Statistical Fading of a Spherical Optical Wave in Atmospheric Turbulence

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STATISTICAL FADEING OF A SPHERICAL OPTICAL
WAVE IN ATMOSPHERIC TURBULENCE

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RESEARCH REPORT
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ABSTRACT

A formula for the average fade time of the intensity of a spherical optical wave travelling through atmospheric turbulence is developed. The model employed involves isotropic, homogeneous statistics using a lognormal distribution for the channel. The analysis is based on the fact that the logarithm of the irradiance is normally distributed and uses the work of S. O. Rice who developed such an expression for a zero mean, Gaussian process. The analysis employs the covariance function and the Taylor frozen turbulence hypothesis which results in an expression for the autocorrelation function.
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I. INTRODUCTION

The evolution of defense systems in this century has been from "labor-intensive systems", whose basis was human muscle-power supplemented by machinery, towards "capital-intensive systems", using fewer men to operate expensive and complex weapons systems. The factor of cost and the abandonment of labor-intensive systems has led to the practice of replacing volumes of misdirected fire with expensive but accurate missiles with a high probability of a first time hit. Despite the high cost of these missiles the exchange value is overwhelmingly in their favor.¹

For a missile to be accurate it must have an accurate tracking system. Recently, lasers have been employed in these tracking systems. A laser seeker in the missile's nose can lock onto a target designated by a laser operating from the ground or from another aircraft (see figure 1). The pilot receives visual confirmation of the lock through a video display in the cockpit, then launches the missile. The Pave Penny target identification
GROUND FORWARD OBSERVER EQUIPPED WITH LASER

HELICOPTER EQUIPPED WITH STABILIZED LASER

Figure 1: Source; Bidwell. Reprinted with permission of Prentice Hall.
sets, which track targets designated by a remote laser are the first operational laser trackers in the United States military forces.\textsuperscript{2} The Pave Penny sets are manufactured by the Martin Marietta Aerospace Corporation at the company's Orlando division. Pave Penny sets allow the aircraft on which they are deployed to acquire targets at greater distances than were previously possible, and to remain locked to a target while executing evasive maneuvers (see figure 2).\textsuperscript{3}

A problem of primary importance in these systems is the interaction of the laser beam with the turbulent atmosphere. This interaction leads to random amplitude and phase variations in the laser beam. The objective of this paper is to present a formula for the average fade time of the intensity of an optical spherical wave passing through atmospheric turbulence, measured at a point detector. The atmospheric model employed here is the one described in the paper by Lawrence and Strohbehn, which means that the turbulence is weak and the propagation path short in order for the analysis to take place in the linear region where the so-called Rytov perturbation method is valid. The statistical works of S. O. Rice
Figure 2: Target Identification with the Pave Penny system. Designation is by a remote laser. Source: Laser Focus. Reprinted with permission of Advanced Technologies Publishers.
and W. C.-Y. Lee are adapted for use at optical frequencies and for a lognormal rather than a normal channel. From these the desired formula and several graphs are derived.
II. EFFECTS OF TURBULENCE

Atmospheric turbulence is the product of conduction between the Earth, heated by sunlight, and the surface layer of air. This sheet of warmed air decreases in density and then rises to mix turbulently with the cooler layer of air above it. This mixing results in a random variation in air temperature from point to point. Refractive index fluctuations are caused almost exclusively by fluctuations in temperature, pressure variations being relatively small and rapidly dispersed; henceforth we use the word turbulence to refer to random changes in temperature only.

A smooth and featureless optical wave travelling through more than a few meters of ordinarily turbulent atmosphere will have its energy redistributed and will display random amplitude and phase variations. The physical mechanism through which turbulence induces beam break-up is the changes in focusing produced by the inhomogeneities in the path of the wave (see figure 3). The inhomogeneities in the path of the beam result in constructive and
destructive self-interference, creating fading in the beam. The effects of fading include the following: 6

Beam steering - angular deviation of the beam from the line-of-sight path, resulting in the beam's missing the receiver.

Image dancing - variations in the beam-arrival angle, causing the focus point to move about in the image plane.

Beam spreading - small angle scattering, increasing the beam divergence and causing a decrease in spatial power density at the receiver.

Spatial coherence degradation - losses in phase coherence across the beam phase fronts, degrading the photomixing performance.

Polarization fluctuations - variations in the polarization state.

Beam scintillation - smallscale destructive interference within the beam cross section, causing a decrease in spatial power density at the receiver (see figure 4).

This last effect, beam scintillation, will be considered in detail in this work.
transmitter
turbulence
receiver plane

Figure 3: An illustration of beam break up due to turbulence. Blobs of differing refractive indices redistribute the energy of the beam, causing regions of high intensity and regions of low intensity. Source: Pratt, p. 134.

Figure 4: An illustration of beam scintillation. In the receiver plane, with time frozen by the photograph, light and dark regions show the regions of high and low intensity caused by the turbulence. Source: Pratt, p. 139.
III. THE ATMOSPHERIC MODEL

The atmospheric model we use here assumes the atmosphere to be composed of discrete, homogeneous, turbulent eddies, which differ in refractive indices. Physically, the motion of the larger turbulent eddies causes the formation and transfer of energy to the smaller eddies. These eddies then form and transfer energy to still smaller eddies, and so forth. Eventually, for very small eddies, viscous effects become important and the energy is dissipated.\(^7\)

The scale length \(L_0\) is called the outer scale of turbulence. In general, the turbulence is no longer isotropic for the larger scale sizes and will depend on direction. The scale length \(l_0\) is called the inner scale of turbulence representing the smallest eddy size that can exist due to viscosity. The region between these two scale sizes is called the inertial subrange. The wave number of the optical wave is denoted \(k\), \(\kappa_0\) is the spatial wave number associated with the outer scale of turbulence, \(\kappa_0 = 2\pi/L_0\), and \(\kappa_m\) is the spatial wave number associated with the inner scale of turbulence,
\[ \kappa_m = 2\pi/\lambda_\infty. \]

We use the spherical wave case when the curvature of the light wave is greater than the curvature of the largest eddy, the plane wave case otherwise.

The degree of atmospheric turbulence and its relationship to the optical properties of the atmosphere are characterized by Tatarski's structure constant for refractive index fluctuations, \( C_n^2(\tau) \), which we will assume to be constant for the region of interest and will henceforth be denoted \( C_n^2 \).

The characteristic frequency, \( f_\infty = \nu/(2\pi\lambda\Lambda)_\infty \), is the ratio of the mean wind speed to the width of the first Fresnel zone. The first Fresnel zone is the disk perpendicular to the straight line between two points so that the phase difference between two optical beams, one travelling between the points directly, the other via the rim of the disk will be 180°. Clifford demonstrated that the Fresnel zone size for a spherical wave is smaller than the Fresnel zone size for a plane wave.

The atmospheric model employs the Taylor hypothesis of frozen-in turbulence, which means that temperature fluctuations, and thereby, refractive index fluctuations, at a point are generated by the different sized eddies being swept past the point
by the mean wind speed perpendicular to the path of the beam. This means that the internal motions of the atmosphere are assumed so slow that the only motion of interest is perpendicular to the path. This hypothesis is questionable if the angle between the wind and the propagation direction is less than fifty degrees. When the Taylor hypothesis is reasonable, the time autocorrelation function for the log-amplitude of the signal equals the spatial correlation function evaluated at the product of the mean wind speed and the time difference.

The modified von Karman spectral density function for the refractive index fluctuations,

\[ \phi_n(\kappa) = 0.033C_n^2(\kappa^2 + \kappa_0^2)^{-11/6}\exp(-\kappa^2/\kappa_m^2) \]

is used instead of the Kolmogorov spectrum,

\[ \phi_n(\kappa) = 0.033C_n^2 \kappa^{-11/3}. \]

The Kolmogorov spectrum arises in the Rytov perturbation method of solution of the wave equation, but the modified von Karman spectrum is a better model of the physical situation since it does not have a singularity at zero and it drops off rapidly for large values of the independent variable. Also,
it can be shown that the second derivative of the Kolmogorov spectrum, which is related to the width of the spectrum in a manner similar to the relationship between the width of a statistical distribution and its second moment, does not exist at zero. The Kolmogorov spectrum is too wide and should be terminated as is the modified von Karman spectrum. The two spectra are compared in figure 5.16
Figure 5: Comparison of Spectra. The modified von Karman (solid) and the Kolmogorov (dashed) spectra are plotted to show the differences in termination. Parameter values are taken as $\kappa_0 = 1$ and $\kappa_m = 1000$. The horizontal axis is used for the variable representing the wave number associated with eddy size. 
Source: O'Hara.
IV. PROPAGATION STATISTICS

The logarithm of the intensity of the signal is assumed to have a normal distribution.\textsuperscript{17} We define $\chi$ such that $\chi = \frac{1}{2} \ln(I/I_0)$, where $I_0$ is the turbulence-free value of the intensity. The variance of the log-intensity is denoted $\sigma_{\ln(I)}^2$, and is assumed to be smaller than 2.5; a fact which limits the allowed propagation distance.\textsuperscript{18} This assumption is forced by the perturbation method used in the derivations and the results are for short distances.

For observation periods of a few minutes, and for sufficiently small separations, variations in temperature can be considered as a stationary random process. We therefore treat relevant statistical averages as stationary in time and locally constant in space.\textsuperscript{19}

We also assume a point detector, that is, a detector with an aperture radius which is small when compared to the first Fresnel-zone size, and we ignore the effects of aperture averaging, which would decrease the variance.\textsuperscript{20} Thus, the analysis considered here is, in this sense, a worst-case analysis.
V. AVERAGE FADES AND FADE TIMES

The average fade time is the average amount of time the signal level drops below some particular level $I_t$. Rice\textsuperscript{21} calculated the mean number of zero crossings for a stationary, ergodic, Gaussian process. Lee\textsuperscript{22} used Rice’s and Kac’s results to calculate the average number of level crossings. We will now review Lee’s derivation of the average number of level crossings, $n(I_t)$, of a given signal level $I_t$, and the average duration of fade time below the given signal level.

Assume a random function, $f$, which is temporally statistically stationary and for which the joint probability density function of $f$ and its slope, $\dot{f}$, is $p(f, \dot{f})$. Any given slope $\dot{f}$ can be obtained by

$$\dot{f} = \frac{df}{\tau}$$

where $\tau$ is the time required for a change of ordinate $df$, as shown in figure 6. The expected number of crossings of a random function $f$ in the interval $(F, F-df)$ for a given slope $\dot{f}$ in time $dt$ is
Figure 6: The notation used in obtaining the expected number of level crossings, $n(F)$, and the average duration of fades, $t(F)$. Source: Lee.
the expected amount of time spent in the interval df for a given f in time dt

the time required to cross once for a given f in the interval df at f = F

\[ \frac{E(t)}{\tau} = \frac{p(f,f)dfdfdt}{df/f} = \frac{fp(F,f)dfdt}{df/f} \text{ at } f = F \]

The expected number of crossings for a given f in time T is

\[ \int_0^T fp(F,f)dfdt = \frac{fp(F,f)dt}{df} \]

Since the expected number of crossings at a particular level F per second can also be stated as:

\[ n(F) = \begin{cases} \text{the expected amount of time where the function is below level } F \text{ in 1 second.} \\ \text{average duration of fades below level } F. \end{cases} \]

\[ = \frac{P(f<F)}{t(F)} \]

hence, the average duration of fades below level F is

\[ t(F) = \frac{P(f<F)}{n(F)} \]
So the results will be derived from the joint probability density function, \( p(f,f) \). It remains to find expressions for \( n(F) \) and \( P(f<F) \). For the present case, we need to find expressions for \( n(I_t) \) and \( P(I<I_t) \), where \( I_t \) is the threshold value of \( I \) for the detector, below which the signal is lost.

The quotient \( I/I_0 \) is assumed lognormal, and therefore properties of the lognormal distribution must be applied. The particular properties are included in equations 2-51 and 2-52 of Panter.\(^{23}\) Equation 2-52 shows that 
\[
\ln\left(\frac{A}{A_0}\right) = \langle \ln(A/A_0) \rangle + \sigma^2_{\chi_1}/2, \quad \text{where} \quad \chi_1 = \chi - \langle \chi \rangle.
\]
But \( \langle A/A_0 \rangle = 1 \), since \( A_0 \) is the average value of \( A \), so that
\[
-\sigma^2_{\chi_1} = 2\langle \ln(A/A_0) \rangle = \langle \ln(A^2/A_0^2) \rangle = \langle \ln(I/I_0) \rangle.
\]
Therefore, the average value of the log-intensity is equal to the negative value of the variance of the log-amplitude. Equation 2-51 demonstrates that the cumulative distribution function of the log-normal distribution can be written in terms of the error function. The result is that

\[
P(I < I_t) = P(I/I_0 < I_t/I_0) =
\]
\[ (2\sqrt{2\pi}\sigma_{\chi_1})^{-1} \int_{0}^{I_t/I_o} (I/I_o)^{-1} \exp \left\{ \frac{-\ln(I/I_o) + \sigma^2_{\chi_1}}{8\sigma^2_{\chi_1}} \right\} \, d(I/I_o) \]

= \kappa \left\{ 1 + \text{erf} \left[ \frac{\ln(I_t/I_o) + \sigma^2_{\chi_1}}{2\sqrt{2}\sigma_{\chi_1}} \right] \right\}

using the fact that the variance of the log-intensity if four times the variance of the log-amplitude.

To calculate \( n(Y) \), Rice\textsuperscript{24} showed that

\[ n(Y) = \int_{0}^{\infty} y p(Y, y) \, dy \]

where \( y \) is the random process with threshold \( T \), and

\( p \) is the joint density function of \( y \) and its first derivative; here, the bivariate normal density.

Assuming a stationary, ergodic, and normal process with autocorrelation function \( R(\tau) \) yields after the integration;

\[ n(Y) = \frac{1}{2\pi} \left( \frac{-R(0)}{R(0) - m_Y^2} \right) \exp \left\{ \frac{-(Y - m_Y)^2}{2(R(0) - m_Y^2)} \right\} \]
where \( m_y \) is the mean of \( y \). For the present case,
\[
Y = \ln \left( \frac{I_t}{I_0} \right), \quad m_y = \sigma_Y^2, \quad R(0) - m_x^2 = 4\sigma_Y^2 = 4R_Y(0),
\]
and \( R(0) = 4R_Y(0) \), using the fact that the intensity is the square of the amplitude. Therefore,

\[
n(I_t) = \frac{1}{2\pi} \left( \begin{array}{c} -R_x(0) \\ R_x(0) \end{array} \right) \exp \left\{ \frac{\ln(I_t/I_0) + \sigma_Y^2}{-8\sigma_Y^2} \right\}
\]

and finally,

\[
t(I_t) = \frac{\pi}{2} \left( \begin{array}{c} 1 + \text{erf} \left( \frac{\ln(I_t/I_0) + \sigma_Y^2}{2\sqrt{2}\sigma_Y} \right) \\ -R_x(0) \\ R_x(0) \end{array} \right)^\frac{1}{2} \exp \left\{ \frac{(\ln(I_t/I_0) + \sigma_Y^2)}{-8\sigma_Y^2} \right\}
\]

Figure 7 shows plots of \( t(I_t) \) as a function of \( I_t/I_0 \) with variances of .6 and .1, with \( -R_x(0)/R_x(0) \) given an arbitrary value of \( 10^6 \).

Physically \( -R_x(0)/R_x(0) \) is a measure of the width of the spectra. When a Fourier transform
Figure 7: Non-normalized average fade time as a function of the threshold to turbulence-free intensity ratio.
is applied to $R_\chi(\tau)$ the result involves the power spectra, denoted $S(\omega)$. $-\ddot{R}_\chi(\tau)$ is then related to the second moment of the power spectra, which is a measure of the width of the spectra. Taking the ratio normalizes the relationship. Expressed in terms of the power spectra, we have that

$$
\frac{-\ddot{R}_\chi(0)}{R_\chi(0)} \propto \frac{\int_{-\infty}^{\infty} \omega^2 S(\omega) \, d\omega}{\int_{-\infty}^{\infty} S(\omega) \, d\omega}.
$$
VI. EXAMINATION OF RESULTS

Using the approximations developed in appendix 3 and simplifying,

\[
-\frac{\dot{R}_X(0)}{R_X(0)} = \frac{1729 \pi k \lambda f_0^2 (k/L \kappa_m^2)^{-\frac{1}{2}} (\kappa_0^2 L/k)^{\frac{1}{2}} (\kappa_0^2 / \kappa_m^2)^{-\frac{1}{2}}}{216}
\]

where \( \lambda \) is the wavelength of the laser light; \( k = 2\pi/\lambda \) is the wave-number associated with the wavelength; \( f_0^2 \) is the characteristic frequency which depends upon the wavelength, the propagation length, \( L \), and the mean wind speed; \( \kappa_m \), the spatial wave-number associated with the inner scale of turbulence, \( l_0 \); \( \kappa_0 \) is the spatial wave-number associated with the outer scale of turbulence, \( l_o \). It is desired to use this expression in the equation for the average fade time,

\[
t(I_t) = \frac{\pi \left\{ 1 + \text{erf} \left[ \frac{\ln(I_t/I_0) + \sigma_\chi^2}{2\sqrt{2}\sigma_\chi_1} \right] \right\}}{-\frac{\dot{R}_X(0)}{R_X(0)} \exp \left\{ \frac{(\ln(I_t/I_0) + \sigma_\chi^2)}{-8\sigma_\chi_1} \right\}}
\]
to examine some of the properties of the average fade time expression for various parameter values.

Two facts about the expression for $t(I_t)$ should be noted at this point. First, $t(I_t)$ is not dependent on the value of Tatarski's structure constant for refractive index fluctuations, $C_n^2$. This fact indicates that the strength of the turbulence doesn't change the shape of the spectrum, that is, the relative magnitudes of the energy distribution. Thus an accurate value for this parameter is not necessary. Secondly, it can be seen after some manipulation, that the product of $t(I_t)$ and $f_0$ is both unitless and independent of the mean wind speed. In the graphs which follow, $t(I_t)f_0$ will be used as the dependent variable in order to normalize and eliminate uncertainty in the expression due to the measured value of $v$.

The approximation formulas for $-R_X(0)/R_X(0)$ for the spherical wave and the plane wave case were compared. The values of the parameters are those derived from an experiment which was run in England by Dr. Ronald Phillips of the University of Central Florida. The experiment is described more fully in Belkerdid [1980]. For our purposes it is sufficient
to note that a Helium-Neon laser was used so that 
\( \lambda = 0.6328 \times 10^{-6} \) meters and \( k = 10^7 \) while the experimental conditions were clear and sunny on a flat and unobstructed path with a mean wind speed of 3 meters per second, \( \kappa_o = 1, \kappa_m = 1000, \) and \( L = 208 \) meters.

We find that \( -R(0)/R(0) \) for the spherical wave is approximately ten times the value of that of the plane wave. So in the formulae for \( t(I_t)f_o \) for which other parameters are comparable, the spherical wave case will then be about one-third the value of the plane wave case.

Figures 8, 9, and 10 compare \( t(I_t)f_o \) for the spherical and plane wave cases as specified parameters are allowed to vary. The graphs for the plane wave were derived by O'Hara.

Figure 8 shows \( t(I_t)f_o \) plotted against \( I_t/I_o \) for a variance of .6 for the plane and spherical waves. As expected, the two cases differ by only a constant factor.

Figure 9 shows \( t(I_t)f_o \) plotted against \( I_t/I_o \) for various wavelengths. While the effect of wavelength on the plane wave case is very slight, causing a measureable difference only at the smaller values of \( I_t/I_o \), the effect on the spherical wave is nil.
Looking at figure 10, we see that while propagation length has a small effect on average fade time for the plane wave case, the spherical wave case is completely independent of propagation length. This is due to the particular approximation formula used for the spherical wave case:

\[
\frac{-R_0}{\chi} = \frac{1729}{216} \pi k \lambda f_0^2 (k/L \kappa_m^2)^{-\frac{1}{2}} (\kappa_0^2 L/k)^{\frac{1}{2}} (\kappa_0^2 / \kappa_m^2)^{-\frac{1}{2}}
\]

where \( f_0^2 = v^2 / 2\pi \lambda L \), and the dependence on \( L \) is seen to be removed. Had more terms been used in the approximation there would undoubtedly be some slight dependence of the normalized average fade time on the propagation length.

In summary, for the spherical wave case, the variance and the threshold to turbulence-free intensity ratio are the parameters which have the strongest effect on average fade time, as they had for the plane wave case. Average fade time is insensitive to changes in propagation length and the wavelength of the laser light, provided the turbulence is weak.
Figure 8: Average fade time versus the ratio of threshold to turbulence-free intensity for a plane and for a spherical wave. The variance is chosen to be .6.
Figure 9: Normalized average fade time versus the ratio of threshold to turbulence-free intensity for various values of the wavelength of the plane and spherical waves.
Figure 10: Average fade time versus propagation length for the plane and spherical waves. The variance has been chosen to be 0.6. The turbulence in the channel is weak.
APPENDIX 1: FORMULAE AND FACTS USED IN DERIVATIONS

In the following, \( W \) is the Whittaker function.

1. From Bateman Manuscript Project, Tables of Integral Transforms, volume II, p. 234, formula number 12:

\[
\int_{0}^{\infty} x^\lambda \exp(-ax) (x + y)^{-\rho} \, dx = \Gamma(\lambda+1) a^{\frac{1}{2}\rho - \frac{1}{2}\lambda - 1} y^{\frac{1}{2}\lambda - \frac{1}{2}\rho} \exp(ay) W_{k, m}(ay)
\]

\[2k = -\lambda - \rho \quad 2m = \lambda - \rho + 1 \quad \text{Re} \alpha > 0, \]

\[|\text{Arg}y| < \pi\]

2. From Gradshteyn and Ryzhik, Tables of Integrals, Series, and Products, p. 284, Section 3.191, formula number 3:

\[
\int_{0}^{1} x^{\nu-1} (1-x)^{\mu-1} \, dx = \int_{0}^{1} x^{\nu-1} (1-x)^{\mu-1} \, dx = B(\nu, \mu) \quad \text{Re} \nu > 0 \quad \text{Re} \mu > 0
\]

3. From Speigel, Complex Variables, p. 143

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\]
4. From Speigel, *Complex Variables*, p. 142:

A power series can be integrated term by term along any curve $C$ which lies entirely inside its circle of convergence.

5. From Slater, *Confluent Hypergeometric Functions*, p. 14:

$$W_{k,m}(x) = \frac{\Gamma(-2m)}{\Gamma\left(\frac{5}{2}-m-k\right)} \exp\left(-\frac{k}{x}\right) \frac{x^{\frac{5}{2}+m}}{\Gamma(2m)}$$

$$\frac{1}{\Gamma}\left[\frac{5}{2}+m-k; 1+2m; x\right] + \frac{\Gamma(2m)}{\Gamma\left(\frac{5}{2}+m-k\right)} \exp\left(-\frac{k}{x}\right)$$

$$x^{\frac{5}{2}-m} \frac{1}{\Gamma}\left[\frac{5}{2}-m-k; 1-2m; x\right]$$


$$pFq\left(a_1, a_2, \ldots; b_1, b_2, \ldots; z\right) =$$

$$\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}$$

where $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ is Pochhammer's symbol.
APPENDIX 2: DERIVATIONS OF FORMULAE FOR THE AUTOCORRELATION FUNCTION AND ITS SECOND DERIVATIVE

a. The autocorrelation function, as given by Lawrence and Strohbehn is:

\[ R_X(\tau) = 4\pi^2 k^2 C^2_n(0.033) \int_0^\infty \int_0^L \kappa J_0(\kappa(2\pi\lambda L)^{1/2}f_0\tau) \exp(-\kappa^2/\kappa_m^2)(\kappa_m^2+\kappa^2)^{-11/6}\sin^2(\kappa^2\eta(L-\eta)/2kL) \, d\eta d\kappa \]

b. Note that when \( \tau = 0 \), the above equation reduced to:

\[ R_X(0) = 4\pi^2 k^2 C^2_n(0.033) \int_0^\infty \int_0^L \exp(-\kappa^2/\kappa_m^2)(\kappa_m^2+\kappa^2)^{-11/6}\sin^2(\kappa^2\eta(L-\eta)/2kL) \, d\eta d\kappa \]

c. Substituting \( u = \kappa^2 \), so that \( du = 2\kappa d\kappa \)

\[ R_X(0) = 2\pi^2 k^2 C^2_n(0.033) \int_0^\infty \int_0^L \exp(-u/\kappa_m^2)(u+\kappa_0^2)^{-11/6}\sin^2(u\eta(L-\eta)/2kL) \, d\eta du \]
d. Substituting \( \eta = L \nu \), so that \( d\eta = L \, d\nu \)

\[
R_\chi(0) = 2\pi^2 k^2 C_n^2 (0.033) L \int_0^\infty \int_0^1 \exp\left(-u/\kappa_m^2\right) \nu \exp\left(-\nu L /(2k)\right) \, d\nu \, du
\]

\((u+\kappa^2)^{-11/6} \sin^2 (u \nu (1-v)/2k) \, d\nu \, du\)

e. Substituting \( x = ku/L \), so that \( dx = k \, du/L \)

\[
R_\chi(0) = 2\pi^2 k^2 C_n^2 (0.033) \int_0^\infty \int_0^1 \exp\left(-kx/L \kappa_m^2\right) \nu \exp\left(-\nu (1-v)/2\right) \, dv \, dx
\]

\((kx/L + \kappa_0^2)^{-11/6} \sin^2 (x \nu (1-v)/2) \, dv \, dx\)

f. Simplifying:

\[
R_\chi(0) = 2\pi^2 k^7/6 L^{11/6} C_n^2 (0.033)
\]

\[
\int_0^\infty \int_0^1 \exp\left(-kx/L \kappa_m^2\right) (x + \kappa_0^2 L/k)^{-11/6} \nu \exp\left(-\nu (1-v)/2\right) \, dv \, dx
\]

g. Utilizing the fact that \( \sin^2 (\theta) = \frac{1}{2} (1-\cos(2\theta)) \)

\[
R_\chi(0) = \pi^2 k^7/6 L^{11/6} C_n^2 (0.033)
\]

\[
\int_0^\infty \int_0^1 \exp\left(-kx/L \kappa_m^2\right) (x + \kappa_0^2 L/k)^{-11/6} \times
\]

\[(1-\cos(xv(l-v))) \, dv \, dx\]

h. Separating the integral into terms, and integrating over the variable, \(v\), in the first term;

\[R_x(0) = \pi^2 k^7/6 \, L^{11}/6 \, C_n^2(.033)\]

\[\left[ \int_0^\infty \exp\left(-kx/L\mu_m^2\right) (x + \kappa_0^2 L/k)^{-11/6} \, dx \right.\]

\[\int_0^1 \int_0^1 \exp\left(-kx/L\mu_m^2\right) (x + \kappa_0^2 L/k)^{-11/6} \]

\[\cos(xv(l-v)) \, dv \, dx \]

i. Applying formula 1 from Appendix 1,

\[\int_0^\infty x^\lambda \exp(-\alpha x) (x+y)^{-\rho} \, dx = \Gamma(\lambda+1) \alpha^{\frac{1}{2}(\lambda+\rho-1)} \]

\[y^{\frac{1}{2}(\lambda-\rho)} \exp\left(\frac{1}{2}(\alpha y)\right) W_{\lambda, m}(\alpha y)\]

\[2k = -\lambda - \rho \quad \quad 2m = \lambda - \rho + 1\]

to the first integral results in;

\[R_x(0) = \pi^2 k^7/6 \, L^{11}/6 \, C_n^2(.033) \times\]
\[ \Gamma (1) \frac{(k/Lk_m^2)^{-1/12}}{(\kappa_0^2L/k)^{-11/12}}. \]

\[ \exp \frac{\kappa_0^2}{k_m^2} W_{-11/12, -5/12} (\kappa_0^2/k_m^2) - \]

\[ \int_0^\infty \int_0^1 \exp \left(-kx/Lk_m^2\right) (x + \kappa_0^2L/k)^{-11/6} \cos(xv(1-v)) \, dv \, dx \]

\[ j. \text{Now write the cosine factor as an expanded series;} \]

\[ R_\chi (0) = \pi^2 k^{7/6} L^{11/6} C_n^2 (.033) \]

\[ (k/Lk_m^2)^{-1/12} (\kappa_0^2L/k)^{-11/12} \exp \frac{\kappa_0^2}{k_m^2} \]

\[ W_{-11/12, -5/12} (\kappa_0^2/k_m^2) - \int_0^\infty \int_0^1 \exp (-kx/Lk_m^2) \]

\[ (x + \kappa_0^2L/k)^{-11/6} \sum_{n=0}^\infty \frac{(-1)^n x^n v^n (1-v)^n}{(2n)!} \, dv \, dx \]

\[ k. \text{Setting up to integrate over the variable, v;} \]

\[ R_\chi (0) = \pi^2 k^{7/6} L^{11/6} C_n^2 (.033) \times \]
\[
\frac{(k/L\kappa_m^2)^{-1/2}}{L \sqrt{\pi}} \left( \frac{\kappa_0^2 L}{k} \right)^{-1/2} \text{exp} \int \frac{k^2}{\kappa_m^2} \exp \left( \frac{-k^2}{\kappa_m^2} \right)
\]

\[
W_{-11/12, -5/12} \left( \frac{\kappa_0^2}{\kappa_m^2} \right) = \int_0^\infty \exp \left( -\frac{k^2}{\kappa_m^2} \right)
\]

\[
(x + \frac{\kappa_0^2 L}{k})^{-11/6} \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!} B(2n+1, 2n+1) \ dx
\]

1. Using Formula 2 of Appendix 1;

\[
\int_0^1 x^{\nu-1} (1-x)^{\mu-1} \ dx = \int_0^1 x^{\mu-1} (1-x)^{\nu-1} \ dx = B(\mu, \nu)
\]

results in;

\[
R_\chi(0) = \pi^2 k^7 / 6 L^{11/6} \text{C}_n^2 (0.033)
\]

\[
\frac{(k/L\kappa_m^2)^{-1/2}}{L \sqrt{\pi}} \left( \frac{\kappa_0^2 L}{k} \right)^{-1/2} \text{exp} \int \frac{k^2}{\kappa_m^2} \exp \left( \frac{-k^2}{\kappa_m^2} \right)
\]
m. Simplifying and setting up to integrate over $x$;

$$R_{X}(0) = \pi^2 k^{7/6} L^{11/6} C_n^2 (.033)$$

$$(k/L\kappa_m^2)^{-1/12} (\kappa_o^2 / Lk)^{-1/12} \exp \frac{1}{2} (\kappa_o^2 / \kappa_m^2)$$

$$W_{-11/12, -5/12} (\kappa_o^2 / \kappa_m^2) - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$$

$$B(2n+1, 2n+1) \int_0^\infty \exp (-kx/L\kappa_m^2) x^{2n}$$

$$(x + \kappa_o^2 L/k)^{-11/6} dx$$

n. Applying Formula 1 of Appendix 1

$$\int_0^\infty x^\lambda \exp (-ax) (x + y)^{-\rho} dx = \Gamma(\lambda + 1) a^{\frac{1}{2}\rho - \frac{1}{2}\lambda - 1}$$

$$y^{\frac{1}{2}\lambda - \frac{1}{2}\rho} \exp \frac{1}{2} (ay) W_{k, m}(ay)$$

$$2k = -\lambda - \rho \quad 2m = \lambda - \rho + 1$$

to the remaining integral results in;

$$R_{X}(0) = \pi^2 k^{7/6} L^{11/6} C_n^2 (.033)$$

$$(k/L\kappa_m^2)^{-1/12} (\kappa_o^2 L/k)^{-11/12} \exp \frac{1}{2} (\kappa_o^2 / \kappa_m^2) \times$$
\[ W_{-11/12, -5/12} \left( \frac{\kappa_0^2}{\kappa_m^2} \right) = - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \] 

\[ B(2n+1, 2n+1) \Gamma(2n+1) \left( \frac{k/L \kappa_m^2}{\kappa_0^2} \right)^{-1/12} \] 

\[ \text{Note that when } n = 0 \text{ the first terms will cancel. The final result is that;} \]

\[ R(0) = \pi^2 k^7/6 L^{11/6} \text{C}^2_n (.033) \]

\[ \chi \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B(2n+1, 2n+1) \Gamma(2n+1)}{(2n)!} \]

\[ \left( \frac{k/L \kappa_m^2}{\kappa_0^2} \right)^{-1/12-n} \left( \frac{\kappa_0^2 L/k}{\kappa_0^2} \right)^{n-11/12} \exp \frac{1}{2} \left( \frac{\kappa_0^2}{\kappa_m^2} \right) \]

\[ W_{-n-11/12, n-5/12} \left( \frac{\kappa_0^2}{\kappa_m^2} \right) \]

The second derivative is then computed with respect to the variable \( \tau \). The integral is then evaluated by the same procedure as was employed to evaluate the integral for \( R_\chi(0) \).
a. $R_X(\tau) = 2\pi^2k^2C_n^2(0.033)\int_0^L \int_0^L \kappa^3(2\pi\lambda L)f_0^2$

$\left(J_2(\kappa(2\pi\lambda L)^{1/2}f_0\tau) - J_0(\kappa(2\pi\lambda L)^{1/2}f_0\tau)\right)$

$\exp(-\kappa^2/\kappa_m^2)(\kappa^2 + \kappa_o^2)^{-1/6} \sin^2(\kappa^2\eta(L-\eta)) /
2kL) \, d\eta d\kappa$

b. When $\tau = 0$ the above equation reduces to:

$R_X(0) = -4\pi^3k^2C_n^2(0.033)\lambda Lf_0\int_0^L \int_0^L \kappa^3$

$\exp(-\kappa^2/\kappa_m^2)(\kappa^2 + \kappa_o^2)^{-11/6} \sin^2(\kappa^2\eta(L-\eta)) /
2kL) \, d\eta d\kappa$

c. Substituting $\kappa^2 = u$, so that $2\kappa d\kappa = du$

$R_X(0) = -2\pi^3k^2C_n^2(0.033)\lambda Lf_0^2 \int_0^L \int_0^L u\exp(-u/\kappa_m^2)$

$(u + \kappa_o^2)^{-11/6} \sin^2(u\eta(L-\eta)/2kL) \, d\eta du$

d. Substituting $\eta = Lv$, so that $d\eta = Ldv$
\[ R_\chi(0) = -2\pi^9 k^2 C_n^2(.033)\lambda L^2 f_o^2 \int_0^1 \int_0^1 u \exp(-u/\kappa_m^2) \]
\[(u + \kappa_o^2)^{-11/6} \sin^2(uLv(l-v)/2k) \, dv \, du\]

e. Substituting \( x = L/k \, u \), so that \( dx = L/k \, du \);

\[ R_\chi(0) = -2\pi^3 k^4 C_n^2(.033)\lambda f_o^2 \int_0^1 \int_0^1 x \exp(-kx/L\kappa_m^2) \]
\[(kx/L + \kappa_o^2)^{-11/6}\sin^2(xv(l-v)/2) \, dv \, dx\]

f. Factoring and simplifying;

\[ R_\chi(0) = -2\pi^3 k^{13/6} L^{11/6} C_n^2(.033)\lambda f_o^2 \int_0^1 \int_0^1 x \]
\[ \exp(-kx/L\kappa_m^2) \, (x + \kappa_o^2 L/k)^{-11/6} \, \sin^2(xv(l-v)/2) \, dv \, dx\]

g. Utilizing the fact that \( \sin^2(\alpha) = \frac{1}{2}(1-\cos(2\alpha)) \)

\[ R_\chi(0) = \pi^3 k^{13/6} L^{11/6} C_n^2(.033)\lambda f_o^2 \int_0^1 \int_0^1 x \]
\[ \exp(-kx/L\kappa_m^2) \, (x + \kappa_o^2 L/k)^{-11/6} \times \]
\[ \left[ 1 - \cos(xv(l-v)) \right] \, dv \, dx \]

h. Separating into two integrals and expending the cosine factor into a series:

\[
R_X(0) = -\pi^3 k^{13/6} L^{11/6} C_n^2(0.033) \lambda f_0^2
\]

\[
\int_{0}^{\infty} \int_{0}^{1} x \exp\left(-kx/L_\kappa^2\right) (x + \kappa^2 L/k)^{-11/6} \, dv \, dx - \int_{0}^{\infty} \int_{0}^{1} x \exp\left(-kx/L_\kappa^2\right) (x + \kappa^2 L/k)^{-11/6} \, dv \, dx
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} v^{2n} (l-v)^{2n}}{(2n)!} \, dx
\]

i. In the first term, integrate over the variable, \( v \), in the second term prepare to integrate over the variable \( v \).

\[
R_X(0) = -\pi^3 k^{13/6} L^{11/6} C_n^2(0.033) \lambda f_0^2
\]

\[
\int_{0}^{\infty} x \exp\left(-kx/L_\kappa^2\right) (x + \kappa^2 L/k)^{-11/6} \, dx - \int_{0}^{\infty} x \exp\left(-kx/L_\kappa^2\right) (x + \kappa^2 L/k)^{-11/6} \times
\]
\[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \int_0^1 v^{2n}(1-v)^{2n} \, dv \, dx \]

j. To the first integral apply formula 1 of Appendix 1:

\[ \int_0^\infty x^\lambda \exp(-ax)(x+y)^{-\rho} \, dx = \Gamma(\lambda+1)a^{\frac{\lambda}{\rho}-\frac{1}{2}} \lambda \]

\[ y^{\frac{\lambda}{\rho}-\frac{1}{2}} \exp\frac{1}{2}(ay)W_{k,m}(ay) \]

2k = -\lambda - \rho \\
2m = \lambda - \rho + 1

and the result is

\[ R_X(0) = -\pi^3 k^{1.3/6} L^{1.1/6} C_n^2 (0.033) \lambda f_0^2 \]

\[ \Gamma(2)(k/Lk^2)^{-7/12}(k^2L/k)^{-5/12}\exp\frac{1}{2}(k^2/k^2) \]

\[ W_{-17/12,1/12}(k^2/k^2) = \int_0^\infty x \exp(-kx/Lk^2) \]

\[ (x + \frac{k^2}{L/k})^{-11/6} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \int_0^1 v^{n^2}(1-v)^{2n} \, dv \, dx \]
k. To the second term, apply to the integral over the variable, \( v \), formula 2 of Appendix 1;

\[
\int_0^1 x^{v-1} (1-x)^{\mu-1} \, dx = B(v, \mu)
\]

and the result is that

\[
\mathcal{R}_X(0) = -\pi^3 k_1^{13/6} L^{11/6} C_n (0.033) \lambda f^2_0
\]

\[
\Gamma(2) (k/Lk_m^2)^{-7/12} (\kappa_o^2 L/k)^{-5/12} \exp \frac{\ln^2}{2} (\kappa_o^2/k_m^2)
\]

\[
\mathcal{W} = -17/12, 1/12 (\kappa_o^2/k_m^2) - \left[ \frac{\exp (-kx/Lk_m^2)}{0} \right]
\]

\[
(x + \kappa_o^2 L/k)^{-11/6} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} B(2n+1, 2n+1)
\]

1. Preparing the remaining integral for integration over the variable, \( x \);

\[
\mathcal{R}_X(0) = -\pi^3 k_1^{13/6} L^{11/6} C_n (0.033) \lambda f^2_0
\]

\[
(k/Lk_m^2)^{-7/12} (\kappa_o^2 L/k)^{-5/12} \exp \frac{\ln^2}{2} (\kappa_o^2/k_m^2)
\]
\[ W_{-17/12, 1/12} \left( \frac{\kappa^2_o}{\kappa^2_m} \right) - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} B(2n+1, 2n+1) \]

\[ \int_0^\infty x^{2n+1} \exp(-kx/L\kappa^2_m) (x + \kappa^2_o L/k)^{-11/6} \, dx \]

m. Applying to the remaining integral formula 1 of Appendix 1;

\[ \int_0^\infty x^\lambda \exp(-ax) (x+y)^{-\rho} \, dx = \Gamma(\lambda+1) \alpha^\frac{\lambda \rho}{2} - ^\frac{\lambda - 1}{2} \]

\[ y^{\frac{\lambda}{2} - \frac{\rho}{2}} \exp(ay) W_{k,m}(ay) \]

\[ 2k = -\lambda - \rho \quad \quad 2m = \lambda - \rho + 1 \]

the result is that

\[ R_x(0) = -\pi^3 k^{13/6} L^{11/6} \sigma^2_n (0.033) \lambda f^2_o \]

\[ (k/L\kappa^2_m)^{-7/12} (\kappa^2_o L/k)^{-5/12} \exp(\frac{\kappa^2_o}{\kappa^2_m}) \]

\[ W_{-17/12, 1/12} \left( \frac{\kappa^2_o}{\kappa^2_m} \right) - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} B(2n+1, 2n+1) \]

\[ (2n+2) (k/L\kappa^2_m)^{-7/12} - n \left( \frac{\kappa^2_o L/k}{n} \right)^{n-5/12} \times \]
\[ \exp^{\frac{\kappa^2}{\mu^2}} W_{-17/12-n, n+1/12} \left( \frac{\kappa^2}{\mu^2} \right) \]

n. Note that when \( n = 0 \), again the first terms cancel and the final result is that

\[ R(0) = -\pi^3 \frac{1}{6} L^{11/6} C_n^{2} (0.033) \lambda f^2 \]

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B(2n+1, 2n+1) \Gamma(2n+2)}{(2n)!} \]

\[ (k/L \kappa^2)^{-7/12-n} (\kappa^2 L/k)^{n-5/12} \exp^{\frac{\kappa^2}{\mu^2}} \left( \frac{\kappa^2}{\mu^2} \right) \]

\[ W_{-17/12-n, n+1/12} \left( \frac{\kappa^2}{\mu^2} \right) \]
Representative values are taken to be $K_0 = 1$, $K_m = 100$, $k = 10^7$, and $L = 208$. Consider the equation developed for the autocorrelation function in Appendix 2,

\begin{equation}
R_X(0) = \pi^2 k^7/6 L^{11/6} C_n^2 \cdot 0.033
\end{equation}

\[ \sum_{n=1}^{\infty} (-1)^{n+1} B(2n+1, 2n+1) \Gamma(2n+1) \]

\[ (2n)! \]

\[ (k/L k_m^2)^{-1/12} - n(k_0^2 L/k) n^{-11/12} \exp \frac{1}{2} (\kappa_0^2 / \kappa_m^2) \]

\[ W_{-n-11/12, n-5/12} (\kappa_0^2 / \kappa_m^2) \]

Utilizing the definition of Euler's Beta function, $B(x, y) = \Gamma(x) \Gamma(y) / \Gamma(x+y)$, and the definition of the gamma function for integer arguments, $\Gamma(n) = (n-1)!$, we have that

\begin{equation}
R_X(0) = \pi^2 k^7/6 L^{11/6} C_n^2 \cdot 0.033
\end{equation}

\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n)!}{(4n+1)!} \times \]
\[
\left(\frac{k}{L\kappa_m^2}\right)^{-1/12} - \beta \left(\frac{\kappa_O^2}{\kappa_m^2}\right)^{n-1/12}
\]

\[
\exp\frac{1}{2} \left(\frac{\kappa^2}{\kappa_m^2}\right) W_{-n-1/12, n-5/12} \left(\frac{\kappa_O^2}{\kappa_m^2}\right)
\]

Now using formula 5 of Appendix 1:

\[
W_{k,m}(x) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)} \exp\left(-\frac{1}{2}x\right) x^{\frac{1}{2}+m}
\]

\[
1F1 \left(\frac{1}{2}+m-k;1+2m;x\right) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} \exp\left(-\frac{1}{2}x\right)
\]

\[
x^{\frac{3}{2}-m} 1F1 \left(\frac{3}{2}-m-k;1-2m;x\right)
\]

to the Whittaker function expressed above results in:

(3)

\[
W_{-n-11/12, n-5/12} \left(\frac{\kappa_O^2}{\kappa_m^2}\right) = \frac{\Gamma(5/6-2n)}{\Gamma(11/6)}
\]

\[
\exp\frac{1}{2} \left(\frac{\kappa^2}{\kappa_m^2}\right) \left(\frac{\kappa^2}{\kappa_m^2}\right)^{n+1/12}
\]

\[
1F1 \left(2n+1;2n+1;\kappa_O^2/\kappa_m^2\right) + \frac{\Gamma(2n-5/6)}{\Gamma(2n+1)}
\]

\[
\exp\frac{1}{2} \left(\frac{\kappa^2}{\kappa_m^2}\right) \left(\frac{\kappa^2}{\kappa_m^2}\right)^{11/12-n}
\]

\[
1F1 \left(11/6;11/6-2n;\kappa_O^2/\kappa_m^2\right)
\]
Applying formula 6 of Appendix 1,
\[ p^F_q(a_1, a_2, \ldots; b_1, b_2, \ldots; z) = \]
\[ \frac{(a_1)_k(a_2)_k \ldots (a_p)_k}{(b_1)_k(b_2)_k \ldots (b_q)_k} \frac{z^k}{k!} \]

where \( (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \) is Pochhammer's symbol.

we have that;

\[ (3a) \quad _1F_1 \quad 2n+1; 2n+1/6; \kappa_0^2/\kappa_m^2 = \]
\[ \sum_{m=0}^{\infty} \frac{(2n+1+m) (2n+1/6)}{(2n+1) (2n+1/6+m)} \frac{10^{-4m}}{m!} \]

since \( \kappa_0^2/\kappa_m^2 = 1/10^4 \), and

\[ (3b) \quad _1F_1 \quad 11/6; 11/6-2n; \kappa_0^2/\kappa_m^2 = \]
\[ \sum_{m=0}^{\infty} \frac{\Gamma(11/6+n) \Gamma(11/6-2n)}{\Gamma(11/6) \Gamma(11/6-2n+m)} \frac{10^{-4m}}{m!} \]

The first term equals 1 and the second term is on the order of \( 10^{-4} \), while further terms are on the order of
$10^{-8}$ or less. The above functions can be approximated by 1, with an error of .014% or less. So an approximation for \( W_{-n-11/12,n-5/12}(\kappa_o^2/\kappa_m^2) \) is

\[
(3c) \quad \frac{\Gamma(5/6-2n)}{\Gamma(11/6)} \exp -\frac{1}{2}(\kappa_o^2/\kappa_m^2)(\kappa_o^2/\kappa_m^2)^{n+1/12} +
\frac{\Gamma(2n-5/6)}{\Gamma(2n+1)} \exp -\frac{1}{2}(\kappa_o^2/\kappa_m^2)(\kappa_o^2/\kappa_m^2)^{11/12-n}
\]

Again, noting that \( \kappa_o^2/\kappa_m^2 = 1/10^4 \), we have that

\[
(\kappa_o^2/\kappa_m^2)^{n+1/12} = (10)^{-4n-1/3}
\]

and that

\[
(\kappa_o^2/\kappa_m^2)^{11/12-n} = (10)^{4n-11/3}
\]

So the first term in the approximation of the Whittaker function can be discarded with an error of .02%, leaving the approximation

\[
(3d) \quad W_{-n-11/12,n-5/12}(\kappa_o^2/\kappa_m^2) = \frac{\Gamma(2n-5/6)}{\Gamma(2n+1)} \exp -\frac{1}{2}(\kappa_o^2/\kappa_m^2)(\kappa_o^2/\kappa_m^2)^{11/12-n}
\]

and substitution into equation (2) results in;

\[
(4) \quad R_X(0) = \pi^2 k^7/6 L^{11/6} c_n^2 (.033) \times
\]
\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n+1)!!}{(4n+1)} \left( \frac{k/L\kappa_{m}^{2}}{\kappa_{o}^{2}/\kappa_{m}^{2}} \right)^{-1/12-n}
\]

\[
(k_{o}^{2}L/k)^{n-11/12} \Gamma(2n-5/6) \left( \frac{\kappa_{o}^{2}/\kappa_{m}^{2}}{\Gamma(2n+1)} \right)^{11/12-n}
\]

The first term is on the order of \(10^{9}C_{n}^{2}\), while the following terms are on the order of \(10^{5}C_{n}^{2}\) or smaller and can be neglected. The final approximation is that

\[
R_{x}(0) = \pi^{2}k/5L^{11/6}C_{n}^{2}(.033)
\]

\[
1/30 \left( k/L\kappa_{m}^{2}\right)^{-13/12} (\kappa_{o}^{2}L/k)^{1/12}
\]

\[
\frac{\Gamma(7/6)}{2} (\kappa_{o}^{2}/\kappa_{m}^{2})^{-1/12}
\]

For \(R_{x}(0)\) we use the same representative values for the parameters, \(\kappa_{o} = 1\), \(\kappa_{m} = 100\), \(k = 10^{7}\), and \(L = 208\), where \(\kappa_{o}\) is the spatial wave number associated with the outer scale of turbulence, \(\kappa_{m}\) is the spatial wave number associated with the inner scale of turbulence, \(k\) is the wave number of the optical wave, and \(L\) represents the path length.

From Appendix 2 we had derived that

\[
R_{x}(0) = -\pi^{3}k^{13/6}L^{11/6}C_{n}^{2}(.033)\lambda f_{o}^{2} \times
\]
Now reduce Euler's Beta function and the gamma function to factorials and the result is that above. Applying formula 5 of Appendix 1,

\[ W_{-17/12-n,n+1/12} \left( \frac{\kappa^2_o}{\kappa^2_m} \right) \]

To the Whittaker function expressed above results in;

\[ W_{-17/12-n,n+1/12} \left( \frac{\kappa^2_o}{\kappa^2_m} \right) = \frac{\Gamma(-2n-1/6)}{\Gamma(11/6)} \]

\[ \exp^{-\frac{1}{2} \left( \frac{\kappa^2_o}{\kappa^2_m} \right)} \left( \frac{\kappa^2_o}{\kappa^2_m} \right)^{n+1/12} \]

\[ \left( \frac{\kappa^2_o}{\kappa^2_m} \right)^{n+1/12} \]

\[ \left( \frac{\kappa^2_o}{\kappa^2_m} \right)^{n+1/12} \]

\[ \Gamma(2n+1/6) \left( \frac{\kappa^2_o}{\kappa^2_m} \right)^{n+1/12} \]
As before, the hypergeometric series, \(_1F_1\ 2n+2; 2n+7/6; k^2/k^2\), and \(_1F_1\ 11/6; 11/6-2n; k^2/k^2\) decreases sufficiently rapidly so that the first term only can be taken with very small error.

Noting that

\[
\left(\frac{k^2}{\kappa^2_m}\right)^{n+7/12} = 10^{-4n-7/3}
\]

\[
\left(\frac{k^2}{\kappa^2_m}\right)^{5/12} = 10^{4n-5/3}
\]

we see that again, the first term of the Whittaker is the only one necessary, leaving the approximation;

\[
W_{-17/12-n, n+1/12} \left(\frac{k^2}{\kappa^2_m}\right) = \frac{\Gamma(2n+1/6)}{\Gamma(2n+2)}
\]

\[
\exp^{-\frac{1}{2}} \left(\frac{k^2}{\kappa^2_m}\right) \left(\frac{\kappa^2}{\kappa^2_m}\right)^{5/12-n}
\]

whose substitution into equation (6) leads to the equation;

\[
\hat{R}_x(0) = -\pi^3 k^{11/6} L^{11/6} C_n^2 (0.033) \lambda f_0^2
\]

\[
\sum (-1)^{n+1} \frac{(2n)! (2n+1)!}{(4n+1)!} \frac{(k/Lk^2_m)^{-7/12-n}}{(2n+1/6)} \left(\frac{\kappa^2}{\kappa^2_m}\right)^{n-5/12} \left(\frac{k^2}{\kappa^2_m}\right)^{5/12-n}
\]
The first term is on the order of $10^{16}C_\pi^2(.033)\lambda f_\Omega^2$
while the following terms are on the order of $10^{13}C_\pi^2(.033)\lambda f_\Omega^2$ or less and can be discarded with an
error of about .22%. The final approximation, therefore, is about

$$
\text{(8)} \hspace{1cm} R_{\chi}(0) = -\tau^3 k^{13/6} L^{11/6} C_\pi^2(.033)\lambda f_\Omega^2
$$

$$
\frac{1}{10} \left( \frac{k}{L k_m^2} \right)^{-1.9/12} \left( \frac{k_o^2 L}{k} \right)^{7/12}
$$

$$
\frac{\Gamma(25/6)}{6} \left( \frac{k_o^2}{k_m^2} \right)^{-7/12}
$$
FOOTNOTES


3Ibid.


6Pratt, p. 139.


8Lutomirski, p. 8.

9Pratt, p. 133.


12Lutomirski, p. 2.

14 Lawrence, p. 1538.


17 Pratt, p. 134.

18 Lawrence, p. 1529.

19 Lutomirski, p. 3.

20 Lawrence, p. 1535.


24 Rice, p. 51.

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