 0.$$

Just as in the proof of statement (1), (2.17) defines t as a function of s_1 . Differentiating (2.17) with respect to s_1 gives

$$(2.18) \quad \frac{dt}{ds_1} = - \frac{[\dot{y}(s_1), \dot{y}(t), y(t) - y(s_0)]}{[y(s_1) - y(s_0), \ddot{y}(t), y(t) - y(s_0)]}.$$

The denominator in (2.18) does not vanish. Otherwise, together with (2.17) this gives $\Phi(t, s_0) = [y(t) - y(s_0), \dot{y}(t), \ddot{y}(t)] = 0$, which contradicts Proposition 2.3. Here we have used the fact that $y(s_1) - y(s_0)$ and $y(t) - y(s_0)$ are not parallel (cf. (2.9)). Hence we can consider the function $t(s)$ for some $s \leq s_1$ using the fact that $\Phi(t, s_0) \neq 0$ for $t \in (s_0, s_1]$. As s decreases from s_1 towards s_0 , one of the following must happen:

- (a) $s, t \rightarrow s^* \neq s_0$. Replacing s_1 with s and t with $t(s)$ in (2.17) gives $[y(s^*) - y(s_0), \dot{y}(s^*), \ddot{y}(s^*)] = 0$, which contradicts Proposition 2.3.
- (b) $t \rightarrow s_0, s \rightarrow s^* > s_0$. Then (2.17) implies $[y(s^*) - y(s_0), \dot{y}(s_0), \ddot{y}(s_0)] = 0$, which is again a contradiction.
- (c) $s, t \rightarrow s_0$. Now (2.17) implies $[\dot{y}(s_0), \ddot{y}(s_0), \ddot{y}(s_0)] = 0$, i.e., $\tau(s_0) = 0$. This contradicts the assumption $\tau(s_0) > 0$.

Our argument proves that (2.17) does not happen, so statement (2) is established.

To prove statement (3), first consider $H(s_0, q)$ for $q - s_0 > 0$ sufficiently small. As follows from statements (1) and (2), $\hat{C}(s_0, q)$ is contained between the rays $R_+(s_0, q)$ and $R_-(s_0, q)$, which are close to each other. As q increases towards s_1 , the two rays cannot collapse into one. Because of the containment, $\hat{C}(s_0, q)$ is always located between the rays. So if the two rays collapse into one for some $q > s_0$, then $C(s_0, q)$ is a planar curve, which contradicts the assumption $\tau > 0$. Hence $Q(s_0, s_1) = 0$ if and only if $R_+(s_0, s_1)$ and $R_-(s_0, s_1)$ point in opposite directions (see Figure 5).

Statements (1)–(3) imply that (i) whenever a line through O intersects $\hat{C}(s_0, s_1)$, then all of the intersection points (IPs) are on one side of O and (ii) neither $R_+(s_0, s_1)$ nor $R_-(s_0, s_1)$ intersects the interior of $\hat{C}(s_0, s_1)$. By (i) we can replace “line” with “ray” in statement (4). Suppose there is a ray γ with a vertex at O , which intersects $\hat{C}(s_0, s_1)$ at two interior points. Clearly, by rotating γ around O towards either $R_+(s_0, s_1)$ or $R_-(s_0, s_1)$ we can make the two IPs collide. As soon as the IPs collide,

we get a ray tangent to $\hat{C}(s_0, s_1)$ at an interior point, which contradicts statement (2). $\square$

COROLLARY 2.5. *No plane intersects $C(s_0, s_1)$ at more than three points.*

Proof. Suppose there is a plane Π that has at least four IPs with $C(s_0, s_1)$: $s_0 \leq t_1 < t_2 < t_3 < t_4 \leq s_1$. Consider $C(t_1, t_4)$, and project it onto the plane perpendicular to $H(t_1, t_4)$ (as was done in the proof of Proposition 2.4). As before, let O denote the projection of $H(t_1, t_4)$. The projection of Π gives the line through O which intersects $\hat{C}(t_1, t_4)$ at least at two points, which contradicts statement (3) of Proposition 2.4. $\square$

COROLLARY 2.6. *Pick any $x \in H(s_0, s_1)$ and $s \in (s_0, s_1)$. Consider a plane Π rotating around the line through x and $y(s)$. The number of IPs of Π and $C(s_0, s_1)$ changes from one to three when Π passes through $H(s_0, s_1)$.*

Proof. Consider the critical case when Π contains $H(s_0, s_1)$. As follows from Proposition 2.4, the vectors $\dot{y}(s_0)$ and $-\dot{y}(s_1)$ point into the opposite half-planes relative to Π . Hence, a small rotation of Π around $\beta(s, x)$ in one direction gives one IP and in the opposite direction three IPs. See section 4 in [15] for more details. $\square$

3. Establishing uniqueness of PI lines. To establish uniqueness of PI lines, we generalize the standard argument from helices [17, 8, 7] to general curves.

Fix some reconstruction point $x \in \mathbb{R}^3 \setminus C$. For each $s \in I$, fix a vector $N(s)$, $|N(s)| \equiv 1$ (a specific $N(s)$ will be chosen later). Define the functions $q(s)$ and $\lambda(s)$ so that $q(s) > s$, $H(s, q(s))$ is a PI segment, $0 < \lambda(s) < 1$, and the point

$$(3.1) \quad x(s) := y(s) + \lambda(s)(y(q(s)) - y(s)) \in H(s, q(s))$$

has the property

$$(3.2) \quad x(s) - x \parallel N(s).$$

We assume that the functions $q(s)$ and $\lambda(s)$ with the required properties exist. Later (see (3.11) and the proof of Proposition 3.3) we find an open set U such that for any $x \in U$ the functions $q(s)$ and $\lambda(s)$ do exist.

Condition (3.2) means that the parallel projection of $x(s)$ onto the plane through x with normal vector $N(s)$ coincides with x . Note that the vector-valued function $N(s)$ is determined independently of $q(s)$ and $\lambda(s)$. A similar idea is used in proving the uniqueness of PI lines for the standard helix, the difference being that the vector $N(s)$ is constant and directed along the axis of the helix.

Figure 4 illustrates the setup: The functions $q(s)$ and $\lambda(s)$ are defined in such a way as to ensure that the parallel projection of $x(s)$ onto the plane through x with normal $N(s)$ always coincides with x . Denote $\Delta y(s) := y(q(s)) - y(s)$. Thus,

$$(3.3) \quad \varepsilon(s) := N(s) \cdot \{(y(s) + \lambda(s)\Delta y(s)) - x\}$$

is the signed distance from $y(s) + \lambda(s)\Delta y(s)$ to x , i.e., $\varepsilon(s) = 0$ if and only if the chord $H(s, q(s))$ passes through x . We are interested in calculating $\varepsilon'(s)$.

Combining (3.1)–(3.3) gives

$$(3.4) \quad y(s) + \lambda(s)(y(q(s)) - y(s)) = x + \varepsilon(s)N(s).$$

Differentiate (3.4) with respect to s :

$$(3.5) \quad \dot{y}(s) + \lambda'(s)\Delta y(s) + \lambda(s)(\dot{y}(q(s))q'(s) - \dot{y}(s)) = \varepsilon'(s)N(s) + \varepsilon(s)\dot{N}(s).$$

FIG. 4. Parallel projection onto the plane $N^\perp(s)$ through x .

Computing the dot product of (3.5) with $\Delta y(s) \times \dot{y}(q)$ on both sides gives the following expression:

$$(3.6) \quad \varepsilon'(s) = A(s) + \varepsilon(s)B(s),$$

$$A(s) := -(1 - \lambda(s)) \frac{Q(s, q(s))}{[N(s), \Delta y(s), \dot{y}(q(s))]}, \quad B(s) := -\frac{[\dot{N}(s), \Delta y(s), \dot{y}(q(s))]}{[N(s), \Delta y(s), \dot{y}(q(s))]},$$

where we have used (2.2).

The goal is to obtain the uniqueness of PI lines. We start by choosing a vector $N(s)$ in such a way as to ensure that the denominator in (3.6) is never zero as long as $H(s, q(s))$ is a PI line. Denote the supremum (respectively, infimum) of all q such that $H(s, q)$ is a PI line by $q_{\max}(s)$ (respectively, $q_{\min}(s)$). Since $I = [a, b]$ is a compact interval, $q_{\max}(s)$ and $q_{\min}(s)$ are well-defined.

Now we study the properties of the function $q_{\max}(s)$. Pick any $s_0 \in (a, b)$ such that $q_{\max}(s_0) < b$. Consider the equation

$$(3.7) \quad Q(q_{\max}(s), s) = [y(q_{\max}(s)) - y(s), \dot{y}(s), \dot{y}(q_{\max}(s)))] = 0$$

for s in a neighborhood of s_0 . In particular, $q_{\max}(s)$ satisfies (3.7) when $s = s_0$. Formally differentiating (3.7) with respect to s gives

$$(3.8) \quad q'_{\max}(s_0) = -\frac{[y(q_{\max}(s_0)) - y(s_0), \ddot{y}(s_0), \dot{y}(q_{\max}(s_0))]}{[y(q_{\max}(s_0)) - y(s_0), \dot{y}(s_0), \ddot{y}(q_{\max}(s_0))]}.$$

By assumption C2, $y(q_{\max}(s_0)) - y(s_0)$ and $\dot{y}(s_0)$ are not parallel. Hence, if the denominator in (3.8) is zero, together with (3.7) this implies

$$(3.9) \quad [y(q_{\max}(s_0)) - y(s_0), \dot{y}(q_{\max}(s_0)), \ddot{y}(q_{\max}(s_0))] = 0,$$

which contradicts Proposition 2.3. By the implicit function theorem, in a neighborhood of the point $(s_0, q_{\max}(s_0))$ there exists a locally unique solution $q(s)$ to (3.7), and this solution satisfies $q(s_0) = q_{\max}(s_0)$. By construction, $q(s) < b$ in a neighborhood of s_0 . Clearly, $q_{\max}(s)$ is continuous at s_0 . Otherwise we can find $s_j \rightarrow s_0$ such that

$q_j := q_{\max}(s_j) \not\rightarrow q_{\max}(s_0)$. Thus we can choose a subsequence q_{j_k} which converges to some $\bar{q} \neq q_{\max}(s_0)$. By the definition of $q_{\max}$, $q_{\max}(s_j) \leq q(s_j) < b$, so $Q(q_j, s_j) = 0$. By the continuity of Q , $Q(\bar{q}, s_0) = 0$. Since $\bar{q} \neq q_{\max}(s_0)$, by the definition of $q_{\max}$ we must have $\bar{q} > q_{\max}(s_0)$. This is a contradiction, because, by using the continuity of $q(s)$, there must exist K such that $q_{\max}(s_{j_k}) > q(s_{j_k})$ for all $k > K$.

Using (3.7) and assumption C2 gives $\dot{y}(q_{\max}(s_0)) = c_1 \dot{y}(s_0) + c_2(y(q_{\max}(s_0)) - y(s_0))$, $c_1 \neq 0$. Substituting into (3.8) we obtain after simple transformations: $q'_{\max}(s_0) = -c_1^2 \Phi(s_0, q_{\max}(s_0)) / \Phi(q_{\max}(s_0), s_0)$. By Proposition 2.3, $q'_{\max}(s_0) > 0$.

Our argument implies that the function $q_{\max}(s)$ has the following properties: (1) $q_{\max}(s)$ is continuous on $[a, b]$. If $q_{\max}(a) < b$, then there exists $s_{\max}^* \in (a, b)$ such that (2) $q_{\max}(s) \in C^\infty([a, s_{\max}^*])$, $q_{\max}(s) < b$ and $q'_{\max}(s) > 0$ on $[a, s_{\max}^*)$, and $q_{\max}(s) \equiv b$ on $[s_{\max}^*, b]$, and (3) to compute $q_{\max}(s)$ on $[a, s_{\max}^*]$ we can find $q_{\max}(s_0)$ at any $s_0 \in [a, s_{\max}^*]$ and then extend it to the entire interval by solving (3.7).

Properties of $q_{\min}(s)$ are completely analogous: (1) $q_{\min}(s)$ is continuous on $[a, b]$. If $q_{\min}(b) > a$, then there exists $s_{\min}^* \in (a, b)$ such that (2) $q_{\min}(s) \in C^\infty([s_{\min}^*, b])$, $q_{\min}(s) > a$ and $q'_{\min}(s) > 0$ on $(s_{\min}^*, b]$, and $q_{\min}(s) \equiv a$ on $(a, s_{\min}^*]$, and (3) to compute $q_{\min}(s)$ on $[s_{\min}^*, b]$ we can find $q_{\min}(s_0)$ at any $s_0 \in [s_{\min}^*, b]$ and then extend it to the entire interval by solving (3.7) (with $q_{\max}$ replaced by $q_{\min}$).

From the monotonicity of $q_{\max}(s)$ we immediately obtain the following.

PROPOSITION 3.1. *Pick any $s_0 \in [a, b)$. If $s, q \in (s_0, q_{\max}(s_0)]$ and $s \neq q$, then $H(s, q)$ is a PI line.*

Proposition 3.1 is a generalization of a similar property for helices: Any line segment connecting two different points within one turn of a helix is a PI line [4, 5, 7]. Note also the following immediate corollary to Proposition 3.1: $q_{\min}(q_{\max}(s_0)) = s_0$ if $q_{\max}(s_0) < b$.

Define

$$(3.10) \quad N_{\max}(s) := \frac{y(q_{\max}(s)) - y(s)}{|y(q_{\max}(s)) - y(s)|}, \quad N_{\min}(s) := \frac{y(q_{\min}(s)) - y(s)}{|y(q_{\min}(s)) - y(s)|}, \quad s \in (a, b).$$

Thus, $N_{\max}(s)$ (respectively, $N_{\min}(s)$) is the unit vector along $H(s, q_{\max}(s))$ (respectively, $H(q_{\min}(s), s)$).

PROPOSITION 3.2. *Pick any $t \in (s, q_{\max}(s))$. One has $[y(t) - y(s), \dot{y}(t), N_{\max}(s)] \neq 0$, and the curve segments $C(s, t)$ and $C(t, q_{\max}(s))$ are located on opposite sides of the plane containing $H(s, q_{\max}(s))$ and $y(t)$. Similarly, pick any $t \in (q_{\min}(s), s)$. One has $[y(t) - y(s), \dot{y}(t), N_{\min}(s)] \neq 0$, and the curve segments $C(t, s)$ and $C(q_{\min}(s), t)$ are located on opposite sides of the plane containing $H(q_{\min}(s), s)$ and $y(t)$.*

Proof. We prove only the statements concerning $q_{\max}(s)$. The other half of the proposition is completely analogous.

The assertion $[y(t) - y(s), \dot{y}(t), N_{\max}(s)] \neq 0$ follows immediately from statement (2) of Proposition 2.4 (see also its proof). This proposition also implies that any line which contains O and passes between the rays $R_+(s, q_{\max}(s))$ and $R_-(s, q_{\max}(s))$ divides $\hat{C}(s, q_{\max}(s))$ into two segments located in the opposite half-planes (see Figure 5). This means that the curve segments $C(s, t)$ and $C(t, q_{\max}(s))$ are located on opposite sides of the plane containing $H(s, q_{\max}(s))$ and $y(t)$. $\square$

Next we determine the region where PI lines, if exist, are unique. Even though the curve C is well-behaved locally, very little can be said about the global behavior of C . So we choose a “local” piece of C : $I_0 := [a_0, b_0] \subset (a, b)$. The word local is made precise later. For each $s \in I_0$ consider the curve $\hat{C}(s, q_{\max}(s))$ in the plane $N_{\max}^\perp(s)$. By construction, $\hat{C}(s, q_{\max}(s))$ is closed. Let $Cyl_{\max}(s)$ be the infinite open cylinder with

FIG. 5. Projection onto the plane $N_{\max}(s)^\perp$ in the case $q_{\max}(s) < b$.

axis $N_{\max}(s)$, whose base is the interior of $\hat{C}(s, q_{\max})$. In the same fashion we define the cylinders $Cyl_{\min}(s)$ using $\hat{C}(q_{\min}, s)$ and $N_{\min}(s)$. Define U as the intersection of all such open cylinders:

$$(3.11) \quad U := \cap_{s \in I_0} (Cyl_{\min}(s) \cap Cyl_{\max}(s)).$$

If the curve turns too much, U can be empty. As an example, imagine a “slinky” toy. Locally it looks like a section of a helix. However if the slinky twists too much and the interval I_0 is sufficiently large, there can be no x that belongs to all of the cylinders. We assume that a sufficiently “local” piece of C is taken, so $U \neq \emptyset$. In other words, the only condition we assume for the interval I_0 is that the set U defined by (3.11) be nonempty. Note that in the case of a helix all cylinders $Cyl_{\min}(s)$ and $Cyl_{\max}(s)$ are identical, so (3.11) gives the usual domain inside the helix.

PROPOSITION 3.3. *Pick $x \in U$. If x admits a PI line, it is unique in the sense that there is no other PI line with an end point inside I_0 .*

Proof. Choose $N(s) := N_{\max}(s)$ in (3.2). Since $x \in U$, x projects along $N(s)$ into the interior of $\hat{C}(s, q_{\max}(s))$ for any $s \in I_0$. Hence the functions $q(s)$ and $\lambda(s)$ and the map $s \rightarrow x(s)$ (cf. (3.1), (3.2)) are well-defined on I_0 . By Proposition 3.2, $[\Delta y(s), \dot{y}(q(s)), N(s)] \neq 0$ for any $s \in I_0$. By Proposition 3.1, $H(s, q(s))$ are PI segments, so $Q(s, q(s)) \neq 0$ on I_0 . By construction, $\lambda(s) < 1$ on I_0 .

Our argument implies that $A(s)$ (cf. (3.6)) is bounded away from zero and of constant sign on I_0 . Consider now $B(s)$ (cf. (3.6)). As we already know, the denominator is bounded away from zero. Differentiating (3.10) gives

$$(3.12) \quad \begin{aligned} \dot{N}_{\max}(s) = & \frac{1}{|y(q_{\max}(s)) - y(s)|} \\ & \times \{[\dot{y}(q_{\max}(s))q'_{\max}(s) - \dot{y}(s)] - N(N \cdot [\dot{y}(q_{\max}(s))q'_{\max}(s) - \dot{y}(s)])\}. \end{aligned}$$

By assumption C1, C has no self-intersections, so $|y(q_{\max}) - y(s)|$ is bounded away from zero. From (3.8) and the subsequent discussion, it follows that $q'_{\max}(s)$ is bounded away from zero. Hence, $\dot{N}_{\max}(s)$ is bounded, and $B(s)$ is bounded as well.

From the properties of $A(s)$ and $B(s)$ we get that $\varepsilon(s)$ cannot have more than one root on I_0 . This follows immediately from the fact that the signs of $\varepsilon'(s)$ and $A(s)$ in a neighborhood of any s where $\varepsilon(s) = 0$ are the same. Hence x cannot have more than one PI segment with $s_b(x) \in I_0$.

Choosing $N(s) := N_{\min}(s)$ in (3.2) and repeating the same argument gives that x cannot have more than one PI segment with $s_t(x) \in I_0$. $\square$

4. Reconstruction algorithm. In order to derive an inversion formula we need to study the curve C some more.

PROPOSITION 4.1. *Let $H(s_0, s_1)$ be a (possibly maximal) PI segment of C . Then $\hat{C}(s_0, s_1)$ has everywhere nonvanishing curvature.*

Proof. Recall that $\hat{C}(s_0, s_1)$ is smooth by Proposition 2.4. Pick any $t \in (s_0, s_1)$. Up to a nonzero factor the curvature of $\hat{C}(s_0, s_1)$ at the point t is given by $[y(s_1) - y(s_0), \dot{y}(t), \ddot{y}(t)]$. Using (2.2) we see that this expression equals $\Phi(t, s_0) - \Phi(t, s_1)$. Since $s_0 < t < s_1$, Proposition 2.3 gives the desired result. We can think of $\Phi(t, s)$ as a signed “distance” from $y(s)$ to $\Pi_{osc}(t)$, so Proposition 2.3 also gives that the line segment $H(s_0, s_1)$ intersects $\Pi_{osc}(t)$ for any $t \in (s_0, s_1)$. $\square$

COROLLARY 4.2. *Let $H(s_0, s_1)$ be a (possibly maximal) PI segment of C . For any $x \in H(s_0, s_1)$ and $t \in (s_0, s_1)$, the vectors $\dot{y}(t)$ and $x - y(t)$ are not collinear.*

Proof. By Proposition 2.3, $\hat{C}(s_0, q_{max}(s_0))$ is strictly convex. $x \in H(s_0, s_1)$ implies that x projects into the domain bounded by $\hat{C}(s_0, q_{max}(s_0))$. Thus $\dot{y}(t)$ and $x - y(t)$ are not collinear. $\square$

PROPOSITION 4.3. *Let $H(s_0, s_1)$ be a (possibly maximal) PI segment of C . For any $x \in H(s_0, s_1)$ there exists the unique $s^*(x) \in (s_0, s_1)$ such that $x \in \Pi_{osc}(s^*(x))$.*

Proof. As follows from the proof of Proposition 4.1, $\Pi_{osc}(t)$ intersects $H(s_0, s_1)$ for any $t \in [s_0, s_1]$. Hence we can write

$$(4.1) \quad y(s_0) + \lambda(t)(y(s_1) - y(s_0)) = y(t) + a(t)\dot{y}(t) + b(t)\ddot{y}(t)$$

for some scalar functions λ , a , and b . Differentiate (4.1) with respect to t , multiply the resulting equation by $\dot{y}(t) \times \ddot{y}(t)$, and solve for λ' :

$$(4.2) \quad \lambda'(t) = b(t) \frac{[\dot{y}(t), \ddot{y}(t), \ddot{\ddot{y}}(t)]}{[y(s_1) - y(s_0), \dot{y}(t), \ddot{y}(t)]}.$$

Since the torsion of C is nonzero, the numerator in (4.2) does not vanish. From the proof of Proposition 4.1, the denominator in (4.2) is nonzero. By Corollary 4.2, $b(t) \neq 0, t \in (s_0, s_1)$. Hence $\lambda(t)$ is a smooth monotone function on $[s_0, s_1]$. Obviously, $\Pi_{osc}(s_0)$ (respectively, $\Pi_{osc}(s_1)$) intersects $H(s_0, s_1)$ at $y(s_0)$ (respectively, $y(s_1)$). Thus $\lambda(s_0) = 0$, $\lambda(s_1) = 1$, and the proposition is proven. $\square$

Due to the containment property (statement (1) of Proposition 2.4), the curve $C(s, q_{max}(s))$ (respectively, $C(s, q_{min}(s))$) is on one side of the plane passing through $y(s)$ and parallel to $\dot{y}(s)$ and $N_{max}(s)$ (respectively, $N_{min}(s)$). This makes it very convenient to project $C(s, q_{max}(s))$ (respectively, $C(s, q_{min}(s))$) onto a plane parallel to $\dot{y}(s)$ and $N_{max}(s)$ (respectively, $N_{min}(s)$). The corresponding projections turn out to be smooth. Let $DP_+(s)$ (respectively, $DP_-(s)$) denote a plane not passing through $y(s)$ and parallel to $\dot{y}(s)$ and $N_{max}(s)$ (respectively, $N_{min}(s)$). We think of $DP_+(s)$ and $DP_-(s)$ as detector planes, so they are chosen on the same side of $y(s)$ as the set U . More precisely, the rays with vertex $y(s)$ passing through U intersect $DP_+(s)$ and $DP_-(s)$. The stereographic projection of $C(s, q_{max}(s))$ onto $DP_+(s)$ is denoted Γ_+ , while the stereographic projection of $C(q_{min}(s), s)$ onto $DP_-(s)$ is denoted Γ_- .

PROPOSITION 4.4. *Γ_+ and Γ_- are smooth and have nonvanishing curvature at every point.*

Proof. We consider only Γ_+ . The statement about Γ_- is proven analogously. Suppose, for simplicity, that the origin is at $y(s)$ and the equation of $DP_+(s)$ is $x_3 = 1$. Thus, x_1 and x_2 are the coordinates on $DP_+(s)$. Let $x_1(t)$ and $x_2(t)$ be the coordinates of the projection of $y(t), t \in (s, q_{max}(s))$, onto $DP_+(s)$. Then

$$(4.3) \quad x_1(t) = \frac{y_1(t)}{y_3(t)}, \quad x_2(t) = \frac{y_2(t)}{y_3(t)}.$$

Applying (2.3) to a planar curve gives

$$(4.4) \quad \kappa(t) = \frac{\dot{x}_1^2}{(\dot{x}_1^2 + \dot{x}_2^2)^{3/2}} \left(\frac{\dot{x}_2}{\dot{x}_1} \right)'.$$

Differentiating (4.3) gives

$$(4.5) \quad \begin{aligned} \left(\frac{\dot{x}_2}{\dot{x}_1} \right)' &= \left(\frac{\dot{y}_2 y_3 - \dot{y}_3 y_2}{\dot{y}_1 y_3 - \dot{y}_3 y_1} \right)' \\ &= \frac{(\ddot{y}_2 y_3 - \ddot{y}_3 y_2)(\dot{y}_1 y_3 - \dot{y}_3 y_1) - (\dot{y}_2 y_3 - \dot{y}_3 y_2)(\ddot{y}_1 y_3 - \ddot{y}_3 y_1)}{(\dot{x}_1 y_3^2)^2} \\ &= \frac{1}{(\dot{x}_1 y_3^2)^2} \begin{vmatrix} y_1 & y_2 & y_3 \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 \\ \ddot{y}_1 & \ddot{y}_2 & \ddot{y}_3 \end{vmatrix}. \end{aligned}$$

Substituting (4.5) into (4.4) and using (4.3) (recall that $y(s) = 0$ is the origin) gives the curvature of Γ_+ :

$$(4.6) \quad \kappa(t) = \frac{\Phi(t, s)}{y_3^4(t) (\dot{x}_1^2(t) + \dot{x}_2^2(t))^{3/2}}.$$

By the properties of $C(s, q_{\max}(s))$ mentioned prior to this proposition, $y_3(t) \neq 0, t \in (s, q_{\max}(s))$. Also, $y_3(s) = 0$, and, if $H(s, q_{\max}(s))$ is maximal, $y_3(q_{\max}(s)) = 0$. It remains to show that $\dot{x}_1^2(t) + \dot{x}_2^2(t) \neq 0$. This would also imply that Γ_+ is smooth. We argue by contradiction. Suppose $\dot{x}_1(t) = \dot{x}_2(t) = 0$. Then $\dot{y}_2 y_3 = \dot{y}_3 y_2, \dot{y}_1 y_3 = \dot{y}_3 y_1$. Consequently, $y(t) \times \dot{y}(t)$ is parallel to the x_3 -axis. Thus, either both $y(t)$ and $\dot{y}(t)$ are parallel to $DP_+(s)$ or $y(t)$ and $\dot{y}(t)$ are parallel to each other. Both cases are impossible because of the convexity of $\hat{C}(s, q_{\max}(s))$ (cf. Proposition 4.1). Since $\Phi(t, s) \neq 0$ for $t \in [s, q_{\max}(s))$ (cf. Proposition 2.3), the desired assertion is proven. $\square$

Denote $L_0^+ := DP_+(s) \cap \Pi_{osc}(s)$. It is clear that L_0^+ is an asymptote of Γ_+ : $\text{dist}(\hat{y}(t), L_0^+) \rightarrow 0$ as $t \rightarrow s^+$. Similarly, $L_0^- := DP_-(s) \cap \Pi_{osc}(s)$ is an asymptote of Γ_- : $\text{dist}(\hat{y}(t), L_0^-) \rightarrow 0$ as $t \rightarrow s^-$.

Fix $x \in U$, which admits a PI line. Let $I_{PI}(x) = [s_b(x), s_t(x)]$ be the PI interval of x . Let $\hat{x}$ denote the projection of x onto a detector plane. Frequently it is convenient to identify detector planes by introducing systems of coordinates that depend smoothly on s . This allows one to identify all $DP_+(s)$ and, separately, all $DP_-(s)$. Since $x \in U$, x does not belong to any plane passing through $y(s)$ and parallel to $DP_+(s)$ or $DP_-(s)$, where $s \in I_{PI}(x)$. Hence Propositions 4.3 and 3.3 immediately imply the following statement.

COROLLARY 4.5. *As s moves along $I_{PI}(x)$, the point $\hat{x}$ traces smooth curves on $DP_+(s)$ and $DP_-(s)$. $\hat{x}$ is between $\Gamma_+(s)$ and L_0^+ on $DP_+(s)$ if and only if $s \in (s_b(x), s^*(x))$, and $\hat{x}$ is between L_0^- and $\Gamma_-(s)$ on $DP_-(s)$ if and only if $s \in (s^*(x), s_t(x))$.*

Loosely speaking, Corollary 4.5 can be stated as follows: $\hat{x}$ is between $\Gamma_+(s)$ and $\Gamma_-(s)$ if and only if $s \in I_{PI}(x)$.

Following [9, 12], choose any $\psi \in C^\infty(\mathbb{R}^+)$ with the properties

$$(4.7) \quad \begin{aligned} \psi(0) &= 0; \quad 0 < \psi'(t) < 1, \quad t \geq 0, \\ \psi'(0) &= 0.5; \quad \psi^{(2k+1)}(0) = 0, \quad k \geq 1. \end{aligned}$$

FIG. 6. Detector planes $DP_+(s)$ (left panel) and $DP_-(s)$ (right panel).

Suppose s , s_1 , and s_2 are related by

$$(4.8) \quad s_1 = \begin{cases} \psi(s_2 - s) + s, & s_2 \geq s, \\ \psi(s - s_2) + s_2, & s_2 < s. \end{cases}$$

From (4.7), $s_1 = s_1(s, s_2)$ is a C^∞ function of s and s_2 . Conditions (4.7) are easy to satisfy. One can take, for example, $\psi(t) = t/2$, and this leads to

$$(4.9) \quad s_1 = (s + s_2)/2.$$

Denote also

$$(4.10) \quad u(s, s_2) = \frac{(y(s_1) - y(s)) \times (y(s_2) - y(s))}{|(y(s_1) - y(s)) \times (y(s_2) - y(s))|} \text{sgn}(s_2 - s),$$

$$q_{\min}(s) < s_2 < q_{\max}(s), s_2 \neq s,$$

$$u(s, s_2) = \frac{\dot{y}(s) \times \ddot{y}(s)}{|\dot{y}(s) \times \ddot{y}(s)|}, s_2 = s.$$

In the same way as in [12], we prove that $u(s, s_2)$ is a C^∞ vector function of its arguments. Let $\Pi(s, s_2)$ be the plane through $y(s)$, $y(s_2)$, and $y(s_1(s, s_2))$. The intersection of $\Pi(s, s_2)$ with $DP_+(s)$ if $s < s_2 < q_{\max}(s)$ or with $DP_-(s)$ if $q_{\min}(s) < s_2 < s$ is called a filtering line and denoted $L(s, s_2)$.

Fix $x \in U$, which admits a PI line, and $s \in I_{PI}(x)$. Find $s_2 \in I_{PI}(x)$ such that $\Pi(s, s_2)$ contains x . More precisely, we have to solve for s_2 the following equation:

$$(4.11) \quad (x - y(s)) \cdot u(s, s_2) = 0, \quad s_2 \in I_{PI}(x).$$

Recall that $\dot{y}(s)$ is parallel to $DP_+(s)$ and $DP_-(s)$. For convenience, we choose the x_1 - and x_2 -axes so that

1. $\dot{y}(s)$ and the x_1 -axis are parallel and point in the same direction;
2. the equation of $\Pi_{osc}(s)$ is $x_2 = 0$;
3. on $DP_+(s)$, Γ_+ is located in the half-plane $x_2 > 0$;
4. on $DP_-(s)$, Γ_- is located in the half-plane $x_2 < 0$.

Figure 6 illustrates the two detector planes.

The advantage of planes $DP_+(s)$ and $DP_-(s)$ is that the segments $C(s, q_{\max}(s))$ and $C(q_{\min}(s), s)$ are projected onto them as continuous curves with positive curvature. If C is a helix, the two segments become the usual 2π -segments $C(s, s + 2\pi)$ and $C(s - 2\pi, s)$. This makes it very convenient when describing how to choose filtering lines in a shift-invariant FBP algorithm. On the other hand, the disadvantage is that

the two segments are projected onto two different planes. This makes it difficult to adapt the proofs from [12, 9] to the present more general situation. Fortunately, the difficulty can be resolved. Given $x \in U$ with the PI interval $I_{PI}(x) = [s_b(x), s_t(x)]$, we can find a family of “detector planes” such that for any $s \in I_{PI}(x)$ the entire PI segment of x , $C(s_b(x), s_t(x))$, projects onto them in exactly the same way as in the case of a regular constant-pitch helix. There is no guarantee that the larger segment $C(q_{min}(s), q_{max}(s))$ (which is equivalent to two adjacent turns of a helix) projects well onto the planes, but this is not needed.

Let $DP(s)$, $s \in I_{PI}(x)$, be a plane not passing through $y(s)$ and parallel to $\dot{y}(s)$ and $N_{max}(s_b(x))$. Using the convexity of $\hat{C}(s_b(x), s_t(x)) \subset \hat{C}(s_b(x), q_{max}(s_b(x)))$ (cf. Proposition 4.1 and Figure 5) and repeating the proof of Proposition 4.4, we establish that the stereographic projection of $C(s_b(x), s_t(x))$ onto $DP(s)$ has all of the usual properties as in the constant-pitch helix case. More precisely, the projections of $C(s_b(x), s)$ and $C(s, s_t(x))$ are concave down and up, respectively, they share the usual asymptote $DP(s) \cap \Pi_{osc}(s)$, they are located on the opposite sides of the latter, etc. Thus, using the same argument as in [12, 7], we immediately obtain the following result.

PROPOSITION 4.6. *The solution s_2 to (4.11) exists, is unique, and depends smoothly on s .*

The following result shows that filtering lines are shared by sufficiently many points $x \in U$. The planes $DP(s)$ used for the proof of Proposition 4.6 are selected separately for each x , so they do necessarily work for all x in a large subset of U . Thus we have to go back to the planes $DP_+(s)$ and $DP_-(s)$.

PROPOSITION 4.7. *All $x \in U$ that project onto any line $L(s, s_2)$, $s < s_2 < q_{max}(s)$, on $DP_+(s)$ to the left of s_2 or onto $L(s, s_2)$, $q_{min}(s) < s_2 < s$, on $DP_-(s)$ to the right of s_2 share $L(s, s_2)$ as their filtering line.*

Proof. We consider only the case when $s_2 > s$, i.e., $\hat{x} \in DP_+(s)$. The other case can be considered analogously. We have $s_t(x) \in \Gamma_+$. By Corollary 4.5, $\hat{x}$ appears between L_0^+ and Γ_+ . From the proof of Proposition 4.1, $\Pi_{osc}(s)$ intersects the PI segment of x , $H(s_b(x), s_t(x))$. Let $z_{osc}(s)$ denote the point of intersection. Let $\Pi_{max}(s)$ be the plane through $y(s)$ and parallel to $\dot{y}(s)$ and $N_{max}(s)$. Our first goal is to show that the line segment $[z_{osc}(s), y(s_t(x))]$ lies on one side of $\Pi_{max}(s)$ and, therefore, projects well onto $DP_+(s)$. Let $z_{max}(s)$ denote the intersection of the line through $L_{PI}(x)$ and $\Pi_{max}(s)$. Clearly, $z_{osc}(s) = z_{max}(s)$ when $s = s_b(x)$. From the proof of Proposition 4.3, $z_{osc}(s)$ moves toward $y(s_t(x))$ along $L_{PI}(x)$ as s increases from $s_b(x)$ to $s_t(x)$. From the convexity of $\hat{C}(s, q_{max}(s))$ (cf. Figure 5), it is easy to obtain that in a neighborhood of $s = s_b(x)$ the point $z_{max}(s)$ moves away from $H(s_b(x), s_t(x))$ as s increases. If for some $s \in (s_b(x), s_t(x))$ the points $z_{osc}(s)$ and $y(s_t(x))$ are on opposite sides of $\Pi_{max}(s)$, then the point $z_{max}(s)$ enters the line segment $[z_{osc}(s), y(s_t(x))]$ for some $s = s_0 \in (s_b(x), s_t(x))$. Hence, either (i) $z_{osc}(s_0) = z_{max}(s_0)$ or (ii) $y(s_t(x)) = z_{max}(s_0)$. From Proposition 2.3, $[y(q_{max}(s_0)) - y(s_0), \dot{y}(s_0), \ddot{y}(s_0)] \neq 0$, so (i) implies that $z_{osc}(s_0) - y(s_0)$ and $\dot{y}(s_0)$ are collinear, which contradicts Corollary 4.2. In case (ii), $y(s_t(x)) \in \Pi_{max}(s_0)$, which contradicts the containment property.

Hence $\hat{L}_{PI}(x)$, the projection of $H(s_b(x), s_t(x))$ onto $DP_+(s)$, intersects L_0^+ . More precisely, the projection of the line segment $[z_{osc}(s), y(s_t(x))]$ $\subset L_{PI}(x)$ is a continuous line segment that connects Γ_+ and L_0^+ (see Figure 6). Note that Proposition 4.3 implies $x \in [z_{osc}(s), y(s_t(x))]$ if $s < s^*(x)$. It turns out that $\hat{L}_{PI}(x)$ does not intersect Γ_+ at any point other than $s_t(x)$. Suppose there is an additional intersection point t . Thus the plane through $y(s)$ and $H(s_b(x), s_t(x))$ intersects $C_{PI}(x)$ at four points: $s_b(x)$, t , s ,

and $s_t(x)$, and this contradicts Corollary 2.5. Here $C_{PI}(x)$ is the section of the curve corresponding to the parametric interval $I_{PI}(x)$, i.e., $C_{PI}(x) := C(s_b(x), s_t(x))$.

If x projects onto $L(s, s_2)$ to the left of s_2 , we make two observations: (i) $\hat{x}$ is between L_0^+ and Γ_+ on $DP_+(s)$, and (ii) $s_2 < s_t(x)$ (due to the properties of $\hat{L}_{PI}(x)$ that we just established). From (i) and Corollary 4.5, $s \in I_{PI}(x)$. From (ii), $s_2 \in (s, s_t(x))$, so by (i) $s_2 \in I_{PI}(x)$. By construction, s_2 was chosen to satisfy $(x - y(s)) \cdot u(s, s_2) = 0$. We have just shown that $s, s_2 \in I_{PI}(x)$. This proves that $L(s, s_2)$ is the filtering line for x . $\square$

By Proposition 4.7, our construction defines $s_2 := s_2(s, x)$ and, consequently, $u(s, x) := u(s, s_2(s, x))$. Let $D_f(s, \Theta) = \int_0^\infty f(y(s) + t\Theta)dt$, $|\Theta| = 1$, denote the cone beam transform of f . The main result of the paper is the following theorem.

THEOREM 4.8. *Let C be a curve (2.1), which satisfies conditions C1–C4. Let $I_0 \subset I$ be an interval such that the set U defined by (3.11) is nonempty. For any $f \in C_0^\infty(U)$ and $x \in U$ which admits a PI line such that $I_{PI}(x) \subset I_0$, one has*

$$(4.12) \quad f(x) = -\frac{1}{2\pi^2} \int_{I_{PI}(x)} \frac{1}{|x - y(s)|} \int_0^{2\pi} \frac{\partial}{\partial q} D_f(q, \Theta(s, x, \gamma)) \Big|_{q=s} \frac{d\gamma}{\sin \gamma} ds,$$

where $\beta(s, x) = (x - y(s))/|x - y(s)|$, $e(s, x) := \beta(s, x) \times u(s, x)$, and $\Theta(s, x, \gamma) := \cos \gamma \beta(s, x) + \sin \gamma e(s, x)$.

Proof. Corollaries 2.5, 2.6, and 4.5 and Propositions 4.1, 4.3, 4.4, and 4.6 imply that locally, i.e., in a neighborhood of $I_{PI}(x)$, the curve C behaves in essentially the same way as the usual helix. Hence the same argument as in [12, 7] can be used to prove that (4.12) holds. For the convenience of the reader we recall the key steps in the proof. Fix $x \in U$ such that $I_{PI}(x) \subset I_0$ and $s \in I_{PI}(x)$. Suppose $s < s^*(x)$ (cf. Proposition 4.3). Consider the detector plane $DP(s)$ introduced prior to Proposition 4.6. Let d_0 be the unit vector perpendicular to $DP(s)$ and pointing from the source position $y(s)$ towards the detector. By the choice of U and the detector plane, this implies $\beta(s, x) \cdot d_0 > 0$ and $\dot{y}(s) \cdot d_0 = 0$. In the same way as in [10] we show that the stereographic projection of all of the relevant vectors onto $DP(s)$ preserves the sign of dot products. Let Π be a generic plane containing x and $y(s)$. By Corollary 2.5, there can be only either one or three IPs in the set $\Pi \cap C_{PI}(x)$. If s is the only IP or the middle of the three IPs, then the slope of the line $\Pi \cap DP(s)$ on $DP(s)$ is greater than the slopes of L_0 ($= \Pi_{osc}(s) \cap DP(s)$) and the filtering line through $\hat{x}$. Recall that the filtering line intersects the projection of $C(s, s_t(x))$ onto $DP(s)$ twice, and the projected curve is convex. Hence the IP s gets weight 1. Since $s < s^*(x)$, the only remaining alternative is that s is the smallest of the three IPs. In this case s gets weight 1 or -1 depending on the location of $\Pi \cap DP(s)$ relative to the filtering line through $\hat{x}$. Using exactly the same argument as in section 3 of [12], we prove that the largest of the three IPs gets weight -1 or 1 , respectively. Hence the inversion formula (4.12) is exact. $\square$

Proposition 4.7 implies that (4.12) is of the efficient shift-invariant FBP form. This means that filtering in (4.12) is convolution-based and is performed along a one-parametric family of lines.

5. Numerical experiments. Numerical experiments are conducted using flat detector geometry. The simulation parameters are summarized in Table 1. The detector is located at the distance of 600 mm from the axis of rotation opposite to the source position. The algorithm is implemented in the native coordinates following [18]. The clock phantom (see, e.g., [7]) is chosen for reconstructions. The background cylinder is at 0 HU, the spheres are at 1000 HU, and the air is at -1000 HU.

TABLE 1
Simulation parameters.

Parameter	Value	Units
Views per rotation	1000	
Number of detector columns	1101	
Number of detector rows	111	
Actual detector pixel size	1×1	mm ²
Isocenter to detector distance	600	mm

FIG. 7. Projection of the source trajectory in (5.1) onto the xy -plane.

Two source trajectories have been used. The first one is a variable radius helix given by the formula:

$$(5.1) \quad y(s) = \left(R(s) \cos s, R(s) \sin s, \frac{h_0}{2\pi} s \right), \quad R(s) = R(1 + 0.3 \sin(s/3)),$$

where $R = 600$ mm, and the table feed per turn is $h_0 = 35$ mm. The projection of this trajectory onto the plane $x_3 = 0$ for $s \in [-2\pi, 2\pi]$ is shown in Figure 7.

Because of the variable radius, detector usage is different for different source positions. The detector parameters given in Table 1 were chosen so as to accommodate all source positions. The horizontal size of the detector varied from 894 to 1098 mm; the average was 945 mm. The vertical size of the detector varied from 49 to 75 mm; the average was 59 mm. These values were calculated for the segment of the trajectory necessary to reconstruct the clock phantom.

The boundary of the set U is calculated according to (3.11). The cross section of the boundaries of cylinders $Cyl_{min}(s)$ and $Cyl_{max}(s)$ with the plane $x_3 = 0$ is shown in Figure 8 (left panel). The solid circle of radius $r = 240$ mm shows the boundary of the clock phantom, and the dashed circle is of the maximum radius $r \approx 374$ mm that fits inside the cross section of U . The result of reconstruction is shown in Figure 9. Here and in the experiment below we use voxels of size 1mm in each direction.

The second experiment is carried out using the variable-radius and variable-pitch helix given by:

$$(5.2) \quad y(s) = \left(R(s) \cos s, R(s) \sin s, \frac{h(s)}{2\pi} s \right), \quad h(s) = h_0 \left(1 + \frac{\sin(s/2)}{s} \right).$$

Here $R(s)$ and h_0 are the same as in (5.1). Because of the variable radius/variable pitch, detector usage is different for different source positions. The horizontal size

FIG. 8. Cross section of boundaries of cylinders $Cyl(s)$ from (3.11) for trajectory (5.1) (left panel) and trajectory (5.2) (right panel).

FIG. 9. Reconstruction of the clock phantom from trajectory (5.1): slice $x_3 = 0$, $WL = 0$ HU, $WW = 100$ HU.

of the detector varied from 894 to 1098 mm; the average was 945 mm. The vertical size of the detector varied from 38 to 88 mm; the average was 71 mm. These values were calculated for the segment of the trajectory necessary to reconstruct the clock phantom. The cross section of the boundaries of cylinders $Cyl_{min}(s)$ and $Cyl_{max}(s)$ with the plane $x_3 = 0$ is shown in Figure 8 (right panel). Again, the solid circle of radius $r = 240$ mm shows the boundary of the clock phantom, and the dashed circle is of the maximum radius $r \approx 348$ mm that fits inside the cross section of U . The results of the reconstruction are shown in Figure 10.

FIG. 10. *Reconstruction of the clock phantom from trajectory (5.2): slice $z = 0$, $WL = 0$ HU, $WW = 100$ HU.*

REFERENCES

- [1] C. BONTUS, T. KÖHLER, AND R. PROKSA, *EnPiT: Filtered back-projection algorithm for helical CT using an n -Pi acquisition*, IEEE Trans. Medical Imaging, 24 (2005), pp. 977–986.
- [2] C. BONTUS, P. KOKEN, T. KÖHLER, AND R. PROKSA, *Circular CT in combination with a helical segment*, Physics in Medicine and Biology, 51 (2007), pp. 107–120.
- [3] G.-H. CHEN, T. ZHUANG, B. E. NETT, AND S. LENG, *A showcase of exact cone-beam image reconstruction algorithms for circle-based trajectories*, in Proceedings of the Eighth International Meeting on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine, F. Noo, L. G. Zeng, and H. Kudo, eds., Salt Lake City, Utah, 2005, pp. 295–299. Online access: www.ucair.med.utah.edu/3D05/proceedings.html.
- [4] P. E. DANIELSSON, P. EDHOLM, J. ERIKSSON, AND M. M. SEGER, *Towards exact reconstruction for helical cone-beam scanning of long objects. A new detector arrangement and a new completeness condition*, in Proceedings of the 1997 Meeting on Fully 3D Image Reconstruction in Radiology and Nuclear Medicine, Pittsburgh, D. W. Townsend and P. E. Kinahan, eds., 1997, Institute of Physics, Bristol, pp. 141–144.
- [5] M. DEFRISE, F. NOO, AND H. KUDO, *A solution to the long-object problem in helical cone-beam tomography*, Physics in Medicine and Biology, 45 (2000), pp. 623–643.
- [6] M. KAPRALOV AND A. KATSEVICH, *A one-PI algorithm for helical trajectories that violate the convexity condition*, Inverse Problems, 22 (2006), pp. 2123–2143.
- [7] A. KATSEVICH, S. BASU, AND J. HSIEH, *Exact filtered backprojection reconstruction for dynamic pitch helical cone beam computed tomography*, Physics in Medicine and Biology, 49 (2004), pp. 3089–3103.
- [8] A. KATSEVICH AND G. LAURITSCH, *Filtered backprojection algorithms for spiral cone beam CT*, in Sampling, Wavelets, and Tomography, J. Benedetto and A. Zayed, eds., Birkhauser, Boston, 2003, pp. 255–287.
- [9] A. KATSEVICH, *Analysis of an exact inversion algorithm for spiral cone-beam CT*, Physics in Medicine and Biology, 47 (2002), pp. 2583–2598.
- [10] A. KATSEVICH, *Theoretically exact filtered backprojection-type inversion algorithm for spiral CT*, SIAM J. Appl. Math., 62 (2002), pp. 2012–2026.

- [11] A. KATSEVICH, *Image reconstruction for the circle and line trajectory*, Physics in Medicine and Biology, 49 (2004), pp. 5059–5072.
- [12] A. KATSEVICH, *An improved exact filtered backprojection algorithm for spiral computed tomography*, Adv. Appl. Math., 32 (2004), pp. 681–697.
- [13] A. KATSEVICH, *On two versions of a 3π algorithm for spiral CT*, Physics in Medicine and Biology, 49 (2004), pp. 2129–2143.
- [14] A. KATSEVICH, *Image reconstruction for the circle and arc trajectory*, Physics in Medicine and Biology, 50 (2005), pp. 2249–2265.
- [15] A. KATSEVICH, *3PI algorithms for helical computer tomography*, Adv. Appl. Math., 21 (2006), pp. 213–250.
- [16] T. KÖHLER, C. BONTUS, AND P. KOKEN, *The radon-split method for helical cone-beam CT and its application to nongated reconstruction*, IEEE Trans. Medical Imaging, 25 (2006), pp. 882–897.
- [17] H. KUDO, F. NOO, AND M. DEFRISE, *Quasi-exact filtered backprojection algorithm for long-object problem in helical cone-beam tomography*, IEEE Trans. Medical Imaging, 19 (2000), pp. 902–921.
- [18] F. NOO, J. PACK, AND D. HEUSCHER, *Exact helical reconstruction using native cone-beam geometries*, Physics in Medicine and Biology, 48 (2003), pp. 3787–3818.
- [19] J. D. PACK, F. NOO, AND R. CLACKDOYLE, *Cone-beam reconstruction using the backprojection of locally filtered projections*, IEEE Trans. Medical Imaging, 24 (2005), pp. 1–16.
- [20] J. D. PACK AND F. NOO, *Cone-beam reconstruction using 1D filtering along the projection of M-lines*, Inverse Problems, 21 (2005), pp. 1105–1120.
- [21] E. SIDKY, Y. ZOU, AND X. PAN, *Minimum data image reconstruction algorithms with shift-invariant filtering for helical, cone-beam CT*, Physics in Medicine and Biology, 50 (2005), pp. 1643–1657.
- [22] H. YANG, M. LI, K. KOIZUMI, AND H. KUDO, *Exact cone beam reconstruction for a saddle trajectory*, Physics in Medicine and Biology, 51 (2006), pp. 1157–1172.
- [23] Y. YE, S. ZHAO, H. YU, AND G. WANG, *A general exact reconstruction for cone-beam CT via backprojection-filtration*, IEEE Trans. Medical Imaging, 24 (2005), pp. 1190–1198.
- [24] Y. YE, J. ZHU, AND G. WANG, *Geometric studies on variable radius spiral cone-beam scanning*, Medical Physics, 31 (2004), pp. 1473–1480.
- [25] H. YU, S. ZHAO, Y. YE, AND G. WANG, *Exact BPF and FBP algorithms for nonstandard saddle curves*, Medical Physics, 32 (2005), pp. 3305–3312.
- [26] T. ZHUANG, S. LENG, B. E. NETT, AND G. CHEN, *Fan-beam and cone-beam image reconstruction via filtering the backprojection image of differentiated projection data*, Physics in Medicine and Biology, 49 (2004), pp. 1643–1657.
- [27] Y. ZOU, X. PAN, D. XIA, AND G. WANG, *PI-line-based image reconstruction in helical cone-beam computed tomography with a variable pitch*, Medical Physics, 32 (2005), pp. 2639–2648.