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STATISTICAL FADING OF A PLANE OPTICAL WAVE IN ATMOSPHERIC TURBULENCE

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RESEARCH REPORT

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ABSTRACT

A formula for the average fade time of the intensity of a plane optical wave traveling through atmospheric turbulence is developed. The model employed involves isotropic, homogeneous statistics using a lognormal distribution for the channel. The analysis is based on the fact that the logarithm of the irradiance is normally distributed and uses the work of S. O. Rice who developed such an expression for a zero mean, Gaussian process. The analysis employs the covariance function and the Taylor frozen turbulence hypothesis which results in an expression for the autocorrelation function.
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I. INTRODUCTION

Atmospheric turbulence results from the absorption of incident sunlight energy by the Earth and the heating of the surface layer of air by conduction. The decrease in the density of this air causes it to rise and mix turbulently with the cooler air above it, producing a random variation of the air temperature from point to point. Since the index of refraction of air depends on the temperature of the air, there will also be a random variation of the index of refraction from point to point. A laser beam passing through this region of turbulence will be partially or totally deviated from its path, depending upon the beam size relative to the sizes of the temperature inhomogeneties. The interaction of the beam with the turbulent medium leads to random amplitude and phase variations of the laser light.¹

The physical mechanism through which turbulence induces beam break-up is the focusing produced by the inhomogeneities in the path of the wave (see figure 1).² The constructive and destructive self-interference of the beam causes random fading in the carrier optical beam intensity, which can lead to a decrease in the signal to noise ratio at the optical receiver, as well as possibly
causing spectral interference between carrier information modulation and beam intensity fluctuations.$^{3,4,5}$ The effect of interest here is beam scintillation, which is small scale interference within the beam cross-section causing variations in the spatial power density at the receiver (see figure 2).$^6$

The objective of this paper is to present a formula for the average fade time of the intensity of an optical plane wave signal passing through atmospheric turbulence, measured at a point detector. The atmospheric model described in the paper by Lawrence and Strohbehn is employed and, therefore, the turbulence is assumed to be weak and the propagation distance to be short in order for the analysis to take place in the linear region where the so-called Rytov perturbation method is valid. The statistical works of S. O. Rice and W. C.-Y. Lee are adapted for use at optical frequencies and for a lognormal, vice normal, channel. From these the desired formula is derived and, using representative values for the various parameters, graphs are generated.
Figure 1: An illustration of beam break up due to turbulence. Blobs of differing refractive indices redistribute the energy of the beam, causing regions of high intensity and regions of low intensity. Source: Pratt, p. 134.

Figure 2: An illustration of beam scintillation. In the receiver plane, with time frozen by the photograph, light and dark regions show the regions of high and low intensity caused by the turbulence. Source: Pratt, p. 139.
II. THE ATMOSPHERIC MODEL

The atmospheric model we use here assumes the turbulent medium to be composed of discrete, homogeneous blobs, or turbulent eddies, which differ in refractive indices. The inner scale of turbulence, denoted $l_0$, represents the smallest eddy size that can exist due to viscosity and $k_m$ is its associated spatial wave-number. The outer scale of turbulence, denoted $L_0$, represents the largest scale over which the temperature fluctuations remain reasonably correlated and $k_o$ is its associated spatial wave-number.\(^7\) The wave-number of the optical wave is denoted $k$. Tatarski's structure constant for refractive index fluctuations, $C_n(r)$, while varying in general with altitude and time of day, is assumed to be constant for the region of interest.\(^8\) The characteristic frequency, $f_o = v/(2\pi\lambda L)^{1/2}$, is the ratio of the mean wind speed to the width of the first Fresnel-zone.\(^9\) The first Fresnel-zone is the disk perpendicular to the straight line between two points so that the phase difference between two light beams traveling between the points, one directly and the other via the rim of the zone, will be $180^\circ$. 
The modified von Karman spectral density function for the refractive index fluctuations\(^{10}\)

\[
\phi_n(\kappa) = 0.033 \ C_n^2 \ (\kappa^2 + \kappa_0^2)^{-11/6} \ \exp(-\kappa^2/\kappa_m^2), \quad (1)
\]

is employed instead of the Kolmogorov spectrum\(^{11}\)

\[
\phi_n(\kappa) = 0.033 \ C_n^2 \ \kappa^{-11/3}. \quad (2)
\]

Although (2) arises in the perturbation solution of the wave equation, (1) better represents the physical situation, since it is bounded at zero and drops off rapidly for large values of \(\kappa\). Furthermore, the second derivative of the autocorrelation function associated with (2), which is related to the width of the spectrum in a manner similar to the relationship between the width of a statistical distribution and its second moment, does not exist at zero. This means that the Kolmogorov spectrum is too wide and needs to be terminated like the modified von Karman spectrum. Use of (2) drives the average fade time to zero, which is not observed experimentally. The spectra are compared in figure 3. It should be noted that in the limit as \(\kappa_0\) goes to zero so that \(L_0\) goes to infinity, and \(\kappa_m\) goes to infinity so that \(l_0\) goes to zero, the autocorrelation function in step 4 of appendix 4,
Figure 3: Comparison of Spectra. The modified von Karman (solid) and the Kolmogorov (dashed) spectra are plotted to show the differences in termination. Parameter values are taken as $\kappa_0 = 1$ and $\kappa_m = 1000$, for example. The horizontal axis is used for the variable representing the wave number associated with eddy size.
\[ R_x(0) = 0.614 \, k^{7/6} \, C_n^2 \, L^{11/6}, \]
differs only by a multiplicative constant from the approximation given in step 3c of appendix 3,

\[ R_x(0) = \frac{10.9 \, k^{7/6} \, C_n^2 \, L^{5/6}}{\kappa_m^2} \]

\[ X \left( 0.141 \, L \, \kappa_m^2 + 0.54 \, \kappa_0^{1/3} \, L^{1/6} \, k^{5/6} - \kappa_m^{1/3} \, L^{1/6} \, k^{5/6} + 0.527 \, k \right). \]

The Taylor hypothesis of frozen-in turbulence is adopted, meaning that temperature fluctuations and, thereby, the refractive index fluctuations at a point are generated by the different sized blobs being swept past the point by the mean wind perpendicular to the path.\(^12,13\) This means that the internal motions of the atmosphere are taken to be negligibly slow and that the only motion of interest is perpendicular to the path. This hypothesis is questionable if the angle between the wind and the propagation direction is less than fifty degrees.\(^14\) When the Taylor hypothesis is reasonable, the time autocorrelation function for the log-amplitude of the signal equals the spatial correlation function evaluated at the product of the mean wind speed and the time difference.\(^15\)
III. PROPAGATION STATISTICS

The log-amplitude of the signal, $x = \ln(A/A_0)$, and the log-intensity, $\ln(I/I_0)$, are assumed to be normally distributed.\textsuperscript{16} We define $\chi_1 = x - \langle x \rangle$ to be the fluctuating portion of the log-amplitude. $A_0$ and $I_0$ are the no-turbulence values of the amplitude and intensity, respectively, and $2\chi = \ln(I/I_0)$ since intensity is the square of amplitude. The variance of the log-intensity will, therefore, equal four times the variance of the log-amplitude, so that $\sigma_{\ln(I)}^2 = 4\sigma_{\chi}^2$. These variances are assumed to be small, less than 2.5 for log-intensity and .64 for log-amplitude, which limits the allowed propagation distance.\textsuperscript{17} This assumption is forced by the perturbation method used in the derivations and the results are for short distances.

In the case considered here, a point detector is assumed, which is a detector with an aperture radius that is small compared to the first Fresnel-zone size, and we do not consider the effect of aperture averaging, which would tend to decrease the variance.\textsuperscript{18} In this sense, this analysis is a worst-case analysis.

For periods of observation on the order of a few minutes, the temperature variations can be considered as
a stationary random process for sufficiently small separations. Thus it can be assumed that relevant statistical averages are stationary in time and locally constant in space.\textsuperscript{19}
IV. THE AVERAGE FADE TIME

According to Lee, the desired expression for the average fade time of the intensity of an optical plane wave signal in the atmospheric model described in part II will have the form:

$$t(I_t) = P(I < I_t) / n(I_t),$$

where $I$ is the random signal intensity, $I_t$ is the threshold value for the detector below which the signal is lost, $n(I_t)$ is the average number of crossings of the level $I_t$ per second, and $t(I_t)$ is the average duration of fades below $I_t$. It remains to find expressions for $n(I_t)$ and $P(I < I_t)$.

Since $I/I_o$ is assumed to be lognormal, properties of the lognormal distribution will need to be applied. The particular properties are included in equations 2-51 and 2-52 of Panter. Equation 2-52 shows that

$$\ln(<A/A_o>) = <\ln(A/A_o)> + \sigma^2_{\chi^2}/2. \quad \text{But} \quad <A/A_o> = 1 \text{ since } A_o \text{ is the average value of } A, \text{ so that } <\ln(A/A_o)> = -\sigma^2_{\chi^2}/2. \quad \text{Thus } -\sigma^2_{\chi^2} = 2<\ln(A/A_o)> = <\ln(A^2/A_o^2)> = <\ln(I/I_o)>.$$

Therefore, the average value of the log-intensity equals the negative of the variance of the log-amplitude. Equation 2-51 shows that the cumulative distribution function of the lognormal
distribution can be written in terms of the error function. Combining all of this yields

\[
P(I < I_t) = P(I/I_0 < I_t/I_0) =
\]

\[
(2\sqrt{2\pi}\sigma_{x_1})^{-1} \int_0^{I_t/I_0} (I/I_0)^{-1} \exp \left\{ \frac{-(\ln(I/I_0) + \sigma^2 x_1)^2}{8\sigma^2 x_1} \right\} d(I/I_0)
\]

\[
= \chi_2 \left\{ 1 + \text{erf} \left( \frac{\ln(I_t/I_0) + \sigma^2 x_1}{2\sqrt{2}\sigma x_1} \right) \right\}
\]

using the fact that, as before, the variance of the log-intensity equals four times the variance of the log-amplitude.

The work of Rice\textsuperscript{22} shows that

\[
n(Y) = \int_0^\infty \dot{Y} p(Y, \dot{Y}) d\dot{Y}
\]

where \(Y\) is the random process with threshold \(Y\) and \(p\) is the joint density function of \(Y\) and its first derivative, which in this case is the bivariate normal density. Assuming a stationary, ergodic, and normal process with autocorrelation function \(R(\tau)\) yields after the integration\textsuperscript{23}

\[
n(Y) = \frac{1}{2\pi \left( \frac{-R(0)}{R(0) - m_Y^2} \right)^{\chi_2}} \exp \left( \frac{-(Y - m_Y)^2}{2(R(0) - m_Y^2)} \right)
\]
where $m_y$ is the mean of $y$. For the present case, $y = \ln(I_t/I_0)$, $m_y = -\sigma_{x_1}^2$, $R(0) - m_y = 4\sigma_{x_1}^2 = 4R_x(0)$, and $\tilde{R}(0) = 4\tilde{R}_x(0)$, using the fact that the intensity is the square of the amplitude. Therefore,

$$n(I_t) = \frac{1}{2\pi} \left[ \frac{-\ddot{R}_x(0)}{R_x(0)} \right]^{1/2} \exp \left\{ \frac{(\ln(I_t/I_0) + \sigma_{x_1}^2)^2}{-8\sigma_{x_1}^2} \right\},$$

and finally,

$$t(I_t) = \frac{\pi \left\{ 1 + \text{erf} \left[ \frac{\ln(I_t/I_0) + \sigma_{x_1}^2}{2\sqrt{2}\sigma_{x_1}} \right] \right\}}{\left\{ \frac{-\ddot{R}_x(0)}{R_x(0)} \right\}^{1/2} \exp \left\{ \frac{(\ln(I_t/I_0) + \sigma_{x_1}^2)^2}{-8\sigma_{x_1}^2} \right\}}.$$
V. EXAMINATION OF RESULTS

Using the approximations developed in appendix 3 and simplifying,

\[
\frac{-\dot{R}_X(0)}{R_X(0)} = 6.28 \lambda L \kappa_m^2 f_0^2 \chi
\]

\[
= \left\{ \begin{array}{l}
0.863 \kappa_0^{-1/3} + 9.66 k^{1/6} L^{-1/6} + 0.833 \kappa_m^{1/3} \\
0.141 L^{5/6} \kappa_m^2 k^{-5/6} + 0.54 \kappa_0^{1/3} - \kappa_m^{1/3} + 0.527 k^{1/6} L^{-1/6}
\end{array} \right\}
\]

where \( \lambda \) is the wavelength of the laser light; \( k = 2\pi/\lambda \) is the wave-number associated with the wavelength; \( L \) is the propagation length; \( \kappa_m \) is the spatial wave-number associated with the inner scale of turbulence, \( \kappa_0 \); \( \kappa_0 \) is the spatial wave-number associated with the outer scale of turbulence, \( L_0 \); and \( f_0 \) is the characteristic frequency which depends upon the wavelength, the propagation length, and the square of the mean wind speed. It is desired to use this expression in the formula for the average fade time,
to examine some of the properties of the average fade time expression for various parameter values. Experimental results will be used to obtain typical values of the parameters. The experiment from which this data is taken was run in England by Dr. Ronald Phillips of the University of Central Florida and is described more fully in Belkerdid [1980]. For the present purpose, it is sufficient to note that a Helium-Neon laser was used so that $\lambda = 0.6328 \times 10^{-6}$ meters and $k = 10^7$ while the experimental conditions were clear and sunny on the flat and unobstructed path with a mean wind speed of 3 meters per second, $\kappa_0 = 1$, $\kappa_m = 1000$, and $L = 208$ meters.

Two facts about the expression for $t(I_t)$ can be seen at this point. First, $t(I_t)$ is not a function of Tatarski's structure constant for refractive index fluctuations, $C_n$. Thus an accurate value for this parameter is not needed. Second, after a little
algebra, it can be seen that the product of $t(I_t)$ and $f_0$ is both unitless and independent of the mean wind speed. In the theoretical graphs which follow, $t(I_t)f_0$ will be used as the dependent variable in order to normalize and eliminate uncertainty in the expression due to the measured value of $v$. It is also worth repeating at this point that in the limit as $\kappa_0$ goes to zero and $\kappa_m$ goes to infinity, the derived approximation for $R_x(0)$ using the modified von Karman spectrum differs from the expression for $R_x(0)$ using the Kolmogorov spectrum by only a multiplicative constant, so that results here should not contradict previous work.

Figure 4 shows $t(I_t)f_0$ plotted against $I_t/I_0$ for various values of $\sigma^2$, while figure 5 shows it plotted against $I_t/I_0$ for various values of $\lambda$, and figure 6 shows it plotted against $L$ for various values of the variance.

Looking at figure 4, the slope of the curve is significantly less for small values of the threshold to average intensity value ratio. For example, when the variance is .3, as $I_t/I_0$ goes from 1 to 10, $t(I_t)f_0$ goes from .0572 to .327, a factor of 5.72, while as $I_t/I_0$ goes from .001 to .01, $t(I_t)f_0$ goes from .00583 to .00867, a factor of 1.49. This effect
is more dramatic for small variances and less dramatic for large variances. It can be seen that the result of decreasing $I_t$, which is controlled by the receiver, or increasing $I_0$, which is controlled by the sender, become less and less effective in reducing $t(I_t)f_0$. If a reduction in average fade time is needed, a more promising approach is to try to reduce the variance. The variance is in large part controlled by atmospheric conditions, but may be reduced by such techniques as spatial diversity and aperture averaging, which are beyond the scope of this paper which deals with a single point detector.

Looking at figure 5, the most obvious feature is the lack of effect of wavelength change on average fade time. This is not entirely unexpected, since wavelength does not appear in the argument of the error function or the exponential function. It would appear that there is no real advantage in changing wavelength just to try to reduce average fade time.

Looking at figure 6, apparently propagation length has a very small effect on average fade time, at least for the short distances where this linear analysis is appropriate. The curve is almost straight and parallel to the horizontal axis.
It can also be seen that the effect of changing the variance is much greater. This is to be expected, since the variance appears in the argument of the exponential function and the error function while the propagation length does not.

In summary, the variance and the threshold to average intensity ratio are the parameters which have the strongest effect on the average fade time. The expression for average fade time seems relatively insensitive to changes in propagation length and wavelength of the laser light.

Figure 7 shows data points measured experimentally. Also shown is the curve which results when the average intensity is taken to be 2.86 and the variance is taken to be .01. The curve for these typical values shows agreement with the experimental points.
Figure 4: Average fade time versus the ratio of threshold intensity to average intensity for values of the variance equal to .01, .1, .3, .4, and .5.
Figure 5: Average fade time versus the ratio of threshold intensity to average intensity for values of the wavelength equal to $0.5 \times 10^{-6}$ m, $0.6 \times 10^{-6}$ m, $0.8 \times 10^{-6}$ m, $0.9 \times 10^{-6}$ m and $1.0 \times 10^{-6}$ m. Note that the curves for the three largest values of the wavelength are indistinguishable and that variation of the wavelength does not strongly affect the average fade time.
Figure 6: Average fade time versus propagation length for values of the variance equal to .01, .1, .3, .4, and .6. Note that the effect in a change of variance is small when the change is on the order of .1.
Figure 7: Experimental points compared with the curve generated with the mean intensity equal to 2.86 and the variance equal to .01.
APPENDIX 1: FORMULAE AND FACTS USED IN DERIVATIONS.

In the following, W is the Whittaker function and G is Meijer's function.

1. From Bateman Manuscript Project, Tables of Integral Transforms, volume II, p. 234, formula number 12:

\[
\int_0^\infty x^\lambda e^{-ax}(x+y)^{-\rho} dx = \Gamma(\lambda+1) a^{\frac{1}{2}\rho-\frac{1}{2}\lambda-\frac{1}{2}y} y^{-\frac{1}{2}\lambda-\frac{1}{2}\rho} e^{\frac{1}{2}\alpha y} W_{2k,-\lambda-\rho} (ay)
\]

\[2k=-\lambda-\rho, \quad 2m=\lambda-\rho+1, \quad \text{Re}a>0, \quad |\text{arg}y|<\pi.\]

2. From Bateman Manuscript Project, Tables of Integral Transforms, volume II, p. 442:

\[
x^k e^{\frac{k}{2}x} W_{k,m}(x) = \frac{1}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)} G^{21}_{12} \left( \begin{array}{c} x \\ \frac{k+\lambda+1}{2} \end{array} \mid \begin{array}{c} \frac{k-\lambda-\rho}{2} \frac{k+m+\lambda}{2} \\ \frac{k+\lambda}{2} \end{array} \right)
\]

3. From Bateman Manuscript Project, Tables of Integral Transforms, volume II, p. 417, formula number 1:

\[
\int_0^1 x^{\sigma-1}(1-x)^{\rho-1} G_{mn}^{pq} \left( \begin{array}{c} ax \\ b_1, \ldots, b_q \\ \alpha \end{array} \mid \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \\ \alpha \end{array} \right) dx
\]

\[= \Gamma(\sigma) G_{p+1,q+1}^{m,n+1} \left( \begin{array}{c} \alpha \\ 1-\rho \end{array} \mid \begin{array}{c} 1-\rho, a_1, \ldots, a_p \\ b_1, \ldots, b_q, 1-\rho-\sigma \end{array} \right)
\]

\[p+q<(m+n), \quad |\text{arg}a|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi, \quad \text{Re}(\rho+b_j)>0, \quad j=1, \ldots, m, \quad \text{Re}\sigma>0.\]
4. From Bateman Manuscript Project, *Higher Transcendental Functions*, volume I, p. 208, formula number 5:

If no two \( b_j, j = 1, \ldots, m \), differ by an integer, all poles are of the first order and

\[
G_{pq}^{mn}(x \mid a_r \mid b_r) = \sum_{h=1}^{m} \sum_{j=1}^{n} \frac{\pi \Gamma(b_j-b_h) \prod_{j=1}^{m} \Gamma(l+b_j-a_j)}{\prod_{j=m+1}^{n} \prod_{j=1}^{n} \Gamma(l+b_h-b_j) \prod_{j=m+1}^{n} \Gamma(a_j-b_h)} x^{b_h} 
\]

\[
X \prod_{p=1}^{P} (1+b_h-a_1, \ldots, l+b_h-a_p; l+b_h-b_1, \ldots, *; \ldots, p q^{-1} l+b_h-b_q; (-1)^{P-m-n} x) \quad p < q \text{ or } p = q \text{ and } |x| < 1.
\]

5. From Bateman Manuscript Project, *Higher Transcendental Functions*, volume I, p. 182:

\[
pFq\left[a_1, \ldots, a_p; z \right] = \sum_{n=1}^{\infty} \frac{(a_1)_n \cdots (a_p)_n z^n}{(\rho_1)_n \cdots (\rho_q)_n n!}
\]

6. From Gradshteyn and Ryshik, *Table of Integrals, Series, and Products*, p. 1069:

Meijer's \( G\)-function is analytic.

7. From Gradshteyn and Ryshik, *Table of Integrals, Series, and Products*, p. 1069:

\[
G_{p,q}^{m,n}(x \mid a_1, \ldots, a_p \mid b_1, \ldots, b_q) \text{ is symmetric with respect to the parameter sets } a_1, \ldots, a_n; a_{n+1}, \ldots, a_p; b_1, \ldots, b_m; \text{ and } b_{m+1}, \ldots, b_q.
\]
8. From Bateman Manuscript Project, Higher Transcendental Functions, volume I, p. 209, formula number 7:

If one of the \(a_j, j=1,\ldots,n\), is equal to one of the \(b_j, j=m+1,\ldots,q\), (or one of the \(b_j, j=1,\ldots,m\) equals one of the \(a_j, j=n+1,\ldots,p\)), then the \(G\)-function reduces to one of lower order: \(p, q,\) and \(n\) decrease by unity.

\[
G_{pq}^{mn} \left( x \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_{q-1}, a_1 \end{array} \right| n, p, q > 1 \right) = G_{p-1,q-1}^{m,n-1} \left( x \left| \begin{array}{c} a_2, \ldots, a_p \\ b_1, \ldots, b_{q-1} \end{array} \right| n, p, q > 1 \right)
\]

is such a reduction formula and all others are similar.

9. From Gradshteyn and Ryshik, Table of Integrals, Series, and Products, p. 447, formula number 3.823:

\[
\int_0^\infty x^{\mu-1} \sin^2 ax \, dx = \frac{-\Gamma(\mu) \cos \frac{a \pi}{2}}{2^{\mu+1} a^\mu} \quad a > 0, -2 < \Re \mu < 0.
\]
APPENDIX 2: DERIVATIONS OF FORMULAE FOR THE AUTOCORRELATION FUNCTION AND ITS SECOND DERIVATIVE.

Preliminary Results:

Preliminary Result 1:

\[ \int_{0}^{\infty} e^{-au(u+\beta)} \left( \frac{-11}{6} \right) \sin^2(u) \, du = \frac{\beta^{-5/6} G_{21} \left( a \beta \right)}{2 \Gamma(11/6)} - \frac{\beta^{-5/6} \text{Re} G_{21} \left( (a-j2) \beta \right)}{2 \Gamma(11/6)}. \]

Proof: a. Using a trigonometric identity to write the square of the sine function in terms of a cosine function and the fact that the cosine function can be written as the real part of a complex exponential, we obtain:

\[ \int_{0}^{\infty} e^{-au(u+\beta)} \left( \frac{-11}{6} \right) \sin^2(u) \, du = \frac{1}{2} \int_{0}^{\infty} e^{-au(u+\beta)} \left( \frac{-11}{6} \right) (1 - \cos(2u)) \, du. \]

b. To obtain a formula to evaluate the integrals in part a, we employ successively formulae 1,

\[ \int_{0}^{\infty} x^\lambda e^{-ax(x+y)} \, dx = \Gamma(\lambda+1) a^{-\lambda-1} y^\lambda \frac{\partial}{\partial y} \left[ \Gamma(\lambda+1) e^{ay} \Psi_{k,m}(ay) \right]. \]
and 2,

\[ x^\frac{1}{2} e^{\frac{1}{2}x} W_{\nu, m}(x) = \frac{1}{\Gamma(\frac{1}{2}+m-k) \Gamma(\frac{1}{2}-m-k)} G_{12}^{21} \left( x \left| \begin{array}{c} k+m+1 \\ \frac{1}{2}-m+\frac{1}{2}, m+\frac{1}{2} \end{array} \right. \right), \]

from appendix 1 to obtain:

\[ \int_0^\infty e^{-\gamma u} (u+\beta)^{-11/6} du = \gamma^{-1/12} \beta^{-11/12} e^{\frac{1}{2} \gamma \beta} W_{\gamma \beta} (\gamma \beta), \]

\[ = \frac{\beta^{-5/6} G_{12}^{21} \left( \gamma \beta \left| \begin{array}{c} 0 \\ 5/6, 0 \end{array} \right. \right)}{\Gamma(11/6)}. \]

c. Applying the result in part b to the integrals in part a yields:

\[ \int_0^\infty e^{-\alpha u} (u+\beta)^{-11/6} \sin^2(u) du = \frac{\alpha^{-5/6} G_{12}^{21} \left( \alpha \beta \left| \begin{array}{c} 0 \\ 5/6, 0 \end{array} \right. \right) - \beta^{-5/6} \Re G_{12}^{21} \left( \alpha-j2 \beta \left| \begin{array}{c} 0 \\ 5/6, 0 \end{array} \right. \right)}{2 \Gamma(11/6)}. \]
Preliminary result 2:

\[
\int_0^\infty u e^{-au} (u+\beta) ^{-11/6} \sin^2(u) \, du
\]

\[
= \beta^{1/6} \frac{G_{12}^{21} \left( \alpha \beta \mid -1/6, 0 \right)}{2 \Gamma (11/6)} - \beta^{1/6} \Re \frac{G_{12}^{21} \left( (\alpha-j2) \beta \mid -1/6, 0 \right)}{2 \Gamma (11/6)} .
\]

Proof: a. Using the trigonometric identity to write the square of the sine function in terms of the cosine function and the fact that the cosine function can be written as the real part of a complex exponential:

\[
\int_0^\infty u e^{-au} (u+\beta) ^{-11/6} \sin^2(u) \, du
\]

\[
= \frac{1}{2} \int_0^\infty u e^{-au} (u+\beta) ^{-11/6} (1-\cos(2u)) \, du
\]

\[
= \frac{1}{2} \int_0^\infty u e^{-au} (u+\beta) ^{-11/6} \, du - \frac{1}{2} \Re \int_0^\infty u e^{-\alpha-j2u} (u+\beta) ^{-11/6} \, du
\]

b. To obtain a formula to evaluate the integrals in part a, we employ successively formulae 1,

\[
\int_0^\infty x^\lambda e^{-ax} (x+y)^{-\rho} \, dx = \Gamma (\lambda+1) a^{\frac{k+1}{2}} \lambda^\frac{1}{2} \gamma \frac{1}{2} \lambda - \frac{1}{2} \rho e^{\frac{1}{2}} ay W_{k,m} (\alpha y),
\]

and two,

\[
x^\ell e^{-\frac{k}{2}x} W_{k,m} (x) = \frac{1}{\Gamma (\frac{k+1}{2}) \Gamma (\frac{k-1}{2})} G_{12}^{21} \left( \begin{array}{c} \ell+1 \\ \ell-m+\frac{1}{2}, m+\ell+\frac{1}{2} \end{array} \mid x \right)
\]
from appendix 1 to obtain:

\[ \int_0^\infty u e^{-\gamma u} (u+\beta)^{-1/6} du = \gamma^{-7/12} e^{-5/12} e^{\frac{1}{2} \gamma \beta} W_{-17/12, 1/12} \]

\[ = \beta^{1/6} G_{12}^{21} \begin{pmatrix} \gamma \beta & -1 \\ -1/6, 0 \end{pmatrix} \frac{2 \gamma (11/6)}{r (11/6)} \]

c. Applying the result in part b to the integrals in part a yields:

\[ \int_0^\infty u e^{-\alpha u} (u+\beta)^{-1/6} \sin^2(u) du \]

\[ = \beta^{1/6} G_{12}^{21} \begin{pmatrix} \alpha \beta & -1 \\ -1/6, 0 \end{pmatrix} \frac{2 \gamma (11/6)}{2 \gamma (11/6)} - \beta^{1/6} R G_{12}^{21} \begin{pmatrix} \alpha - j2 \beta & -1 \\ -1/6, 0 \end{pmatrix} \frac{2 \gamma (11/6)}{2 \gamma (11/6)} . \]
Preliminary result 3:

\[
\int_0^1 G_{12}^{21}\left(\frac{\kappa_0^2}{\kappa_m^2} - jL\kappa_0^2 y/k \right|_{b_1,b_2}^{a_1} dy
\]

= \left(-j k/(L\kappa_m^2)\right) G_{23}^{22}\left(\frac{\kappa_0^2}{\kappa_m^2} \right|_{b_1,b_2,-1}^{0,a_1} \\
+ \left(1+j k/(L\kappa_m^2)\right) G_{23}^{22}\left(\frac{\kappa_0^2}{\kappa_m^2} - jL\kappa_0^2 /k \right|_{b_1,b_2,-1}^{0,a_1}\right)

Proof:  

a. Form a triangular contour composed of the straight line segments \(\Gamma_1\) from \(\kappa_0^2/\kappa_m^2 - jL\kappa_0^2 /k\) to \(\kappa_0^2/\kappa_m^2\), \(\Gamma_2\) from \(\kappa_0^2/\kappa_m^2\) to 0, and \(\Gamma_3\) from 0 to \(\kappa_0^2/\kappa_m^2 - jL\kappa_0^2 /k\).

We will employ the fact that \(G\) is analytic and Cauchy's Theorem for complex contour integrals to evaluate the desired integral. (See figure 8).

b. Substituting \(z = \kappa_0^2/\kappa_m^2 - jL\kappa_0^2 y/k\) so that

\[
dz = -jL\kappa_0^2 /k \, dy,
\]

and using part a:

\[
\int_0^1 G_{12}^{21}\left(\frac{\kappa_0^2}{\kappa_m^2} - jL\kappa_0^2 y/k \right|_{b_1,b_2}^{a_1} dy = \left(j k/L\kappa_0^2\right) \int G_{12}^{21}\left(z \right|_{b_1,b_2}^{a_1} dz
\]

\[
= -j k/(L\kappa_0^2) \int_{\Gamma_1} G_{12}^{21}\left(z \right|_{b_1,b_2}^{a_1} dz
\]

\[
= j k/(L\kappa_0^2) \int_{\Gamma_2} G_{12}^{21}\left(z \right|_{b_1,b_2}^{a_1} dz + \int_{\Gamma_3} G_{12}^{21}\left(z \right|_{b_1,b_2}^{a_1} dz
\]
Figure 8: The contour used in the integration done in preliminary result 3.
c. On $\Gamma_2$, $z = -(\kappa_0^2/\kappa_m^2)(v-1)$ so that $dz = -(\kappa_0^2/\kappa_m^2)dv$
and $v = 1-x$ so that $dv = -dx$, yields

$$\int_{\Gamma_2} G_{12}^{21} \left( z \left| a_1 \right. \right) b_1, b_2 \right) dz = -(\kappa_0^2/\kappa_m^2) \int_0^1 G_{12}^{21} \left( (\kappa_0^2/\kappa_m^2)(v-1) \left| a_1 \right. \right. \right) b_1, b_2 \right) dv$$

$$= -(\kappa_0^2/\kappa_m^2) \int_0^1 G_{12}^{21} \left( (\kappa_0^2/\kappa_m^2)x \left| a_1 \right. \right. \right) b_1, b_2 \right) dx.$$  

d. On $\Gamma_3$, $z = (\kappa_0^2/\kappa_m^2 - j\Lambda k_0^2/k)x$ so that $dz = (\kappa_0^2/\kappa_m^2 - j\Lambda k_0^2/k)dx$ yields:

$$\int_{\Gamma_3} G_{12}^{21} \left( z \left| a_1 \right. \right) b_1, b_2 \right) dz$$

$$= (\kappa_0^2/\kappa_m^2 - j\Lambda k_0^2/k) \int_0^1 G_{12}^{21} \left( (\kappa_0^2/\kappa_m^2 - j\Lambda k_0^2/k)x \left| a_1 \right. \right. \right) b_1, b_2 \right) dx$$

e. Substituting the results of parts c and d into part b yields:

$$\int_0^1 G_{12}^{21} \left( \kappa_0^2/\kappa_m^2 - j\Lambda k_0^2/y/k \left| a_1 \right. \right. \right) b_1, b_2 \right) dy$$

$$= (j/k/(\Lambda k_0^2)) \left[ -(\kappa_0^2/\kappa_m^2) \int_0^1 G_{12}^{21} \left( \kappa_0^2x/\kappa_m^2 \left| a_1 \right. \right. \right) b_1, b_2 \right) dx$$

$$+ (\kappa_0^2/\kappa_m^2 - j\Lambda k_0^2/k) \int_0^1 G_{12}^{21} \left( (\kappa_0^2/\kappa_m^2 - j\Lambda k_0^2/k)x \left| a_1 \right. \right. \right) b_1, b_2 \right) dx \right]$$

$$= -(j/k/(\Lambda k_m^2)) \int_0^1 G_{12}^{21} \left( \kappa_0^2x/\kappa_m^2 \left| a_1 \right. \right. \right) b_1, b_2 \right) dx$$

$$+ (1+jk/(\Lambda k_m^2)) \int_0^1 G_{12}^{21} \left( (\kappa_0^2/\kappa_m^2 - j\Lambda k_0^2/k)x \left| a_1 \right. \right. \right) b_1, b_2 \right) dx$$
f. By formula 3 of appendix 1,

\[
\int_0^1 x^{\rho-1}(1-x)^{\sigma-1} G_{mn}^{pq}(\alpha, a_1, \ldots, a_p, b_1, \ldots, b_q) \, dx
\]

\[
= r(\sigma) G_{m,n+1}^{p,q+1}(\alpha, 1-\rho, a_1, \ldots, a_p),
\]

we get:

\[
\int_0^1 G_{12}^{21}(\delta x, a_1, b_1, b_2) \, dx = G_{23}^{22}(\delta, 0, a_1, b_1, b_2, -1).
\]

g. Applying the result in part f to the integrals in part e yields:

\[
\int_0^1 G_{12}^{21}(\kappa_0^2/\kappa_m^2 - jL\kappa_0^2 y/k, a_1, b_1, b_2) \, dy
\]

\[
= -(jk/(L\kappa_m^2)) G_{23}^{22}(\kappa_0^2/\kappa_m^2, a_1, a_2, b_1 b_2 , -1)
\]

\[
+ (1+jk/(L\kappa_m^2)) G_{23}^{22}(\kappa_0^2/\kappa_m^2 - jL\kappa_0^2/k, 0, a_1, b_1, b_2, -1)
\]
Preliminary result 4:

Examination of the forms of formulae 4,

\[ G_{pq}^{mn} \left( \begin{array}{c} x \\ a_r \\ b_s \end{array} \right) = \frac{1}{x} \prod_{h=1}^{m} \frac{\Gamma \left( b_j - b_h \right) \prod_{j=1}^{n} \Gamma \left( 1 + b_h - a_j \right)}{\prod_{j=m+1}^{q} \Gamma \left( 1 + b_h - b_j \right) \prod_{j=n+1}^{p} \Gamma \left( a_j - b_h \right)} x \]

where \( (.)^n \) denotes the Pockhammer symbol, it is clear that \( G_{23}^{22} \left( x \begin{array}{c} 0, a_1 \\ b_1, b_2, -1 \end{array} \right) \) is real when \( x \) is real. From this it may be concluded that when \( x \) is real,

\[ \text{Re} \left( -jk/(Lc^3) \right) G_{23}^{22} \left( x \begin{array}{c} 0, a_1 \\ b_1, b_2, -1 \end{array} \right) = 0. \]
Main Results: In order to evaluate the autocorrelation function and its second derivative at zero, we first express them in terms of the $G$ function and then, via the hypergeometric function, in terms of an infinite series.

Main result 1:

$$R_\chi(0) = \frac{6\pi^2 k^2 C_n^2 \Gamma(1/6) (.033)}{\kappa_0^{5/3}} \sum_{n=0}^{\infty} \left( \frac{((\Gamma(n+1/6))^{-1}}{\Gamma(n+1/6)(n+1)} \right)$$

$$-\left(\frac{\kappa_0}{\kappa_m}\right)^{5/3} n! \left(\frac{\kappa_0}{\kappa_m}\right)^{2n} - \frac{r^n \cos(n\theta)}{\Gamma(n+1/6)(n+1)}$$

$$+ \frac{kr^n \sin(n\theta)}{L_k^2(n+1/6)(n+1)} - \frac{kr^{n+5/6} \sin(n\theta+5\theta/6)}{L_k^2(n+11/6)n!}$$

$$+ \frac{r^{n+5/6} \cos(n\theta+5\theta/6)}{(n+11/6)n!}$$

Proof: a. $R_\chi(\tau) = 4\pi^2 k^2 C_n^2 (.033)$

$$\chi \int \int \kappa J_\nu \left( \kappa (2\pi L \lambda)^{2f_0} \right) e^{-\kappa^2 / \kappa_m^2 (\kappa^2 + \kappa_0^2)^{-11/6}}$$

$$\chi \sin^2 (\kappa^2 (L-n)/(2k)) \, d\kappa d\eta$$

b. $R_\chi(0) = 4\pi^2 k^2 C_n^2 (.033)$

$$\chi \int \int \kappa e^{-\kappa^2 / \kappa_m^2 (\kappa^2 + \kappa_0^2)^{-11/6}} \sin^2 (\kappa^2 (L-n)/(2k)) \, d\kappa d\eta$$
c. Substituting $L y = L - n$, so that $L \, dy = -dn$:

$$R_x(0) = 4\pi^2 k^2 c_n^2 L (.033)$$

$$X \int \int e^{-\kappa^2/\kappa_m^2 (\kappa^2 + \kappa_o^2)}^{-11/6} \sin^2 (L y^2/(2k)) \kappa \, d\kappa \, dy$$

d. Substituting $u = L y^2/(2k)$, so that $du = (L y/k) \kappa \, d\kappa$:

$$R_x(0) = 4\pi^2 k^2 c_n^2 (.033) (L/(2k))^{11/6}$$

$$X \int \int \frac{y^{5/6}}{\sin^2 (u)} dudy.$$  

e. Letting $\alpha = (2k)/(L y^2 m)$ and $\beta = (L y^2 O)/(2k)$ while applying preliminary result 1,

$$\int e^{-\alpha u} (u + \beta)^{-11/6} \sin^2 (u) \, du$$

$$= \beta^{-5/6} G_{12}^{21} \left( \alpha \beta \left| \begin{array}{c} 0 \\ 5/6, 0 \end{array} \right. \right) - \beta^{-5/6} \text{Re} G_{12}^{21} \left( (\alpha-j2) \beta \left| \begin{array}{c} 0 \\ 5/6, 0 \end{array} \right. \right),$$

to the result in part d yields:

$$R_x(0) = \pi^2 k^2 c_n^2 (.033) L$$

$$\frac{1}{\Gamma (11/6) \kappa_o^{5/3}}$$

$$X \int \int G_{12}^{21} \left( \alpha \beta \left| \begin{array}{c} 0 \\ 5/6, 0 \end{array} \right. \right) \text{Re} G_{12}^{21} \left( (\alpha-j2) \beta \left| \begin{array}{c} 0 \\ 5/6, 0 \end{array} \right. \right) \, dy$$
\[ R_x(0) = \frac{\pi^2 k^2 C_n^2 L(.033)}{\Gamma(11/6) \kappa_0^{5/3}} \left[ G_{12}^{21} \left( \kappa_0^2 / \kappa_m^2 \right| 5/6, 0 \right) \right.

- \text{Re} \int_0^1 G_{12}^{21} \left( \kappa_0^2 / \kappa_m^2 - j L \kappa_0^2 y / k \right| 5/6, 0 \right) dy \]

f. Applying preliminary result 3,

\[ \int_0^1 G_{12}^{21} \left( \kappa_0^2 / \kappa_m^2 - j L \kappa_0^2 y / k \right| a_1 \right. \]

\[ b_1, b_2, -1 \]

\[ = -j k / (L \kappa_m^2) G_{23}^{22} \left( \kappa_0^2 / \kappa_m^2 \right| 0, a_1 \right. \]

\[ b_1, b_2, -1 \]

\[ + (1 + j k / (L \kappa_m^2)) G_{23}^{22} \left( \kappa_0^2 / \kappa_m^2 - j \kappa_0^2 L / k \right| 0, a_1 \right. \]

\[ b_1, b_2, -1 \]

to the result in part e yields:

\[ R_x(0) = \frac{\pi^2 k^2 C_n^2 L(.033)}{\Gamma(11/6) \kappa_0^{5/3}} \left[ G_{12}^{21} \left( \kappa_0^2 / \kappa_m^2 \right| 5/6, 0 \right) \right.

- \text{Re} \left( -j k / (L \kappa_m^2) G_{23}^{22} \left( \kappa_0^2 / \kappa_m^2 \right| 0, 0 \right) \right.

\[ 5/6, 0, -1 \]

\[ + (1 + j k / (L \kappa_m^2)) G_{23}^{22} \left( \kappa_0^2 / \kappa_m^2 - j \kappa_0^2 L / k \right| 0, 0 \right. \]

\[ 5/6, 0, -1 \right) \right] \right]

g. Applying preliminary result 4,

when \( x \) is real, \( \text{Re}(-j k / (L \kappa_m^2)) G_{23}^{22} \left( x \right| 0, a_1 \right. \]

\[ b_1, b_2, -1 \right) = 0, \]

to the result in part f yields:
\[ R_x(0) = \frac{\pi^2 k^2 C_{n}^2 L(0.033)}{r(11/6) \kappa_{6}^{5/3}} \left( G_{21}^{12} \left( \frac{\kappa_{o}^{2}}{\kappa_{m}^{2}} \right)_{5/6,0}^{0,0} \right) \]

- \( \text{Re}(1 + jk/(L\kappa_{m}^{2})) G_{23}^{22} \left( \frac{\kappa_{o}^{2}}{\kappa_{m}^{2}} - jL\kappa_{o}^{2}/k \right)_{5/6,0,-1}^{0,0} \)

\[
h = (\kappa_{o}^{4}/\kappa_{m}^{4} + L^{2}\kappa_{o}^{4}/k^{2})^{1/2} \quad \text{and} \quad \theta = \tan^{-1}(-L\kappa_{m}^{2}/k).
\]

i. Using formulae 4,

\[
G_{pq}^{mn}(x | a_{r} b_{s}) = \sum_{h=1}^{m} \prod_{j=1}^{n} \Gamma(b_{j} - b_{h}) \prod_{j=1}^{n} \Gamma(1 + b_{h} - a_{j}) \frac{b_{h}}{x} \prod_{j=m+1}^{q} \Gamma(1 + b_{h} - b_{j}) \prod_{j=n+1}^{p} \Gamma(a_{j} - b_{h})
\]

\[
X_{F}(1 + b_{h} - a_{1}, \ldots, 1 + b_{h} - a_{p}; 1 + b_{h} - b_{1}, \ldots, *, \ldots, 1 + b_{h} - b_{q}; (-1)^{p-m-n}x),
\]

and 5,

\[
p_{F}^{q} \left[ \begin{array}{c} a_{1}, \ldots, a_{p} \\ \rho_{1}, \ldots, \rho_{q} \end{array} \right] = \sum_{n=1}^{\infty} (a_{1})_{n} \cdots (a_{p})_{n} z^{n} \frac{(\rho_{1})_{n} \cdots (\rho_{q})_{n}}{n!},
\]

\[
G_{23}^{22} \left( \frac{\kappa_{o}^{2}}{\kappa_{m}^{2}} - jL\kappa_{o}^{2}/k \right)_{5/6,0,-1}^{0,0} \]

\[
= 6 \Gamma(-5/6) \Gamma(11/6) / 11r^{5/6} e^{j5\theta}/6_{2}F_{3}(11/6,11/6;11/6,17/6; r e^{j\theta})
\]

\[
+ \Gamma(5/6)_{2}F_{3}(1,1;1/6,2; r e^{j\theta})
\]

\[
= \Gamma(-5/6) \Gamma(11/6) r^{5/6} \sum_{n=0}^{\infty} \frac{r^{n} e^{j(n+5/6)\theta}}{(n+11/6)n!}
\]

\[
+ \Gamma(5/6) \Gamma(11/6) \sum_{n=0}^{\infty} \frac{r^{n} e^{j\theta}}{\Gamma(n+1/6)(n+1)}
\]
j. Using the result in part a and simplifying,
\[
\begin{align*}
(1+jk/\langle \mathbf{L}\mathbf{k}_m^2 \rangle) G_{23}^{22} \left( \frac{\kappa_0^2/\kappa_m^2 - jL\kappa_0^2/k}{5/6,0,-1} \right) \\
= \Gamma(-5/6) \Gamma(11/6) r^{5/6} \sum_{n=0}^{\infty} \frac{r^n e^{j(n+5/6)\theta}}{(n+11/6)n!} \\
+ \Gamma(5/6) \Gamma(1/6) \sum_{n=0}^{\infty} \frac{r^n e^{jn\theta}}{\Gamma(n+1/6)(n+1)} \\
+ \Gamma(-5/6) \Gamma(11/6) k/(\mathbf{L}\mathbf{k}_m^2) r^{5/6} \sum_{n=0}^{\infty} \frac{r^n e^{j(n\theta + 5\theta/6 + \pi/2)}}{(n+11/6)n!} \\
+ \Gamma(5/6) \Gamma(1/6) k/(\mathbf{L}\mathbf{k}_m^2) \sum_{n=0}^{\infty} \frac{r^n e^{j(\pi/2)}}{\Gamma(n+1/6)(n+1)}
\end{align*}
\]

k. Using the result in part b and simplifying:
\[
\begin{align*}
\text{Re}(1+jk/\langle \mathbf{L}\mathbf{k}_m^2 \rangle) G_{23}^{22} \left( \frac{\kappa_0^2/\kappa_m^2 - jL\kappa_0^2/k}{5/6,0,-1} \right) \\
= \Gamma(1/6) \Gamma(5/6) \sum_{n=0}^{\infty} \left( \frac{r^n \cos(n\theta)}{\Gamma(n+1/6)(n+1)} - \frac{kr^n \sin(n\theta)}{\mathbf{L}\mathbf{k}_m^2 \Gamma(n+1/6)(n+1)} \right) \\
+ \frac{kr^{n+5/6} \sin(n\theta + 5\theta/6)}{\mathbf{L}\mathbf{k}_m^2(n+11/6)n!} - \frac{r^{n+5/6} \cos(n\theta + 5\theta/6)}{(n+11/6)n!}
\end{align*}
\]

l. Using formulae 4,
\[
G_{pq}^{mn}(\mathbf{x} | \mathbf{a}_r) = \sum_{h=1}^{m} \Pi_{j=m+1}^{m} \Gamma(b_j - b_h) \Pi_{j=1}^{n} \Gamma(l + b_h - a_j) x^{b_h} \\
\sum_{h=1}^{q} \Pi_{j=m+1}^{q} \Gamma(l + b_h - b_j) \Pi_{j=1}^{P} \Gamma(a_j - b_h) x^{b_h}
\]
\[ X \prod_{q=1}^{q} (1+b_{h}^{a_1}, \ldots, 1+b_{h}^{a_p}, 1+b_{h}^{b_1}, \ldots, 1+b_{h}^{b_q}; (1)^{P-m-n_1}), \]

and 5,

\[ p^F_q \begin{bmatrix} a_1, \ldots, a_p; z \\ \rho_1, \ldots, \rho_q \end{bmatrix} = \sum_{n=1}^{\infty} \frac{(a_1)_n \cdots (a_p)_n z^n}{(\rho_1)_n \cdots (\rho_q)_n n!} , \]

we get: \[ G_{12}^{21} \left( \frac{\kappa_0^2}{\kappa_m^2} \right| 0 \right) 5/6, 0 \]

\[ = \Gamma(-5/6) \Gamma(11/6) \left( \frac{\kappa_0}{\kappa_m} \right)^{5/3} {}_1F_1(11/6; 11/6; \kappa_0^2/\kappa_m^2) \]

\[ + \Gamma(5/6) {}_1F_1(1; 1/6; \kappa_0^2/\kappa_m^2) \]

\[ = \Gamma(1/6) \Gamma(5/6) \sum_{n=0}^{\infty} \left( \frac{\left( \Gamma(n+5/6) \right)^{-1} \left( \frac{\kappa_0}{\kappa_m} \right)^{5/3}}{n!} \right) \left( \frac{\kappa_0}{\kappa_m} \right)^{2n} \]

m. Combining parts k and l with part g produces:

\[ R_0^l = 6\pi^2 k^2 C_n^2 L \left( \frac{1/6}{0.033} \right)^5 \frac{\sum_{n=0}^{\infty} \left( \left( \Gamma(n+1/6) \right)^{-1} \left( \frac{\kappa_0}{\kappa_m} \right)^{5/3} \right) \left( \frac{\kappa_0}{\kappa_m} \right)^{2n}}{\Gamma(n+1/6)(n+1)} \]

\[ + \frac{kr^n \sin(n\theta)}{L \kappa_m^2 \Gamma(n+1/6) (n+1)} - \frac{kr^{n+5/6} \sin(n\theta + 5\theta/6)}{L \kappa_m^2 (n+11/6) n!} \]

\[ + \frac{r^{n+5/6} \cos(n\theta + 5\theta/6)}{(n+11/6) n!} \]
Main result 2:

\[ \ddot{R}_x(0) = -6\pi^3\lambda k^2 C_n^2 L^2 f_0^2 \kappa^{1/3} \Gamma(1/6) / 5 \]

\[
\begin{align*}
X \sum_{n=0}^{\infty} & \left( \frac{r^n \cos(n\theta)}{\Gamma(n+7/6)} - \frac{kn^n \sin(n\theta)}{L\kappa_m^2 \Gamma(n+7/6)} + \frac{kr^{n-1/6} \sin(n\theta - \theta/6)}{L\kappa_m^2 n!} \right) \\
- & \frac{r^{n-1/6} \cos(n\theta - \theta/6)}{n!} + \frac{n+5/6 (\kappa_0^2 / \kappa_m^2)^{n-1/6}}{n!} - \frac{(n+1) (\kappa_0^2 / \kappa_m^2)^{2n}}{5\Gamma(n+7/6)}
\end{align*}
\]

Proof:

a. \[ \ddot{R}_x(0) = 2\pi^2 k^2 C_n^2 (.033) \]

\[
X \int \int_{0}^{L} \kappa^3 (2\pi\lambda L) f_0^2 (J_1 (\kappa (2\pi\lambda L)^{1/2} f_0) - J_0 (\kappa (2\pi\lambda L)^{1/2} f_0))
\]

\[ X e^{-\kappa^2 / \kappa_m^2 (\kappa^2 + \kappa_0^2)} -11/6 \sin^2 (\kappa^2 (L-n)/(2k)) d\kappa dn \]

b. \[ \ddot{R}_x(0) = -2\pi^2 k^2 C_n^2 (.033) \]

\[
X \int \int_{0}^{L} \kappa^3 (2\pi\lambda L) e^{-\kappa^2 / \kappa_m^2 (\kappa^2 + \kappa_0^2)} -11/6 \sin^2 (\kappa^2 (L-n)/(2k)) d\kappa dn \]

c. Substituting Ly=L-n, so that L dy = -dn,

\[ \ddot{R}_x(0) = -4\pi^3 k^2 C_n^2 \lambda L^2 f_0^2 (.033) \]

\[
X \int \int_{0}^{L} \kappa^3 e^{-\kappa^2 / \kappa_m^2 (\kappa^2 + \kappa_0^2)} -11/6 \sin^2 (Ly^2/(2k)) d\kappa dy \]

d. Substituting \[ u = Ly \kappa^2 / (2k) \], so that \[ du = Ly \kappa / k \ dk \],

\[ \ddot{R}_x(0) = -2^{7/6} \pi^3 k^{13/6} C_n^2 L^{11/6} \lambda f_0^2 (.033) \]

\[
X \int \int_{0}^{\infty} u^{-1/6} \exp (-2ku/Ly \kappa_m^2) (u+Ly \kappa_0^2 /k)^{-11/6} \sin^2 (u) du dy \]

e. Letting \[ \alpha = (2k)/(Ly \kappa_m^2) \] and \[ \beta = (Ly \kappa_0^2) / (2k) \] while applying preliminary result 2,
\[
\int_0^\infty u e^{-au} (u+\beta)^{-11/6} \sin^2(\beta) \, du
\]
\[
= \frac{\beta^{1/6} G_{12}^{21} (\alpha \mid -1/6, 0)}{2\Gamma(11/6)} - \frac{\beta^{1/6} \Re G_{12}^{21} (\alpha-j2 \beta \mid -1/6, 0)}{2\Gamma(11/6)},
\]
and simplifying yields:
\[
\hat{R}_x (0) = -\pi^3 k^2 C_n^2 L^2 \lambda f_0^2 \kappa_0^{1/3} (0.033) \frac{G_{12}^{21} (\kappa_2^2 / \kappa_m^2 \mid -1)}{\Gamma(11/6)} \]
\[
- \Re \int_0^1 G_{12}^{21} (\kappa_2^2 / \kappa_m^2 - jL \kappa_2^2 y / k \mid -1/6, 0) \, dy
\]
f. Applying preliminary result 3,
\[
\int_0^1 G_{12}^{21} (\kappa_2^2 / \kappa_m^2 - jL \kappa_2^2 y / k \mid b_1, b_2) \, dy
\]
\[
= -j k / (L \kappa_m^2) \int_0^1 G_{12}^{21} (\kappa_2^2 x / \kappa_m^2 \mid b_1, b_2) \, dx
\]
\[
+ (1+jk / (L \kappa_m^2)) \int_0^1 G_{12}^{21} (\kappa_2^2 / \kappa_m^2 - jL \kappa_2^2 / k) x \mid a_1 \, b_1, b_2) \, dx,
\]
yields:
\[
\hat{R}_x (0) = -\pi^3 k^2 C_n^2 L^2 \lambda f_0^2 \kappa_0^{1/3} (0.033) \frac{G_{12}^{21} (\kappa_2^2 / \kappa_m^2 \mid -1)}{\Gamma(11/6)}
\]
\[
- \Re \left( -j k / (L \kappa_m^2) G_{23}^{22} (\kappa_2^2 / \kappa_m^2 \mid 0, -1) \right)
\]
\[
+ (1+jk / (L \kappa_m^2)) G_{23}^{22} (\kappa_2^2 / \kappa_m^2 - jL \kappa_2^2 / k \mid -1/6, 0, -1) \right)
\]
g. Applying preliminary result 4,
when \(x\) is real, \(\Re \left( -j k / (L \kappa_m^2) G_{23}^{22} (x \mid 0, a_1, b_1, b_2, -1) = 0,\right)\)
formula 7,

g_{PQ}^{mn}(x \mid a_1, \ldots, a_p) \mid b_1, \ldots, b_q) \text{ is symmetric with respect to the parameter sets } a_1, \ldots, a_n; a_{n+1}, \ldots, a_p; b_1, \ldots, b_m; \text{ and } b_{m+1}, \ldots, b_q,

and formula 8,

\[ g_{PQ}^{mn}(x \mid a_1, a_2, \ldots, a_p) = g_{P-1, Q-1}^{m, n-1}(x \mid b_1, \ldots, b_{q-1}, a_1) \]

yields:

\[ \ddot{R}_x(0) = -\pi^3 k^2 C^2 \frac{L^2 \lambda f_0^2 \kappa^{1/3}}{\rho} \left( \frac{G_2^{21}}{G_{12}^{21}} \left( \left| \frac{\kappa_0^2}{\kappa_m^2} \right| \begin{array}{c} -1 \\ 1/6, 0 \end{array} \right) \right) \]

- \text{Re} \left( \frac{1+jk}{(L\kappa_m^2)} \right) G_{23}^{22} \left( \left| \frac{\kappa_0^2}{\kappa_m^2} \right| - jL\kappa_0^2/k \begin{array}{c} 0, -1 \\ -1/6, 0, -1 \end{array} \right) \right).

\[ \ddot{R}_x(0) = -\pi^3 k^2 C^2 \frac{L^2 \lambda f_0^2 \kappa^{1/3}}{\rho} \left( \frac{G_2^{21}}{G_{12}^{21}} \left( \left| \frac{\kappa_0^2}{\kappa_m^2} \right| \begin{array}{c} -1 \\ 1/6, 0 \end{array} \right) \right) \]

- \text{Re} \left( \frac{1+jk}{(L\kappa_m^2)} \right) G_{23}^{22} \left( \left| \frac{\kappa_0^2}{\kappa_m^2} \right| - jL\kappa_0^2/k \begin{array}{c} 0 \\ -1/6, 0 \end{array} \right) \right).

Using formulae 4,

\[ g_{PQ}^{mn}(x \mid a_r) = \prod_{h=1}^{m} \prod_{j=1}^{n} \Gamma(b_j - b_h) \prod_{j=1}^{n} \Gamma(l + b_h - a_j) \]

\[ \prod_{j=m+1}^{p} \prod_{j=n+1}^{q} \Gamma(l + b_h - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - b_h) \]

\[ \prod_{j=n+1}^{q} \Gamma(l + b_h - a_1, \ldots, l + b_h - a_p; l + b_h - b_1, \ldots, (*) \prod_{j=n+1}^{q} \Gamma(l + b_h - b_q; (-1)^{p-m-n} x), \]
and 5,
\[
p^F_q \left[ \frac{\alpha_1, \ldots, \alpha_p}{\rho_1, \ldots, \rho_q} \right] = \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n z^n}{(\rho_1)_n \cdots (\rho_q)_n n!},
\]
yields:
\[
G_{12}^{21} \left( \frac{\kappa^2_o / \kappa_m^2}{j L \kappa^2_o / k} \right)|_{-1/6, 0} = \Gamma(-1/6) {}_1 F_1 \left( 1; 7/6; r e^{j\theta} \right)
\]
\[
+ \Gamma(1/6) \Gamma(5/6) r^{-1/6} e^{-j\theta/6} {}_1 F_1 \left( 5/6; 5/6; r e^{j\theta} \right)
\]
\[
= \Gamma(1/6) \Gamma(5/6) \sum_{n=0}^{\infty} \left( \frac{r^{n-1/6} e^{j(n\theta - \theta/6)}}{n!} - \frac{r^n e^{jn\theta}}{\Gamma(n+7/6)} \right)
\]
as well as
\[
G_{12}^{21} \left( \frac{\kappa^2_o / \kappa_m^2}{-1/6, 0} \right) = \Gamma(-1/6) \Gamma(2) {}_1 F_1 \left( 2; 7/6; \frac{\kappa_o^2}{\kappa_m^2} \right)
\]
\[
+ \Gamma(1/6) \Gamma(11/6) \left( \frac{\kappa^2_o / \kappa_m^2}{-1/6, 0} \right) {}_1 F_1 \left( 11/6; 5/6; \frac{\kappa^2_o}{\kappa_m^2} \right)
\]
\[
= \Gamma(1/6) \Gamma(5/6) \sum_{n=0}^{\infty} \left( \frac{n+5/6 \left( \frac{\kappa^2_o / \kappa_m^2}{n!} \right)^{n-1/6}}{5 \Gamma(n+7/6)} - \frac{(n+1) \left( \frac{\kappa^2_o / \kappa_m^2}{n!} \right)^n}{\Gamma(n+7/6)} \right)
\]
i. \quad \text{Re} \left( 1 + j k / ( L \kappa_m^2 ) \right) G_{12}^{21} \left( \frac{\kappa^2_o / \kappa_m^2}{j L \kappa^2_o / k} \right)|_{-1/6, 0}
\]
\[
= \text{Re} \left( \Gamma(1/6) \Gamma(5/6) \sum_{n=0}^{\infty} \left( \frac{r^{n-1/6} e^{j(n\theta - \theta/6)}}{n!} - \frac{r^n e^{jn\theta}}{\Gamma(n+7/6)} \right) \right)
\]
\[
+k / ( L \kappa_m^2 ) \Gamma(1/6) \Gamma(5/6) \sum_{n=0}^{\infty} \left( \frac{r^{n-1/6} e^{j(n\theta - \theta/6 + \pi/2)}}{n!} - \frac{r^n e^{j(n\theta + \pi/2)}}{\Gamma(n+7/6)} \right) = \Gamma(1/6) \Gamma(5/6) \sum_{n=0}^{\infty} \left( \frac{r^{n-1/6} \cos(n\theta - \theta/6)}{n!} \right)
\]
\[
- \frac{r^n e^{j(n\theta + \pi/2)}}{\Gamma(n+7/6)}
\]
\[
- \frac{r^n \cos(m\theta)}{\Gamma(n+7/6)} + \frac{k}{(L\kappa_m^2)} \Gamma(1/6) \Gamma(5/6) \sum_{n=0}^{\infty} \frac{r^n \sin(m\theta)}{\Gamma(n+7/6)} \\
- \frac{r^{n-1/6} \sin(m\theta - \theta/6)}{n!}
\]

j. Combining the results of parts h and i with part g yields:

\[
\mathbb{R}_x(0) = -6\pi^3 \lambda k^2 C_h^2 L^2 f_0^2 \kappa^{1/3} \Gamma(1/6) \sum_{n=0}^{\infty} \frac{r^n \cos(m\theta)}{\Gamma(n+7/6)} \\
- \frac{kr^n \sin(m\theta)}{L\kappa_m^2 \Gamma(n+7/6)} + \frac{kr^{n-1/6} \sin(m\theta - \theta/6)}{L\kappa_m^2 n!} - \frac{r^{n-1/6} \cos(m\theta - \theta/6)}{n!} \\
+ \frac{n+5/6 (\kappa_m^2/\kappa_0^2)^{n-1/6} - n+1 (\kappa_m^2/\kappa_0^2)^{2n}}{n! 5 \Gamma(n+7/6)}
\]
APPENDIX 3: APPROXIMATIONS AND ESTIMATES.

Representative values are taken to be $\kappa_0 = 1$, $\kappa_m = 1000$, $k = 10000000$, and $L = 200$ and $1000$.

1. $-L\kappa_m^2/k$ is on the order of $-20$ to $-100$. An approximation for $\theta = \tan^{-1}(-L\kappa_m^2/k)$ is $\theta = -\pi/2$ which is about a 3.5% error.

2. $\kappa_m^4/\kappa^4_m$ is on the order of $10^{-12}$ while $L^2\kappa_m^4/k$ is on the order of $10^{-8}$. An approximation to $r = (\kappa_m^4/\kappa_m^4 + L^2\kappa_m^4/k^2)^{1/2}$ is $r = L\kappa_m^2/k$ which is about a .125% error.

3. a. Substituting the representative values into the terms of the series part of the autocorrelation function,

$$R_x(0) = \frac{6\pi^2 k^2 c^2 L \Gamma(1/6)}{\kappa_0^{5/3}} \sum_{n=0}^{\infty} \left( (\Gamma(n+1/6))^{-1} \right.$$

$$-(\kappa_0/\kappa_m)^{5/3}/n!)(\kappa_0/\kappa_m)^{2n} - \frac{r^n \cos(n\theta)}{\Gamma(n+1/6)(n+1)}$$

$$+ \frac{kr^n \sin(n\theta)}{L\kappa_m^2 \Gamma(n+1/6)(n+1)} - \frac{kr^{n+5/6} \sin(n\theta + 5\theta/6)}{L\kappa_m^2 (n+11/6)n!}$$

$$+ \frac{r^{n+5/6} \cos(n\theta + 5\theta/6)}{(n+11/6)n!} \left) \right)$$

shows that the first term is on the order of $10^{-5}$, the second is on the order of $10^{-6}$, and the rest are on the order of $10^{-9}$ or less. Thus the autocorrelation function will be approximated by the first two terms of its series.
b. The second term of the series is
\[-\kappa_0^{11/3}k^{11/6} + 0.54\kappa_0^{2}\kappa_m^{5/3}k^{11/6} + 0.09L^{5/6}\kappa_0^{11/3}\kappa_m^{5/3}k
\]
\[-3L^{11/6}\kappa_0^{11/3}\kappa_m^{1/3}
\]
\[\kappa_m^{11/3}k^{11/6} \]

The second term of the numerator is on the order of 10^{1.8} while the others are on the order of 10^{1.6} or less and are discarded.

c. \[R_X(0) = 10.9k^{7/6}C_n^2L^{5/6}/\kappa_m^2 \]
\[X(.141Lk_m^2 + 0.54\kappa_0^{1/3}L^{1/6}k_5^{5/6} - \kappa_m^{1/3}L^{1/6}k_5^{5/6} + 0.527k). \]

4. a. Substituting the representative values into the terms of the series part of the second derivative of the autocorrelation function,
\[R_X(0) = -6\pi^3\kappa^2C_n^2L^2f_0^2\kappa^{1/3}L^{1/6}/5 \sum_{n=0}^{\infty} \frac{r^n\cos(n\theta)}{\Gamma(n+7/6)} \]
\[-kr^{n-1/6}\sin(n\theta - \theta/6) - \frac{r^{n-1/6}\cos(n\theta - \theta/6)}{n!} \]
\[+\frac{n+5/6}{n!} (\kappa_0^2/\kappa_m^2)^{n-1/6} - \frac{n+1}{5\Gamma(n+7/6)} (\kappa_0^2/\kappa_m^2)^{2n} \]
shows that the first term is on the order of 10 while the rest are on the order of 10^{-4} or less. Within the first term, its second term is negligible.

b. \[\ddot{R}_X(0) = -68.4\kappa^2C_n^2L^{15/6}f_0^2(0.863L^{7/6}\kappa_0^{1/3}
\[+9.66k_1^{1/6}L + 0.833L^{7/6}\kappa_m^{1/3}). \]
APPENDIX 4: CALCULATION OF $R_\chi(0)$ WITH THE KOLMOGOROV SPECTRUM.

1. $R_\chi(\tau) = 4\pi^2 k^2 C_n^2(0.033) \int \int J_0(\kappa \sqrt{2 \pi \lambda \ell_f \tau}) \kappa^{-11/3} \sin^2(\kappa^2 (L-\eta)/(2k)) \kappa \, d\kappa \, d\eta$

2. Substituting $Lw = L-\eta$ so that $L \, dw = -d\eta$,

$$R_\chi(\tau) = 4\pi^2 k^2 C_n^2 L(0.033) \int \int J_0(\kappa \sqrt{2 \pi \lambda \ell_f \tau}) \kappa^{-11/3} \sin^2(Lw \kappa^2/(2k)) \kappa \, d\kappa \, dw.$$  

3. Substituting $z = Lw \kappa^2/(2k)$ so that $\kappa \, d\kappa = k/(Lw) \, dz$,

$$R_\chi(0) = 4\pi^2 k^3 C_n^2 (0.033) (L/(2k))^{11/6} \int \int w^5/6 \, dw \, f_0^1 z^{-11/6} \sin^2(z) \, dz.$$  

4. By formula 9,

$$\int x^{\mu-1} \sin^2 ax \, dx = -\Gamma(\mu) \cos(\mu x/2)/(2^{\mu+1} a^\mu),$$  

$$R_\chi(0) = -(12/11) \pi^2 k^7/6 C_n^2 L^{11/6}(0.033) \Gamma(-5/6) \cos(5\pi/12)$$  

$$R_\chi(0) = 0.614 k^7/6 C_n^2 L^{11/6}.$$
FOOTNOTES


3 Pratt, p. 139.


6 Pratt, p. 133.

7 Lutomirski, p. 8.

8 Pratt, p. 133.


11 Lawrence, p. 1526.

12 Lutomirski, p. 2.

13 Fante, p. 1669.

14 Lawrence, p. 1538

15 Ibid., p. 1537.

16 Pratt, p. 134.
17 Lawrence, p. 1529.

18 Ibid., p. 1535.

19 Lutomirski, p. 3.


24 Ibid.
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