Measures Of Concordance Of Polynomial Type

2004

Heather Edwards
University of Central Florida

MEASURES OF CONCORDANCE OF POLYNOMIAL TYPE

by

HEATHER H. EDWARDS
B.S. University of Florida, 1994
M.Ed. University of Florida, 1995
M.S. University of Central Florida, 2002

A dissertation submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Arts and Sciences
at the University of Central Florida
Orlando, Florida

Fall Term
2004

Major Professor: Michael D. Taylor

ABSTRACT

A measure of concordance, κ, is of polynomial type if and only if $\kappa(tA + (1-t)B)$ is a polynomial in t where A and B are 2-copulas. The degree of such a type of measure of concordance is simply the highest degree of the polynomial associated with κ.

In previous work [2][3], properties of measures of concordance preserving convex sums (equivalently measures of concordance of polynomial type degree one) were established; however, a characterization was not made. Here a characterization is made using approximations involving doubly stochastic matrices. Other representations are provided from this characterization leading naturally to two interpretations of degree one measures of concordance.

The existence of a family of measures of concordance of polynomial type having higher degree generated by a certain family of Borel measures on $(0,1)^{2n}$ is also shown. The representation of this family immediately leads to a probabilistic interpretation for all finite measures in d_n. Also, higher degree analogs of commonly known degree one measures of concordance are given as examples. For the degree 2 case in particular, we see there is no finite $\mu \in d_2$ generating Kendall's tau. Finally, another family of measures of concordance is given containing those generated by finite measures in d_2 as well as Kendall's tau.

Em yêu anh

ACKNOWLEDGMENTS

Dr. Taylor and Dr. Mikusiński, my gratitude goes to you both for getting me through my first experience in publishing a paper. The foundation for this dissertation would not exist without having the opportunity to co-author with you.

Dr. Taylor, my thanks goes to you for your undying patience and kindness. You have been an incredible mentor to me for many years and I am looking forward to continuing research with you after my graduation. Emma would also like to thank you for introducing her to limes, *The Space Child's Mother Goose*, and seminars on doubly stochastic measures.

The time and attention I received from my entire advisement committee, Dr. Taylor, Dr. Mikusiński, Dr. Li, and Dr. Schott is much appreciated and valued.

Dr. Mohapatra, thanks for being so caring and supportive.

Mehrdad, your notes on measures of concordance defined on doubly stochastic matrices were very helpful. Thank you.

June, my appreciation for answering my many LaTex questions.

Last but not least, I owe a huge acknowledgment to my family. Dad, thank you for your loving interest and concern in my education all the way from grade school to graduate school. You have no idea how your influence motivated me to get to this point. Mom, you are the most wonderful mother I could ever have. Robert, you continued to push me when I would get discouraged. Thank you. And of course, Emma, my sweet baby girl, your hugs and kisses mean everything to me.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

When considering two random variables it can be useful to know to what degree large values of the random variables correspond to each other. The stronger this correspondence is, the more concordant a random vector, (X, Y), is thought to be. Similarly, the stronger the correspondence is between large values of one random variable and small values of the other random variable, the more discordant a random vector, (X, Y), is thought to be. The concordance and discordance of (X, Y) is gauged by a measure of concordance and was axiomatically formulated by Scarsini in a way which is invariant under almost surely increasing transformations of X and Y [8].

As a result of Sklar's Theorem, for a continuous random vector, (X, Y), with joint distribution function, $F_{X,Y}$, there exists a unique 2-copula, C, such that

$$F(x, y) = C(F_X(x), F_Y(y)) \tag{1}$$

where F_X and F_Y are the distribution functions of X and Y respectively. Some common examples of this relationship are as follows. If Y is an almost surely increasing function of X, then $M(x, y) = \min(x, y)$ is the 2-copula satisfying (1). If Y is an almost surely decreasing function of X, then $W(x, y) = \max(x + y - 1, 0)$ is the 2-copula satisfying (1). Finally, if X and Y are independent, then $\Pi(x, y) = xy$ is the 2-copula satisfying (1). One can see by the role a 2-copula, C, plays in forming a joint distribution function that it induces a doubly stochastic measure, say μ_C. In other words for a Borel set, A, in $(0, 1)$ we have $\mu_C(A \times (0, 1)) = \mu_C((0, 1) \times A) = \lambda(A)$ where λ is the one-dimensional Lebesgue measure.

Suppose C is the 2-copula associated with the continuous random vector, (X, Y), in the manner set forth in (1). It can be shown for any almost surely increasing transformations, g_1 and g_2, that C is associated with $(g_1(X), g_2(Y))$ in the same way [6]. Otherwise stated, a 2-copula, C, associated with (X, Y) holds the information regarding the dependence between X and Y which is invariant under almost surely increasing transformations. Because of this, we may use C to determine the value for a measure of concordance instead of (X, Y).

A measure of concordance has many equivalent variations of its definition. The following definition is an adaptation of the one found in [6]. Here and in all that follows we will let Cop(2) denote the set of all 2-copulas.

Definition 1. Let $\kappa : \mathrm{Cop}(2) \to [-1, 1]$. If C is the 2-copula for the continuous random vector, (X, Y), then we shall also write $\kappa_{X,Y}$ for $\kappa(C)$. κ is a measure of concordance if the following conditions are satisfied.

1. $\kappa_{X,X} = 1$,
2. $\kappa_{-X,Y} = -\kappa_{X,Y}$,
3. $\kappa_{Y,X} = \kappa_{X,Y}$,
4. $\kappa(C_1) \leq \kappa(C_2)$ whenever $C_1 \leq C_2$ pointwise, and
5. $\kappa(C_n) \to \kappa(C)$ whenever $C_n \to C$ pointwise.

Definition 2. A measure of concordance, κ, is of *polynomial type* if for every choice of $A, B \in \mathrm{Cop}(2)$ the mapping $t \mapsto \kappa(tA + (1-t)B)$ is a polynomial in t for $t \in [0, 1]$.

Definition 3. The *degree* of a measure of concordance of polynomial type, κ, is defined as $\deg \kappa = \sup\{\deg \kappa(tA + (1-t)B) | A, B \in \mathrm{Cop}(2)\}$.

Table 1: Some Common Measures of Concordance [6]

Spearman's rho	$\rho(C) = 12\int_{(0,1)^2} Cd\Pi - 3$
Blomqvist's beta	$\beta(C) = 4C\left(\frac{1}{2},\frac{1}{2}\right) - 1$
Gini's measure of association	$\gamma(C) = 8\int_{(0,1)^2} Cd\left(\frac{M+W}{2}\right) - 2$
Kendall's tau	$\tau(C) = 4\int_{(0,1)^2} CdC - 1$

Referring to Table 1 one can see that Spearman's rho, Blomqvist's beta, and Gini's measure of association all are of polynomial type degree one. Also observe that Kendall's tau is of polynomial type degree two. In the second chapter we will give a characterization of degree one measures of concordance using approximations which involve the use of doubly stochastic matrices. The third chapter will use the characterization given in the second chapter to form other representations of degree one measures of concordance which lead naturally to some interesting interpretations. The fourth and last chapter gives a family of measures of concordance of polynomial type having higher degree. This family is generated by a particular set of Borel measures on $(0,1)^{2n}$ to be denoted as d_n. Then we place a special focus a family of functions defined on $(0,1)^4$ which in many cases generates measures of concordance of polynomial type degree 2.

2. CHARACTERIZING DEGREE ONE MEASURES OF CONCORDANCE

In this chapter we will examine the mapping

$$C \mapsto \int_{(0,1)^2} (C - \Pi) d\mu. \tag{2}$$

We wish to find what properties are necessary and sufficient for μ to possess so a characterization of degree one measures of concordance can be made. This is accomplished by defining any fixed measure of concordance as a sequence of linear functionals. By examining the limiting behavior (with the use of doubly stochastic matrices) of this sequence we see these functionals are each bounded and therefore may be extended. We then acquire an increasing sequence of measures via the Riesz Representation Theorem associated with the sequence of functionals. This sequence of measures converges to a measure which generates a degree one measure of concordance in the manner reflected in (2).

2.1 Defining Measures of Concordance on Other Spaces

Imagine any grid placed on $(0,1)^2$ where the mass in each cell of the grid is spread uniformly over each cell. If this is done in such a manner that induces a doubly stochastic measure then what results is called a checkerboard 2-copula [1]. In particular, if the grid is one of $n \times n$ squares then we can show there is a one-to-one correspondence between $n \times n$ checkerboard 2-copulas and $n \times n$ doubly stochastic matrices. Because there exists a mapping of the set of doubly stochastic matrices into Cop(2), we can adapt the properties of a measure of concordance so it may be defined on the set of doubly stochastic matrices. In fact, we may do so for any set having a one-to-one correspondence with the set of doubly stochastic matrices. In the case of this section we will give a definition for a measure of concordance on a particular translation of doubly stochastic matrices.

Let $\check{\text{C}}\text{op}(2^n)$ be the set of $2^n \times 2^n$ checkerboard 2-copulas and $\text{dsm}(2^n)$ be the set of $2^n \times 2^n$ doubly stochastic matrices. In order to allow these matrices and any matrices to follow in this paper to be compatible with the rectangular coordinate system in $(0,1)\times(0,1)$, the numbering of the rows will go from bottom to top. With this in mind, a bijection between these two sets is easily seen since for any $R = (r_{i,j}) \in \text{dsm}(2^n)$, $\gamma_{i,j} = 2^n r_{i,j}$ where $\gamma_{i,j}$ is the density in the cell, $\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] \times \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]$, for some $C \in \check{\text{C}}\text{op}(2^n)$. $\phi_n : \text{dsm}(2^n) \to \check{\text{C}}\text{op}(2^n)$ will be this bijection. Also, define $\theta_n : \text{Cop}(2) \to \check{\text{C}}\text{op}(2^n)$ such that $\theta_n(A)$ is the $2^n \times 2^n$ checkerboard approximation of A where $\theta_n(A)\left(\frac{i}{2^n}, \frac{j}{2^n}\right) = A\left(\frac{i}{2^n}, \frac{j}{2^n}\right)$ for $1 \le i, j \le 2^n$.

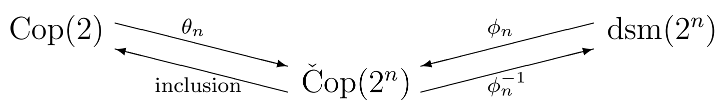

Figure 1: Mappings Between Cop(2), $\check{\text{C}}\text{op}(2^n)$, and $\text{dsm}(2^n)$

Recall that $M(x,y) = \min(x,y)$, $W(x,y) = \max(x+y-1, 0)$, and $\Pi(x,y) = xy$. For $M_n = (m_{i,j}) \in \text{dsm}(2^n)$ where $m_{i,j} = \delta_{ij}$ and δ_{ij} is Kronecker's delta, we have $\phi_n(M_n) = \theta_n(M)$. Similarly, for $W_n = (w_{i,j}) \in \text{dsm}(2^n)$ where $w_{i,j} = \delta_{i,2^n+1-j}$, we have $\phi_n(W_n) = \theta_n(W)$. Finally, for $P_n = (p_{i,j}) \in \text{dsm}(2^n)$ where $p_{i,j} = \frac{1}{2^n}$, we have $\phi_n(P_n) = \Pi$ for every n.

If (X, Y) is the random vector associated with some checkerboard 2-copula, $\phi_n(R)$, for some $R = (r_{i,j}) \in \text{dsm}(2^n)$ then it is easily seen that $\phi_n(RW_n)$ where $RW_n = (\hat{r}_{i,j})$ and

$\hat{r}_{i,j} = r_{i,2^n+1-j}$ is associated with $(-X, Y)$. Similarly, $\phi_n(R^T)$ is associated with (Y, X) where R^T is the transpose of R.

Given $C_1, C_2 \in \check{\text{C}}\text{op}(2^n)$, let $R_p = (r_{k,m}(p)) = \phi_n^{-1}(C_p)$ for $p = 1, 2$. If $C_1 \leq C_2$ pointwise, then we say that $R_1 \leq R_2$. Note that if $R_1 \leq R_2$, then it is equivalent that for every i and j we have $\sum_{k=1}^{i} \sum_{m=1}^{j} r_{k,m}(1) \leq \sum_{k=1}^{i} \sum_{m=1}^{j} r_{k,m}(2)$.

For any measure of concordance, $\kappa : \text{Cop}(2) \to [-1, 1]$ we may define a sequence $\{\kappa_n\}$ where $\kappa_n : \text{dsm}(2^n) \to [-1, 1]$ is defined as $\kappa_n = \kappa \circ \phi_n$. Let us reformulate Definition 1 so we may see what a measure of concordance is in context of $\text{dsm}(2^n)$.

Definition 4. Let $\kappa_n : \text{dsm}(2^n) \to [-1, 1]$ and $R, S \in \text{dsm}(2^n)$. $\{\kappa_n\}$ is a measure of concordance if the following hold:

1. $\kappa_n(M_n) \to 1$ as $n \to \infty$,
2. $\kappa_n(RW_n) = -\kappa_n(R)$,
3. $\kappa_n(R^T) = \kappa_n(R)$,
4. $\kappa_n(R) \leq \kappa_n(S)$ whenever $R \leq S$, and
5. $\kappa_n(R^N) \to \kappa_n(R)$ whenever $R^N \to R$.

Definition 5. A measure of concordance, $\kappa_n : \text{dsm}(2^n) \to [-1, 1]$, is of *polynomial type* if for every choice $A, B \in \text{dsm}(2^n)$ the mapping $t \mapsto \kappa_n(tA + (1-t)B)$ is a polynomial in t for $t \in [0, 1]$.

Definition 6. The *degree* of a measure of concordance of polynomial type, $\kappa_n : \text{dsm}(2^n) \to [-1, 1]$, is defined as $\deg \kappa_n = \sup\{\deg \kappa_n(tA + (1-t)B) | A, B \in \text{dsm}(2^n)\}$.

Let $T_n = \{A - P_n | A \in \text{dsm}(2^n)\}$. It is easily seen that the sum of the entries from any row or column for any $T \in T_n$ is zero; however, T_n is not a vector space. For instance, $2T_n + P_n \neq \text{dsm}(2^n)$. Z_n, the vector space made of all $2^n \times 2^n$ matrices whose each row and column sum to zero, contains T_n. More generally defined, in order to cover square matrices whose dimensions are not of powers of 2, let $\hat{Z}_n$ be the vector space of all $n \times n$ matrices whose each row and column sum to zero so that $\hat{Z}_{2^n} = Z_n$.

In the proof of the next lemma the following notation will be useful. Consider a $(q+1) \times (q+1)$ matrix. Separate the matrix into a 2×2 block matrix where the lower left block, B_1, has dimension $(q-1) \times (q-1)$ and B_2, B_3, and B_4 are the three remaining blocks. Also, let Υ_q be the set of $(q+1) \times (q+1)$ matrices whose entries in row $(q+1)$ and column $(q+1)$ are all zero. Define a linear transformation, $\upsilon_q : \Upsilon_q \to \mathbb{R}^{q \times q}$, where for $\upsilon_q(U) = (w_{i,j})$ and $U = (u_{i,j})$ we have $w_{i,j} = u_{i,j}$ for $1 \leq i, j \leq q$. What υ_q is essentially doing is "removing" row $(q+1)$ and column $(q+1)$ (which are both filled with zeros) so a $q \times q$ matrix remains.

Let $E^p_{i,j} = (e_{k,m})$ be a $p \times p$ matrix where

$$e_{k,m} = \begin{cases} 1, & k = i, m = j \text{ or } k = i+1, m = j+1 \\ -1, & k = i, m = j+1 \text{ or } k = i+1, m = j \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

for $1 \leq i, j \leq p-1$. The two following lemmas were formulated and proven by M. Khosravi [4].

Lemma 1. *$\{E^p_{i,j}\}$ for $1 \leq i, j \leq p-1$ form a basis for $\hat{Z}_p$.*

Proof. Since $\dim(\hat{Z}_p) = (p-1)^2$, it suffices to show the matrices $E^p_{i,j}$ are linearly independent for $1 \leq i, j \leq p-1$. We will do this inductively.

Since $\dim(\hat{Z}_2) = 1$, linear independence in this case is immediate. Assume $\{E^q_{i,j}\}$ are linearly independent for $1 \leq i, j \leq q-1$ and let

$$\sum_{i=1}^{q} \sum_{j=1}^{q} c_{i,j} E^{q+1}_{i,j} = 0 \tag{4}$$

or more conveniently for $\Lambda = \{(i,j) \,|\, i = q \text{ or } j = q\}$,

$$\sum_{i=1}^{q-1} \sum_{j=1}^{q-1} c_{i,j} E^{q+1}_{i,j} = - \sum_{(i,j) \in \Lambda} c_{i,j} E^{q+1}_{i,j}. \tag{5}$$

Observe the sum on the right hand side of (5) results in B_1 having all zero entries while the sum on the left hand side of (5) results in all entries of row $q+1$ and column $q+1$ being zero. Finally, since the sums in (5) are that of matrices whose each row and column sum are zero, this results in all entries of row q and column q being zero as well. Then applying υ_q to each side of (5) we get

$$\sum_{i=1}^{q-1} \sum_{j=1}^{q-1} c_{i,j} \upsilon_q(E^{q+1}_{i,j}) = 0_{q \times q} \tag{6}$$

where $0_{q\times q}$ is the $q \times q$ zero matrix. Consequently, $c_{i,j} = 0$ for $1 \leq i, j \leq q-1$ by the induction hypothesis. Furthermore, since B_2, B_3, and B_4 have only zero entries, $c_{q,1} = 0$, which in turn causes $c_{q,2} = 0$ and so on such that $c_{i,j} = 0$ for all $(i,j) \in \Lambda$. □

By Lemma 1, for any $Z = (z_{k,m}) \in \hat{Z}_p$ we can write

$$Z = \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} \alpha^p_{i,j}(Z) E^p_{i,j}. \tag{7}$$

In the proof of the next lemma the following equation will be useful and can be verified by using the matrices in Lemma 1 which form a basis of $\hat{Z}_p$. For $Z = (z_{i,j}) \in \hat{Z}_p$,

$$z_{i,j} = \begin{cases} \alpha^p_{i,j}(Z), & (i,j) = (1,1) \\ \alpha^p_{i,j}(Z) - \alpha^p_{i,j-1}(Z), & i = 1, 1 < j < p \\ \alpha^p_{i,j}(Z) - \alpha^p_{i-1,j}(Z), & j = 1, 1 < i < p \\ \alpha^p_{i,j}(Z) - \alpha^p_{i-1,j}(Z) - \alpha^p_{i,j-1}(Z) + \alpha^p_{i-1,j-1}(Z), & 1 < i,j < p. \end{cases} \tag{8}$$

Lemma 2. *For $Z \in \hat{Z}_p$, if $Z = \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} \alpha^p_{i,j}(Z) E^p_{i,j}$, then*

$$\alpha^p_{i,j}(Z) = \sum_{k=1}^{i} \sum_{m=1}^{j} z_{k,m}. \tag{9}$$

Proof. By (8) $\alpha^p_{1,1}(Z) = z_{1,1}$. Suppose for all $i, j \leq q$ where $q < p-1$ that $\alpha^p_{i,j}(Z) = \sum_{k=1}^{i} \sum_{m=1}^{j} z_{k,m}$. Again by (8),

$$z_{1,q+1} = \alpha^p_{1,q+1}(Z) - \alpha^p_{1,q}(Z). \tag{10}$$

So the induction hypothesis gives us

$$\alpha^p_{1,q+1}(Z) = \sum_{i=1}^{1} \sum_{j=1}^{q+1} z_{i,j}. \tag{11}$$

Continuing in this fashion,

$$z_{2,q+1} = \alpha^p_{2,q+1}(Z) - \alpha^p_{1,q+1}(Z) - \alpha^p_{2,q}(Z) + \alpha^p_{1,q}(Z), \tag{12}$$

which by (11) and the induction hypothesis gives us

$$\alpha^p_{2,q+1}(Z) = \sum_{i=1}^{2} \sum_{j=1}^{q+1} z_{i,j}. \tag{13}$$

Eventually we get

$$\alpha^p_{i,q+1}(Z) = \sum_{k=1}^{i} \sum_{m=1}^{q+1} z_{k,m} \tag{14}$$

for $1 \leq i \leq q$. It is similarly attained that

$$\alpha^p_{q+1,j}(Z) = \sum_{k=1}^{q+1} \sum_{m=1}^{j} z_{k,m} \tag{15}$$

for $1 \leq j \leq q$. Then (14) and (15) with (8) gives us

$$\alpha^p_{q+1,q+1}(Z) = \sum_{k=1}^{q+1} \sum_{m=1}^{q+1} z_{k,m} \tag{16}$$

to complete our result. □

Definition 7. $\overline{\kappa}_n : T_n \to [-1,1]$ is a measure of concordance if there exists a measure of concordance $\kappa_n : \text{dsm}(2^n) \to [-1,1]$ such that for every $Q \in T_n$, $\overline{\kappa}_n(Q) = \kappa_n(Q + P_n)$.

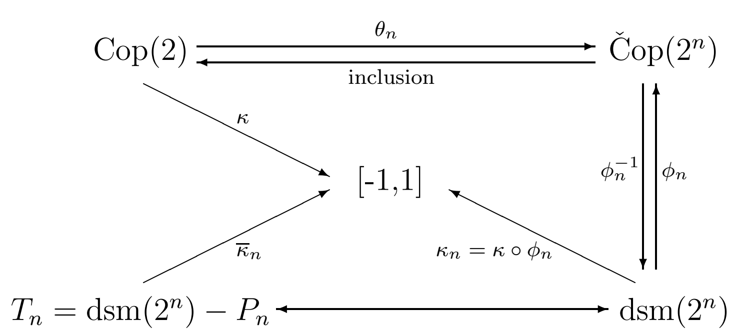

Figure 2: Mappings of Redefined Measures of Concordance

Definition 8. A measure of concordance, $\overline{\kappa}_n : T_n \to [-1,1]$, is of *polynomial type* if for every choice of $A, B \in T_n$ the mapping $t \mapsto \overline{\kappa}_n(tA + (1-t)B)$ is a polynomial in t for $t \in [0,1]$.

Definition 9. The *degree* of a measure of concordance of polynomial type, $\overline{\kappa}_n : T_n \to [-1,1]$, is defined as $\deg \overline{\kappa}_n = \sup\{\deg \overline{\kappa}_n(tA + (1-t)B) | A, B \in T_n\}$.

Note that for any $\overline{\kappa}_n : T_n \to [-1,1]$ of polynomial type degree one, $\overline{\kappa}_n$ can be extended from T_n to Z_n since $T_n \subset Z_n$ and Z_n is of finite dimension.

Lemma 3. $\overline{\kappa}_n : T_n \to [-1,1]$ *is a measure of concordance of polynomial type degree one if and only if there exists* $\{\beta_{i,j}^n\}$ *for* $1 \leq i, j < 2^n$ *such that the following hold:*

1. $\overline{\kappa}_n(Q) = \sum_{i=1}^{2^n-1} \sum_{j=1}^{2^n-1} \alpha_{i,j}^{2^n}(Q) \beta_{i,j}^n$ *for all* $Q \in T_n$,

2. $\overline{\kappa}_n(M_n - P_n) \to 1$ *as* $n \to \infty$,

3. $\beta_{i,j}^n = \beta_{j,i}^n = \beta_{i,2^n-j}^n$, *and*

4. $\beta_{i,j}^n \geq 0$.

Proof. Let us assume $\overline{\kappa}_n$ is a measure of concordance of polynomial type degree one on T_n and $Q \in T_n$. By Lemmas 1 and 2, Q can be written as

$$Q = \sum_{i=1}^{2^n-1} \sum_{j=1}^{2^n-1} \alpha_{i,j}^{2^n}(Q) E_{i,j}^{2^n} \tag{17}$$

and since $\overline{\kappa}_n$ is of degree one and is therefore linear,

$$\overline{\kappa}_n(Q) = \sum_{i=1}^{2^n-1} \sum_{j=1}^{2^n-1} \alpha_{i,j}^{2^n}(Q) \overline{\kappa}_n(E_{i,j}^{2^n}) \tag{18}$$

is obtained. Letting $\beta_{i,j}^n = \overline{\kappa}_n(E_{i,j}^{2^n})$ we have

$$\overline{\kappa}_n(Q) = \sum_{i=1}^{2^n-1} \sum_{j=1}^{2^n-1} \alpha_{i,j}^{2^n}(Q) \beta_{i,j}^n. \tag{19}$$

By Definition 4, $\overline{\kappa}_n(M_n - P_n) = \kappa_n(M_n) \to 1$ as $n \to \infty$.

Note that $\frac{E^{2^n}_{i,j}}{2^n} \in T_n$. Since

$$\begin{aligned} \frac{\beta^n_{i,j}}{2^n} = \frac{\overline{\kappa}_n(E^{2^n}_{i,j})}{2^n} &= \kappa_n\left(\frac{E^{2^n}_{i,j}}{2^n} + P_n\right) \\ &= -\kappa_n\left(\left(\frac{E^{2^n}_{i,j}}{2^n} + P_n\right)W_n\right) = \frac{-\overline{\kappa}_n(-E^{2^n}_{i,2^n-j})}{2^n} = \frac{\beta^n_{i,2^n-j}}{2^n}, \end{aligned} \tag{20}$$

it is easily seen that $\beta^n_{i,j} = \beta^n_{i,2^n-j}$. Similarly,

$$\frac{\beta^n_{i,j}}{2^n} = \frac{\overline{\kappa}_n(E^{2^n}_{i,j})}{2^n} = \kappa_n\left(\frac{E^{2^n}_{i,j}}{2^n} + P_n\right) = \kappa_n\left(\left(\frac{E^{2^n}_{i,j}}{2^n} + P_n\right)^T\right) = \frac{\overline{\kappa}_n(E^{2^n}_{j,i})}{2^n} = \frac{\beta^n_{j,i}}{2^n} \tag{21}$$

gives $\beta^n_{i,j} = \beta^n_{j,i}$.

Finally, noting that $\frac{E^{2^n}_{i,j}}{2^n} \geq 0_{2^n \times 2^n}$ where $0_{2^n \times 2^n}$ is the $2^n \times 2^n$ zero matrix, we have by applying $\overline{\kappa}_n$ to the previous inequality $\beta^n_{i,j} \geq 0$.

Now let us assume that the four aforementioned properties of this lemma hold. Recall for any $R \in \mathrm{dsm}(2^n)$ we have $\kappa_n(R) = \overline{\kappa}_n(R - P_n)$.

Clearly by the form given in the first property, $\overline{\kappa}_n$ is of degree one.

$\kappa_n(M_n) = \overline{\kappa}_n(M_n - P_n) \to 1$ as $n \to \infty$.

Since $\beta^n_{i,j} = \beta^n_{j,i} = \beta^n_{i,2^n-j}$ it is easily seen using changes of index that $\kappa_n(RW_n) = -\kappa_n(R)$ and $\kappa_n(R^T) = \kappa_n(R)$ for any $R \in \mathrm{dsm}(2^n)$.

Because $\beta^n_{i,j} \geq 0$, it is easily seen by its form that $\overline{\kappa}_n$ is monotone.

Finally, since $\overline{\kappa}_n$ is simply a linear combination, we know that $\overline{\kappa}_n$ is continuous. □

In order to compare $T \in T_m$ to $\hat{T} \in T_n$ for $m < n$ we will say $T \equiv \hat{T}$ if $\phi_m(T + P_m) = \phi_n(\hat{T} + P_n)$. This idea can be generalized from elements of T_n to elements of Z_n for any n. Consider in particular $\hat{T} = (\hat{t}_{i,j}) \in T_n$ and $T = (t_{k,l}) \in T_{n+1}$ where $T \equiv \hat{T}$. Since

$\phi_n(\hat{T} + P_n) = \phi_{n+1}(T + P_{n+1})$, then for every i, j such that $1 \le i, j < 2^n$ we have

$$\frac{\alpha_{i,j}^{2^n}(\hat{T})}{2^n} + \frac{ij}{2^{2n}} = \frac{\alpha_{2i,2j}^{2^{n+1}}(T)}{2^{n+1}} + \frac{4ij}{2^{2(n+1)}}.$$

Therefore, $2\alpha_{i,j}^{2^n}(\hat{T}) = \alpha_{2i,2j}^{2^{n+1}}(T)$ whenever $1 \le i, j < 2^n$. Furthermore, since $\phi_{n+1}(T+P_{n+1}) \in \check{\mathrm{C}}\mathrm{op}(2^n)$ we have

$$t_{2i-1,2j-1} = t_{2i-1,2j} = t_{2i,2j-1} = t_{2i,2j}$$

for $1 \le i, j < 2^n$. With this in mind straightforward calculations give us

$$\begin{aligned} E_{i,j}^{2^n} \equiv \frac{1}{2}E_{2i-1,2j-1}^{2^{n+1}} + E_{2i-1,2j}^{2^{n+1}} + \frac{1}{2}E_{2i-1,2j+1}^{2^{n+1}} + E_{2i,2j-1}^{2^{n+1}} + 2E_{2i,2j}^{2^{n+1}} + E_{2i,2j+1}^{2^{n+1}} \\ + \frac{1}{2}E_{2i+1,2j-1}^{2^{n+1}} + E_{2i+1,2j}^{2^{n+1}} + \frac{1}{2}E_{2i+1,2j+1}^{2^{n+1}}. \end{aligned} \tag{22}$$

Lemma 4.

$$\begin{aligned} \beta_{i,j}^{n} = \frac{1}{2}\beta_{2i-1,2j-1}^{n+1} + \beta_{2i-1,2j}^{n+1} + \frac{1}{2}\beta_{2i-1,2j+1}^{n+1} + \beta_{2i,2j-1}^{n+1} + 2\beta_{2i,2j}^{n+1} + \beta_{2i,2j+1}^{n+1} \\ + \frac{1}{2}\beta_{2i+1,2j-1}^{n+1} + \beta_{2i+1,2j}^{n+1} + \frac{1}{2}\beta_{2i+1,2j+1}^{n+1}. \end{aligned}$$

Proof. This is immediately seen by applying any degree one measure of concordance, $\overline{\kappa}_n : T_n \to [-1, 1]$, to (22). □

Let us define a measure, μ_n, on the Borel sets of $(0,1)^2$, denoted $\mathcal{B}((0,1)^2)$, where a point mass of $2^n\beta_{i,j}^n$ is placed at $(\frac{j}{2^n}, \frac{i}{2^n})$ for every i and j such that $1 \le i, j \le 2^{n-1}$.

Here and in all that follows we will use the subsequent notation. Let

$$L_n = \left\{ \left(\frac{i}{2^n}, \frac{j}{2^n}\right) \mid 1 \le i, j \le 2^n - 1 \right\},$$

$$B_n = \{[x_1, y_1] \times [x_2, y_2] | \, (x_1, x_2), (y_1, y_2) \in L_n \text{ and } x_i \le y_i \text{ for } i = 1, 2\},$$

$$L = \cup_{i=1}^{\infty} L_n, \text{ and}$$

$$B = \{[x_1, y_1] \times [x_2, y_2] | \, (x_1, x_2), (y_1, y_2) \in L \text{ and } x_i \le y_i \text{ for } i = 1, 2\}.$$

Notice that B_n and B include degenerate rectangles.

Define $\hat{n} : B \to \mathbb{N}$ where $\hat{n}(H) = \min\{n | H \in B_n\}$.

Lemma 5. *For any $H \in B$, $\{\mu_n(H)\}$ is a convergent sequence in $\mathbb{R}$. Moreover, $\mu_n(H) \geq \mu_{n+1}(H)$ for all $n \geq \hat{n}(H)$.*

Proof. Fix H. To ease notation in this proof, we will write $\hat{n}$ rather than $\hat{n}(H)$. $H = \left[\frac{i}{2^{\hat{n}}}, \frac{j}{2^{\hat{n}}}\right] \times \left[\frac{k}{2^{\hat{n}}}, \frac{m}{2^{\hat{n}}}\right]$ for $1 \leq i \leq j \leq 2^{\hat{n}} - 1$ and $1 \leq k \leq m \leq 2^{\hat{n}} - 1$. Notice for any $n \geq \hat{n}$, $H = \left[\frac{2^{n-\hat{n}} i}{2^n}, \frac{2^{n-\hat{n}} j}{2^n}\right] \times \left[\frac{2^{n-\hat{n}} k}{2^n}, \frac{2^{n-\hat{n}} m}{2^n}\right]$. Straightforward calculations yield for $n \geq \hat{n}(H)$ that

$$\mu_n(H) = 2^n \sum_{q=2^{n-\hat{n}} i}^{2^{n-\hat{n}} j} \sum_{p=2^{n-\hat{n}} k}^{2^{n-\hat{n}} m} \beta^n_{p,q} \tag{23}$$

and

$$\mu_{n+1}(H) = 2^{n+1} \sum_{q=2^{n+1-\hat{n}} i}^{2^{n+1-\hat{n}} j} \sum_{p=2^{n+1-\hat{n}} k}^{2^{n+1-\hat{n}} m} \beta^{n+1}_{p,q}. \tag{24}$$

By Lemma 4 we know

$$\begin{aligned} \mu_n(H) = 2^{n+1} & \sum_{q=2^{n+1-\hat{n}} i}^{2^{n+1-\hat{n}} j} \sum_{p=2^{n+1-\hat{n}} k}^{2^{n+1-\hat{n}} m} \beta^{n+1}_{p,q} \\ & + 2^n \sum_{p=2^{n+1-\hat{n}} k}^{2^{n+1-\hat{n}} m} \left(\beta^{n+1}_{p,2^{n+1-\hat{n}} i-1} + \beta^{n+1}_{p,2^{n+1-\hat{n}} j+1}\right) \\ & + 2^n \sum_{q=2^{n+1-\hat{n}} i}^{2^{n+1-\hat{n}} j} \left(\beta^{n+1}_{2^{n+1-\hat{n}} k-1,q} + \beta^{n+1}_{2^{n+1-\hat{n}} m+1,q}\right) \\ & + 2^{n-1}(\beta^{n+1}_{2^{n+1-\hat{n}} k-1, w^{n+1-\hat{n}} i-1} + \beta^{n+1}_{2^{n+1-\hat{n}} k-1, 2^{n+1-\hat{n}} j+1} \\ & + \beta^{n+1}_{2^{n+1-\hat{n}} m+1, 2^{n+1-\hat{n}} i-1} + \beta^{n+1}_{2^{n+1-\hat{n}} m+1, 2^{n+1-\hat{n}} j+1}). \end{aligned} \tag{25}$$

By (23), (24), (25), and Lemma 3 we see that $\mu_n(H) \geq \mu_{n+1}(H)$ for all $n \geq \hat{n}$. Finally, since $\mu_n(H) \geq 0$ for all n, we know that $\{\mu_n(H)\}$ is convergent in $\mathbb{R}$. We will call this limiting value $\mu_0(H)$. □

2.2 Construction of the Measure, μ

We will construct a measure, μ, from $\{\mu_n\}$. Remember that $\{\mu_n\}$ is a sequence of measures where μ_n is associated with a degree one measure of concordance κ.

Let $R_n = [\frac{1}{2^n}, \frac{2^n-1}{2^n}]^2$. We define

$$\mathrm{Cop}_n(2) = \{C \in \mathrm{Cop}(2) | (C - \Pi)(x,y) = 0 \text{ for all } (x,y) \notin R_n\}. \tag{26}$$

Notice if $\mathcal{E}_n = (\mathrm{Cop}_n(2) - \Pi)$ then $\mathcal{A}_n = \{A|\ A = B|_{R_n} \text{ for some } B \in \mathcal{E}_n\}$ is a convex, compact subset of $\mathcal{C}(R_n)$ (the space of continuous functions defined on R_n) with respect to $||\cdot||_\infty$. Also, we define the linear functional, $J_n : \mathcal{A}_n \to [-1,1]$, to be $J_n(A) = \kappa(B + \Pi)$ where B is the unique element in $\mathcal{E}_n$ such that $A = B|_{R_n}$.

The reader might find it helpful to refer back to Figure 2. Choosing any $C \in \mathrm{Cop}_n(2)$ we have $\theta_{n+k}(C) \in (\check{\mathrm{C}}\mathrm{op}(2^{n+k}) \cap \mathrm{Cop}_n(2))$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned}
|\kappa(\theta_{n+k}(C))| &= |\overline{\kappa}_{n+k}(\phi_{n+k}^{-1}(\theta_{n+k}(C)) - P_{n+k})| \\
&= \sum_{i=1}^{2^{n+k}-1} \sum_{j=1}^{2^{n+k}-1} \alpha_{i,j}^{2^{n+k}} (\phi_{n+k}^{-1}(\theta_{n+k}(C)) - P_{n+k}) \beta_{i,j}^{n+k} \\
&= \sum_{i=1}^{2^{n+k}-1} \sum_{j=1}^{2^{n+k}-1} (\theta_{n+k}(C) - \Pi) \left(\frac{j}{2^{n+k}}, \frac{i}{2^{n+k}} \right) 2^{n+k} \beta_{i,j}^{n+k} \\
&= \int_{(0,1)^2} (C - \Pi) d\mu_{n+k} = \int_{R_n} (C - \Pi) d\mu_{n+k} \le ||C - \Pi||_\infty \mu_{n+k}(R_n) \\
&= ||\,(C - \Pi)|_{R_n}\,||_\infty \mu_{n+k}(R_n).
\end{aligned} \tag{27}$$

Letting $k \to \infty$, we have by Lemma 5 that

$$J_n((C - \Pi)|_{R_n}) = \kappa(C) < ||\,(C - \Pi)|_{R_n}\,||_\infty (\mu_0(R_n) + 1). \tag{28}$$

Therefore, J_n is a bounded linear functional on $\mathcal{A}_n$ and can be extended to $\mathcal{C}(R_n)$.

By the Riesz Representation Theorem, for every n there exists a Borel measure, $\bar{\mu}_n$, on R_n such that

$$J_n(f) = \int_{R_n} f d\bar{\mu}_n \tag{29}$$

for every $f \in \mathcal{C}(R_n)$.

Choose any closed rectangle (possibly degenerate), say $R = [x_1, x_2] \times [y_1, y_2]$. There exists an N such that $R \subset R_N$. Observe the density distribution in following figure. We will define a 2-copula $C_{R,\delta}$ accordingly for any positive $\delta < \frac{1}{2^{N+1}}$. From Figure 3 we see that

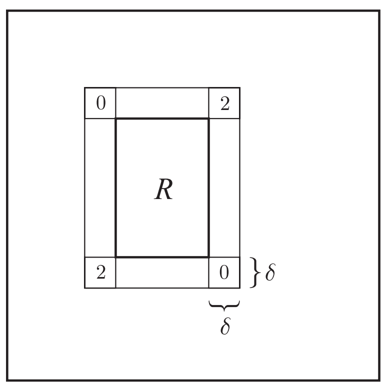

Figure 3: The Density Distribution of $C_{R,\delta}$ [3]

$C_{R,\delta}$ has density 2 in $(((x_1 - \delta, x_1) \times (y_1 - \delta, y_1)) \cup ((x_2, x_2 + \delta) \times (y_2, y_2 + \delta)))$ and density 0 in $(((x_1 - \delta, x_1) \times (y_2, y_2 + \delta)) \cup ((x_2, x_2 + \delta) \times (y_1 - \delta, y_1)))$. Otherwise, $C_{R,\delta}$ has density 1.

For positive $\delta < \frac{1}{2^{N+1}}$ the following are true. $0 \le \frac{C_{R,\delta} - \Pi}{\delta^2} \le 1$. $C_{R,\delta} \in (\mathrm{Cop}_n(2) \cap \mathcal{C}(R_n))$ for all $n > N$. Lastly, $\frac{C_{R,\delta} - \Pi}{\delta^2} \downarrow \chi_R$. Since $J_n\left(\frac{C_{R,\delta} - \Pi}{\delta^2}\right) = J_{n+1}\left(\frac{C_{R,\delta} - \Pi}{\delta^2}\right)$ for all $n > N$

and by the dominated convergence theorem $\lim_{\delta\to 0} J_n\left(\frac{C_{R,\delta}-\Pi}{\delta^2}\right) = \int_{R_n} \chi_R d\bar{\mu}_n = \bar{\mu}_n(R)$, we have $\bar{\mu}_n(S) = \bar{\mu}_{n+1}(S)$ for every $S \in \mathcal{B}(R_n)$.

Finally, if for every n we extend $\bar{\mu}_n$ so that $\bar{\mu}_n((0,1)^2 - R_n) = 0$ we now have an increasing sequence of Borel measures on $(0,1)^2$. Therefore, $\{\bar{\mu}_n\}$ converges to some Borel measure on $(0,1)^2$ which we shall call μ here and in all that follows.

2.3 Establishing $\int_{(0,1)^2}(M-\Pi)d\bar{\mu}_n \to \int_{(0,1)^2}(M-\Pi)d\mu$

Now that we know $\{\bar{\mu}_n\}$ converges setwise to the measure, μ, we must now establish that $\lim \int_{(0,1)^2}(M-\Pi)d\bar{\mu}_n = \int_{(0,1)^2}(M-\Pi)d\mu$ in order to satisfy the continuity property for a measure of concordance. While such a result is immediate when $\mu((0,1)^2) < \infty$, the case where $\mu((0,1)^2) = \infty$ is not so obvious.

Lemma 6. *$\lim_{n\to\infty} \int_{(0,1)^2}(M-\Pi)d\bar{\mu}_n = \int_{(0,1)^2}(M-\Pi)d\mu$.*

Proof. Since $\bar{\mu}_n \le \mu$ setwise we have $\int_{(0,1)^2}(M-\Pi)d\bar{\mu}_n \le \int_{(0,1)^2}(M-\Pi)d\mu$. However, a generalization of Fatou's Lemma given in [7] leaves us with $\int_{(0,1)^2}(M-\Pi)d\mu \le \liminf \int_{(0,1)^2}(M-\Pi)d\bar{\mu}_n$. □

2.4 Proof of the Characterization

Here and in all that follows we will let $\mathcal{M}(1)$ denote the set of all measures of concordance defined on Cop(2) that are of polynomial type degree one. This section establishes a characterization of elements of $\mathcal{M}(1)$. The representation given in this characterization is generated by a collection of Borel measures on $(0,1)^2$ satisfying certain properties which are said to be of *degree one form*. Also in this section a one-to-one correspondence between $\mathcal{M}(1)$ and these generating measures of degree one form is shown.

A property that measures of degree one form share is a certain invariance with regard to the group of symmetries on $(0,1)^2$. This group of symmetries is standardly denoted D_4. Recall that $\mathcal{B}((0,1)^2)$ denotes the collection of Borel sets in $(0,1)^2$.

Definition 10. A measure, μ, is D_4*-invariant* if and only if for every $\xi \in D_4$ and every $S \in \mathcal{B}((0,1)^2)$ we have $\mu(\xi(S)) = \mu(S)$.

Since h (the reflection about $x = \frac{1}{2}$) and T (the reflection about $y = x$) generate D_4 it will always suffice to check invariance with respect to these operations in order to confirm whether a measure is D_4-invariant.

The following three lemmas are results shown in an earlier paper and will be given here without their proofs. (See [3].) If μ_C is the doubly stochastic measure induced by the 2-copula, C, then $(\mu_C \circ h)$ and $(\mu_C \circ T)$ are the doubly stochastic measures induced by C^h and C^T respectively.

Lemma 7. *A Borel measure, μ, on $(0,1)^2$ is D_4-invariant if and only if*

$$\int_{(0,1)^2} (C^h - \Pi) d\mu = -\int_{(0,1)^2} (C - \Pi) d\mu \tag{30}$$

and

$$\int_{(0,1)^2} (C^T - \Pi) d\mu = \int_{(0,1)^2} (C - \Pi) d\mu \tag{31}$$

for all $C \in \mathrm{Cop}(2)$.

Lemma 8. *If μ and ν are regular Borel measures on $(0,1)^2$ such that for every $C \in \mathrm{Cop}(2)$*

$$\int_{(0,1)^2} (C - \Pi) d\mu = \int_{(0,1)^2} (C - \Pi) d\nu, \tag{32}$$

then $\mu = \nu$.

Lemma 9. *Let μ be a Borel measure on $(0,1)^2$.*

$$\kappa(C) = \gamma \int_{(0,1)^2} (C - \Pi) d\mu \tag{33}$$

is a measure of concordance for some $\gamma > 0$ if and only if μ is D_4-invariant, $0 < \int_{(0,1)^2}(M - \Pi) d\mu < \infty$, and $\gamma = (\int_{(0,1)^2}(M - \Pi)\ d\mu)^{-1}$.

Definition 11. A Borel measure, μ, is of *degree one form* if and only if it is D_4-invariant and $\int_{(0,1)^2}(M - \Pi) d\mu = 1$.

Theorem 1. *$\kappa \in \mathcal{M}(1)$ if and only if there exists a unique measure, μ, of degree one form where*

$$\kappa(C) = \int_{(0,1)^2} (C - \Pi) d\mu \tag{34}$$

for all $C \in \mathrm{Cop}(2)$.

Proof. Choose $\kappa \in \mathcal{M}(1)$. For any $C \in \mathrm{Cop}(2)$, we can form a sequence, $\{C_n\}$, such that $C_n \in \mathrm{Cop}_n(2)$ where $C_n \to C$. By Lemma 6 and a generalization of the Lebesgue Convergence Theorem given in [7],

$$\lim_{n\to\infty} \kappa(C_n) = \lim_{n\to\infty} J_n((C - \Pi)|_{R_n}) = \lim_{n\to\infty} \int_{(0,1)^2} (C_n - \Pi) d\bar{\mu}_n = \int_{(0,1)^2} (C - \Pi) d\mu. \tag{35}$$

Noting that for a measure of concordance, $\kappa(C_n) \to \kappa(C)$ whenever $C_n \to C$, we have $\kappa(C) = \int_{(0,1)^2}(C - \Pi)d\mu$ and in particular $\int_{(0,1)^2}(M - \Pi)d\mu = 1$. Furthermore, (30) and (31) are satisfied from Lemma 7 since $\kappa \in \mathcal{M}(1)$. Therefore, μ is D_4-invariant and must be of degree one form. Finally, μ is unique by Lemma 8.

If we assume that $\kappa(C) = \int_{(0,1)^2}(C - \Pi)d\mu$ where μ is of degree one form, then $\kappa \in \mathcal{M}(1)$ by Lemma 9. □

When we write $\kappa(C) = \int_{(0,1)^2}(C - \Pi)d\mu$ for $\kappa \in \mathcal{M}(1)$ we will say μ is the unique measure of degree one form generating κ.

3. OTHER REPRESENTATIONS OF $\mathcal{M}(1)$

This chapter establishes another characterization of elements of $\mathcal{M}(1)$. Interpretations of a probabilistic and measure-theoretic nature via this characterization are given. In addition, a focus is placed on what types of measures and functions generate the forms associated with these interpretations.

3.1 Another Measure Generated Characterization of $\mathcal{M}(1)$

We can write $D_4 = \{e, r, r^2, r^3, h, hr, hr^2, hr^3\}$ as the group of symmetries on $(0,1)^2$ where h is the reflection about $x = \frac{1}{2}$ and r is a $90°$ counterclockwise rotation. For a 2-copula, C, inducing the doubly stochastic measure, μ_C, and associated with the random vector, (X, Y), we define C^ξ as the 2-copula inducing the doubly stochastic measure, μ_{C^ξ}, where $\mu_{C^\xi}(S) = \mu_C(\xi(S))$ for all $S \in \mathcal{B}((0,1)^2)$. Each C^ξ is the 2-copula of a random vector of the form $(\pm X, \pm Y)$ or $(\pm Y, \pm X)$. The table to follow shortly describes this relationship more explicitly.

Table 2: Symmetries of 2-Copulas on $(0,1)^2$ and Associated Random Vectors

D_4	2-copula	random vector
e	C	(X, Y)
r	C^r	$(-Y, X)$
r^2	C^{r^2}	$(-X, -Y)$
r^3	C^{r^3}	$(Y, -X)$
h	C^h	$(-X, Y)$
hr	C^{hr}	(Y, X)
hr^2	C^{hr^2}	$(X, -Y)$
hr^3	C^{hr^3}	$(-Y, -X)$

When observing the table one might recognize how the symmetry properties of Definition 1 lend themselves to manipulating the representation of a degree one measure of concordance. In order to ease notation let us set forth another definition.

Definition 12. For each $\xi \in D_4$, the *order* of ξ is defined as

$$|\xi| = \min\{n|\, \xi = \xi_1\xi_2...\xi_n \text{ where } \xi_i = h, r \text{ for all } i = 1, 2, ..., n\}. \tag{36}$$

Recall in Definition 1 it is stated that $\kappa_{-X,Y} = -\kappa_{X,Y}$ and $\kappa_{Y,X} = \kappa_{X,Y}$. Therefore we may write $\kappa(C^\xi) = (-1)^{|\xi|}\kappa(C)$. With this in mind and being prompted by some helpful notes [11] we have the following theorem.

Theorem 2. $\kappa \in \mathcal{M}(1)$ *if and only if there exists a unique measure,* μ*, of degree one form such that*

$$\kappa(C) = \frac{1}{8}\int_{(0,1)^2} \left\{\sum_{\xi \in D_4} (-1)^{|\xi|} C^\xi \right\} d\mu \tag{37}$$

for all $C \in \mathrm{Cop}(2)$*.*

Proof. Choose $\kappa \in \mathcal{M}(1)$. By Theorem 1 there exists a unique measure of degree one form where

$$\begin{aligned}\kappa(C) = \frac{1}{8}\sum_{\xi \in D_4}(-1)^{|\xi|}\kappa(C^\xi) &= \frac{1}{8}\sum_{\xi \in D_4}\left\{(-1)^{|\xi|}\left(\int_{(0,1)^2}(C^\xi - \Pi)d\mu\right)\right\} \\ &= \frac{1}{8}\int_{(0,1)^2}\sum_{\xi \in D_4}\left\{(-1)^{|\xi|}C^\xi\right\} d\mu.\end{aligned} \tag{38}$$

Now let us assume that $\kappa(C) = \frac{1}{8}\int_{(0,1)^2}\sum_{\xi \in D_4}\left\{(-1)^{|\xi|}C^\xi\right\} d\mu$ where μ is of degree one form. By Theorem 1, $C \mapsto \int_{(0,1)^2}(C - \Pi)d\mu$ is some degree one measure of concordance, say κ_0. Repeating the calculations in (38) on κ_0 gives us $\kappa = \kappa_0$. So $\kappa \in \mathcal{M}(1)$. □

Theorem 3. *Let $\kappa : \mathrm{Cop}(2) \to [-1, 1]$. The following are equivalent.*

1. *κ is a measure of concordance of polynomial type degree one.*

2. *There exists a unique measure, μ, of degree one form such that*

$$\kappa(C) = \int_{(0,1)^2} (C - \Pi) d\mu \tag{39}$$

for all $C \in \mathrm{Cop}(2)$.

3. *There exists a unique measure, μ, of degree one form such that*

$$\kappa(C) = \frac{1}{8} \int_{(0,1)^2} \left\{ \sum_{\xi \in D_4} (-1)^{|\xi|} C^{\xi} \right\} d\mu \tag{40}$$

for all $C \in \mathrm{Cop}(2)$.

Proof. This is immediate by Theorems 1 and 2. □

For any $\kappa \in \mathcal{M}(1)$, if there exists ν where

$$\kappa(C) = \frac{1}{8} \int_{(0,1)^2} \sum_{\xi \in D_4} \left\{ (-1)^{|\xi|} C^{\xi} \right\} d\nu \tag{41}$$

for every $C \in \mathrm{Cop}(2)$ let us also say that ν generates κ. This suggests the measure generating κ is not unique in general when using the representation given in Theorem 2. For instance, when considering Gini's measure of association we have

$$\gamma(C) = \int_{(0,1)^2} \sum_{\xi \in D_4} (-1)^{|\xi|} C^{\xi} d\left(\frac{M + W}{2}\right) = \int_{(0,1)^2} \sum_{\xi \in D_4} (-1)^{|\xi|} C^{\xi} dM. \tag{42}$$

Therefore, $8\mu_{\frac{M+W}{2}}$ and $8\mu_M$ generate γ. However, by Theorem 1 exactly one generator is D_4-invariant for a fixed $\kappa \in \mathcal{M}(1)$. In here and in all that follows let

$$V(\kappa) = \{\nu | \nu \text{ generates } \kappa\}. \tag{43}$$

Remark 1. $V(\kappa)$ is convex. Let $\mu, \nu \in V(\kappa)$ and define

$$S(C,\mu) = \frac{1}{8}\int_{(0,1)^2} \sum_{\xi \in D_4} \left\{(-1)^{|\xi|} C^{\xi}\right\} d\mu. \tag{44}$$

Choose any $t \in [0,1]$.

$$S(C, t\mu + (1-t)\nu) = tS(C,\mu) + (1-t)S(C,\nu) = t\kappa(C) + (1-t)\kappa(C) = \kappa(C). \tag{45}$$

So $(t\mu + (1-t)\nu) \in V(\kappa)$.

Referring to the following table we see that $\sum_{\xi \in D_4} (-1)^{|\xi|} C^{\xi} = \sum_{\xi \in D_4} (C \circ \xi) - 2$.

Table 3: Equations for Symmetries of 2-Copulas, C^{ξ}, in Terms of C

$C^{r}(x,y) = x - C(1-y, x)$
$C^{r^2}(x,y) = x + y - 1 + C(1-x, 1-y)$
$C^{r^3}(x,y) = y - C(y, 1-x)$
$C^{h}(x,y) = y - C(1-x, y)$
$C^{hr}(x,y) = C(y,x)$
$C^{hr^2}(x,y) = x - C(x, 1-y)$
$C^{hr^3}(x,y) = x + y - 1 + C(1-y, 1-x)$

Remark 2. If $\mu \in V(\kappa)$, then $(\mu \circ \zeta) \in V(\kappa)$ for every $\zeta \in D_4$ since

$$\begin{aligned}
\frac{1}{8}\int_{(0,1)^2} \sum_{\xi \in D_4} \left\{(-1)^{|\xi|} C^{\xi}\right\} d(\mu \circ \zeta) &= \frac{1}{8}\int_{(0,1)^2} \sum_{\xi \in D_4} \left\{(C \circ \xi) - 2\right\} d(\mu \circ \zeta) \\
&= \frac{1}{8}\int_{(0,1)^2} \left(\sum_{\xi \in D_4} (C \circ \xi \circ \zeta^{-1}) - 2\right) d\mu = \frac{1}{8}\int_{(0,1)^2} \left(\sum_{\xi \in D_4} (C \circ \xi) - 2\right) d\mu \\
&= \frac{1}{8}\int_{(0,1)^2} \sum_{\xi \in D_4} \left\{(-1)^{|\xi|} C^{\xi}\right\} d\mu = \kappa(C).
\end{aligned} \tag{46}$$

Remark 3. There exists exactly one D_4-invariant element of $V(\kappa)$. This is immediate by Theorem 2.

Remark 4. $V(\kappa)$ is a singleton if and only if κ is Blomqvist's beta.

If $\kappa = \beta$, then $\mu = 4\delta\left(\frac{1}{2}, \frac{1}{2}\right)$ where $\delta(x, y)$ is the unit point mass at (x, y) is the only D_4-invariant measure generating β. For $\nu \in V(\beta)$, we know from the work shown in Remarks 1 and 2 that $\frac{1}{8}\sum_{\xi \in D_4}(\nu \circ \xi) \in V(\kappa)$. Furthermore, since $\frac{1}{8}\sum_{\xi \in D_4}(\nu \circ \xi)$ is D_4-invariant we know by Remark 3 that

$$\frac{1}{8}\sum_{\xi \in D_4}(\nu \circ \xi) = \mu. \tag{47}$$

However, this implies ν has all its mass concentrated at $\left(\frac{1}{2}, \frac{1}{2}\right)$. So $\nu = \mu$.

On the other hand, suppose $V(\kappa)$ is a singleton. If $\kappa \neq \beta$, then for the unique D_4-invariant measure generating κ, say μ, we must have $\mu\left(\left(\left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right]\right)\right) > 0$. Let us define a measure, ν, on $\mathcal{B}((0,1)^2)$ where $\nu(S) = \begin{cases} 2\mu(S), & S \in \mathcal{B}\left(\left(\left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right]\right) \cup \left(\frac{1}{2}, 1\right)^2\right) \\ \mu(S), & S \in \mathcal{B}\left(\left(\left\{\frac{1}{2}\right\} \times (0,1)\right)\right) \\ 0, & \text{otherwise.} \end{cases}$

Notice $\nu \neq \mu$ since $\mu\left(\left(\left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right]\right)\right) > 0$. Recalling that

$$\sum_{\xi \in D_4}(-1)^{|\xi|}C^{\xi} = \sum_{\xi \in D_4}(C \circ \xi) - 2 \tag{48}$$

and keeping in mind the construction of ν as well as the D_4-invariance of μ, straightforward calculations yield

$$\begin{aligned} &\frac{1}{8}\int_{(0,1)^2}\sum_{\xi \in D_4}\left\{(-1)^{|\xi|}C^{\xi}\right\}d\nu \\ &= \frac{1}{8}\left\{2\int_{\left(\left(0,\frac{1}{2}\right)\times\left(0,\frac{1}{2}\right]\right)\cup\left(\frac{1}{2},1\right)^2}\sum_{\xi \in D_4}\left\{(-1)^{|\xi|}C^{\xi}\right\}d\mu + \int_{\left\{\frac{1}{2}\right\}\times(0,1)}\sum_{\xi \in D_4}\left\{(-1)^{|\xi|}C^{\xi}\right\}d\mu\right\} \end{aligned} \tag{49}$$

and

$$
\begin{aligned}
\int_{\left(\left(0,\frac{1}{2}\right)\times\left(0,\frac{1}{2}\right]\right)\cup\left(\frac{1}{2},1\right)^2} \sum_{\xi\in D_4} (-1)^{|\xi|} C^{\xi} d\mu &= \int_{\left(\left(0,\frac{1}{2}\right)\times\left(0,\frac{1}{2}\right]\right)\cup\left(\frac{1}{2},1\right)^2} \left(\sum_{\xi\in D_4} (C\circ\xi) - 2\right) d\mu \\
&= \int_{\left(\left(\frac{1}{2},1\right)\times\left(0,\frac{1}{2}\right]\right)\cup\left(\left(0,\frac{1}{2}\right)\times\left(\frac{1}{2},1\right)\right)} \left(\sum_{\xi\in D_4} (C\circ\xi\circ h) - 2\right) d(\mu\circ h) \\
&= \int_{\left(\left(\frac{1}{2},1\right)\times\left(0,\frac{1}{2}\right]\right)\cup\left(\left(0,\frac{1}{2}\right)\times\left(\frac{1}{2},1\right)\right)} \left(\sum_{\xi\in D_4} (C\circ\xi) - 2\right) d\mu \\
&= \int_{\left(\left(\frac{1}{2},1\right)\times\left(0,\frac{1}{2}\right]\right)\cup\left(\left(0,\frac{1}{2}\right)\times\left(\frac{1}{2},1\right)\right)} \sum_{\xi\in D_4} \{(-1)^{|\xi|} C^{\xi}\} d\mu.
\end{aligned}
\tag{50}
$$

Therefore $\int_{(0,1)^2} \sum_{\xi\in D_4}\{(-1)^{|\xi|}C^{\xi}\}d\nu = \int_{(0,1)^2} \sum_{\xi\in D_4}\{(-1)^{|\xi|}C^{\xi}\}d\mu = \kappa(C)$. But this would cause $\nu \in V(\kappa)$ also so that $V(\kappa)$ would not be a singleton. Therefore $\kappa = \beta$.

3.2 A Form for Elements of $\mathcal{M}(1)$ Generated by Finite Measures

When restricting our consideration to degree one measures of concordance generated by finite measures of degree one form we may establish another representation using the characterization in the previous section. This form leads to two interpretations of such a type of measure of concordance.

Fix any $C \in \text{Cop}(2)$ and any measure of degree one form, μ. Let $X = (X^1, X^2) : (0,1)^4 \to (0,1)^2$ and $Y = (Y^1, Y^2) : (0,1)^4 \to (0,1)^2$ be such that X^1 and X^2 are projection maps in the first and second coordinates respectively and Y^1 and Y^2 are projection maps in the third and fourth coordinates respectively. Letting $Z = (X, Y)$ we can construct a product measure, $\eta = \mu_C \times \mu$, on $\mathcal{B}((0,1)^4)$ such that for every $S \in \mathcal{B}((0,1)^2)$, $\eta(X \in S) = \mu_C(X \in S)\mu((0,1)^2)$ and $\eta(Y \in S) = \mu_C((0,1)^2)\mu(S)$. The ideas in the proof of the following lemma were set forth by [10].

Lemma 10. *Let μ be a finite Borel measure on $(0,1)^2$. For every $\xi \in D_4$ and $C \in \mathrm{Cop}(2)$,*

$$\int_{(0,1)^2} C^{\xi} d\mu = \int_{(0,1)^2} (\bar{G}_\mu \circ \xi^{-1}) dC \tag{51}$$

where $\bar{G}_\mu(x_1, x_2) = \mu([x_1, 1) \times [x_2, 1))$.

Proof. Fix $\xi \in D_4$ and $C \in \mathrm{Cop}(2)$. For $\bar{x} = (x_1, x_2) \in (0,1)^2$,

$$C^{\xi}(\bar{x}) = \mu_{C^\xi}([0, x_1] \times [0, x_2]) = \mu_C(\xi([0, x_1] \times [0, x_2])) = \mu_C((\xi^{-1} \circ X) \leq \bar{x}). \tag{52}$$

Let $\{B_k\}$ and $\{D_m\}$, be sequences in $\mathcal{B}((0,1)^2)$ strictly increasing to $(0,1)^2$.

If we define

$$A_{k,m} = \{(\xi^{-1} \circ X \leq Y) \cap (B_k \times D_m))\} \text{ and } A = \{\xi^{-1} \circ X \leq Y\} \tag{53}$$

we have $A_{k,m} \subseteq A_{p,q}$ whenever $k \leq p$ and $m \leq q$ and $\cup_{k=1}^{\infty} \cup_{m=1}^{\infty} A_{k,m} = A$. Because of this we have

$$\lim_{k,m \to \infty} \eta(A_{k,m}) = \eta(A) = \eta(\xi^{-1} \circ X \leq Y). \tag{54}$$

Letting $\bar{G}_m(\bar{y}) = \mu(Y \geq \bar{y}, Y \in D_m)$, calculations using conditional measure give us

$$\begin{aligned} \eta(A_{k,m}) &= \eta(\xi^{-1} \circ X \leq Y, X \in B_k, Y \in D_m) \\ &= \int_{B_k} \eta(Y \geq \bar{x}, Y \in D_m | \xi^{-1} \circ X = \bar{x}) \, d\mu_{C^\xi}(\bar{x}) = \int_{B_k} \bar{G}_m(\bar{x}) d\mu_{C^\xi}(\bar{x}) \\ &= \int_{B_k} (\bar{G}_m \circ \xi^{-1}) d\mu_C = \int_{B_k} (\bar{G}_m \circ \xi^{-1}) dC = \int_{(0,1)^2} (\bar{G}_m \circ \xi^{-1}) \chi_{B_k} dC. \end{aligned} \tag{55}$$

However, if we let $C_k(\bar{x}) = \mu_C(\xi^{-1} \circ X \leq \bar{x}, X \in B_k)$, calculations give us

$$\begin{aligned} \eta(A_{k,m}) &= \eta(\xi^{-1} \circ X \leq Y, X \in B_k, Y \in D_m) \\ &= \int_{D_m} \eta(\xi^{-1} \circ X \leq \bar{y}, X \in B_k | Y = \bar{y}) \, d\mu(\bar{y}) = \int_{D_m} C_k(\bar{y}) d\mu(\bar{y}) \\ &= \int_{(0,1)^2} C_k \chi_{D_m} d\mu. \end{aligned} \tag{56}$$

Fixing k, an application of the monotone convergence theorem to (55) gives us

$$\lim_{m\to\infty}\int_{(0,1)^2}(\bar{G}_m\circ\xi^{-1})\chi_{B_k}dC=\int_{(0,1)^2}(\bar{G}_\mu\circ\xi^{-1})\chi_{B_k}dC \tag{57}$$

while doing the same to (56) gives us

$$\lim_{m\to\infty}\int_{(0,1)^2}C_k\chi_{D_m}=\int_{(0,1)^2}C_k d\mu. \tag{58}$$

We then obtain from an additional application of the monotone convergence theorem to the right hand side of (57)

$$\lim_{k\to\infty}\int_{(0,1)^2}(\bar{G}_\mu\circ\xi^{-1})\chi_{B_k}dC=\int_{(0,1)^2}(\bar{G}_\mu\circ\xi^{-1})dC \tag{59}$$

while doing the same to (58), (52) gives us

$$\lim_{k\to\infty}\int_{(0,1)^2}C_k d\mu=\int_{(0,1)^2}C^\xi d\mu. \tag{60}$$

From (54)-(60) we have

$$\eta(\xi^{-1}\circ X\le Y)=\int_{(0,1)^2}(\bar{G}_\mu\circ\xi^{-1})dC=\int_{(0,1)^2}C^\xi d\mu \tag{61}$$

which completes our proof. □

Remark 5. For any $\kappa\in\mathcal{M}(1)$ generated by a finite measure of degree one form it can be shown that $\kappa(C)$ is simply a difference of the concordance and discordance of any continuous random vector, (X_1,X_2), associated with C.

Choose any $\kappa\in\mathcal{M}(1)$ generated by a finite measure of degree one form, say μ. Also, let $\bar{Y}=(Y_1,Y_2)$ be any random vector associated with μ and choose any $C\in\mathrm{Cop}(2)$. Let (X_1,X_2) be any continuous random vector associated with C.

Define $\mu^\xi(S) = \mu(\xi(S))$ for every $S \in \mathcal{B}((0,1)^2)$. By Theorem 2 as well as the D_4-invariance and finiteness of μ we have,

$$\kappa(C) = \frac{1}{8}\int_{(0,1)^2} \sum_{\xi \in D_4} \{(-1)^{|\xi|} C^\xi\} d\mu = \frac{1}{8}\sum_{\xi \in D_4} \left\{(-1)^{|\xi|} \int_{(0,1)^2} C^\xi d\mu^\xi\right\}. \tag{62}$$

Having η be defined in the same context as in Lemma 10,

$$\begin{aligned}\kappa(C) &= \frac{1}{8}\sum_{\xi \in D_4} \left\{(-1)^{|\xi|}\eta(\xi^{-1} \circ \bar{X} < \xi^{-1} \circ \bar{Y})\right\} \\ &= \frac{1}{4}(\eta((X_1 - Y_1)(X_2 - Y_2) > 0) - \eta((X_1 - Y_1)(X_2 - Y_2) < 0)).\end{aligned} \tag{63}$$

Therefore, $\kappa(C)$ is simply $\frac{1}{4}$ times the difference of the η measure of concordance and the η measure of discordance of any continuous random vector (X_1, X_2) associated with C.

Here and in all that follows for $\bar{x} = (x_1, x_2) \in (0,1)^2$ we will let

$$S_1(\bar{x}) = \left[\frac{1}{2} - \left|\frac{1}{2} - x_1\right|, \frac{1}{2} + \left|\frac{1}{2} - x_1\right|\right) \times \left[\frac{1}{2} - \left|\frac{1}{2} - x_2\right|, \frac{1}{2} + \left|\frac{1}{2} - x_2\right|\right) \tag{64}$$

and

$$S_2(\bar{x}) = \left[\frac{1}{2} - \left|\frac{1}{2} - x_2\right|, \frac{1}{2} + \left|\frac{1}{2} - x_2\right|\right) \times \left[\frac{1}{2} - \left|\frac{1}{2} - x_1\right|, \frac{1}{2} + \left|\frac{1}{2} - x_1\right|\right). \tag{65}$$

Theorem 4. *If $\kappa \in \mathcal{M}(1)$ is generated by a finite measure of degree one form, say μ, then*

$$\kappa(C) = \frac{1}{8}\int_{(0,1)^2} \operatorname{sgn}\left(\left(\frac{1}{2} - x_1\right)\left(\frac{1}{2} - x_2\right)\right)(\mu(S_1(\bar{x})) + \mu(S_2(\bar{x})))dC \tag{66}$$

for all $C \in \mathrm{Cop}(2)$.

Proof. Fix $\kappa \in \mathcal{M}(1)$ generated by a finite measure of degree one form, μ. By Theorem 2 and Lemma 10

$$\begin{aligned}\kappa(C) &= \frac{1}{8}\int_{(0,1)^2}\left\{\sum_{\xi\in D_4}(-1)^{|\xi|}C^{\xi}\right\}d\mu = \frac{1}{8}\sum_{\xi\in D_4}(-1)^{|\xi|}\int_{(0,1)^2}C^{\xi}d\mu \\ &= \frac{1}{8}\sum_{\xi\in\{e,h,hr^2,r^2\}}(-1)^{|\xi|}\int_{(0,1)^2}(\bar{G}_{\mu}\circ\xi^{-1})dC + \frac{1}{8}\sum_{\xi\in\{hr,r,r^3,hr^3\}}(-1)^{|\xi|}\int_{(0,1)^2}(\bar{G}_{\mu}\circ\xi^{-1})dC.\end{aligned} \tag{67}$$

If one considers $(x_1, x_2) \in \left(0, \frac{1}{2}\right)^2$ it can be seen that

$$\begin{aligned}\sum_{\xi\in\{e,h,hr^2,r^2\}}(-1)^{|\xi|}(\bar{G}_{\mu}\circ\xi^{-1})(x_1,x_2) &= \mu(S_1(\bar{x})), \text{ and} \\ \sum_{\xi\in\{hr,r,r^3,hr^3\}}(-1)^{|\xi|}(\bar{G}_{\mu}\circ\xi^{-1})(x_1,x_2) &= \mu(S_2(\bar{x})).\end{aligned} \tag{68}$$

More generally for $(x_1, x_2) \in (0,1)^2$,

$$\begin{aligned}\sum_{\xi\in\{e,h,hr^2,r^2\}}(-1)^{|\xi|}(\bar{G}_{\mu}\circ\xi^{-1})(x_1,x_2) &= \operatorname{sgn}\left(\left(\frac{1}{2}-x_1\right)\left(\frac{1}{2}-x_2\right)\right)\mu(S_1(\bar{x})), \text{ and} \\ \sum_{\xi\in\{hr,r,r^3,hr^3\}}(-1)^{|\xi|}(\bar{G}_{\mu}\circ\xi^{-1})(x_1,x_2) &= \operatorname{sgn}\left(\left(\frac{1}{2}-x_1\right)\left(\frac{1}{2}-x_2\right)\right)\mu(S_2(\bar{x}))\end{aligned} \tag{69}$$

which completes our proof. □

While it seems reasonable that Theorem 4 should hold for all $\kappa \in \mathcal{M}(1)$, it still remains to be seen if that is the case.

In here and in all that follows let $\overline{\Delta}$ denote the set of all finite measures of degree one form and $\mathcal{M}_{<\infty}(1) = \{\kappa | \kappa \in \mathcal{M}(1) \text{ where } \kappa \text{ is generated by some } \mu \in \overline{\Delta}\}$.

Remark 6. For any $\kappa \in \mathcal{M}_{<\infty}(1)$, if $\nu \in V(\kappa)$ then $\nu((0,1)^2) < \infty$. If we assume that $\nu((0,1)^2) = \infty$ then from Remarks 1, 2, and 3 we have $\mu = \frac{1}{8}\sum_{\xi\in D_4}(\nu\circ\xi)$ as the unique measure of degree one form generating κ. However, this gives us $\mu((0,1)^2) = \infty$. So $\nu((0,1)^2) < \infty$ for all $\nu \in V(\kappa)$.

Remark 7. Notice the representation in Theorem 4 gives us another probabilistic interpretation for any $\kappa \in \mathcal{M}_{<\infty}(1)$ in addition to the one given in Remark 5. Namely,

$$\kappa(C) = E(\Gamma_\mu(\bar{X})) \tag{70}$$

where $\Gamma_\mu(\bar{x}) = \frac{1}{8}\operatorname{sgn}\left(\left(\frac{1}{2} - x_1\right)\left(\frac{1}{2} - x_2\right)\right)(\mu(S_1(\bar{x})) + \mu(S_2(\bar{x})))$, μ is the unique measure of degree one form generating κ, and $\bar{X}$ is any continuous random vector associated with C.

Moreover, by Lemma 10 and Remarks 4 and 6 we also see for $\nu \in V(\kappa)$ that

$$\kappa(C) = \int_{(0,1)^2} \Gamma_\nu dC \tag{71}$$

for every $C \in \mathrm{Cop}(2)$ even when ν is not of degree one form.

Definition 13. Let $\bar{\Gamma}_{V(\kappa)} = \{\Gamma_\nu | \nu \in V(\kappa)\}$. $\cup_{\kappa \in \mathcal{M}_{<\infty}(1)} \bar{\Gamma}_{V(\kappa)}$ is the set of *concordance functions.*

There are some properties that concordance functions share which are noteworthy. Let $\mu \in V(\kappa)$ for some $\kappa \in \mathcal{M}_{<\infty}(1)$. To better illustrate some of the properties to be listed next it will be helpful to keep in mind that for $(x_1, x_2) \in \left(0, \frac{1}{2}\right)^2$

$$\Gamma_\mu(x_1, x_2) = \frac{1}{8}\left(\mu([x_1, 1 - x_1) \times [x_2, 1 - x_2)) + \mu([x_2, 1 - x_2) \times [x_1, 1 - x_1))\right) \tag{72}$$

and for $\bar{x} = (x_1, x_2) \in (0,1)^2$

$$\Gamma_\mu(x_1, x_2) = \frac{1}{8}\operatorname{sgn}\left(\frac{1}{2} - x_1\right)\left(\frac{1}{2} - x_2\right)(\mu(S_1(\bar{x})) + \mu(S_2(\bar{x}))). \tag{73}$$

It will be helpful to note that any concordance function, Γ, has the following properties:

1. for every $\xi \in D_4$ we have $\Gamma(\bar{x}) = (-1)^{|\xi|}\Gamma(\xi(\bar{x}))$ (see (73)),

2. Γ is left-continuous in $\left(0, \frac{1}{2}\right)^2$ (see (72)), and

3. Γ is a survival function on $\left(0, \frac{1}{2}\right)^2$ (again see (72)).

Theorem 5. *If $\mu, \nu \in V(\kappa)$ for some $\kappa \in \mathcal{M}_{<\infty}(1)$, then $\Gamma_\mu = \Gamma_\nu$.*

Proof. Because of the properties held by concordance functions, it will suffice to show uniqueness in $\left(0, \frac{1}{2}\right)^2$.

Choose $\kappa \in \mathcal{M}_{<\infty}(1)$ and $\mu, \nu \in V(\kappa)$. By Theorem 4, for any $C \in \text{Cop}(2)$ and any continuous random vector, $\bar{X}$, associated with C we have $\kappa(C) = E(\Gamma_\mu(\bar{X})) = E(\Gamma_\nu(\bar{X}))$. More conveniently written,

$$E((\Gamma_\mu - \Gamma_\nu)(\bar{X})) = 0 \tag{74}$$

for every continuous random vector $\bar{X}$.

Suppose there exists $(x_1, x_2) \in \left(0, \frac{1}{2}\right)^2$ such that $(\Gamma_\mu - \Gamma_\nu)(x_1, x_2) = \epsilon$ for some $\epsilon > 0$. Since Γ_μ and Γ_ν are both left continuous in each coordinate there exists a $\delta > 0$ such that

$$(x_1 - \delta, x_1) \times (x_2 - \delta, x_2) \subset \left(0, \frac{1}{2}\right)^2 \tag{75}$$

and

$$\inf_{(x_1,x_2)\in(x_1-\delta,x_1)\times(x_2-\delta,x_2)} (\Gamma_\mu - \Gamma_\nu)(x_1, x_2) > \frac{\epsilon}{2}. \tag{76}$$

With this $\delta > 0$ we refer back to the construction of the 2-copula, $C_{R,\delta}$, whose density distribution is described in Figure 3 where $R = [x_1, 1 - x_1] \times [x_2, 1 - x_2]$. Indeed, $C_{R,\delta}$ has density of 2 in $((x_1-\delta,x_1)\times(x_2-\delta,x_2))\cup((1-x_1,1-x_1+\delta)\times(1-x_2,1-x_2+\delta))$ and density 0 in $((x_1-\delta,x_1)\times(1-x_2,1-x_2+\delta))\cup((1-x_1,1-x_1+\delta)\times(x_2-\delta,x_2))$. Otherwise, $C_{R,\delta}$ has density 1. For any continuous random vectors $\bar{Y}$ and $\bar{Z}$ associated with $C_{R,\delta}$ and Π respectively

$$\begin{aligned} 0 = E((\Gamma_\mu-\Gamma_\nu)(\bar{Y})) - E((\Gamma_\kappa-\Gamma_0)(\bar{Z})) &= \int_{(0,1)^2}(\Gamma_\mu-\Gamma_\nu)d(C_{R,\delta}-\Pi) \\ = 4\int_{(x_1-\delta,x_1)\times(x_2-\delta,x_2)}(\Gamma_\mu-\Gamma_\nu)d\Pi &> 2\epsilon\delta^2 > 0. \end{aligned} \tag{77}$$

Therefore $\Gamma_\mu(x_1,x_2)=\Gamma_\nu(x_1,x_2)$. ☐

Blomqvist's beta, Spearman's rho, and Gini's measure of association are all elements of $\mathcal{M}_{<\infty}(1)$ where $\beta(C) = 4C\left(\frac{1}{2},\frac{1}{2}\right)-1$, $\rho(C) = 12\int_{(0,1)^2}Cd\Pi-3$, and $\gamma(C) = 8\int_{(0,1)^2}Cd(\frac{M+W}{2})-2$ [6]. Let us see what are their respective concordance functions.

Example 1. Blomqvist's beta is generated by a measure of degree one form, μ, where a mass of 4 is placed at $\left(\frac{1}{2},\frac{1}{2}\right)$. In this case we have

$$\Gamma_\mu(x_1,x_2) = \begin{cases} 1, & \left(\frac{1}{2}-x_1\right)\left(\frac{1}{2}-x_2\right)>0, \\ -1, & \left(\frac{1}{2}-x_1\right)\left(\frac{1}{2}-x_2\right)<0, \\ 0, & \text{otherwise} \end{cases} \tag{78}$$

as the concordance function of β.

Example 2. Spearman's rho is generated by a measure of degree one form which is $\frac{3}{2}$ times the Lebesgue measure on $\mathcal{B}((0,1)^2)$. Here we will let λ^2 denote the two-dimensional Lebesgue measure. In this case we have

$$\Gamma_{\frac{3}{2}\lambda^2}(x_1,x_2) = 3(1-2x_1)(1-2x_2) \tag{79}$$

as the concordance function of ρ.

Example 3. Gini's measure of association, γ, is generated by a measure of degree one form where a mass of 8 units is distributed uniformly on the line segments $y = x$ and $x + y - 1 = 0$ intersecting with $(0,1)^2$. More explicitly, γ is generated by 8 times the doubly stochastic measure induced by the 2-copula $\frac{M+W}{2}$ (written $8\mu_{\frac{M+W}{2}}$). In this case we have

$$\Gamma_{8\mu_{\frac{M+W}{2}}}(x_1, x_2) = \begin{cases} 0, & (x_1, x_2) = (\frac{1}{2}, \frac{1}{2}) \\ 4 \operatorname{sgn}\left(\frac{\frac{1}{2} - x_2}{\frac{1}{2} - x_1}\right) \left|\frac{1}{2} - x_2\right|, & x_2 \leq x_1 \text{ and } x_1 + x_2 > 1, \text{ or} \\ & x_2 > x_1 \text{ and } x_1 + x_2 \leq 1 \\ 4 \operatorname{sgn}\left(\frac{\frac{1}{2} - x_1}{\frac{1}{2} - x_2}\right) \left|\frac{1}{2} - x_1\right|, & \text{otherwise} \end{cases} \tag{80}$$

as the concordance function of γ.

4. MEASURES OF CONCORDANCE OF HIGHER DEGREE

In this chapter we give a representation for a family of measures of concordance of higher degree. This family is generated by a particular collection of Borel measures on $(0,1)^{2n}$. We will denote this collection of measures on $(0,1)^{2n}$ as d_n. Also, we will place special focus on the degree 2 measure of concordance, Kendall's tau, and show it is not generated by any finite measure in d_2. We then give another form for a family of measures of concordance containing those generated by finite measures in d_2 as well as Kendall's tau.

4.1 Measures of Concordance Generated by d_n

Let $\bar{x}_i \in [0,1]^2$ for $i = 1, \dots, n$. For $C \in \text{Cop}(2)$, we define

$$C(\bar{x}_i) \bigotimes C(\bar{x}_j) = C(\bar{x}_i)C(\bar{x}_j). \tag{81}$$

Furthermore, we will let

$$C^n(\bar{x}_1, \dots, \bar{x}_n) = \bigotimes_{i=1}^{n} C(\bar{x}_i). \tag{82}$$

It is straightforward to see that $C^n \in \text{Cop}(2n)$ [9].

Definition 14. A Borel measure, μ, on $(0,1)^{2n}$ belongs to d_n if and only if

$$0 < \int_{(0,1)^{2n}} (M^n - W^n) d\mu < \infty. \tag{83}$$

Theorem 6. $\kappa(C) = \frac{\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{\int_{(0,1)^{2n}} ((C^\xi)^n - (C^{h\xi})^n) d\mu\}}{\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{\int_{(0,1)^{2n}} ((M^\xi)^n - (M^{h\xi})^n) d\mu\}}$ *is a measure of concordance of polynomial type degree m for some $m \leq n$ if and only if $\mu \in d_n$.*

Proof. If κ having the prescribed representation is a degree m measure of concordance for some $m \leq n$, clearly $\sum_{\xi\in\{e,r^2,hr,hr^3\}} \int_{(0,1)^{2n}} ((M^{\xi})^n - (M^{h\xi})^n) d\mu$ is nonzero and finite. For $C_1 \leq C_2$, it is immediately seen when referring to Table 3 that $(C_1^{\xi} - C_1^{h\xi}) \leq (C_2^{\xi} - C_2^{h\xi})$ whenever $|\xi|$ is even. Because of this fact, the form of κ implies that $\sum_{\xi\in\{e,r^2,hr,hr^3\}}\{\int_{(0,1)^{2n}}((M^{\xi})^n - (M^{h\xi})^n)d\mu\}$ is nonnegative. Finally, since

$$4\int_{(0,1)^{2n}} (M^n - W^n)d\mu = \sum_{\xi\in\{e,r^2,hr,hr^3\}} \left\{\int_{(0,1)^{2n}} ((M^{\xi})^n - (M^{h\xi})^n)d\mu\right\} \tag{84}$$

we see $\mu \in d_n$.

Since $\mu \in d_n$, we know by the form of κ that it is defined for all $C \in \text{Cop}(2)$. The form of κ also makes it clear that $\kappa(M) = 1$ and that $\kappa(C_1) \leq \kappa(C_2)$ whenever $C_1 \leq C_2$.

Recall from Table 2 if C is the 2-copula associated with the continuous random vector (X,Y), then C^h and C^{hr} are the 2-copulas associated with $(-X,Y)$ and (Y,X) respectively. Because of this fact, it is again clear by the form of κ that $\kappa_{-X,Y} = -\kappa_{X,Y}$ and $\kappa_{Y,X} = \kappa_{X,Y}$.

If $\{M_k\}$ is a sequence in Cop(2) such that $M_k \to M$, then it is immediate that $(M_k^{\xi})^n \to M^n$ whenever $|\xi|$ is even and $M_k^{\xi} \to W$ whenever $|\xi|$ is odd. Recall $W \leq C \leq M$ for every $C \in \text{Cop}(2)$ [5]. Therefore for any $|\xi|$ even we have

$$\int_{(0,1)^{2n}} ((M_k^{\xi})^n - (M_k^{h\xi})^n)d\mu \leq \int_{(0,1)^{2n}} (M^n - W^n)d\mu. \tag{85}$$

However, by a generalization of Fatou's Lemma [7] we have

$$\int_{(0,1)^{2n}} (M^n - W^n)d\mu \leq \liminf \int_{(0,1)^{2n}} ((M_k^{\xi})^n - (M_k^{h\xi})^n)d\mu. \tag{86}$$

So $\lim \int_{(0,1)^{2n}}((M_k^{\xi})^n - (W_k^{h\xi})^n)d\mu = \int_{(0,1)^{2n}}(M^n - W^n)d\mu$ for every $|\xi|$ even. This fact combined with a generalization of the Lebesgue Convergence Theorem [7] gives us for any

$C \in \mathrm{Cop}(2)$ and any sequence, $\{C_k\}$, in Cop(2) where $C_k \to C$ that

$$\lim \int_{(0,1)^{2n}} ((C_k^{\xi})^n - (C_k^{h\xi})^n) d\mu = \int_{(0,1)^{2n}} ((C^{\xi})^n - (C^{h\xi})^n) d\mu. \tag{87}$$

Hence, $\kappa(C_k) \to \kappa(C)$.

Finally, κ must be of polynomial type degree m for some $m \leq n$ by its form. □

The next lemma will be of use in the subsequent corollary as well as in some upcoming examples. Its proof can be found in [2].

Lemma 11. *Given $A, B \in \mathrm{Cop}(2)$, $\int_{(0,1)^2} A^{\xi} dB = \int_{(0,1)^2} A dB^{\xi}$ whenever $|\xi|$ is even and $\int_{(0,1)^2} A^{\xi} dB + \int_{(0,1)^2} A dB^{\xi} = \frac{1}{2}$ whenever $|\xi|$ is odd.*

Corollary 1. *Let $\mu \in d_n$ be the multiply stochastic measure induced by the $2n$-copula, $\bigotimes_{i=1}^{n} A$, for some D_4-invariant 2-copula, A. μ generates a degree n measure of concordance for all n odd and generates a degree $(n-1)$ measure of concordance for all n even.*

Proof. For simplicity, let $\alpha = \left(\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{\int_{(0,1)^{2n}} ((M^{\xi})^n - (M^{h\xi})^n) d\mu\}\right)^{-1}$. Combining Fubini's Theorem and Lemma 11 we have for every $|\xi|$ even,

$$\int_{(0,1)^{2n}} ((C^{\xi})^n - (C^{h\xi})^n) d\mu = \left(\int_{(0,1)^2} C dA\right)^n - \left(\frac{1}{2} - \int_{(0,1)^2} C dA\right)^n. \tag{88}$$

Therefore, $\kappa(C) = 4\alpha\left(\left(\int_{(0,1)^2} C dA\right)^n - \left(\frac{1}{2} - \int_{(0,1)^2} C dA\right)^n\right)$. □

It is interesting to note by Corollary 1 that any measure of concordance generated by $(A \bigotimes A)$ for a D_4-invariant 2-copula, A, is identical to the measure of concordance generated by A.

Remark 8. We will show that any measure of concordance, κ, generated by some finite $\mu \in d_n$ takes the form of an expected value. We may easily generalize Lemma 10 to measures on $(0,1)^{2n}$ so that for the survival function, $\bar{G}_\mu$, associated with μ we have

$$\int_{(0,1)^{2n}} (C^\xi(\bar{x}_1, \dots, \bar{x}_n))^n d\mu = \int_{(0,1)^{2n}} \bar{G}_\mu(\xi^{-1}(\bar{x}_1), \dots, \xi^{-1}(\bar{x}_n)) dC^n. \tag{89}$$

Therefore by Theorem 6 we may write $\kappa(C) = \int_{(0,1)^{2n}} \Gamma_\mu dC^n$ where

$$\Gamma_\mu(\bar{x}_1, \dots, \bar{x}_n) = \frac{\sum_{\xi \in D_4} (-1)^{|\xi|} \bar{G}_\mu(\xi^{-1}(\bar{x}_1), \dots, \xi^{-1}(\bar{x}_n))}{4 \int_{(0,1)^{2n}} (M^n - W^n) d\mu}. \tag{90}$$

If we have a collection of continuous random vectors (X_i, Y_i) for $i = 1, \dots, n$ where C is the 2-copula associated with (X_i, Y_i) for each i and each random vector is independently observed from the other, then for $\bar{Z} = (X_1, Y_1, \dots, X_n, Y_n)$ we have

$$\kappa(C) = E(\Gamma_\mu(\bar{Z})). \tag{91}$$

Blomqvist's beta, Spearman's rho, and Gini's measure of association are all generated by measures in d_1. Now let us form analogs of those measures belonging to d_n.

Example 4. Let a mass of $k > 0$ be placed at $\left(\frac{1}{2}, \frac{1}{2}\right)$ in order to form $\mu \in d_1$. From Table 3 it is easily seen that for every $C \in \text{Cop}(2)$ and $|\xi|$ even, $C^\xi\left(\frac{1}{2}, \frac{1}{2}\right) = C\left(\frac{1}{2}, \frac{1}{2}\right)$ and $C^{h\xi}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} - C\left(\frac{1}{2}, \frac{1}{2}\right)$. Therefore it can be calculated directly that

$$\frac{\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{\int_{(0,1)^2} (C^\xi - C^{h\xi}) d\mu\}}{\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{\int_{(0,1)^2} (M^\xi - M^{h\xi}) d\mu\}} = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1 \tag{92}$$

which is Blomqvist's beta. An analog of μ, say $\mu_\beta \in d_n$, can be made by placing a mass of $k > 0$ at $\left(\frac{1}{2}, \dots, \frac{1}{2}\right) \in (0,1)^{2n}$ so that

$$\begin{aligned}\kappa(C) &= \frac{\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{ \int_{(0,1)^{2n}} ((C^\xi)^n - (C^{h\xi})^n) d\mu_\beta \}}{\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{ \int_{(0,1)^{2n}} ((M^\xi)^n - (M^{h\xi})^n) d\mu_\beta \}} \\ &= 2^n \left(\left(C\left(\frac{1}{2}, \frac{1}{2}\right)\right)^n - \left(\frac{1}{2} - C\left(\frac{1}{2}, \frac{1}{2}\right)\right)^n \right)\end{aligned} \tag{93}$$

is a degree n analog of Blomqvist's beta for n odd and is a degree $(n-1)$ analog of Blomqvist's beta for n even.

Example 5. Let $\mu \in d_1$ be the two-dimensional Lebesgue measure. It can also be said that μ is the doubly stochastic measure induced by the 2-copula, $\Pi(x, y) = xy$. Recall $\Pi^\xi = \Pi$ for any $\xi \in D_4$. By Lemma 11, $\int_{(0,1)^2} C^\xi d\mu = \int_{(0,1)^2} C d\Pi$ and $\int_{(0,1)^2} C^{h\xi} d\mu = \frac{1}{2} - \int_{(0,1)^2} C d\Pi$ whenever $|\xi|$ is even. Noting that $\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{ \int_{(0,1)^2} (M^\xi - M^{h\xi}) d\Pi = \frac{2}{3}$ we have

$$\frac{\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{ \int_{(0,1)^2} (C^\xi - C^{h\xi}) d\mu \}}{\sum_{\xi \in \{e, r^2, hr, hr^3\}} \{ \int_{(0,1)^2} (M^\xi - M^{h\xi}) d\mu \}} = 12 \int_{(0,1)^2} C d\Pi - 3 \tag{94}$$

which is Spearman's rho. An analog of μ, say $\mu_\rho \in d_n$, can be made by simply having μ_ρ be the $2n$-dimensional Lebesgue measure. Corollary 1 gives us

$$\kappa(C) = \frac{3^n}{1 - 2^{-n}} \left(\left(\int_{(0,1)^2} C d\Pi \right)^n - \left(\frac{1}{2} - \int_{(0,1)^2} C d\Pi \right)^n \right) \tag{95}$$

is a degree n analog of Spearman's rho for n odd and is a degree $(n-1)$ analog of Spearman's rho for n even.

Example 6. We can use μ_M, the doubly stochastic measure induced by $M(x, y) = \min(x, y)$, to generate Gini's measure of association. For any $|\xi|$ even, $M^\xi = M$ and $M^{h\xi} = W$. With this in mind we can use Lemma 11 to see for any $|\xi|$ even, $\int_{(0,1)^2} C^\xi d\mu_M = \int_{(0,1)^2} C dM$ and

$\int_{(0,1)^2} C^{h\xi} d\mu_M = \frac{1}{2} - \int_{(0,1)^2} CdW$. Since $\sum_{\xi\in\{e,r^2,hr,hr^3\}}\{\int_{(0,1)^2}(M^\xi - M^{h\xi})dM\} = 4\int_{(0,1)^2}(M - W)dM = 1$, straightforward calculations show that

$$\frac{\sum_{\xi\in\{e,r^2,hr,hr^3\}}\{\int_{(0,1)^2}(C^\xi - C^{h\xi})d\mu\}}{\sum_{\xi\in\{e,r^2,hr,hr^3\}}\{\int_{(0,1)^2}(M^\xi - M^{h\xi})d\mu\}} = 8\int_{(0,1)^2} Cd\left(\frac{M+W}{2}\right) - 2 \tag{96}$$

which is Gini's measure of association. One possible analog of μ_M, say $\mu_\gamma \in d_n$, can be made by letting μ_γ be the multiply stochastic measure induced by the $2n$-copula, $M(x_1, \ldots, x_{2n}) = \min(x_1, \ldots, x_{2n})$ so that

$$\begin{aligned}\kappa(C) &= \frac{\sum_{\xi\in\{e,r^2,hr,hr^3\}}\{\int_{(0,1)^{2n}}((C^\xi)^n - (C^{h\xi})^n)d\mu_\gamma\}}{\sum_{\xi\in\{e,r^2,hr,hr^3\}}\{\int_{(0,1)^{2n}}((M^\xi)^n - (M^{h\xi})^n)d\mu_\gamma\}} \\ &= \frac{n+1}{2}\sum_{\xi\in\{e,r^2,hr,hr^3\}}\left\{\int_0^1((C^\xi(x,x))^n - (C^{h\xi}(x,x))^n)dx\right\}\end{aligned} \tag{97}$$

provides a higher degree analog of Gini's measure of association.

Another possible analog of μ_M, say $\bar{\mu}_\gamma \in d_n$, can be made by letting $\bar{\mu}_\gamma$ be the multiply stochastic measure induced by the $2n$-copula, $\bigotimes_{i=1}^n M$, the n-wise tensor product of M, so that

$$\begin{aligned}\kappa(C) &= \frac{\sum_{\xi\in\{e,r^2,hr,hr^3\}}\{\int_{(0,1)^{2n}}((C^\xi)^n - (C^{h\xi})^n)d\bar{\mu}_\gamma\}}{\sum_{\xi\in\{e,r^2,hr,hr^3\}}\{\int_{(0,1)^{2n}}((M^\xi)^n - (M^{h\xi})^n)d\bar{\mu}_\gamma} \\ &= \frac{2^{n-2}}{1-2^{-n}}\sum_{\xi\in\{e,r^2,hr,hr^3\}}\left\{\left(\int_0^1 C^\xi(x,x)dx\right)^n - \left(\int_0^1 C^{h\xi}(x,x)dx\right)^n\right\}\end{aligned} \tag{98}$$

provides a degree n analog of Gini's measure of association for n odd and a degree $(n-1)$ analog for n even.

4.2 A Focus on Kendall's Tau

In this section a focus is placed on the degree 2 measure of concordance, Kendall's tau, which is written $\tau(C) = 4\int_{(0,1)^2} CdC - 1$ [6]. We will find a function, H_τ, generating τ in a certain way.

Let $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ where $\bar{x}, \bar{y} \in (0,1)^2$. We will write $[0, \bar{x}]$ for $[0, x_1] \times [0, x_2]$. We will also write $\bar{x} \leq \bar{y}$ when $x_i \leq y_i$ for $i = 1, 2$. It will be helpful to note for $\omega, \xi \in D_4$ that

$$\begin{aligned}\int_{(0,1)^2} C^\xi dC^\xi &= \int_{(0,1)^4} \chi_{[0,\bar{y}]}(\bar{x}) d\left(C^\xi(\bar{x}) \bigotimes C^\xi(\bar{y})\right) \\ &= \int_{(0,1)^4} \chi_{[0,\xi^{-1}(\bar{y})]}(\xi^{-1}(\bar{x})) d\left(C(\bar{x}) \bigotimes C(\bar{y})\right).\end{aligned} \tag{99}$$

We may use (99) to rewrite Kendall's tau.

$$\begin{aligned}\tau(C) &= \frac{1}{8}\sum_{\xi \in D_4}(-1)^{|\xi|}\tau(C^\xi) = \frac{1}{2}\sum_{\xi \in D_4}(-1)^{|\xi|}\int_{(0,1)^2} C^\xi dC^\xi \\ &= \frac{1}{2}\sum_{\xi \in D_4}(-1)^{|\xi|}\int_{(0,1)^4} \chi_{[0,\xi^{-1}(\bar{y})]}(\xi^{-1}(\bar{x})) d\left(C(\bar{x}) \bigotimes C(\bar{y})\right) \\ &= \sum_{\xi \in D_4}(-1)^{|\xi|}\int_{(0,1)^4} H_\tau(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) d\left(C(\bar{x}) \bigotimes C(\bar{y})\right)\end{aligned} \tag{100}$$

where $H_\tau(\bar{x}, \bar{y}) = \begin{cases} \frac{1}{2}, & \bar{x} \leq \bar{y} \\ 0, & \text{otherwise.} \end{cases}$

Example 7. Kendall's tau is not generated by any finite measure in d_2. For if it were, there would exist a finite $\mu_\tau \in d_2$ such that

$$\tau(C) = \alpha \sum_{\xi \in D_4}(-1)^{|\xi|}\int_{(0,1)^4} \bar{G}_{\mu_\tau}(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) d\left(C(\bar{x}) \bigotimes C(\bar{y})\right) \tag{101}$$

for every $C \in \text{Cop}(2)$ where $\alpha = \left(4\int_{(0,1)^4}(M^2 - W^2)d\mu_\tau\right)^{-1}$ is some positive, finite value and $\bar{G}_{\mu_\tau}$ is the survival function associated with μ_τ.

This gives us $\sum_{\xi \in D_4} (-1)^{|\xi|} H_\tau(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) = \alpha \sum_{\xi \in D_4} (-1)^{|\xi|} (\bar{G}_{\mu_\tau} \circ (\xi^{-1}, \xi^{-1}))(\bar{x}, \bar{y})$. Choose $x \in \left(0, \frac{1}{2}\right)$. Since $\sum_{\xi \in D_4} (-1)^{|\xi|} H_\tau(\xi^{-1}(x,x), \xi^{-1}(x,x)) = 0$,

$$\begin{aligned}
0 &= \alpha \sum_{\xi \in D_4} (-1)^{|\xi|} \int_{(0,1)^4} (\bar{G}_{\mu_\tau}(\xi^{-1}(x,x), \xi^{-1}(x,x))) \\
&= 2\alpha\{\mu_\tau([x, 1-x) \times [x,1)^3) + \mu_\tau([1-x,1) \times [x,1) \times [x,1-x) \times [x,1)) \\
&- \mu_\tau([x,1-x) \times [1-x,1) \times [x,1) \times [1-x,1)) \\
&- \mu_\tau([1-x,1)^2 \times [x,1-x) \times [1-x,1))\}.
\end{aligned} \tag{102}$$

Letting $x \to 0$ gives us $\mu_\tau((0,1)^4) = 0$. This will cause $\int_{(0,1)^4} (M^2 - W^2) d\mu_\tau = 0$ so that $\mu_\tau \notin d_2$.

4.3 Measures of Concordance Generated by 2-Constructive Functions

When considering Kendall's tau represented as

$$\tau(C) = \sum_{\xi \in D_4} (-1)^{|\xi|} \int_{(0,1)^4} H_\tau(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) d\left(C(\bar{x}) \bigotimes C(\bar{y})\right) \tag{103}$$

where $H_\tau(\bar{x}, \bar{y}) = \begin{cases} \frac{1}{2}, & \bar{x} \leq \bar{y} \\ 0, & \text{otherwise} \end{cases}$ it is natural to ask if we can find a family of functions containing H_τ that generate other degree 2 measures of concordance. Let us call such a family *2-constructive*.

Definition 15. Let $\bar{x}, \bar{y} \in (0,1)^2$. $H : (0,1)^4 \to [0, \infty]$ is *2-constructive* if:

1. for fixed $\bar{x}$, $H(\bar{x}, \bar{y})$ and $H(\bar{y}, \bar{x})$ are each either a distribution or survival function associated with Borel measures on $(0,1)^2$,

2. H is bounded, and

3. $\int_{(0,1)^4} \sum_{\xi \in D_4} \left\{ (-1)^{|\xi|} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) \right\} d\left(M(\bar{x}) \bigotimes M(\bar{y})\right) = 1.$

Besides H_τ being 2-constructive, we see any survival function $\bar{G}_\mu$ associated with finite $\mu \in d_2$ is 2-constructive up to a positive scalar multiple.

Let us further clarify what Borel measure is associated with H as mentioned in the first item of Definition 15. If $\bar{y} = (y_1, y_2) \in (0,1)^2$, define $(0, \bar{y}] = (0, y_1] \times (0, y_2]$ and $[\bar{y}, 1) = [y_1, 1) \times [y_2, 1)$. If $H(\bar{x}, \bar{y})$ is a distribution function for fixed $\bar{x}$, then there exists a Borel measure, $\mu_{\bar{x}}$, on $(0,1)^2$ such that $H(\bar{x}, \bar{y}) = \mu_{\bar{x}}((0, \bar{y}])$. Similarly, if $H(\bar{x}, \bar{y})$ is a survival function for fixed $\bar{x}$, then $H(\bar{x}, \bar{y}) = \mu_{\bar{x}}([\bar{y}, 1))$. The same relationship holds for some Borel measure $\nu_{\bar{y}}$ when $\bar{y}$ is held fixed for $H(\bar{x}, \bar{y})$.

Definition 16. For any $C \in \mathrm{Cop}(2)$, $\bar{C}(x, y) = 1 - x - y + C(x, y)$ is the *survival 2-copula of* C [6].

Note that $\bar{C}(x, y) = \mu_C([x, 1] \times [y, 1])$ where μ_C is the doubly stochastic measure induced by C. We define $\bar{C}^\xi(x, y) = \mu_C(\xi([x, 1] \times [y, 1]))$. Also, let $\bar{x}, \bar{y} \in (0,1)^2$ where $\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2)$, and $\bar{x} \leq \bar{y}$ so we may define $\Delta_{\bar{x}}^{\bar{y}} f = f(y_1, y_2) - f(x_1, y_2) - f(y_1, x_2) + f(x_1, x_2)$. We see for any $C \in \mathrm{Cop}(2)$ that $\Delta_{\bar{x}}^{\bar{y}} C = \Delta_{\bar{x}}^{\bar{y}} \bar{C}$. Therefore for any integrable function, f, defined on $(0,1)^2$ we have

$$\int_{(0,1)^2} f dC = \int_{(0,1)^2} f d\bar{C}. \tag{104}$$

Lemma 12. *Let μ be a finite Borel measure on $(0,1)^2$. For every $\xi \in D_4$ and survival 2-copula, $\bar{C}$,*

$$\int_{(0,1)^2} \bar{C}^{\xi} d\mu = \int_{(0,1)^2} (\bar{F}_\mu \circ \xi^{-1}) d\bar{C} = \int_{(0,1)^2} (\bar{F}_\mu \circ \xi^{-1}) dC \tag{105}$$

where $\bar{F}_\mu(x_1, x_2) = \mu((0, x_1] \times (0, x_2])$.

Proof. Fix $\xi \in D_4$ and survival 2-copula, $\bar{C}$, associated with a continuous random vector, X. For $\bar{x} = (x_1, x_2) \in (0,1)^2$,

$$\bar{C}^{\xi}(\bar{x}) = \mu_C([x_1, 1] \times [x_2, 1]) = \mu_C(\xi^{-1} \circ X \geq \bar{x}). \tag{106}$$

Let us define η in the same context as used in Lemma 10. If we find the η measure of the set $A = \{\xi^{-1} \circ X \geq Y\}$ where X is associated with $\bar{C}$ and Y is associated with μ and repeat the calculations in Lemma 10 it is shown that $\int_{(0,1)^2} \bar{C}^{\xi} d\mu = \int_{(0,1)^2} (\bar{F}_\mu \circ \xi^{-1}) d\bar{C}$. Then by (104) $\int_{(0,1)^2} (\bar{F}_\mu \circ \xi^{-1}) d\bar{C} = \int_{(0,1)^2} (\bar{F}_\mu \circ \xi^{-1}) dC$. □

Theorem 7. *If H is 2-constructive, then*

$$\kappa(C) = \int_{(0,1)^4} \sum_{\xi \in D_4} \left\{ (-1)^{|\xi|} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) \right\} d\left(C(\bar{x}) \bigotimes C(\bar{y}) \right) \tag{107}$$

is either a degree 1 or degree 2 measure of concordance.

Proof. Clearly κ is defined for all $C \in \mathrm{Cop}(2)$ and $\kappa(M) = 1$. By the form of κ it is also immediate that κ is of degree 1 or degree 2, $\kappa(C^h) = -\kappa(C)$, and $\kappa(C^{hr}) = \kappa(C)$.

Notice that

$$\begin{aligned} &\left(C_2(\bar{x}) \bigotimes C_2(\bar{y}) \right) - \left(C_1(\bar{x}) \bigotimes C_1(\bar{y}) \right) \\ &\qquad = \left\{ C_2(\bar{x}) \bigotimes (C_2(\bar{y}) - C_1(\bar{y})) \right\} + \left\{ C_1(\bar{y}) \bigotimes (C_2(\bar{x}) - C_1(\bar{x})) \right\}. \end{aligned} \tag{108}$$

Let $C_1, C_2 \in \text{Cop}(2)$ where $C_1 \leq C_2$. Fixing $\bar{x}$, let $H(\bar{x}, \bar{y})$ be associated with the Borel measure, $\mu_{\bar{x}}$. By Lemmas 10 and 12 either

$$\begin{aligned}
&\int_{(0,1)^2} H(\bar{x}, \bar{y}) dC_1(\bar{y}) = \int_{(0,1)^2} C_1 d\mu_{\bar{x}} \leq \int_{(0,1)^2} C_2 d\mu_{\bar{x}} = \int_{(0,1)^2} H(\bar{x}, \bar{y}) dC_2(\bar{y}), \text{ or} \\
&\int_{(0,1)^2} H(\bar{x}, \bar{y}) dC_1(\bar{y}) = \int_{(0,1)^2} \bar{C}_1 d\mu_{\bar{x}} \leq \int_{(0,1)^2} \bar{C}_2 d\mu_{\bar{x}} = \int_{(0,1)^2} H(\bar{x}, \bar{y}) dC_2(\bar{y}).
\end{aligned} \tag{109}$$

Similarly, fixing $\bar{y}$ and having $H(\bar{x}, \bar{y})$ be associated with some Borel measure, say $\nu_{\bar{y}}$, we again have either

$$\begin{aligned}
&\int_{(0,1)^2} H(\bar{x}, \bar{y}) dC_1(\bar{x}) = \int_{(0,1)^2} C_1 d\nu_{\bar{y}} \leq \int_{(0,1)^2} C_2 d\nu_{\bar{y}} = \int_{(0,1)^2} H(\bar{x}, \bar{y}) dC_2(\bar{x}), \text{ or} \\
&\int_{(0,1)^2} H(\bar{x}, \bar{y}) dC_1(\bar{x}) = \int_{(0,1)^2} \bar{C}_1 d\nu_{\bar{y}} \leq \int_{(0,1)^2} \bar{C}_2 d\nu_{\bar{y}} = \int_{(0,1)^2} H(\bar{x}, \bar{y}) dC_2(\bar{x}).
\end{aligned} \tag{110}$$

If $C_1 \leq C_2$, then $C_1^\xi \leq C_2^\xi$ for $|\xi|$ even. Therefore we may say more generally from (109) and (110) that for all $|\xi|$ even

$$\begin{aligned}
&\int_{(0,1)^2} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) dC_1(\bar{x}) \leq \int_{(0,1)^2} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) dC_2(\bar{x}), \text{ and} \\
&\int_{(0,1)^2} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) dC_1(\bar{y}) \leq \int_{(0,1)^2} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) dC_2(\bar{y}).
\end{aligned} \tag{111}$$

Similarly, if $C_1 \leq C_2$, then $C_1^\xi \geq C_2^\xi$ for all $|\xi|$ odd. Therefore for all $|\xi|$ odd,

$$\begin{aligned}
&\int_{(0,1)^2} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) dC_1(\bar{x}) \geq \int_{(0,1)^2} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) dC_2(\bar{x}), \text{ and} \\
&\int_{(0,1)^2} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) dC_1(\bar{y}) \geq \int_{(0,1)^2} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y})) dC_2(\bar{y}).
\end{aligned} \tag{112}$$

Finally, (111), (112), and the form of κ give us $\kappa(C_1) \leq \kappa(C_2)$.

Lastly, we will address the continuity property for a measure of concordance. Choose $C \in \text{Cop}(2)$. Let $\{C_n\}$ be a sequence in Cop(2) where $C_n \to C$. Note that $C_n^\xi \to C^\xi$ for every $\xi \in D_4$ as well since from Table 3 we see that $||C^\xi - C_n^\xi||_\infty = ||C - C_n||_\infty$. Letting K

denote an upper bound of H gives us $\mu_{\bar{x}}((0,1)^2), \nu_{\bar{y}}((0,1)^2) \leq K$ for every $\bar{x}, \bar{y} \in (0,1)^2$. For any $\xi \in D_4$,

$$\begin{aligned}
&\left|\int_{(0,1)^4} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y}))d\left(\left(C(\bar{x})\bigotimes C(\bar{y})\right) - \left(C_n(\bar{x})\bigotimes C_n(\bar{y})\right)\right)\right| \\
&= \left|\int_{(0,1)^4} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y}))d\left((C(\bar{x}) - C_n(\bar{x}))\bigotimes C(\bar{y})\right)\right| \\
&+ \left|\int_{(0,1)^4} H(\xi^{-1}(\bar{x}), \xi^{-1}(\bar{y}))d\left((C(\bar{y}) - C_n(\bar{y}))\bigotimes C_n(\bar{x})\right)\right| \\
&= \left|\int_{(0,1)^2}\left\{\int_{(0,1)^2}(C^{\xi} - C^{\xi}_n)(\bar{x})d\nu_{\xi^{-1}(\bar{y})}(\bar{x})\right\}dC(\bar{y})\right| \\
&+ \left|\int_{(0,1)^2}\left\{\int_{(0,1)^2}(C^{\xi} - C^{\xi}_n)(\bar{y})d\mu_{\xi^{-1}(\bar{x})}(\bar{y})\right\}dC_n(\bar{x})\right| \leq 2K||C - C_n||_{\infty}.
\end{aligned} \tag{113}$$

Choosing $\epsilon > 0$, there exists N such that $||C - C_n||_{\infty} < \frac{\epsilon}{16K}$ for all $n > N$. Therefore, $\kappa(C_n) \to \kappa(C)$. □

4.4 Questions for Further Examination

1. Does there exist an infinite measure in d_2 generating Kendall's tau?
2. Can we relax the condition of 2-generating functions being bounded in order to generate an even larger family of measures of concordance?
3. What characterizes degree 2 measures of concordance?
4. What characterizes degree n measures of concordance?

REFERENCES

[1] Carley, H., and Taylor, M. D., *A new Proof of Sklar's Theorem,* Proceedings of the Conference on Distributions with Given Marginals and Statistical Modelling, (eds: C. M. Cuadras, J. Fostiana, and J. A. Rodriguez-Lallena), Barcelona, 2000, 29-34.

[2] Edwards, H. H., Mikusiński, P., and Taylor M. D., *Measures of Concordance Determined by D_4-Invariant Copulas*, International Journal of Mathematics and Mathematical Sciences, to appear.

[3] Edwards, H. H., Mikusiński, P., and Taylor M. D., *Measures of Concordance Determined by D_4-Invariant Measures on* $(0,1)^2$, Proceedings of the American Mathematical Society, to appear.

[4] Khosravi, M., personal correspondence.

[5] Mikusiński, P., Sherwood, H., and Taylor, M. D., *The Fréchet Bounds Revisited*, Real Analysis Exchange, **17** (1991-92), 759-764.

[6] Nelsen, R., (1999), *An Introduction to Copulas*, Springer-Verlag, New York.

[7] Royden, H. L., (1988), *Real Analysis*, 3rd ed., Prentice-Hall, Englewood Cliffs.

[8] Scarsini, M., *On Measures of Concordance*, Stochastica, VIII, (1984), 201-218.

[9] Schweizer, B., and Sklar, A., (1983), *Probabilistic Metric Spaces*, North-Holland, New York.

[10] Taylor, M. D., unpublished manuscript, 2004.

[11] Taylor, M. D., personal correspondence, 2004.