Fade Time Statistics of Laser Light Propagating Through Turbulent Atmosphere

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Date: Winter 1980

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ABSTRACT

The average fade time of the fluctuations in the intensity of laser light propagation in the atmosphere is analyzed. Experimental results show that the average fade time for a propagation distance of 208 meters, in the weak turbulence regime, is well described by formulas developed from a Log-Normal distribution. A circuit that calculates real time statistics of fade time is designed, then validated using a sine wave input whose fade time statistics are predetermined. The experimental signal is then tested for stationarity, and the average fade time is measured. A curve fitting is then performed for the experimental data and compared theoretical results.
FADE TIME STATISTICS OF LASER LIGHT PROPAGATING THROUGH TURBULENT ATMOSPHERE

BY

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B.S.E., University of Central Florida, 1978

THESIS

Submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering in the Graduate Studies Program of the College of Engineering at the University of Central Florida; Orlando, Florida

Winter Quarter
1980
ACKNOWLEDGEMENTS

The author wishes to express appreciation to the many people who contributed toward the preparation of this thesis. Dr. Ronald Phillips deserves special thanks for his valuable advice, guidance, and time. Dr. Bruce Mathews and Dr. Michael Harris gave advice and helpful suggestions all along the development of this thesis. Finally, much thanks is given to Dian Brandstetter for many hours of typing.
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I - INTRODUCTION

This thesis describes the average fade time statistics of the intensity of a CW laser beam propagating through turbulent atmosphere. The type of turbulence considered is scintillation. Scintillation is caused by changes of the earth's temperature, which cause changes in air temperature. Hot air rises and mixes with colder air and turbulence originates. There is no net energy loss from the beam intensity due to scintillation; but rather the energy is randomly redistributed in space.

Previous experimental and theoretical work showed that the average fade time statistics for weak turbulence were well described by a Log-Normal distribution. In the present investigation, a special circuit that measures this fade time is designed and tested for confidence using a sine wave input. The average fade time of the sine wave was first obtained graphically and then measured using the designated circuit. This circuit is also able to measure the median value of an incoming signal, as well as checking for stationarity.

The experimental input signal, which is proportional to the fluctuating laser beam intensity, is first checked for stationarity. Then the average fade time is measured. Finally, this thesis also
includes a comparison with some theoretical work on the same data. The laser beam was assumed to be an optional plane wave in the theoretical model.
II - EXPERIMENT

1. Experimental Conditions

The experiment was conducted in conjunction with the Royal Signals and Radar Establishment in August, 1978 (Parry and Pusey 1978) over an aircraft runway in Malvern, United Kingdom. The weather was fair with sunshine and some light clouds. The temperature ranges from 15 to 23°C. The wind speed was recorded to be 3 meters per second. The wind direction was normal to the beam direction as shown in Figure 1.

Two hundred meters down range from the transmitter, a high gain photo-multiplier tube with an effective aperture of 50 micrometers was used on a detector. The detected signal was fed to an attenuator, and finally to an FM tape recorder. The signal was also fed to a real time processing system, and the normalized second moment of the signal were measured. A CW (continuous wave) HeNe laser ($\lambda = 633$ nm) with a natural beam divergence of 1 mrad with no additional optics was used as the source.

The results of atmospheric stratification and the normal variations with height of the atmosphere are eliminated. This is assumed because the beam height from the ground was about 1 meter throughout the propagation path. Any absorption by gaseous constituents as well as scattering or absorption by aerosols or humidity were negli-
Figure 1. Experimental Layout
gible. The assumed point detector (50 mm pin hole) was also shielded against the sunlight such that any kind of undesired bias was eliminated.

2. Data Analysis Procedure

In analyzing the signal, the laser fluctuations are seen, heard, and processed. The signal played back from the FM tape is fed through a low pass filter to attenuate any kind of noise superimposed on the tape (the bandwidth of the signal extends from DC to 300 Hertz). Then both the AC and DC portions of the signal are amplified with a gain of 5. The output of the pre-amplifier goes to an oscilloscope for monitoring purposes, the circuit design for fade time measurements, and finally to an AC pre-amplifier. The output of the AC pre-amplifier is coupled to a speaker after going through a power amplifier. The output of the AC-DC pre-amplifier is coupled with the audio signal from the speaker and a TV tape was made. Figure 2 depicts the analysis layout.
Figure 2. Hardware Analysis Set Up
1. Scintillation

The CW laser outputs a constant intensity signal. This signal propagates through the scattering channel and it is then detected as an intensity fluctuating rather than a constant signal. The propagation channel (atmosphere) is characterized by sunshine, clear air, no fog, and no rain. Under these conditions the cause of these fluctuations in laser light is turbulence due to random energy redistribution in the atmosphere. This fluctuating phenomena is known as scintillation (Lawrence and Strohbehn 1970). Typical received intensity signal is depicted in Figure 4.

I(t)
\[ I_0 \]
\[ t \]

Figure 3. Transmitted Signal

I(t)
\[ t \]

Figure 4. Received Intensity Signal
Scintillation can be caused by either random changes in pressure or random changes in temperature of the atmosphere. The beam was about 1 meter above the ground along the entire path, hence the beam deflections due to gradient pressure changes can be neglected. Temperature, however, is the parameter that causes the strongest scintillation. Temperature affects the light beam because of the change in index of refraction of the atmosphere from one point to the other. Therefore, a region with a higher temperature would have a lower refraction index than a region with a lower temperature (higher refraction index) (Strohbehn 1975). Strohbehn showed that a change in refraction index is due to change in temperature. This is given in the following equation

\[ \Delta N = \frac{-79 P}{v-1} \frac{\Delta T}{T} \]

where \( v \) = ratio of specific heat (=1.4 for air)

\[ P = \text{pressure in millibars} \]

\[ T = \text{absolute temperature, } ^\circ\text{K} \]

\[ N = \text{refractivity in parts per million} \]

A medium with a high index of refraction scatters more light intensity than a medium with a lower index of refraction. As the laser beam propagates through the atmosphere it encounters regions with different temperatures. The fluctuating intensity of the light beam is due to the beam interfacing with itself. The speed of these fluctuations depends on many parameters, including wavelength of the laser beam, distance of propagation, and the wind velocity (speed
and direction). The instantaneous random state of these fluctuations, observed in a plane parallel to the wave front, displays spatial frequency components with a predominant scale, $f_{zs}$, known as Fresnel-Zone size,

$$f_{zs} = \sqrt{\frac{\lambda}{L}}$$

where $\lambda$ is the wavelength of the laser beam, and $L$ is the distance of propagation. The frequency of the fluctuations depends on the normal components of the wind that moves the turbulence across the beam. Mainly the frequency of fluctuations is equal to the normal component of the wind speed divided by the Fresnel-Zone size:

$$f_o = \frac{V_n}{\sqrt{\lambda L}}$$ in Hertz

This assumption is called the frozen turbulence hypothesis. It assumes the random fluctuation of the air turbulence flown across the beam, and internal mixing are neglected. This frequency $f_o$ is the maximum frequency of the fluctuations since the Fresnel-Zone size is the predominant scale size (Pratt 1974). For the HeNe laser used $\lambda$ was equal to 633 nanometers and $L$ was equal to 208 meters. The wind speed was $V_n = 3$ m/s and the direction was normal to the beam direction, therefore:

$$f_o = \frac{3 \text{ m/s}}{\sqrt{633 \times 10^{-3} \text{ m} \times 208 \text{ m}}} = 262 \text{ Hertz}$$

This calculated frequency justifies the cut-off frequency of 300 Hertz in the low pass filter of Figure 14. It is also known that the effects of scintillation on the laser beam result only in a
redistribution of the energy in the beam. There is no net energy
lost.

It has been shown that random scintillation causes the amplitude
of the laser signal received by the point detector to be Log-normally
distributed (Parry 1979). To understand this distribution, consider
the turbulent medium as composed of a large number of independent
globes, called turbulent eddies in the propagation path. These
eddies or inhomogenieties act as a continuous random ensemble of
lenses as shown in Figure 5.

Figure 5. Channel Simulation

As the beam propagates through these eddies, some of the beam energy
is scattered and most of it goes through unscattered. This scatter-
ing phenomena is called the law of "L'Effet Proportionnel" (Strohbehn
1975) and is formulated by

\[ a_i = a_{i-1} + \varepsilon_i a_i \]

where \( a_i \) is the amplitude of the signal at globe \( i \), and \( \varepsilon_i \) is the
scattering coefficient assumed much less than 1. The above expres-
sion can be rewritten as \( \frac{a_i - a_{i-1}}{a_i} = \varepsilon_i \). This scattering phenomena
occurs at each eddie. Since we have \( N \) eddies throughout the channel,
at the end of the propagation path the law stated above can be for-
mulated as the sum of all the scatterings due to the $N$ eddies in the channel, i.e.,

$$\sum_{1}^{N} \frac{a_{i} - a_{i-1}}{a_{i}} = \sum_{1}^{N} \varepsilon_i$$

In the limit $a_{i} - a_{i-1} = d_{a_{i}}$ and $\sum \varepsilon_i = E$. But the channel is a continuum and hence the summation sign becomes an integration sign and so:

$$\int_{A_{0}}^{A_{N}} \frac{d_{a_{i}}}{a_{i}} = E$$

where $E$ is a random variable, and $A_{N}$ and $A_{0}$ are the respective amplitudes of the signal at the transmitter and receiver ends. Now using the central limit theorem, it is concluded that $E$ is normally distributed. The expression for $E$ is written as

$$E = \ln A_{N} - \ln A_{0} = \ln \frac{A_{N}}{A_{0}}.$$ 

Therefore, scintillation scatters the smooth amplitude of the CW laser beam into a Log-Normal distribution, because the natural logarithm of the amplitude is normally distributed.

2. Random Variables

In this study two characteristics of random variables are of interest: stationarity and fade time.

a. Stationarity and Non-Stationarity Random Processes

A random process is said to be stationary if all marginal and joint density functions do not depend on the choice of time
origin. Consequently, all the moments of the process are constant and do not depend on the absolute value of time. Checking for stationarity is not simple, especially for non-deterministic processes. In fact, there is no stationary process that can physically exist if we stick rigorously to the above definition, because any process must have started sometime in the past, and must terminate sometime in the future. Taking this last remark into consideration, another less stringent definition is often used. If the mean value of any random variable, \( X(t_1) \), is independent of the choice of \( t_1 \) and that the cross-correlation of two random variables, \( X(t_1) \) and \( X(t_2) \) depends only upon the time difference, \( t_2 - t_1 \). A process that satisfies these two conditions is said to be stationary in the "wide sense" (Panter 1972). This definition of stationarity also means that all the moments do not depend upon the absolute value of time. However, they do depend on the duration of the sample function. For example, a random process can be stationary for 10 minutes, but may not be stationary for 20 minutes. As an illustration, consider recording the average temperature on a sunny day from 12 noon to 12:05. The sample function duration is 5 minutes. Within a few tenths of a degree the temperature remains constant during this period of time, i.e., the mean temperature is constant and insuring stationarity. Now consider recording the temperature under the same conditions, but with the process duration changed. The process starts at noon and ends at 6 p.m. The temperature decreases with sundown and hence the mean temperature is not constant. There-
fore, this process is stationary for 5 minutes, but is not stationary for 6 hours.

b. Fade Time

Since there is no net energy loss due to scintillation only a spatial redistribution, one way to minimize the fluctuations would be to use a large receiver aperture that will average out many of those fluctuations. However, a large receiver aperture may be expensive and may not be field deployable. An alternate technique is to use many small receiver apertures in parallel and then use their separate outputs in an OR-ing digital circuit. These techniques are called spatial diversity techniques. This spatial diversity technique apparently to change the statistics since the central limit theorem states that the distribution of the sum of independent variates must converge to a normal distribution. For the case of Log-Normal distribution, the central limit theorem is weak (Mitchell 1968) because of the skewness and the large tail of the Log-Normal distribution as noted on Figure 9. The actual detector used in the experiment was a point detector where threshold detection is the key.

The receiver will detect only portions of the signal greater in amplitude than its threshold level. If a continuous optical wave is launched from a source and if it then propagates through a turbulent atmosphere, it is received as a fluctuating signal. When the signal drops below this threshold level it cannot be recovered and whatever information there may have been is lost. The time for which the
signal level is lower than the threshold level is called "fade time".

\[ V_r \quad \text{and} \quad V_{th} \]

**Figure 6.** Fade Time of a Random Signal where \( V_r \) is the Received Signal and \( V_{th} \) is the Threshold Level

The purpose of the CW laser is to monitor the channel. Placed parallel to the CW laser is a pulsed laser where useful modulated information is transmitted. The two laser beams are propagating in the same channel and under the same conditions. The pulsed laser beam undergoes the same energy redistribution due to scintillation as the CW laser beam. If we know the statistics of the fade time in the CW laser beam, we can optimize the detector threshold level. Finally the data rate for the pulsed laser can be determined to yield better and more efficient detection. Once the average fade time for a certain threshold level is obtained, the data rate can be set to a certain number of bits to minimize the losses due to fading. The average fade time depends on the threshold level. One usual way to minimize the fade time is to decrease the threshold level for most statistical channels. However, if the threshold level is lowered, the probability of noise instead of signal triggering the detector is increased.
Figure 7. Gaussian Probability Density Function
Figure 8. Gaussian Cumulative Probability Distribution Function

P(x < x₁)
3. Normal Distribution

The most important probability distribution function in statistical analysis applied to communication systems is the Normal distribution which is derived from the binomial distribution $f_N(x)$ as $N \to \infty$.

There are many reasons for the importance of the Gaussian probability function. Mainly it provides a good mathematical description for a large number of different physically observed random phenomena. Another merit to the Gaussian probability function is the existence of a remarkable characteristic summarized by a theorem called "The Central Limit Theorem".

The probability density function of a Gaussian random variable (single variable) is depicted in Figure 7 and is given by

$$P(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$

where $x_0$ is the mean, and $\sigma^2$ is the variance.

The probability distribution function is depicted in Figure 10 and is given by

$$P(x \leq X) = \int_{-\infty}^{X} P(x) \, dx = \int_{-\infty}^{X} \frac{1}{\sqrt{2\pi} \sigma} \exp \left(-\frac{(x-x_0)^2}{2\sigma^2}\right) \, dx$$

This integral cannot be evaluated in closed form, but the error function has been defined as

$$\text{Erf}(u) = \frac{2}{\sqrt{\pi}} \int_{0}^{u} \exp(-y^2) \, dy \quad \text{(Ziemer 1976)}$$

So

$$P(x \leq X) = \int_{-\infty}^{X} P(x) \, dx = \frac{1}{2}(1 + \text{erf}\left(\frac{X-x_0}{2\sigma}\right))$$
4. Log-Normal Distribution

Since the fade time of the beam fluctuations is predicted to be Log-Normally distributed, the Log-Normal density function, as well as the cumulative function explained along with their respective graphs.

Let $x$ be a random variable normally distributed. If we define a new random variable $A$, such that $A = e^x$, this new random variable is said to have a Log-Normal distribution. Taking the natural logarithm of $A$,

$$x = \ln(A)$$

and thus the natural logarithm of $A$ is normally distributed where $A$ is the amplitude of the received signal. Now, using the transformation, $P(A) \ dv = P(x) \ dx$

since $x = \ln(A)$ \quad $\frac{dx}{dA} = \frac{1}{A}$

and (1) becomes

$$P(A) = \frac{1}{\sqrt{2\pi} \sigma_{\ln A} A} \exp \left\{ -\frac{1}{2} \left( \ln(A) - \ln(A_0) \right)^2 + \frac{\sigma^2_{\ln A}}{2} \right\}$$

$$\ln(A) = \ln(A_0) - \frac{\sigma^2_{\ln A}}{2}$$

$A_0$ is the amplitude of signal without scintillation.

$$P(A) = \frac{1}{A \sqrt{2\pi} \sigma_{\ln A}} \exp \left\{ -\frac{1}{2} \left( \ln(A) - \ln(A_0) \right)^2 + \frac{\sigma^2_{\ln A}}{2} \right\}$$
Now rearranging the above equation
\[
P(A) = \frac{1}{A \sqrt{2\pi} \sigma_{\ln(A)}} \exp \left\{ -\left( \frac{\ln \frac{A}{A_0}}{\sigma_{\ln(A)}^2} \right)^2 \right\}
\]

Now defining a new random variable, I, which is the intensity of the laser beam, i.e.,
\[I = A^2, \text{ and } I_0 = A_0^2\]

using the transformation
\[P(I) = P(A) \left| \frac{dA}{dI} \right|\]

where
\[\frac{dA}{dI} = \frac{1}{2A}\]

\[P(I) = \frac{1}{\sqrt{I} \sqrt{2\pi} \sigma_{\ln(A)}} \exp \left\{ -\left( \frac{\ln \left( \frac{I}{I_0} \right)}{2 \sigma_{\ln(A)}^2} \right)^2 \right\} \frac{1}{2 \sqrt{I}}\]

rearranging this expression
\[P(I) = \frac{1}{I \sqrt{2\pi} (2 \sigma_{\ln(A)})} \exp \left[ -\left( \ln \left( \frac{I}{I_0} \right) + \frac{\sigma_{\ln(A)}^2}{2} \right) / 2 \right] \frac{1}{2 \sqrt{I}}\]

The intensity of a signal is equal to its amplitude square mainly
\[I = A^2\]

therefore
\[
\ln I = 2 \ln A
\]

then it follows that
\[
\ln I = 2 \ln A
\]
and
\[\frac{(\ln I)^2}{2} = 4 \left( \frac{\ln A}{2} \right)^2\]
by definition the variance of \(\ln I\) and \(\ln A\) are given by
\[ \sigma^2_{\text{LnI}} = (\ln I)^2 - (\ln I)^2 \]

and

\[ \sigma^2_{\text{LnA}} = (\ln A)^2 - (\ln A)^2 \]

hence

\[ \sigma^2_{\text{LnI}} = 4(\ln A)^2 - (2 \ln A)^2 \]

\[ \sigma^2_{\text{LnI}} = 4(\ln A)^2 - 4(\ln A)^2 \]

\[ \sigma^2_{\text{LnI}} = 4(\ln A)^2 - (\ln A)^2 \]

\[ \sigma^2_{\text{LnI}} = 4 \sigma^2_{\text{LnA}} \]

and

\[ \sigma_{\text{LnI}} = 2 \sigma_{\text{LnA}} \]

Therefore the probability density function of \( I \) becomes

\[ P(I) = \frac{1}{I \sqrt{2\pi} \sigma_{\text{LnI}}} \exp \left[ -\left( \ln \frac{I}{I_0} + \frac{1}{4} \frac{\sigma^2_{\text{LnI}}}{\text{LnI}} \right)^2/2 \right] \]

The numerical values of \( \sigma_{\text{LnI}} \) and \( I_0 \) are calculated in Appendix B and are

\[ \sigma^2_{\text{LnI}} = 0.0256 \]

\[ \sigma_{\text{LnI}} = 0.16 \]

\[ I_0 = 0.68 \]

Then \( P(I) \) becomes

\[ P(I) = \frac{2.5}{I} \exp \left[ -\left( 4.43 \ln \left( \frac{I}{I_0} \right) + 0.028 \right)^2 \right] \]

This density function is depicted in Figure 9. Note that it is skewed to the right. The cumulative probability function is given by:

\[ P(I < I_t) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\ln \left( \frac{I_t}{I_0} \right) + \frac{1}{4} \frac{\sigma^2_{\text{LnI}}}{2 \sigma_{\text{LnI}}}}{\sqrt{2} \sigma_{\text{LnI}}} \right) \right] \]
plugging in the numerical values of $I_0$, $\sigma_{\text{Ln}I}^2$, and $\sigma_{\text{Ln}I}^2$, $P(I < I_t)$ is:

$$P(I < I_t) = \frac{1}{2} + \frac{1}{2} \text{erf}[4.4 \ln\left(\frac{I}{I_0}\right) + 0.028]$$

This cumulative distribution function is depicted in Figure 12.

Also note that the mean value is not $I_0$ but is $I_0 e^{\sigma_{\text{Ln}I}^2}$, because the probability density function is normalized with respect to $A$ (amplitude) and not with respect to $I$ (intensity).

5. Circuit Analysis

In almost all circuit design, a thorough analysis is necessary. The main purpose of this thesis is the analysis of the fade time statistics of a received laser beam intensity signal. The circuit design is a means of achieving this goal, hence, a block diagram analysis along with a functional discussion is given.

Block Diagram

The circuit is designed to measure the fade time statistics of a CW laser fluctuation transmitted over a distance of 200 meters. Low frequency components are associated with short distances of propagation under certain atmospheric conditions (temperature, wind speed and direction, humidity, fog, ...).

To demonstrate the accuracy of the circuit, a 500 Hertz sine wave input is considered.

The circuit was built with on-shelf components, and the total cost was kept under $15. The only requirements are an external clock, and a power supply.
Figure 9. Log-Normal Probability Density Function
Figure 10. Log-Normal Cumulative Probability Distribution Function
A sine wave is fed to the input of the circuit. Then each block is taken separately and the input and output waveforms are drawn below it to clarify the function of each one of the blocks.
Figure 11. Circuit Block Diagram
Low Pass Filter

The cut-off frequency of this low pass filter is set to 500 Hertz so that the applied input will pass through with no distortion. This type of active filter configuration is called the sallien-key model (Roberge 1975).

Figure 12. Low Pass Filter and Input, Output Waveforms
Zero Crossing

The sine wave input is fed directly to the comparator, bypassing the low pass filter (since there is no noise in the sine wave), such that it has zero DC offset. If the voltage reference is set equal to zero, the output of the open loop amplifier responds discontinuously every time the input crosses that reference level (Millman 1972).

![Diagram of Zero Crossing Circuit](image)

Figure 13. Zero Crossings Circuit with Input and Output Waveforms
Inversion

Along with the inversion an amplifier is also needed because the magnitude of the zero crossing pulses depend on the frequency of the incoming signal.

Figure 14. Inverting Amplifier with Input and Output Waveforms
Clipper Limiter

Since the real time signal to be analyzed has more than one frequency component, the amplitude of the pulses is variable; the gain of 2 in the above inversion might make some of these pulses higher than 5 volts (TTL incompatible). The limiter will limit the maximum amplitude to 5 volts; the clipper will pass only positive pulses (Pierce 1972).

![Diagram of clipping and limiting circuit with input and output waveforms](image)

Figure 15. Clipping and Limiting Circuit with Input and Output Waveforms
Counter

The output of the limiter is fed to the first counter as a clock. The output of this counter will be read in binary from an array of twenty LED's. When this number is converted to decimal, the number of times the input went below the reference level is obtained.

Counter 3

This counter is free running; it will start counting when the signal if fed to the low pass filter. The frequency of the clock feeding this counter is set to 10 kHz. In fact, this is nothing but an event counter.

V4(t) is inverted again, clipped and limited to TTL level.

\[ V_4(t) \]

\[ V_6(t) \]

Figure 16. Input and Output Waveforms of Second Clipping Circuit
Buffer

A buffer is a unity gain amplifier that has a high input impedance and a low output impedance. Thus, a buffer will not affect the signal in any way; it is an impedance matcher and therefore will prevent any type of loading.

Figure 17. Buffer
Pulsed Mode Circuit

$V_5(t)$ and $V_6(t)$ are fed to a pulsed mode circuit. The output of this circuit is triggered to a logic high only when $V_6(t)$ is high. This output stays in that state (high) until $V_5(t)$ goes high, which will trigger it to a low state again. It is obvious that, in pulsed mode circuit, both inputs cannot occur at the same time. In the analysis of the signal, this requirement is guaranteed because both negative and positive zero crossings cannot occur at the same time (Hill and Peterson 1975).

Figure 18. Pulsed Mode Circuit
Figure 19. Input and Output Waveforms of Pulsed Mode Circuit. $v_5$ and $v_6$ are Inputs $v_7$ is the Output Switch

$v_7(t)$ is fed to an electronic switch (FET) that will control the clock ($f = 10$ kHz) on whenever $v_7(t)$ is high and turn it off whenever $v_7(t)$ is low (Schilling and Belove 1979).

Figure 20. Input Output of the FET Switch. $v_7$ is the Input, $v_8$ is the Output.

Note that $v_8(t)$ is made of clock pulses that will trigger counter number 2.
Counter 2

\(V_g(t)\) is fed to a counter that will count the clock pulses depicted on Figure 22. Again the number of cumulative count is read from an array of 20 LED's, and knowing the clock frequency, the total time for which the input signal stays below the set reference level is obtained. The average fade time is obtained by dividing the total fade time by the number of negative zero crossings (output of counter).
Figure 21. Fade Time Circuit Analyzer
1. Fade Time of Simple Sine Wave Input

The purpose of a sine wave input, whose fade time can be determined mathematically, is to check the accuracy of the circuit. Let this input be $X(t) = 5 + \sin(wt)$, where $w = 2\pi f$ with $f = 500$ Hertz.

$X(t)$ is well-known, well defined, and deterministic, therefore, it is stationary.

The fade time of a given sine wave as well as the mean value are obtained by two different methods:
a. Theoretical Calculation of Fading of Sine Wave

Since $X(t)$ is periodic, the average fade time is defined as the time for which it stays below $V_r$ for one period.

$$V_r = 5 + \sin(2\pi ft)$$

Since $f$ is set to be 500 Hz, then the equation becomes

$$V_r - 5 = \sin(1000\pi t)$$

For each value of $V_r$ the equation is solved for $t$. There are always two solutions, $t_1$ and $t_2$, because $V_r$ crosses $X(t)$ twice with a period. The average fade time is the difference between $t_1$ and $t_2$. For example for $V_r = 5$ volts we have $5 = 5 + \sin (2\pi ft)$, which leads to $\sin(2\pi ft) = 0$, and the solution is $2\pi ft = 2\pi$ and $2\pi ft = \pi$.
which leads to $t_1 = \frac{2\pi}{2\pi f}$ and $t_2 = \frac{\pi}{2\pi f}$. Then the average fade time is

$$t_1 - t_2 = \frac{\pi}{2\pi f} [2 - 1] = \frac{1}{2f} = \frac{T}{2}$$

The average fade time is calculated and tabulated in Table 1 for each value of $V_r$. 
b. Experimental Measurement of Fading of Sine Wave

The input sine wave is fed to the circuit. The number of negative zero crossings, the cumulative fade time, and the total time is recorded for each value of the reference voltage. Then the reference voltage is incremented from zero to 6 volts. Then the average fade time is the cumulative fade time divided by the number of negative zero crossings. For $V_r = 0$, the average fade time is zero. For $V_r = 6$ volts, the average fade time is equal to the total time of the experiment. Both theoretical and experimental results of the average fade time are tabulated in the following table.
### TABLE 1. Average Fade Time Versus $V_r$

<table>
<thead>
<tr>
<th>$V_r$ (volts)</th>
<th>Average Fade Time Calculated (msecs)</th>
<th>Average Fade Time Measured (msecs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2</td>
<td>0.46</td>
<td>0.456</td>
</tr>
<tr>
<td>4.3</td>
<td>0.52</td>
<td>0.576</td>
</tr>
<tr>
<td>4.6</td>
<td>0.75</td>
<td>0.780</td>
</tr>
<tr>
<td>4.7</td>
<td>0.80</td>
<td>0.750</td>
</tr>
<tr>
<td>4.8</td>
<td>0.90</td>
<td>0.850</td>
</tr>
<tr>
<td>4.9</td>
<td>0.92</td>
<td>0.840</td>
</tr>
<tr>
<td>5.0</td>
<td>1.00</td>
<td>1.020</td>
</tr>
<tr>
<td>5.2</td>
<td>1.15</td>
<td>1.100</td>
</tr>
<tr>
<td>5.4</td>
<td>1.25</td>
<td>1.250</td>
</tr>
<tr>
<td>5.6</td>
<td>1.40</td>
<td>1.460</td>
</tr>
<tr>
<td>5.7</td>
<td>1.55</td>
<td>1.550</td>
</tr>
<tr>
<td>5.9</td>
<td>1.70</td>
<td>1.690</td>
</tr>
</tbody>
</table>
Figure 24. Average Fade Time for Sine Wave

- calculated

x measured
2. Experimental Input Signal

The signal is played back from the tape through an amplifier (gain = 5) before it is fed to the circuit analyzer. The total time duration of the signal was approximately one minute. In order to measure the cumulative fading, the signal was played a number of times while varying $V_r$ each time. Hence, a number of sample functions were generated from the random process. If the process was stationary, all these sample functions exhibit the same statistical properties. On the other hand, if the process was non-stationary, the situation is quite different. For any type of statistical measurement to be valid, one must have the same sample function every time the value of $V_r$ is changed. Physically this is difficult to achieve, since one must start the tape recorder and stop it at exactly the same instant for each sample function.

a. Stationarity

Since the average fade time is being measured for a period of 50 seconds, it was necessary to determine whether the process was stationary for the same period of time or not.

The reference level $V_r$ was set to 4.1 volts. The average fade time of the signal was measured ten times with the special circuit. Each sample function starts and stops at slightly different random times. Ten sample functions were thus generated from the random process or the ensemble. The following is a listing of average fade time of the different sample functions.
<table>
<thead>
<tr>
<th>Sample Function</th>
<th>Average Fade Time (msec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.63</td>
</tr>
<tr>
<td>2</td>
<td>6.14</td>
</tr>
<tr>
<td>3</td>
<td>5.47</td>
</tr>
<tr>
<td>4</td>
<td>5.77</td>
</tr>
<tr>
<td>5</td>
<td>4.67</td>
</tr>
<tr>
<td>6</td>
<td>5.98</td>
</tr>
<tr>
<td>7</td>
<td>4.72</td>
</tr>
<tr>
<td>8</td>
<td>4.56</td>
</tr>
<tr>
<td>9</td>
<td>4.43</td>
</tr>
<tr>
<td>10</td>
<td>4.01</td>
</tr>
</tbody>
</table>

From this table of values, it is difficult to decide whether the process is stationary or not. One way of looking at this problem is to consider the average fade time obtained as a new random variable with a mean and a standard deviation. Then one can look at the variance, which measures the spread around the mean, and then make the necessary assumption. For laboratory results, this latter
method does not need to be considered, and the average fade time is assumed to be fairly constant. Hence, the process is considered stationary.

b. Average Fade Time

The average fade time was measured for different values of $V_r$ and is tabulated in Appendix C.

The circuit also measured the median. The median is defined as the value of $V_r$ for which the cumulative fade time is equal to half the total time duration of the sample function being analyzed. For a random variable Log-normally distributed the median is different from the mean. More details are given in Appendix B.

3. Comparison with Theoretical Work

The laser beam was modeled by an optical plane wave and the average fade time was found to be (O'Hara 1980)

$$t(V_r) = 4.3 \times 10^{-6} \left\{ \frac{1 + \text{erf} \left[ \frac{\ln \left( \frac{V_r}{I_0} \right) + \frac{\sigma^2_{\text{LnA}}}{2.83}}{\frac{\sigma_{\text{LnA}}}{2}} \right]}{\exp \left[ \frac{(\ln \left( \frac{V_r}{I_0} \right) + \frac{\sigma^2_{\text{LnA}}}{2})^2}{-8 \frac{\sigma^2_{\text{LnA}}}{2}} \right]} \right\}$$

Where $V_r$ is the set threshold level and $I_0$ is the laser beam intensity without scintillation.

Figure 15 compares the experimental results with the theoretical results.
In the above expression \( \sigma^2_{\text{LnA}} \) is the variance of the Log Amplitude of the laser beam. \( I_0 \) and \( \sigma^2_{\text{LnA}} \) were not recorded when the data was put on tape due to a shift of the DC bias in the FM tape. A non-linear regression was performed to fit the experimental data to the theoretical expression. The subroutine used was extracted from "BMPD" - Biomedical Computer Programs (Engelman 1977). BMPD was developed at the Health Sciences computing facility at the University of California at Los Angeles. The routine used was BMD PAR. PAR is a derivative-free non-linear regression that estimates the parameters of any function that can be specified by FORTRAN statements. The partial derivatives with respect to the parameters do not need to be specified (Engelman 1977). The parameters are estimated by a pseudo-Gauss-Newton iterative algorithm. At each iteration, the program prints the residual sum of squares and estimates of the parameters. The residuals, predicted values, as well as the observed values of all the variables are also listed. The print outs as well as a listing of the program are tabulated in Appendix C.
Figure 25. Average Fade Time Versus Threshold Level $V_r$

- experimental results
- theoretical Results
Experimental results proved that the fade time statistics of laser beam intensity fluctuations due to scintillation over short distances, are well described by Log-Normal statistics. In parallel with this work, the average fade time was obtained analytically (O'Hara 1980); the laser beam was modeled as an optical plane wave, and although this may not be an exact approximation, both the analytical model and the experimental results were found to agree. It was noted that the equation describing the average fade time was very sensitive to the variance of the log-intensity, which depends on the turbulent channel. The average fade time, however, showed very little variations with variations of $I_0$. So it is concluded that increasing the laser intensity will not automatically decrease fading.

The circuit was able to measure three parameters: median, average fade time, and check for stationarity. The duration of the data was about 1 minute, but the circuit can handle approximately 2 minutes of data which is the maximum time duration. This restriction is due to the fixed maximum number of counts on the five four bit counters (74193's) and the clock frequency. The designed circuit can also be used for long propagation distances where the
spectrum of the fluctuations gets smaller due to the Fresnel-Zone size.
VI - RECOMMENDATIONS

Since the designed circuit is field deployable, it would be advantageous to have a built-in power supply as well as a built-in clock. To measure statistics for longer distances, more counters could be cascaded with the existing ones. A digital read-out would simplify the output read-out.

To make the circuit more desirable, a second moment measuring circuit could be added to the system.
Log-Normal Distribution Parameters

Let \( I \) be a random variable Log-normally distributed.

\[ x = \ln I \]

where

\[ P(x) \text{ is Gaussian.} \]

\[ P(x) = \frac{1}{\sqrt{2\pi \sigma_x}} \exp \left\{ -\frac{(x - \bar{x})^2}{2 \sigma_x^2} \right\} \]

and

\[ P(x < x_1) = \frac{1}{\sqrt{2\pi \sigma_x}} \int_{-\infty}^{x_1} \exp \left\{ -\frac{(x - \bar{x})^2}{2 \sigma_x^2} \right\} \, dx \]

Since

\[ x = \ln I \]

we have

\[ dx = \frac{dI}{I} \]

and

\[ P(x) \, dx = P(I) \, dI \]

Then using this transformation

\[ P(I) = \frac{1}{\sqrt{2\pi \sigma_I}} \exp \left\{ -\frac{(\ln I - \bar{x})^2}{2 \sigma_I^2} \right\} \]
\[ P(I < I_1) = \int_{-\infty}^{I_1} P(I) \, dI = \frac{1}{2} \left[ 1 + \text{erf} \frac{\ln \frac{I}{I_1}}{\sqrt{2} \sigma_x} \right] \]

by definition the first moment is given by

\[ \bar{I} = \int_{-\infty}^{\infty} I \, P(I) \, dI = \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} \exp \left\{ -\frac{(x - \bar{x})^2}{2 \sigma_x^2} \right\} e^x \, dx \]

\[ \bar{I} = \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} \exp \left\{ \frac{2x \sigma_x^2 - x^2 - \bar{x}^2 + 2x \bar{x}}{2 \sigma_x^2} \right\} dx \]

\[ \bar{I} = \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} \exp \left\{ \frac{\left[ -x^2 - 2x(\bar{x} + \sigma_x^2) + \bar{x}^2 \right]}{2 \sigma_x^2} \right\} dx \]

\[ \bar{I} = \exp \left( \frac{\sigma_x^4 + 2\bar{x} \sigma_x^2}{2 \sigma_x^2} \right) \cdot \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} \exp \left\{ \frac{\left[ x - (\bar{x} + \sigma_x^2) \right]^2}{2 \sigma_x^2} \right\} dx \]

Since \( x \) is Gaussian

\[ \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} \exp \left\{ \frac{\left[ x - (\bar{x} + \sigma_x^2) \right]}{2 \sigma_x^2} \right\} dx = 1 \]

and

\[ \bar{I} = \exp \left( \frac{\bar{x} + \sigma_x^2}{2} \right) \]

\[ \bar{I}^2 = \int_{-\infty}^{\infty} \bar{I}^2 \, P(I) \, dI = \int_{-\infty}^{\infty} e^{2x} \, P(x) \, dx \]

\[ \bar{I}^2 = \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} \exp \left( 2x - \frac{(x - \bar{x})^2}{2 \sigma_x^2} \right) dx \]
Following the same steps for $\bar{I}$

$$\bar{I}^2 = \exp \{ 2\bar{x} + 2 \sigma_x^2 \}$$

$$\sigma_{I}^2 = \bar{I}^2 - (\bar{I})^2 = \exp \{ 2\bar{x} + 2 \sigma_x^2 \} - \exp \{ 2\bar{x} + \sigma_x^2 \}$$

$$\sigma_{I}^2 = e^{2\bar{x}} e^{\sigma_x^2} [e^{\sigma_x^2} - 1]$$

$$\sigma_{I}^2 = e^{(2\bar{x} + \sigma_x^2)} [e^{\sigma_x^2} - 1]$$

$$\sigma_{I}^2 = (\bar{I})^2 [e^{\sigma_x^2} - 1]$$

$$\frac{\sigma_{I}^2}{(\bar{I})^2} + 1 = e^{\sigma_x^2}$$

$$e^{\sigma_x^2} = \frac{\sigma_{I}^2 + (\bar{I})^2}{(\bar{I})^2} \quad \sigma_x^2 = \ln \left( \frac{\sigma_{I}^2 + (\bar{I})^2}{(\bar{I})^2} \right)$$

$$x = \ln I$$

$$\frac{\sigma_x^2}{\ln I} = \ln \left( \frac{\sigma_{I}^2 + (\bar{I})^2}{(\bar{I})^2} \right)$$
Mean & Median

The mean is defined as

\[ \bar{I} = \int_{\infty}^{\infty} I P(I) \, dI \]

The median is the threshold level for which the cumulative fade time is half the total time.

For a Gaussian random variable, the mean is equal to the median. In the case of Log-Normal random variable the median is less than the mean. The median is the value for which the probability density function is maximum.

![Figure 26. Mean and Median of a Log-Normal Distribution](image)
From Appendix A

\[ \bar{I} = \exp \left( \bar{x} + \frac{\sigma_x^2}{2} \right) \]

\[ I_{\text{med}} = e^{x_{\text{med}}} \]

but \( x \) is Gaussian

Therefore

\[ x_{\text{med}} = \bar{x} \]

\[ I_{\text{med}} = e^{\bar{x}} \]

\[ \frac{\bar{I}}{I_{\text{med}}} = \frac{e^{(\bar{x} + \frac{\sigma_x^2}{2})}}{e^{\bar{x}}} = e^{\frac{\sigma_x^2}{2}} + \frac{\sigma_x^2}{2} \]

\[ \bar{I} = e^{\frac{\sigma_x^2}{2}} I_{\text{med}} \]

\( I_{\text{med}} \) is the value measured experimentally

We also have from Appendix A

\[ \sigma_x^2 = \ln \left( \frac{\sigma_{\bar{I}}^2 + (\bar{I})^2}{(\bar{I})^2} \right) \]

\[ \bar{I} = I_{\text{med}} \exp \left[ \frac{1}{2} \ln \left( \frac{\sigma_{\bar{I}}^2 + (\bar{I})^2}{(\bar{I})^2} \right) \right] \]

\[ \bar{I} = I_{\text{med}} \sqrt{\frac{\sigma_{\bar{I}}^2 + (\bar{I})^2}{(\bar{I})^2}} \]

\[ \bar{I}^2 = I_{\text{med}}^2 \left[ \frac{\sigma_{\bar{I}}^2 + (\bar{I})^2}{(\bar{I})^2} \right] \]
\[ \overline{I}^4 = I^2_{\text{med}} (\sigma_I^2 + (\overline{I})^2) \]

\[ \overline{I}^4 - I^2_{\text{med}} (\overline{I})^2 - I^2_{\text{med}} \sigma_I^2 = 0 \]

\[ I_{\text{med}} = 0.67 \]

\[ \sigma_{\text{LnI}} = 2 \quad \sigma_{\text{LnA}} = .16 \]

\[ \sigma_{\text{LnI}}^2 = .0256 \]

From Appendix A

\[ \sigma_{\text{LnI}}^2 = \ln \frac{\sigma_I^2 + (\overline{I})^2}{(\overline{I})^2} \]

\[ e^{0.0256} = \frac{\sigma_I^2 + (\overline{I})^2}{(\overline{I})^2} \]

\[ 1.026 = \frac{\sigma_I^2 + (\overline{I})^2}{(\overline{I})^2} \]

\[ \overline{I}^4 - 0.45(\overline{I})^2 - 0.45 \sigma_I^2 = 0 \]

\[ \overline{I} = 0.68 \]

\[ \sigma_I^2 = 0.012 \]
APPENDIX C

Non Linear Regression

BMDP Curve Fitting Routine

FORTRAN IV & LEVEL 21

SUBROUTINE TRANSF(X,KASE,NPROB,USE,NVAR,XMIS)
DIMENSION X(NVAR)

C X = VECTOR OF DATA FOR ONE CASE
C KASE = SEQUENTIAL CASE NUMBER
C NPROB = PROBLEM NUMBER
C USE = CASE USE INDICATOR
C NVAR = THE GREATER OF THE NUMBER OF VARIABLES READ AND
C THE NUMBER OF VARIABLES AFTER TRANSFORMATIONS
C XMIS = BMDP INTERNAL MISSING VALUE INDICATOR,
C 16.**31 FOR IBM AND SIMILAR SYSTEMS
C
C******************************************************************************
C IF YOU HAVE NO FORTRAN STATEMENTS, USE THE BMED PROCEDEURE.
C******************************************************************************
0003 RETURN
0004 END
SUBROUTINE FUN(F,P,X,N,KASE,NVAR,NPAR,IPASS,XLUGS)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION X(NVAR),P(NPAR)

THE DOUBLE PRECISION VERSIONS OF BUILT-IN FORTRAN FUNCTIONS
(DEXP, DLOG, ETC.) MUST BE USED IN THIS SUBROUTINE.

DOUBLE PRECISION A,A,C,Y,Z
D=DABS(X(2)/P(1))
Z=DLOG(B)+F(Z)*P(2)
A=F/(2.83*P(2))
Y=DABS(A)
C=DERF(Y)
IF(Z.LT.0.0) C=-C
F=0.43*DEXP(Z*Z/(8*P(2)*P(2)))*(1+C)
RETURN
END
/* JUPAPHT RUN OF UEF
   EXEG PUMECFBR1A
   AU55. USH=FUU.UU.P199.1NUL.DISP=(ULG.DELETE)
   EXEG CALCULATE, PGM=APRAN, LPS=4.3, USE=REGION=290K
   FUN NO A
     DOUBLE PRECISION B,T,Y,A
     B=DMAG((X(2)/P(1))
     Z=DLNG(1+B(2)T(2))
     A=DF(2, 6*P(2))
     Y=DMAG(A)
   ENDIF IT1(7)
   G0.45*UEXPO(ZA/((c-P(2)*P(2)))*(1+c)
   ENDIF SYMSOM UG ON NSN=FIU.UU.P1974.LOAD(U.MD)UNIT=333=12
   VOLEXP=FUPAK, SPACE=(IN, IO, 2), LT, disp=(KEA, CATLG, DELETE)
   GU.SYM IN UN *
   PARAMETERS TITLE IS 'FADE TIME ANALYSIS'.
   /INPUT VARIABLES ARE 6.
   /PARAMETER NAMES ARE COUNT,TIME.
   /NECESS DEPENDENT IS COUNT.
   /PARAMETER INITIAL ARE 3.0,0.1.
   /MINIMUM=2.5,0.08.
   /MAXIMUM=4.2,0.2.

    /ENDU
    007400.353
    009100.344
    011200.350
    011100.357
    011200.349
    005400.340
    001100.361
    009400.342
    011200.342
    011800.344
    000400.345
    001600.349
    000400.347
    011400.349
    011400.349
    000700.370
    011200.340
    011200.340
    001100.340
    003400.340
    001200.341
    001500.410
    003400.420
    003400.430
    011400.450
    011400.460
    015400.470
    024100.480
    015400.490
    015400.500
    034500.510
    046700.520
    046600.530
    069700.540
    079100.550
    145800.560
    126900.570
    207900.580
    759200.590
    012400.600
READ                  TITLE IS 'FADE TIME ANALYSIS'.
READ                  INPUT VARIABLE NAMES ARE COUNT, TIME.
READ                  REGRRESS DEPENDENT IS COUNT.
READ                  PARAMETER INITIAL ARE 3.0, 0.1.
READ                  MINIMUM=2.5, 0.08.
READ                  MAXIMUM=4.2, 0.2.
READ
0007600353
009100354
0011200356
0011000357
0012800359
0009300360
0015000361
0012200362
0018000363
0011800364
0007000365
0016900366
0009000367
0014300368
0015200369
0047900370
0012100380
0010200390
0013800400
0021500410
0035900420
0054000430
0017800450
0019800460
0015400470
0026100480
0035000490
0031900500
0036500510
0048700520
0049600530
0068700540
0079100550
1358100560
2656100570
2647000580
7392000590
9129400600
PROGRAM CONTROL INFORMATION

/PROGRAM  TITLE IS 'FAUETIME ANALYSIS'.
/INPUT   VARIABLES ARE 2.
/VARIABLE NAMER ARE COUNT, TIME.
/REGRESS DEPENDENT IS COUNT.
/PARMETER PARAMETERS ARE 2.
/INITIAL ARE 3.0, 0.1.
/MINIMUMS 5.0, 0.0.
/MAXIMUMS 2.5, 0.2.
/END

/PROGRAM TITLE ... ... 'FAUETIME ANALYSIS'

NUMBER OF VARIABLES TO READ IN: 2
NUMBER OF VARIABLES ADDED BY TRANSFORMATIONS: 0
TOTAL NUMBER OF VARIABLES: 2
NUMBER OF CASES TO READ IN: 1000000
CASE LABELING VARIABLES:
LIMITS AND MISSING VALUE CHECKED BEFORE TRANSFORMATIONS
BLANKS ARE:
INPUT UNIT NUMBER:
REMAIN INPUT UNIT PRIOR TO READING: DATA: NO

INPUT FORMAT
(2F5.2)

VARIABLES TO BE USED
count  2 time
**REGRESSION TITLE**

**REGRESSION NUMBER**

**DEPENDENT VARIABLE**

**WEIGHTING VARIABLE**

**NUMBER OF PARAMETERS**

**NUMBER OF CONSTRAINTS**

**TOLERANCE FOR PIVOTING**

**TOLERANCE FOR CONVERGENCE**

**MAXIMUM NUMBER OF ITERATIONS**

**MAXIMUM NUMBER OF INCREMENT HALVINGS**

### PARAMETERS TO BE ESTIMATED

<table>
<thead>
<tr>
<th></th>
<th>1 P(1)</th>
<th>2 P(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MINIMUM</td>
<td>2.500000</td>
<td>0.000000</td>
</tr>
<tr>
<td>MAXIMUM</td>
<td>4.200000</td>
<td>0.000000</td>
</tr>
<tr>
<td>INITIAL</td>
<td>3.000000</td>
<td>0.100000</td>
</tr>
</tbody>
</table>

### NUMBER OF CASES READ

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>MEAN</th>
<th>STANDARD DEVIATION</th>
<th>MINIMUM</th>
<th>MAXIMUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 COUN...</td>
<td>73.301511</td>
<td>192.181087</td>
<td>0.640000</td>
<td>912.489990</td>
</tr>
<tr>
<td>2 TIME</td>
<td>4.314729</td>
<td>0.631064</td>
<td>3.530000</td>
<td>6.000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ITEM, INCH, RESIDUAL SUM OF SQUARES</th>
<th>PARAMETERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>NO. HALV.</td>
<td>1 P(1)</td>
</tr>
<tr>
<td>0</td>
<td>4325802.414200</td>
</tr>
<tr>
<td>1</td>
<td>1084313.014488</td>
</tr>
<tr>
<td>2</td>
<td>441504.704848</td>
</tr>
<tr>
<td>3</td>
<td>257696.737478</td>
</tr>
<tr>
<td>4</td>
<td>55704.530142</td>
</tr>
<tr>
<td>5</td>
<td>43024.077437</td>
</tr>
<tr>
<td>6</td>
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<tr>
<td>7</td>
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<tr>
<td>8</td>
<td>43023.295338</td>
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<td>9</td>
<td>43023.295338</td>
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<tr>
<td>10</td>
<td>43023.295338</td>
</tr>
<tr>
<td>11</td>
<td>43023.295338</td>
</tr>
</tbody>
</table>
The residual sum of squares \( = 45023.3 \) was smallest with the following parameter values:

\[
\begin{align*}
1 & \quad P(1) = 3.32412 \\
2 & \quad P(2) = 1.9999980 - 02
\end{align*}
\]

Correlations and standard deviations are conditioned upon

\[
P(2) = 0.00000D - 02
\]

Estimate of asymptotic correlation matrix:

\[
\begin{pmatrix}
1 & P(2) \\
P(1) & 2
\end{pmatrix}
\]

The estimated mean square error is 1.163.

Estimates of asymptotic standard deviations of parameter estimates with 57 degrees of freedom are:

\[
\begin{align*}
1 & \quad P(1) = 4.01920D - 03 \\
2 & \quad P(2) = 0.0
\end{align*}
\]
BIBLIOGRAPHY


