Theoretical Modeling for Detectivity and Resolution Comparison of Single Aperture and Multiple Aperture Optical Imaging Systems

1982

Steven C. Kellogg
University of Central Florida

Find similar works at: https://stars.library.ucf.edu/rtd

University of Central Florida Libraries http://library.ucf.edu

Part of the Engineering Commons

STARS Citation

https://stars.library.ucf.edu/rtd/639

This Masters Thesis (Open Access) is brought to you for free and open access by STARS. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of STARS. For more information, please contact lee.dotson@ucf.edu.
THEORETICAL MODELING
FOR
DETECTIVITY AND RESOLUTION COMPARISONS
OF SINGLE APERTURE AND MULTIPLE APERTURE
OPTICAL IMAGING SYSTEMS

BY

STEVEN C. KELLOGG
B.A., Graceland College, 1972

RESEARCH REPORT

Submitted in partial fulfillment of the requirements
for the Degree of Master of Science
in the Graduate Studies Program of the College of Engineering
University of Central Florida
Orlando, Florida

Fall Term
1982
ABSTRACT

The detectivity and resolution of single aperture and multiple aperture optical imaging systems are compared for single point sources in optically background limited environments. The single aperture system assumes a single large diameter lens with a detector array at the focal plane. The multiple aperture system assumes an independent detector array at the focal plane of each of the apertures of the multiaperture system. The multiaperture lenses are arranged in a rectangularly symmetric pattern within the perimeter that a single large aperture would occupy.

Due to the presence of a constant signal plus an optical noise field whose amplitude is Rayleigh distributed, Rician squared statistics are used to model the detector voltage random variable. The detectivity is analyzed assuming a detector optical amplitude threshold is chosen such that the signal is considered present when the optical amplitude exceeds threshold and considered absent when the optical amplitude falls below threshold. The optimum threshold is found to be given by

$$I_o(\alpha T/n^2) \propto \exp \left[ \frac{A^2}{2n^2} \right]$$
where \( \frac{A^2}{n^2} \) is the signal to noise power ratio, \( T \) is the optical amplitude threshold, and \( I_0 \) is the modified Bessel function of order zero.

Detector size is found to be the predominant factor in resolution, due to the minute size of an Airy disc image from a point source. The resolution angle (\( \gamma_{\text{resolution}} \)) can be approximated by \( \gamma_{\text{resolution}} = \frac{A_e}{f} \) where \( A_e \) is the distance between detector centers and \( f \) is the imaging system focal length.

Single aperture and multiple aperture systems are found to be equal in detectivity performance when optically background limited. For equal detector sizes and spacings, and equal imaging system focal lengths, multiple aperture systems are found to provide resolution improvement over single aperture systems. This resolution improvement depends on the overlap of the field of view between the detector arrays of the individual apertures in the multiple aperture system.
ACKNOWLEDGEMENTS

I have learned, as I am sure every graduate student can appreciate, that any graduate thesis is a team effort. Although I may receive the most attention for its completion, it would not have been possible without the following:

The previous investigations of Dr. Roy Walters and Lee Wayne Schrock.

The sponsorship of Dr. Ronald L. Phillips whose insight and questions stimulated the transition of my thinking from a collection of courses to a cohesive application of varied disciplines.

The patience of my family, Roxy, Ryan, and Erin, who accepted my need to spend nights and weekends studying.

The employment of International Laser Systems, Incorporated, in an environment pertinent to my course of study.

The secretarial assistance of Laurie Cardinell and Cathy Hughes who transformed my rough notes into a finished document.

The Christian heritage of my parents who instilled my desire to utilize my full potential.

Jesus Christ is LORD of all.
# TABLE OF CONTENTS

List of Tables. ................................................................. vi
List of Figures ...................................................................... vii
List of Symbols ..................................................................... ix
Introduction. ................................................................. 1
Single Aperture Analysis. .................................................. 10
  Detectivity. ................................................................. 19
  Threshold Considerations .............................................. 24
  Resolution ...................................................................... 30
Multiple Aperture Analysis. ............................................... 37
  Overlapping Fields of View ............................................ 40
  Focal Plane Intensity Distribution ................................. 49
  Detectivity. ................................................................. 54
  Resolution ...................................................................... 54
Conclusion. ......................................................................... 59
Appendices. ......................................................................... 64
  A. Computer Literature Search Technique. ......................... 65
  B. Spatial Frequency Concepts. ....................................... 67
  C. Fourier Transforming Properties of Lenses ................. 73
  D. Optical Background Noise Considerations ................. 94
  E. Solution of \( \int_{0}^{\infty} \int_{0}^{\infty} \left[ 2J_1(ar)/(ar) \right]^2 r \, dr \, d \) . . . . . . 97
  F. Derivation of Error Probabilities and Optimum
     Threshold ................................................................. 99
G. Computer Calculation of Optimum Threshold and Detectivity ........................................... 113

H. Calculation of the Fraction of Total Point Source Signal Power Reaching a Detector as a Function of the Distance from the Airy Disc Centroid to the Detector Edge. .......................... 128

I. Determination of the Field of View Containing Coherent Noise for Two Apertures. ............. 147

Bibliography. ............................................................................................................................ 159
LIST OF TABLES

1. Summary of Detectivity and Factors Affecting Detectivity ........................................ 62
2. Summary of Resolution and Factors Affecting Resolution ........................................... 63
C1. Values of the Bessel Function for an Airy Disc ....................................................... 92
G1. Computer Program for Calculating Optimum Threshold in Terms of Threshold to Signal Ratio, Given Signal to Noise Ratio and Signal Probability ................................................................. 114
G2. Computer Output Data for Optimum Threshold Calculation ........................................ 117
G3. Computer Program for Calculating Detectivity Given Threshold (in terms of T/A), Signal to Noise Ratio, and Signal Probability .......................................................... 121
G4. Computer Output Data for Detectivity Calculation, P(signal) = 0.5 ................................ 125
H1. Computer Program for Calculating the Fraction of Airy Disc Power on a Detector when the Airy Disc Centroid is a Distance B (in terms of λ f/D) from the Edge of the Detector 138
H2. Computer Output Data for the Fraction of Airy Disc Power on a Detector ....................... 140
# LIST OF FIGURES

1. Optical Imaging System ........................................... 2
2. Detectors Dividing a Focal Plane into Four Quadrants. ........... 3
3. Focal Plane Detector Array ..................................... 17
4. Detectivity as a Function of Airy Disc Centroid Position Relative to the Edge of a Detector for Optimum Threshold, \( P(\text{signal}) = 0.5 \) ......................... 32
5. Multiple Apertures in Front of a Single Large Lens .................. 38
6. Multiple Apertures in Front of Multiple Lenses .................... 38
7. Position and Orientation of Multiple Apertures Relative to a Single Large Aperture. .................. 39
8. Analysis of Overlapping Fields of View .......................... 41
9. Variation of the Overlap Field of View \( (\theta ') \) with Distance from the Aperture Plane \( (R) \) as a Function of Aperture Spacing \( (D/N_S) \) and Aperture Field of View \( (\theta) \). ......................... 43
10. Coherent Noise Portion of the Overlapping Field of View for Two Apertures. .......................... 45
11. Illustration of the Focal Plane Spatial Frequency Position Overlaps of Two Detector Arrays for Two Apertures of a Multiple Aperture System. .................. 56

B-1. Plane Waves Propagating at Angles \( \alpha, \beta, \) and \( \gamma \) Relative to the \( x, y, \) and \( z \) Axes Respectively. ..................... 68

B-2. Plane Wave at a Distance \( \lambda \) from the Coordinate System Origin Intersecting the Coordinate Axes. .......................... 69

B-3. \( P_3(x,y,f) \). ........................................ 71
C-1. Two Planes \((x_2, y_2)\) and \((x_f, y_f)\) Separated by a Distance \(z\) .............. 74

C-2. Lens Configuration ............... 76

C-3. Point Source at a Distance \(d_0\) from a Lens of Focal Length \(f\) .................. 82

C-4. Plot of \([2j_1(\pi Ds)/(\pi Ds)]^2\) for Varying Values of \((Ds)\) ..................... 93

G-1. Optimum Threshold in Terms of the Threshold to Signal Amplitude Ratio \((T/A)\) as a Function of Signal to Noise Ratio \((A^2/n^2)\) and Signal Probability ....................... 119

G-2. Threshold Voltage as a Function of Detector Voltage (due to signal plus noise) Assuming the Noise Power Only Produces a Detector Voltage of 1 ......................... 120

G-3. Detectivity as a Function of Threshold \((T)\) for Various SNR's, \(P(signal) = 0.5\) ............ 126

G-4. Detectivity at Optimum Threshold as a Function of SNR, \(P(signal) = 0.5\) ................. 127

H-1. Fraction of the Total Point Source Signal Power Reaching a Detector as a Function of the Distance (in terms of \(\lambda f/D\)) from the Airy Disc Centroid to the Detector Edge .................... 141

I-1. Illustration for Computing the Angular Field of View of Coherent Noise .................. 148
LIST OF SYMBOLS

A  
rms value of the signal at a detector

$A_A^2$  
area of a detector array

$A_D^2$  
area of an individual detector

$A_e^2$  
effective area of each detector in a detector array. $A_e^2 = A_A^2/ND^2$

$A_o^2$  
total area of overlapped spatial frequency positions by the detectors in the image plane

$A_p$  
point source amplitude

$a_o$  
opical attenuation coefficient for optical signal losses in the atmosphere

B  
distance from an airy disc centroid to a detector boundary

$B(d,u)$  
Beta function of d, and u
$B(d,u) = \Gamma(d) \Gamma(u)/\Gamma(d+u)$

C  
constant of proportionality

$\text{circ} (x)$  
circle function

$\text{comb} (x)$  
\[
\sum_{n=-\infty}^{\infty} \delta(x-n)
\]

D  
opical system limiting aperture diameter

$D_a$  
aperture diameter for each of the apertures of a multiple aperture system

d  
distance between two points
\( d_0 \) distance from a point source to a lens aperture

\( D^* \) spectral D star

\( D_\lambda \) spectral detectivity

\( E_1(t) \) electric field strength of point 1

\( E_2(t) \) electric field strength of point 2

\( E^*(t) \) complex conjugate of the electric field

\( E_{1a}(t) \) source A contribution to the electric field strength at point 1

\( E_{1b}(t) \) source B contribution to the electric field strength at point 1

\( E_{2a}(t) \) source A contribution to the electric field strength at point 2

\( E_{2b}(t) \) source B contribution to the electric field strength at point 2

\( \exp(i\Delta_a(t)) \) random phase difference between points 1 and 2 due to a source at point A propagating over a difference in distance \( r_{2a} - r_{1a} \) from the source to the two points, and a time delay \( \tau \) between the two points

\( \exp(i\Delta_b(t)) \) random phase difference between points 1 and 2 due to a source at point B propagating over a difference in distance \( r_{2b} - r_{1b} \) from the source to the two points, and a time delay \( \tau \) between the two points

\( F \) Fourier Transform operator

\( _2F_3(a_1,a_2;b_1,b_2,b_3;z) \) the generalized hypergeometric series

\( f \) optical system focal length

\( f_a \) focal length of each aperture of a multiple aperture optical system
<table>
<thead>
<tr>
<th>Fraction coherent noise</th>
<th>fraction of the total optical noise in a field of view $\theta$, which is due to the coherent noise in the coherent noise field of view $\theta_c$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_x, f_y$</td>
<td>spatial frequencies in the x and y directions respectively</td>
</tr>
<tr>
<td>$H_1(x)$</td>
<td>Sturve function</td>
</tr>
<tr>
<td>$I(x_f, y_f)$</td>
<td>optical intensity distribution in the focal plane</td>
</tr>
<tr>
<td>$I_0(x)$</td>
<td>modified Bessel function of the first kind of order zero</td>
</tr>
<tr>
<td>$I_1$</td>
<td>time averaged optical intensity at point 1</td>
</tr>
<tr>
<td>$I_1(x)$</td>
<td>modified Bessel function of the first kind of order one</td>
</tr>
<tr>
<td>$I_2$</td>
<td>time averaged optical intensity at point 2</td>
</tr>
<tr>
<td>$J_0(x)$</td>
<td>Bessel function of the first kind of order zero</td>
</tr>
<tr>
<td>$J_1(x)$</td>
<td>Bessel function of the first kind of order one</td>
</tr>
<tr>
<td>$L$</td>
<td>distance across the field of view $\theta$ at a distance $R$ from the aperture plane</td>
</tr>
<tr>
<td>$L'$</td>
<td>distance across the field of view $\theta'$ at a distance $R$ from the aperture plane</td>
</tr>
<tr>
<td>$L_c$</td>
<td>distance across the field of view $\theta_c$ at a distance $R$ from the aperture plane</td>
</tr>
<tr>
<td>$M$</td>
<td>$P$(signal)</td>
</tr>
<tr>
<td>$N$</td>
<td>optical noise power reaching an aperture in the field of view of the total detector array</td>
</tr>
<tr>
<td>$N_a$</td>
<td>number of apertures in the multiple aperture system</td>
</tr>
<tr>
<td>$N_D^2$</td>
<td>total number of detectors in a detector array</td>
</tr>
<tr>
<td>$N_{ND}^2$</td>
<td>noise power at the detector illuminated by signal</td>
</tr>
</tbody>
</table>

$x_i$
\( N_0 \)  
optical noise power uniformly distributed through a field of view

\( N_s \)  
number of multiple aperture spacings that occur across a single large aperture diameter

\( \text{NEP}_\lambda \)  
spectral noise equivalent power

\( n^2 \)  
optical noise variance at the detector

\( P(x_2, y_2) \)  
pupil function of a lens aperture

\( P_{\text{noise total}} \)  
total noise power reaching an aperture from a field of view

\( P_{\text{coherent noise}} \)  
total noise power reaching an aperture from a coherent noise field of view \( \theta_c \)

\( P(\text{error/noise}) \)  
conditional probability of making an error given that noise only is present (false alarm)

\( P(\text{error/signal}) \)  
conditional probability of making an error given that signal and noise are both present (miss)

\( P(\text{noise}) \)  
a priori probability of a noise only event (also \( Q \))

\( P(\text{signal}) \)  
a priori probability of a signal plus noise event (also \( M \))

\( Q \)  
\( P(\text{noise}) \)

\( R \)  
distance from the plane of an aperture to some point in the field of view

\( R_1 \)  
radius of curvature of one lens half

\( R_2 \)  
radius of curvature of second lens half

\( r \)  
distance from the center of the xy plane  
\( r^2 = x^2 + y^2 \)

\( r_{1a} \)  
distance from source A to point 1

\( r_{1b} \)  
distance from source B to point 1

\( r_{2a} \)  
distance from source A to point 2
distance from source B to point 2
optical signal power reaching an aperture in the field of view of the total detector array
source A
source B
actual signal to noise power ratio at a detector \( (A^2/n^2) \)
optical power signal to noise ratio approximation for a detector illuminated by signal
signal to noise voltage ratio
distance from the center of the \( f_x, f_y \) plane
\( s^2 = f_x^2 + f_y^2 \)
detector optical field amplitude threshold for determining the presence of a signal
optical signal amplitude at the lens aperture
optical signal amplitude at the focal plane
optical signal amplitude immediately prior to passing through a lens
optical signal amplitude immediately after passing through a lens
random variable of detector voltage due to optical signal power and optical noise power reaching a detector
possible value of the random variable \( V \)
angular frequency for optical noise, \( w = (2\pi \text{ frequency}) \)
coordinates in the focal plane
coordinates in the lens aperture plane
angle relative to the x axis
angle relative to the y axis
\( \Delta(x_2,y_2) \) lens thickness at \((x_2,y_2)\)

\( \Delta_0 \) thickness of the thickest part of a lens

\( \delta(x,y) \) point source in the xy plane

\( \gamma \) angle relative to the z axis

\( \gamma_{\text{resolution}} \) resolution angle

\( \gamma_{\text{FOV}} \) angular field of view

\( \gamma_{12}(\tau) \) correlation coefficient between points 1 and 2

\( \lambda \) optical signal wavelength

\( \eta \) refractive index

\( \sigma_\theta \) rms angular measurement error

\( \theta \) field of view for detector, lens, aperture combination

\( \theta' \) overlapping field of view for two detector, lens, aperture combinations

\( \theta_C \) overlapping field of view containing coherent noise for two detector, lens, aperture combinations

\( \theta_f \) polar coordinate angle corresponding to the point \((x,y)\)

\( \theta_i \) polar coordinate angle corresponding to the point \((f_x,f_y)\)

\( \tau \) variable time delay

\( \phi(x_2,y_2) \) phase delay an optical signal incurs when passing from \(U_1(x_2,y_2)\) to \(U_1'(x_2,y_2)\)

\( \ast \) convolution operator
INTRODUCTION

The technology of sensing an object's angular position frequently employs a quadrant detector placed in the image plane of an optical system as illustrated in Figure 1.

The quadrant detector divides the image plane into four quadrants as shown in Figure 2. The image position relative to the x and y axes is found by differencing the detector signals on each side of the axis of interest. For example, the angular displacement in the x direction is found by subtracting the sum of the signals from detectors II and III from the sum of the signals from detectors I and IV.

A recent dissertation (Schrock, 1981) proposes the use of a multiaperture optical system (patterned after a fly's eye) to provide a much broader field of view while retaining accuracy and resolution comparable to a single aperture optical system. After fabricating an optical detector assembly consisting of 7 GRIN (Graded Index of Refraction) lenses equally spaced around a 7 cm diameter circle on a 7 cm tall right circular cone, 7 optical fibers were bonded in a random arrangement to the back of each
Figure 1. Optical imaging system
Figure 2. Detectors dividing a focal plane into four quadrants.
lens. The system achieved a 58.7 degree overall field of view with an accuracy of 2% to 5% of the field of view (1.17 degrees to 2.94 degrees).

The advantages of this system over the single aperture approach include:

1. Potential ease in fabrication and reduction in cost.
2. Improvement in seeker aerodynamics by conforming the multiaperture arrangement to airframe design (this can also reduce the sensitivity of aperture geometry to temperature and pressure variations).
3. Improved reliability since the loss of a single aperture does not prohibit continued system operation.

This study develops the mathematical models for and compares the theoretical performance of the single aperture and multiple aperture systems.

The services of State Technology Applications Center were employed to conduct a computer search of technical and scientific journal titles and abstracts for key words relating to multiple apertures and imaging system position measurement errors. The method of conducting the search is detailed in Appendix A.
The search revealed that image position errors associated with quadrant detectors have recently been modeled by Tyler and Fried (1982) using an ideal quadrant detector (no gaps between the quadrants) at the image plane of an optical system. The accuracy of the angular position measurement was found to be inversely proportional to the signal to noise voltage ratio of the four quadrants summed to act as a single detector. For a point source the rms angular measurement error, $\sigma_\theta$, is given by

$$\sigma_\theta = \pi (3/16) (\lambda / D) / \text{SNR}_v$$

Where

$\lambda$ = wavelength of the light used

$D$ = diameter of the limiting aperture in the optical system under consideration

$\text{SNR}_v$ = signal to noise voltage ratio

The use of multi-aperture systems has been limited to X-ray and gamma-ray photography in astronomy and tomography (Dicke, 1968). Since diffraction is usually inconsequential to X-rays and gamma-rays, pinhole apertures are required for sharp resolution but result in very low level signals. The signal level can be increased by employing a large number of pinholes without severely degrading the
resolution, provided an appropriate technique to "unscramble" the overlapping images is employed. The optimum size and positioning of pinholes and appropriate computer techniques for image reconstruction have been extensively studied for use in photographing laser driven compressions of deuterium filled microballoons for laser fusion research. (Fenimore, 1978; Fenimore and Cannon, 1978; Fenimore, Cannon, Van Hulsteyn, and Lee, 1979; Fenimore, 1980; Fenimore and Cannon 1981). This work has coined the terms coded apertures, non-redundant arrays, and uniformly redundant arrays to classify aperture positioning techniques.

Since optical signals diffract, the pinhole technique is inappropriate for optical imaging. The use of multiple apertures in an optical imaging system can still be used to some advantage.

In order to analyze optical signal behavior in an optical imaging system, it is convenient to consider the optical disturbance at a point as a sum of plane waves traveling in different directions. The signal can be decomposed into its plane wave components by finding the Fourier Transform of the complex optical field amplitude across a plane through which the optical signal is passing. Each of these plane wave components has a unique spatial frequency which completely describes its direction of propagation as
shown in Appendix B. By examining how the optical systems respond to various spatial frequencies, comparisons between their angular accuracy can be found for varying signal to noise ratio, aperture diameter, focal length, and light wavelength.

One of the most useful properties of a converging lens in its ability to perform a two-dimensional Fourier Transform. The theory behind this property is described in Appendix C. By performing the Fourier Transform, the lens instantaneously converts the optical signal from an object into its spatial frequency components, thus establishing the angle to the object of interest. The ability of the lens system to resolve specific spatial frequencies (and therefore object angles) is a function of the aperture diameter, focal length, optical wavelength, signal to noise ratio, detector size, and detector spacing.

For the purposes of this research, the object generating the optical signal will be a point source of amplitude $A_p$. All other sources could be similarly analyzed by dividing them into collection of point sources of appropriate amplitudes. Since the Fourier Transform is a linear operator, the super-position of these point sources would yield the correct result for the entire object.
Noise enters into optical signal processing just as it does in electronic signal processing. Since the optical signal is sensed by some photoelectric device and processed electronically, both optical and electronic noise must be considered. As discussed in Appendix D, photoelectric detectors are typically optical back-ground noise limited rather than electronically noise limited. Further, optical detectors are sensitive to optical intensity (power) which is proportional to the square of the optical amplitude. Detector output is a voltage or amperage proportional to detected power. For an optical system with an optical band-pass filter to reject optical wavelengths outside the band of interest, the statistics which will best describe system operation are those which add the constant signal power to the noise power. When the optical noise field is a Gaussian distributed random variable, then the noise amplitude is a Rayleigh distributed random variable, Rician statistics are appropriate for describing the total optical field amplitude reaching the detector, and Rician squared statistics appropriately model the detector output voltage distribution. This study will compare only the optical signal and optical noise at the detector for the single aperture and multiaperture systems. It will be shown that the optical signal to noise ratio for each of the individual detectors of a multiaperture system is the same as the optical signal to noise
ratio for the single large aperture of a single aperture system if the detector sizes and optical system focal lengths (and therefore the fields of view for the detectors) in the two systems are the same. If the detectors are optically background limited, the addition of electronic noise in post detector signal processing should have the same effect on both systems.

The criteria for establishing the presence or absence of a signal shall be the establishment of a threshold optical field amplitude at the detector such that a field amplitude at or above the threshold will indicate the presence of the signal, and a field amplitude below the threshold will indicate the absence of the signal. The square of this threshold field amplitude will be proportional to the detector threshold voltage.
SINGLE APERTURE ANALYSIS

Traditionally, seekers have employed optical imaging systems which take the optical signal striking a lens aperture and focus it on a detector array in the lens focal plane as shown in Figure 1. As is well known (and reiterated in Appendix C), for a point source of amplitude $A_p$, the intensity distribution at the focal plane is proportional to the square of the Fourier Transform of the lens aperture per equation C.14.

$$I(x_f, y_f) = [A_p/(\lambda f)]^2 C^2 |F\{P(x_2, y_2)\}|^2$$

Where

$A_p$ = Amplitude of the point of source

$\lambda$ = Wavelength of the optical signal

$f$ = Focal length of the lens

$C$ = A constant of proportionality

$P(x_2, y_2)$ = Pupil function of the lens aperture

$P(x_2, y_2) = \begin{cases} 1 & \text{in the clear aperture of the lens} \\ 0 & \text{outside the clear aperture of the lens} \end{cases}$
When the lens aperture is circular with diameter $D$ the intensity distribution is a familiar Airy disc described by equation 1.16 as

$$I(x_f, y_f) = \left[ \frac{A_p}{(\lambda f)} \right]^2 \frac{C^2}{(D/2)^4} \pi^2 \left[ 2J_1(\pi D s)/(\pi D s) \right]^2$$

(1.1)

Where

- $D =$ Diameter of the lens aperture
- $s = \sqrt{f_x^2 + f_y^2}$ is the spatial frequency in a radial direction
- $s = \left[ \frac{1}{(\lambda f)} \right] \sqrt{x^2 + y^2} = \left[ \frac{r}{(\lambda f)} \right]$
- $r = \sqrt{x^2 + y^2}$

We can see from equation 1.1 that the magnitude of the intensity at any point depends upon the aperture area $[\pi (D/2)^2]$, the optical signal wavelength ($\lambda$), and the lens focal length ($f$). The total power reaching the focal plane due to the signal can be found by integrating the signal intensity over the detector area.
Total Signal Power = \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x_f, y_f) \, dx_f \, dy_f \)

\[ = \int_{0}^{2\pi} \int_{0}^{\infty} I(r, \theta) \, r \, dr \, d\theta \]

Substituting in the results of equation 1.1 yields

\[ \text{Total Signal Power} = \int_{0}^{2\pi} \int_{0}^{\infty} \left\{ \frac{A_p}{(\lambda f)} \right\}^2 c^2 \left( \frac{D}{2} \right)^4 \pi^2 \]

\[ \times \left\{ 2J_1 (\pi D r / (\lambda f)) / (\pi D r / (\lambda f)) \right\}^2 r \, dr \, d\theta \]

(1.2)

Which will reduce (per equation E.1) to

\[ \text{Total Signal Power} = \left\{ \frac{A_p}{(\lambda f)} \right\}^2 c^2 \left( \frac{D}{2} \right)^4 \pi^2 \left[ 2 / \left( \pi D / (\lambda f) \right) \right]^2 \]

\[ = A_p^2 \, c^2 \pi \left( \frac{D}{2} \right)^2 \]

(1.3)
Since \( \pi(D/2)^2 \) is simply the lens clear aperture area, the total signal power reaching the focal plane depends on optical signal strength and aperture area. It should be noted that this integration over an infinite area is valid even though the focal plane is not infinite. This is due to the fact that the majority of the signal intensity is contained within the first zero of the Bessel function as can be seen qualitatively in Figure C-4 where the radius to the first zero from Table C1 is

\[
Ds = 1.22 = D[r/(\lambda f)]
\]

\[
r = 1.22\lambda f/D
\]

\[
r = \lambda (1.22f/D)
\]  

(1.4)

thus the majority of the signal intensity is contained within an area a few wavelengths in diameter which is orders of magnitude smaller than practical system detector arrays.

Since the optical noise reaching the lens aperture may be represented as a superposition of many noise point sources, the total noise reaching the detector plane is also directly proportional to the clear aperture area.

The two variables of interest in this study are the signal detectivity and angular accuracy or resolution. Detectivity shall be defined as the probability of correctly evaluating the presence or absence of a signal given a
composite signal and noise input or noise only input. Angular accuracy and resolution will be used interchangeably. For an imaging system, the resolution or resolving power is a measure of the ability of the optical system to separate the images of two neighboring objects.

Traditionally, the Rayleigh criterion has been used as a measure of optical system resolution or resolving power. According to this criterion, two images are just resolved when the principal maximum of one coincides with the first minimum of the other. Using the criterion, for an optical imaging system of focal length $f$ and circular aperture diameter of $D$, two point source images are resolved when separated by a distance $r$ equal to the distance to the first zero of the $J_1$ Bessel function. From 1.4 this is

$$r = \lambda (1.22f/D)$$

The corresponding angular resolution (using the development of Appendix B) is

$$\sin \gamma_{\text{resolution}} = \lambda s = \lambda \left[ \frac{r}{\lambda f} \right] = \frac{r}{f}$$
Substituting from equation 1.4 gives

\[ \sin \gamma \text{ resolution} = \frac{\lambda (1.22f/D)}{f} \]

\[ \sin \gamma \text{ resolution} = 1.22 \frac{\lambda}{D} \quad (1.5) \]

Thus, according to this criterion, as the aperture diameter increases the angular separation required for resolution decreases or two objects can be brought closer together and still be resolved.

As will be seen in this study, unless detector size approaches the Airy disc size, detector size will be the limiting factor for resolution. Once the threshold criterion for a detector is satisfied, the object is considered present in the field of view defined by that detector. Two objects whose images fall on the same detector cannot be resolved. An object which moves within the field of view of a detector can be no further resolved by that detector within its field of view. Only when an image moves from one detector to another is an object's position resolved to a new location.
A useful term for describing a system's optical signal acceptance angle is the field of view. The field of view will be defined as that set of angles relative to the optical system axis for which spatial frequencies in the optical system image plane can be detected. Due to the Fourier Transforming properties of lenses, a detector in the optical system image plane will cover an area corresponding to a specific set of spatial frequencies. The angles corresponding to these spatial frequencies represent the detector's field of view. When there is more than one detector at the focal plane, the union of the fields of view for all the detectors is the imaging system's field of view.

Traditionally, the single aperture model shown in Figure 1 incorporates a quadrant detector, or four detectors which divide the focal plane into quadrants as shown in Figure 2.

To take advantage of the Fourier Transforming properties of lenses as discussed in Appendix B, the model for this study will divide the focal plane into a number of detectors as shown in Figure 3.
\[ A_A^2 = \text{AREA OF THE DETECTOR ARRAY} \]
\[ A_D^2 = \text{AREA OF AN INDIVIDUAL DETECTOR} \]
\[ N_D^2 = \text{TOTAL NUMBER OF DETECTORS IN THE ARRAY} \]
\[ A_{e}^2 = \text{EFFECTIVE AREA OF EACH DETECTOR} = \frac{A_A^2}{N_D^2} \]

Figure 3. Focal plane detector array
Since real detectors are separated by gaps, the area of each individual detector \( (A_D^2) \) will not necessarily equal the total array area \( (A_A^2) \) divided by the total number of detectors \( (N_D^2) \). Instead each detector will occupy an effective area

\[
A_e^2 = \frac{A_A^2}{N_D^2}
\]

Various infrared detector manufacturers have been queried about the size of the detectors in their detector arrays. In the information received to date, the smallest detectors are made by EG&G Reticon. One model features detectors in a 100 x 100 array on 60 micron centers. For light in the infrared region where wavelengths are on the order of one micron, all of an Airy disc due to a point source could easily fall on one detector even when the smallest detectors from today's technology are used.

The advantage of a detector array scheme employing a large number of small detectors is that each detector represents a much smaller set of spatial frequencies and therefore a much narrower field of view than each detector in an array containing a few large detectors. One can approximate the angular spread of each detector's field of view using B.5 to yield
\[ \sin \gamma_{\text{FOV}} \approx \frac{A_D}{f} \]

Where

\[ f = \text{lens focal length} \]
\[ A_D = \text{length, width, or diameter of the detector} \]
\[ \gamma_{\text{FOV}} = \text{Angular field of view} \]

For

\[ A_D \ll f \]
\[ \gamma_{\text{FOV}} \approx \frac{A_D}{f} \quad (1.6) \]

Consequently as the number of detectors per unit area increase (causing the effective size of each detector to decrease) the band of spatial frequencies that each detector represents decreases, and each detectors field of view decreases.

**Detectivity**

To evaluate a signal's detectability using the definition that detectivity is the probability of correctly evaluating the presence or absence of a signal given a composite signal plus noise or noise only input, one must have the probability distribution for detector voltages in terms of signal and noise powers.
As described in the introduction, since infrared electro-optical sensors produce a voltage corresponding to optical signal intensity or power, Rician squared statistics appropriately model the detector output voltage distribution. The Rician squared probability distribution is given by

\[ f_V(v) = \frac{1}{2n^2} \exp\left(-\frac{A^2+v}{2n^2}\right) I_0\left(\frac{A\sqrt{v/n^2}}{n^2}\right) \quad v \geq 0 \]

Where

- \( v \) = a possible value of the random variable \( V \).
- \( V \) = the random variable of interest. In this case \( V \) is the detector voltage due to the power from signal and noise reaching the detector.
- \( A \) = the rms value of the signal at the detector.
- \( n^2 \) = the noise variance
- \( I_0 \) = modified Bessel function of order zero

The signal to noise power ratio (SNR) is then given by \( (A^2/n^2) \). When noise only is present, the distribution reduces to

\[ f_V(v) = \frac{1}{2n^2} \exp\left(-\frac{v}{2n^2}\right) \quad v \geq 0 \]
This is a familiar Negative Exponential distribution, which properly gives the detector voltage distribution for a noise only input.

For determining the presence or absence of signal, a threshold optical field amplitude \( T \) must be chosen such that when the threshold amplitude is exceeded the signal is considered present, and when the threshold is not exceeded, the signal will be considered absent.

Once the threshold is selected, the false alarm probability for making an error (due to the optical amplitude exceeding threshold) given that noise only is present is

\[
P(\text{error/noise}) = \frac{1}{\sqrt{2\pi}} \int_{-T_0}^{\infty} e^{-v^2/2} \, dv
\]

Shifting to the Rayleigh field amplitude distribution for computational convenience by replacing \( v \) with \( v^2 \) and \( T^2 \) with \( T \)

\[
P(\text{error/noise}) = \int_{T}^{\infty} \frac{v}{\sqrt{2\pi n^2}} e^{-v^2/(2n^2)} \, dv \quad (1.7)
\]
Where

\( T = \) the threshold optical field amplitude

Then

\[
P(\text{error/noise}) = \left. -\exp\left[-\frac{v^2}{2n^2}\right] \right|_T = \exp\left[-\frac{T^2}{2n^2}\right]
\]

Similarly, the miss probability for making an error (due to the optical amplitude being below threshold) given that signal and noise are both present is

\[
P(\text{error/signal}) = \int_0^T \left( \frac{v}{n^2} \right) \exp\left[-\left(\frac{A^2 + v^2}{2n^2}\right)\right] \left( A^2/n^2 \right) dv
\]

As shown in Appendix F, equation F.6, this reduces to

\[
P(\text{error/signal}) = 1 - \exp\left[-\left(\frac{A^2 + T^2}{2n^2}\right)\right]
\]

\[
\sum_{j=0}^{\infty} (A/T)^j I_j \left( \frac{AT}{n^2} \right)
\]

Consequently, the detectivity, or probability of correctly evaluating the presence or absence of a signal given a composite signal plus noise or noise only input is
Detectivity = \( 1 - \left\{ \frac{P(\text{error/noise}) \cdot P(\text{noise})}{1 - \text{exp} \left( -\frac{T^2}{2n^2} \right)} + \frac{P(\text{error/signal}) \cdot P(\text{signal})}{1 - \text{exp} \left( -\frac{(A^2 + T^2)}{2n^2} \right)} \right\} \)

\[
= 1 - \text{exp} \left[ -\frac{T^2}{2n^2} \right] P(\text{noise}) \tag{1.9}
- P(\text{signal}) \left\{ \sum_{j=0}^{\infty} (A/T)^j I_j \left( \frac{AT}{n^2} \right) \right\}
\]

If signal and noise are equally probable

\[ P(\text{signal}) = P(\text{noise}) = 0.5 \]

Then

\[
\text{Detectivity} = 0.5 \left\{ 1 + \text{exp} \left[ -\frac{(A^2 + T^2)}{2n^2} \right] \right\} \sum_{j=0}^{\infty} (A/T)^j I_j \left( \frac{AT}{n^2} \right) \right\} - \text{exp} \left[ -\frac{T^2}{2n^2} \right] \tag{1.10}
\]
Threshold Considerations

Clearly the detectivity is a function of threshold in addition to the rms signal and noise variance. To properly evaluate a system's detectivity, it should be evaluated at its optimum threshold. The optimum threshold as shown in Appendix F equation F.11 is given by:

\[ I_o(\frac{AT}{n^2}) = \left( \frac{Q}{M} \right) \exp \left( \frac{A^2}{2n^2} \right) \]  

(1.11)

Where

- \( A \) = rms Signal
- \( T \) = Threshold Optical Amplitude
- \( n^2 \) = Noise Variance
- \( Q \) = Probability of Noise \([P(noise)]\)
- \( M \) = Probability of Signal \([P(signal)]\)
- \( \frac{A^2}{n^2} \) = Signal to Noise Ratio (SNR)

Attempts to develop a simplified expression for detectivity (equation 1.9) by substituting the expression for optimum threshold (equation 1.11) into the detectivity formula have been unsuccessful in eliminating the threshold as a variable.

Optimum threshold and detectivity were evaluated numerically using a computer as described in Appendix G.
The results are shown graphically in Figures G-1, G-2, G-3 and G-4. Detectivities of 0.6, 0.7, 0.8, 0.9 and 0.95 at optimum threshold with a signal probability of 0.5 correspond to SNR's of approximately 1.3, 3.1, 5.7, 10.4, and 15.3 respectively.

In order to establish the optimum threshold, the signal to noise ratio must be known. Because a point source signal can be focused onto a single detector, the detector array provides a means of determining the signal to noise ratio for the detector receiving the signal. To clarify this discussion it should be noted that a point source signal will illuminate the optical system aperture with a given amount of signal power \( S \). This power will be focused onto the detector at the image plane. As shown previously, the Airy disc image will be only a few wavelengths in diameter and can easily fall on only one detector.

Optical noise, on the other hand, is generated throughout the optical system's field of view and will tend to be relatively constant across the field of view. If the total noise striking the optical system aperture that is within the detector array's field of view is \( N \), and the detector array is divided into \( N_D^2 \) detectors of equal area, then the noise striking each detector is approximately \( N/N_D^2 \). Note that the total optical noise striking the aperture will be greater than or equal to \( N \) since the total
field of view of the detector is less than or equal to $2\pi$ steradians.

One method of determining the SNR is to assume that the highest reading detector is being illuminated by the signal and the remaining ($N_D^2 - 1$) detectors are being illuminated by noise only. In this case the SNR of the detector illuminated by signal can be approximated by

$$SNR_{\text{DETECTOR}} = \frac{[S + N_{ND}^2]/[(1/(N_D^2-1))\sum_{i=1}^{N_D^2-1} N_i]} - 1$$

Where

- $N_{ND}^2$ is the noise at the detector illuminated by signal.
- $N_i$ is the noise of the $i$th detector
- $\sum_{i=1}^{N_D^2-1}$ is the sum of the noise experienced by all the detectors not illuminated by signal
- $N_D^2 - 1$ is the total number of detectors not illuminated by signal
Now Approximating

\[ N_{ND}^2 = (1/N_D^2)N \]

\[ \sum_{i=1}^{N_D-1} N_i \approx \left( (N_D^2 - 1)/N_D^2 \right)N \]

\[ \left[ 1/(N_D^2 - 1) \right] \sum_{i=1}^{N_D^2 - 1} N_i \approx (1/N_D^2)N \]

Then

\[ SNR_{DETECTOR} \approx [S + (1/N_D^2)N]/[(1/N_D^2)N]-1 \]

\[ \approx [(S/N) + (1/N_D^2)] N_D^2]-1 \]

\[ \approx (S/N) N_D^2 \]

(1.12)

Equation 1.12 shows that the signal to noise ratio approximation for the detector is proportional to the systems optical signal to noise ratio and compensated for the number of detectors in the system. Thus a method is available for determining SNR so the optimum threshold can be used to improve detectivity.
Note that for a given detector array field of view, as the number of detectors \((N_D^2)\) for the same field of view increases, the SNR for the detector illuminated by signal increases proportionately. This is due to the reduction in detector size and therefore noise. Since each detector in the focal plane of an imaging system covers an area corresponding to a specific set of spatial frequencies, as the number of detectors per unit area increases, the detector is illuminated by a smaller field of view, the noise reaching the detector is reduced proportionally to the detector size reduction, and the SNR increases.

To compensate for when the signal energy falls across two detectors, the algorithm could be modified to choose only the highest reading detector unless the two highest reading detectors are within 30% of each other. In this case the SNR would be approximated by taking the average output of the two highest reading detectors and dividing it by the average noise in the remaining detectors.
For example

\[ \text{SNR}_{\text{DETECTOR}} = \frac{(1/2) [S_{\text{ND}^2} + N_{\text{ND}^2} + S_{\text{ND}^2-1} + N_{\text{ND}^2-1}]}{N_{\text{D}^2-2}} \times \frac{1}{(N_{\text{D}^2-2})} \sum_{i=1} N_i \]

Where

- \( S_{\text{ND}^2} \) is the Signal in the \( N_{\text{D}^2} \) detector
- \( S_{\text{ND}^2-1} \) is the Signal in the \( N_{\text{D}^2-1} \) detector
- \( N_{\text{ND}^2} \) is the Noise in the \( N_{\text{D}^2} \) detector
- \( N_{\text{ND}^2-1} \) is the Noise in the \( N_{\text{D}^2-1} \) detector
- \( N_i \) is the Noise in the detectors not illuminated by signal

Approximating

\[ S_{\text{ND}^2} + S_{\text{ND}^2-1} \approx S \]

\[ N_{\text{ND}^2} + N_{\text{ND}^2-1} \approx \left(\frac{2}{N_{\text{D}^2}}\right) N \]

\[ \sum_{i=1}^{N_{\text{D}^2-2}} N_i \approx \left(\frac{(N_{\text{D}^2-2})}{N_{\text{D}^2}}\right) N \]
\[
\sum_{i=1}^{N_D^2-2} \frac{1}{N_i} = \frac{1}{N_D^2} N
\]

Then

\[
\text{SNR}_{\text{DETECTOR}} \approx \frac{1}{2} \frac{S + (2/N_D^2)N}{[(1/N_D^2)N] - 1}
\]

\[
\frac{1}{2} (S/N) N_D^2 + 1 - 1 = \frac{1}{2} (S/N) N_D^2
\]

Which yields one-half the single illuminated detector SNR as would be expected.

The purpose of this discussion is to illustrate that a reasonable approximation of the SNR can be made for use in calculating optimum threshold. The optimum implementation has not been investigated and will not be addressed.

Resolution

A consideration of resolution must also include detectivity since the angular displacement of an object can only be resolved when its image is detected on detectors which correspond to different spatial frequencies.
When the Airy disc image of a point source illustrated in Figure C-4 falls totally on a detector the total signal power reaching the detector is given by equation 1.3. As long as the Airy disc remains totally on the detector face, the signal to noise ratio does not change. Consequently, the detectivity remains constant over the spatial frequencies corresponding to the position of the detector in the image plane.

As the Airy disc approaches the edge of the detector, a portion of the signal power in the Airy disc falls outside the detector boundary and the signal to noise ratio begins to decrease which in turn decreases the detectivity.

Appendix H analyzes the fraction of the Airy disc energy on each side of a straight line boundary and provides a graph (Figure H-1) of the fraction of the Airy disc power reaching the detector as a function of the distance from the Airy disc's centroid to the edge of the detector.

Graphically combining the results of the detectivity at optimum threshold vs. SNR (Figure G-3) and the fraction of the energy reaching the detector as a function of Airy disc position (Figure H-1) yields Figure 4.
Figure 4 Detectivity as a function of Airy disc centroid position relative to the edge of a detector for optimum threshold, \(P\ (signal) = 0.5\)
As Figure 4 shows, for very high SNR's the detectivity remains fairly high (>90%) even after the Airy disc centroid moves off the detector. This widens the band of spatial frequencies for which the detectivity is high and the resolution becomes less precise.

It should be noted that the distance over which the Airy disc moves from the point at which it falls completely on the detector to the point at which it falls completely off the detector is approximately $8(\lambda f/D)$. Unless $(f/D)$ is very large this distance is on the order of fractions of a millimeter, and typically only a small percentage of the total detector length. As a result, the band of spatial frequencies for which the detectivity is high while the Airy disc is totally on the detector will be much larger than those corresponding to the transition off the edge of the detector. Resolution will then be dominated by the detector size rather than the uncertainty at the detector's edge.

At worst the resolution in a single dimension must account for this transition and is then given by

$$SIN \ \gamma_{\text{resolution}} = \frac{[\text{(Detector Length)} + 8(\lambda f/D)]}{f}$$

$$SIN \ \gamma_{\text{resolution}} = \left(\frac{A_D}{f}\right) + 8(\lambda/D)$$
Assuming $A_D \ll f$

$$\gamma_{\text{resolution}} = \frac{A_D}{f} + 8(\lambda/D)$$

Again, the second term in a typical application is negligible compared to the first. As a result

$$\gamma_{\text{resolution}} = \frac{A_D}{f} \quad (1.13)$$

If detectors are placed adjacent to each other, as the Airy disc moves off one detector it will begin to move onto another detector. If the signal to noise ratio is high, the detectivity will be large for both detectors simultaneously. In order to tell to what degree the Airy disc is on each detector, the detectors may be differenced, the difference being proportional to the position of the Airy disc relative to the detectors' boundary.

Since the detectors see different fields of view, the optical noise striking each of them will be independent and the difference signal will contain a random variable component which is a function of the noise variance. This will produce uncertainty in the image position. The rms angular measurement error ($\sigma_\theta$) of a point source has been derived by Tyler and Fried (1982) as
\[ \sigma_0 = \pi (3/16) (\lambda/D)/\text{SNR}_v \]

Where

\[ \lambda = \text{wavelength of the incident light} \]

\[ D = \text{diameter of the limiting aperture in the optical system under consideration} \]

\[ \text{SNR}_v = \text{signal to noise voltage ratio} \]

In actual practice there will be a gap between detectors which would typically be wider than the Airy disc diameter. The differencing technique would then become valid only for extended sources with large SNR's so sufficient power overlaps each detector.

In the worst case, when the gap between detectors exceeds the maximum effective airy disc diameter \((8\lambda f/D)\), an object's angular motion will not be detected until its image moves from one detector to another. Since the distance between the Airy disc centroid and the detector edge required for a given detectivity is the same for either detector, the worst case resolution becomes

\[ \text{SIN } \gamma_{\text{resolution}} = \frac{[\text{distance between detector centers (A}_e \text{)]}}{f} \]

If

\[ A_e \ll f \]
Then
\[ \gamma_{\text{resolution}} \approx \frac{A_e}{f} \quad (1.14) \]

Where

\[ A_e = \text{distance between detector centers} \]
\[ f = \text{optical system focal length} \]

Now the distance between detector centers can be approximated by
\[ A_e \approx \sqrt{\frac{\text{detector array area}}{\# \text{ of detectors}}} = \sqrt{\frac{A_A^2}{N_D^2}} = \frac{A_A}{N_D} \]

And
\[ \gamma_{\text{resolution}} \approx \frac{A_A}{(fN_D)} \]

Where

\[ A_A = \text{The square root of the detector array area} \]
\[ N_D = \text{The square root of the number of detectors in the array} \]
\[ f = \text{Focal length of the imaging system} \]

While the expression of 1.14 appears different than 1.13, if the detectors touch, and therefore have no gaps between them, the distance between detector centers will be detector length and the two definitions are consistent.
MULTIPLE APERTURE ANALYSIS

Until now the discussion has been confined to an optical system with a single large circular aperture of diameter D as shown previously in Figure 1. It will now consider the effect of viewing the object through a pattern of smaller circular apertures confined within the perimeter of the single large aperture previously analyzed. This aperture arrangement may be placed in front of a single large lens in lieu of the single aperture of the previous analysis as shown in Figure 5, or a separate lens may be placed behind each of the multiapertures as shown in Figure 6. These two approaches are assumed to employ a rectangular array of multiple apertures within an area that the single large aperture would occupy as shown in Figure 7. For this discussion the following notation will be used

\[ D = \text{diameter of a single large aperture} \]
\[ D_a = \text{diameter of a multiple aperture} \]
\[ f = \text{focal length of a single aperture lens} \]
\[ f_a = \text{focal length of a multiaperture lens} \]
\[ N_S = \text{number of multiple aperture spacings that occur across a single large aperture diameter} \]
Figure 5. Multiple apertures in front of a single large lens

Figure 6. Multiple apertures in front of multiple lenses
MULTIPLE APERTURES

DISTANCE BETWEEN MULTI-APERTURE CENTERS

MULTI-APERTURE PERIMETER

MULTIPLE APERTURES

SINGLE LARGE APerture DIAMETER

DEFINITION OF SYMBOLS

D = DIAMETER OF A SINGLE LARGE APERTURE
Dₐ = DIAMETER OF A MULTIPLE APERTURE
f = FOCAL LENGTH OF A SINGLE APERTURE LENS
fₐ = FOCAL LENGTH OF A MULTI-APERTURE LENS
Nₛ = NUMBER OF MULTIPLE-APERTURE SPACINGS WITHIN A SINGLE APERTURE DIAMETER

Figure 7. Position and orientation of multiple apertures relative to a single large aperture.
Circular apertures have been chosen again due to their common use in seeker applications. The rectangular array of the multiple apertures was chosen for ease of analysis but could be any array of the designers choice.

**Overlapping Fields of View**

Before analyzing the detectivity and resolution of a multi aperture system one must consider the extent of overlap between the aperture fields of view (which affects resolution) and the coherence of the optical noise between apertures (which affects the detectivity since it will determine the statistical dependence or independence of the noise).

If two apertures whose centers are separated by a distance $D/N_S$ have fields of view equal to $\theta$, they will overlap when the distance $R$ from their mounting plane becomes large enough. This is illustrated in Figure 8.

The value of interest is the overlap field of view ($\theta'$) relative to $\theta$. For $\theta$ less than approximately $15^\circ$

$$\theta \approx L/R$$

and since

$$\theta' < \theta, \quad \theta' \approx L'/R$$
Figure 8. Analysis of overlapping fields of view
now \[ \frac{L'}{2} = \left( \frac{L}{2} - \frac{D}{2N_S} \right) + \left( \frac{L}{2} - \frac{D}{2N_S} \right) \]
\[ = L - \frac{D}{N_S} \]

And
\[ \theta' = \frac{L'}{R} \]
\[ \theta = \frac{L - (D/N_S)}{R} \]
\[ \theta' = \frac{L}{R} - \frac{D}{(N_S R)} \]

Since
\[ \theta = \frac{L}{R} \]
\[ \theta' = \theta - \frac{D}{N_S R} \]

Since \( D/N_S \) is the distance between the aperture centers, as \( R \) becomes large relative to the distance between aperture centers \( \theta' \) approaches \( \theta \). For example if \( \theta \) is 1 degree and \( D/N_S \) is 10 cm, an \( R \) of 50 meters will result in approximately only a 1% deviation of \( \theta' \) from \( \theta \).

The variation in \( \theta' \) with distance from the aperture plane as a function of aperture spacing \( (D/N_S) \) and aperture field of view \( \theta \) is shown graphically in Figure 9.

Consider next the coherence of the optical noise reaching two apertures from equal angular fields of view \( \theta \). As shown in Appendix I, the field of view containing coherent noise for two apertures separated by a distance \( D/N_S \) is given by I.11 as
Figure 9. Variation of the overlap field of view ($\theta'$) with distance from the aperture plane ($R$) as a function of aperture spacing ($D/N_S$) and aperture field of view ($\theta$).
Typically \( \theta_c \) is smaller than \( \theta \) since \( \lambda/(D/N_s) \) will be much smaller than the detector length divided by the focal length. As shown in Figure 10, the field of view for coherent noise will occupy only a fraction of the field of view for the aperture. The parameter of interest is what fraction of the noise power in the field of view is coherent with another aperture.

To find the fraction of coherent noise one must first find the total noise power reaching the aperture from the field of view. The fields of view for two apertures are virtually identical beyond relatively short ranges as illustrated in the discussion on the overlap field of view. Additionally, the further from the aperture plane the optical noise sources are, the more noise sources will fall in the aperture field of view. Since optical signals undergo an exponential attenuation with distance, and assuming optical noise power \( (N_0) \) is uniformly distributed through the field of view, the total noise power from the field of view reaching the aperture is given by

\[
P_{\text{noise total}} = \int \int \int N_0 \exp(-\alpha r)(-r^2) \sin \phi \, dr \, d\phi \, d\gamma
\]

\[
\theta_c = \lambda/(D/N_s)
\]
Figure 10. Coherent noise portion of the overlapping field of view for two apertures.
Where

\[ r = \text{distance from the aperture plane} \]
\[ a_0 = \text{optical attenuation coefficient} \]
\[ \theta/2 = \text{deviation of the field of view from the optical axis} \]

\[
P_{\text{noise total}} = \int_0^{2\pi} d\gamma \int_0^{\theta/2} (-\sin \phi) d\phi \int_0^\infty N_0 r^2 \exp(-a_0 r) dr
\]

From Gradshteyn & Ryzhik (1980) equation 3.351.3

\[
\int_0^\infty x^2 \exp(-a_0 x) dx = 2a_0^{-3}
\]

Thus

\[
P_{\text{noise total}} = N_0 2\pi [1-\cos(\theta/2)] 2a_0^{-3}
\]

Similarly the total coherent noise power is given by

\[
P_{\text{coherent noise}} = N_0 2\pi [1-\cos (\theta_c/2)] 2a_0^{-3}
\]
And the fraction of noise reaching the aperture which is coherent is given by

\[ \text{Fraction}_{\text{coherent noise}} = \frac{N_0^2 \pi [1 - \cos(\theta_c/2)]2a_0}{(N_0^2 \pi[1-\cos(\theta/2)]2a_0^3)} \]

\[ = \frac{1-cos(\theta_c/2)}{1-cos(\theta/2)} \]

To evaluate the fraction let \( \theta = m \theta_c \)

Then

\[ \text{Fraction}_{\text{coherent noise}} = \frac{1-cos(\theta_c/2)}{1-cos(m \theta_c/2)} \]

Expressing the cosines in terms of their series representations

\[ \text{Fraction}_{\text{coherent noise}} = \frac{\sum_{k=0}^{\infty} (-1)^k (\theta_c/2)^{2k}/(2k)!}{\sum_{k=0}^{\infty} (-1)^k (m)^{2k}(\theta_c/2)^{2k}/(2k)!} \]
Fraction_{coherent noise} = \left[ \sum_{k=1}^{\infty} (-1)^k (\theta_c/2)^{2k} / (2k)! \right] /
\left[ \sum_{k=1}^{\infty} (-1)^k (m)^{2k} (\theta_c/2)^{2k} / (2k)! \right]

All the terms in the denominator will be larger than the terms of the numerator by at least a factor of m^2 thus

Fraction_{coherent noise} \leq (1/m^2) = (\theta_c/\theta)^2

And for \theta \geq 10 \theta_c less than one percent of the noise power reaching the multiple apertures will be coherent. For apertures separated by 1 cm at a wavelength of one micron

\theta_c = (1 \times 10^{-6} m) / (10^{-2} m) = 10^{-4}

If these apertures have fields of view greater than 10^{-3} radians (.06 degrees) virtually all the noise power will be incoherent and therefore statistically independent.
**Focal Plane Intensity Distribution**

The signal intensity distribution in the focal plane will be a function of the method of implementing the multi-aperture system. When a single large lens of focal length $f$ has a multiaperture screen placed in front of it as shown in Figure 5, the intensity distribution in the focal plane is given by C.14 as

$$I(x_f,y_f) = \left[\frac{A_p}{\lambda f}\right]^2 C^2 |F\left\{P(x_2,y_2)\right\}|^2$$

Where

- $A$ = amplitude of the point source
- $\lambda$ = wavelength of the optical signal
- $f$ = focal length of the lens
- $C$ = a constant of proportionality
- $P(x_2,y_2)$ = pupil function of the lens aperture

The multiaperture pupil function is given by

$$P(x_2,y_2) = \left\{\text{circ} \left[\frac{r}{(D_a/2)}\right]*\left[\frac{1}{(D/N_S)^2}\right] \text{comb} \left[\frac{x}{(D/N_S)}\right] \text{comb} \left[\frac{y}{(D/N_S)}\right]\right\} \text{circ} \left[\frac{r}{(D/2)}\right]$$
Where
\[
\text{circ } (r/k) = \begin{cases} 
1 & \text{for } r \leq k \\
0 & \text{for } r > k 
\end{cases}
\]
\[
\text{comb } (x/k) = \left| k \right| \sum_{n=-\infty}^{\infty} \delta(x-nk)
\]
\(* = \text{the convolution operator} \]
\(\delta = \text{the dirac delta function} \]

Then
\[
F \left\{ P(x_2,y_2) \right\} = F \left\{ \left[ \text{circ } (r/(D_a/2)) \ast \left[ \frac{1}{(D/N_s)^2} \right. \right. \right. \\
\text{comb } (x/(D/N_s)) \text{ comb } (y/(D/N_s)) \left. \left. \right) \text{ circ } (r/(D/2)) \right] \right\}
\]
\[
F \left\{ P(x_2,y_2) \right\} = \left[ F \left\{ \text{circ } (r/(D_a/2)) \right\} \\
F \left\{ \frac{1}{(D/N_s)^2} \text{ comb } (x/(D/N_s)) \text{ comb } (y/(D/N_s)) \right\} \right] \\
* F \left\{ \text{circ } (r/(D/2)) \right\} \]

\[
(2.1)
\]
Now the Fourier Transforms on the right hand side of Equation 2.1 are

\[ F \left\{ \text{circ} \left[ r/(D_a/2) \right] \right\} = (D_a/2)^2 \pi \left[ 2J_1(\pi D_a s)/(\pi D_a s) \right] \quad (2.2) \]

\[ F \left\{ (1/(D/N_S)^2) \text{comb}(x/(D/N_S))\text{comb}(y/(D/N_S)) \right\} = \text{comb} (f_xD/N_S) \text{comb} (f_yD/N_S) \quad (2.3) \]

\[ F \left\{ \text{circ} \left[ r/(D/2) \right] \right\} = (D/2)^2 \pi \left[ 2J_1(\pi D s)/(\pi D s) \right] \quad (2.4) \]

Substituting 2.2, 2.3, and 2.4 into 2.1 yields

\[ F \left\{ P(x_2, y_2) \right\} = \left\{ (D_a/2)^2 \pi \left[ 2J_1(\pi D_a s)/(\pi D_a s) \right] \right\} \]

\[ \times \left\{ (D/2)^2 \pi \left[ 2J_1(\pi D s)/(\pi D s) \right] \right\} \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ 2J_1(\pi D_a \sqrt{m^2 + n^2})/(\pi D_a \sqrt{m^2 + n^2}) \right] \]

\[ \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left( (1/(D/N_S)^2) \delta (m-i/(D/N_S), n-j/(D/N_S)) \right) \]

\[ [2J_1(\pi D \sqrt{(f_x - m)^2 + (f_y - n)^2})/(\pi D \sqrt{(f_x - m)^2 + (f_y - n)^2})] \]

\[ \text{dmdn} \]
\[ F \left\{ P(x_2, y_2) \right\} = \left( \frac{D_a}{2} \right)^2 \pi \left( \frac{D}{2} \right)^2 \pi \]

\[
\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \frac{2J_1(\pi Da \sqrt{i^2 + j^2/(D/NS)})/(\pi Da \sqrt{i^2 + j^2/(D/NS)})}{(D/NS)} \left( \frac{1}{(D/NS)} \right)^2
\]

\[
\frac{2J_1(\pi D \sqrt{(fx - i/(D/NS))^2 + (fy - j/(D/NS))^2})}{(2 \cdot 5)}
\]

This resultant is an infinite plane of Bessel functions spaced \((D/NS)\) apart in the \(fx\) and \(fy\) directions and modulated by another \(J_1\) Bessel function. The peak amplitude is

\[
\left( \frac{D_a}{2} \right)^2 \pi \left( \frac{D}{2} \right)^2 \pi \left( \frac{NS}{D} \right)^2 = \left( \frac{D_a}{2} \right)^2 \pi \left( \frac{NS}{2} \right)^2 \pi
\]

In the limit as \(D_a\) becomes small and \(NS\) becomes large such that the multiple apertures are small enough and close enough together to cover the area a single large aperture would occupy, the peak amplitude would approach \(\pi(D/2)^2\), the same as the peak amplitude for a single aperture of diameter \(D\).
This analysis shows that a multiaperture in front of a single large lens has a lower peak amplitude than the single aperture case and achieves no improvement in image size. Noise would experience the same reduction in peak amplitude and the signal to noise ratio would remain unchanged. As a result there is nothing to be gained from a signal processing standpoint using this implementation of the multiaperture. At the same time there is nothing lost except total power (signal plus noise) reaching the detector would be lower.

A second method of implementing the multiaperture is by having a series of small lenses arranged in the geometry of Figures 6 and 7. Each of these lenses may then be analyzed separately using the single aperture analysis techniques already presented. The focal plane will contain the composite of the images created by each of these lenses individually, and the equivalent of equation 1.1 for the multiaperture case would be

\[
I(x_f, y_f) = \sum_{m} \sum_{n} \left[ \frac{A_p}{(\lambda f_a)} \right]^2 C^2 \left( \frac{D_a}{2} \right)^4 \pi^2 \\
\frac{2J_1\left(\frac{\pi D_a}{\lambda f_a}\right) \sqrt{(x_f - m/(D/N_S))^2 + (y_f - n/(D/N_S))^2}}{[(\pi D_a/\lambda f_a) \sqrt{(x_f - m/(D/N_S))^2 + (y_f - n/(D/N_S))^2} ]^2}
\]

(2.6)
where m and n only include apertures centered within the original single aperture of diameter D.

**Detectivity**

For each individual lens of the multiple aperture, the total signal reaching the focal plane will be less than the single lens of the single aperture due to the reduction in aperture area. Since this applies to both signal and noise, the signal to noise ratio will not change. As a result, the detectivity for each of the multiple apertures will be the same as a single large aperture unless the aperture becomes so small that the drop in optical signal and noise strength causes the detectors to become self noise limited.

**Resolution**

Similarly, if the ratio of \( \frac{f_a}{D_a} \) for each of the multiple apertures does not change from the \( \frac{f}{D} \) of the single large aperture, the Airy disc diameter will not change, and unless the detector size is reduced to close to the Airy disc size, the resolution will not change.
The one improvement which does result is a function of the detector geometry rather than the aperture. As noted in the previous section on the single aperture, the resolution is primarily dependent upon detector size and focal length. When there are gaps between detectors, the resolution is further degraded. If the detectors in the multi-aperture system are each shifted by a different amount from the aperture center, the detectors behind some apertures will lie over spatial frequency positions that may correspond to a gap between detectors for another aperture. Even if there are no gaps between detectors, the detectors behind one aperture will overlap spatial frequencies corresponding to two or more detectors behind another aperture. Consider the illustration of Figure 11.

Figure 11 shows three complete detectors which effectively divide the spatial frequencies of the focal plane up into four subsets:

1. The left portion of the left detector for aperture 1 only.
2. The intersection of the left detector for aperture 1 and the detector for aperture 2.
3. The intersection of the right detector for aperture 1 and the detector for aperture 2.
4. The right half of the right detector for aperture 1 only.
Figure 11. Illustration of the focal plane spatial frequency position overlaps of two detector arrays for two apertures of a multiple aperture system.
The more overlapping detectors there are, the more finely the spatial frequencies become subdivided by intersections of the spatial frequency sets belonging to each detector. If the detectors overlap each other by exactly the same distance, the effective length of the smallest subset may be approximated by

$$\text{effective detector length} \approx \frac{(\sqrt{A_0^2} - A_D)}{(\sqrt{N_D^2} - 1)}$$

Where

$$A_0^2 = \text{total area of overlapped spatial frequencies by the detectors in the image plane.}$$

$$A_D = \text{length of one detector}$$

$$N_D^2 = \text{total number of detectors}$$

As $$A_0^2$$ and $$N_D^2$$ become large

$$\text{effective detector length} \approx \frac{A_0}{N_D}$$

And the resolution may be approximated by

$$\gamma \text{ resolution} \approx \frac{A_0}{f_a N_D}$$
If the field of view for each of the multiapertures is the same, \( A_0 \) will not change with the number of multiple apertures. If the number of detectors in the focal plane of each aperture is the same, then

\[
\gamma \text{ resolution} = \frac{A_0}{(f_a N_D \sqrt{N_a})}
\]

Where

- \( N_a \) = the number of apertures in the multiple aperture system
- \( f_a \) = the focal length of the lenses in the multi-aperture array
- \( N_D^2 \) = the number of detectors in the focal plane of each aperture
- \( A_o^2 \) = the area in the focal plane of each aperture and the resolution improves inversely with the square root of the number of apertures.
CONCLUSION

This study has analyzed the theoretical detectivity and resolution performance of single aperture and multiple aperture optical imaging systems.

The designer controlled variables of the imaging system which affect the detectivity and resolution are optical signal wavelength ($\lambda$), optical system aperture diameter ($D$ for a single aperture and $D_a$ for each of the multiple apertures), optical system focal length ($f$), detector area ($A_D^2$ for the actual area of a single detector, $A_e^2$ for the effective area of a single detector, and $A_o^2$ for the overlap area of multiple aperture detectors), and detector threshold ($T$).

The relationships between these variables are illustrated in Tables 1 and 2. In summary, the detectivity is a function of the signal to noise ratio (SNR), the signal and noise probabilities, and detector threshold. When the system is optimized, the threshold is a function of the SNR, and the detectivity becomes a function of the SNR and signal and noise probabilities only. The only variable the designer controls which affects these parameters is detector field of view, which is a function of detector size, and imaging system focal length. By reducing the detector
size, or increasing the system focal length, the field of view for the detector decreases causing a reduction in the background noise the detector senses and improving the SNR which improves the detectivity. Although the designer also affects the aperture diameter, it affects the signal and noise proportionally, and thus does not affect the SNR or detectivity. If a multiple aperture system and a single aperture system have the same detector size and focal length, there will be no difference in detectivity due to the difference in aperture diameters. Further summing or differencing the multiple apertures will provide no SNR improvement, or detectivity improvement, since the noise between each of the multiple apertures is incoherent and therefore statistically independent.

Resolution is ultimately limited by wavelength and aperture diameter, however, typical detector sizes are orders of magnitude greater than wavelength and thus are usually the practical limiting factor in resolution measurements. Additionally, real detectors are not immediately adjacent and the effective detector dimension which forms the limit to resolution is the distance between detector centers. Where the detector size or effective detector size is limiting, the focal length becomes a factor affecting resolution as well. As detector size decreases and focal length increases, the resolution angle decreases. When
detector size is limiting, variation of these parameters affects both the single aperture and multiple aperture systems proportionally and does not result in any change in resolution between the two systems. When detector size is not limiting, resolution of the multiple aperture system is degraded relative to the single aperture system unless the diameter of each of the multiple apertures is increased to match the single aperture diameter.

When detector size is limiting, the use of multiple apertures does provide a resolution improvement over a single aperture system if the detectors of one multiple aperture overlap rather than exactly match the spatial frequency bands of the other apertures' detectors. This improvement is due to the opportunity to form the intersection of several detectors' spatial frequency bands. These intersections subdivide the detector size into small subsets corresponding to the overlap of specific detectors. By forming the intersection of the fields of view for these detectors, the angular position of the source can be more finely resolved. In the limit, however, this angular resolution can be no better than the non-detector size limited case for either the single aperture or multiple aperture system. The resolution limit will still be a function of the Airy disc size which reduces to a ratio of the optical signal wavelength divided by the aperture diameter.
TABLE 1

SUMMARY OF DETECTIVITY AND THE FACTORS AFFECTING DETECTIVITY

Detectivity = 1 - \( \frac{P(\text{error}/\text{signal})P(\text{signal}) - P(\text{error}/\text{noise})P(\text{noise})}{1 - \exp\left(-\frac{(A^2+T^2)}{2n^2}\right)} \)

\[ = 1 - \left[1 - \exp\left(-\frac{(A^2+T^2)}{2n^2}\right)\right] \]

\[ \sum_{j=0}^{\infty} \left(\frac{A}{T}\right)j I_j(\frac{AT}{n^2})P(\text{signal}) \]

\[ -\exp\left[-\frac{T^2}{2n^2}\right]P(\text{noise}) \]

At optimum threshold \( I_0(\frac{AT}{n^2}) \approx \exp[\frac{A^2}{2n^2}] \)

Consequently, at optimum threshold Detectivity is a function of SNR \( (\frac{A^2}{n^2}) \)

Factors Affecting \( (\frac{A^2}{n^2}) \)

\( D(\text{aperture diameter}) \quad \text{Affects signal and noise proportionally, consequently no effect on SNR or detectivity} \)

\( f(\text{focal length}) \)

\( A_D^2 \quad \text{or} \quad A_e^2 \quad \text{(detector size)} \)

Affect the field of view. As the field of view decreases toward the source solid angle, the noise decreases and the SNR and detectivity increase

Each of these factors affect the single aperture and multiple aperture designs proportionally and result in no relative improvement in performance
TABLE 2

SUMMARY OF RESOLUTION AND THE FACTORS AFFECTING RESOLUTION

\[ \sin \gamma \text{resolution} = \frac{\text{detector size}}{(\text{focal length})} + 8 \frac{\lambda}{D} \]

Detector size not limiting

resolution is limited by the second term which is dependent upon the optical wavelength and aperture diameter

Detector size limiting

resolution is limited by the first term which is affected by

\[ f \text{ (focal length)} \]

affects single aperture and multiple aperture designs proportionally

\[ A_D^2 \text{(detector area)} \]

or

\[ A_e^2 \text{(effective detector area)} \]

affects single aperture and multiple aperture designs proportionally

\[ A_0^2 \text{(detector overlap area)} \]

for the multiple aperture case, overlapping detector fields of view permits the formation of detector intersections which reduce the effective detector size and improve resolution
Appendices
APPENDIX A

COMPUTER LITERATURE SEARCH TECHNIQUE

The State Technology Applications Center was utilized to access the Lockheed DIALOG Data Bank. This data bank is a computerized file of approximately 3500 scientific and professional journals as well as conference and symposium abstracts. The data bank does not include classified Government sponsored research.

Computer searches were conducted for abstracts containing the following key word combinations

1. (multiaperture) and (optic or camera or photography or image)
2. (multiaperture) and (mosaic or random or coded or distributed or array)
3. (multiaperture) and (lens)
4. (tracker or tracking or seeker or seeking) and (aperture) and (mosaic or random or coded or distributed or array or multi) and (optic or camera or photograph or image)
5. (image correlation or electronic sensing or multi) and (aperture) and (mosaic or random or distributed or array or multi) and (optic or camera photograph or image)
6. (aperture) and (mosaic or random or coded or distributed or multi) and (optic or camera or photograph or image) and (optical data processing or imaging techniques or image processing)

The search identified 104 abstracts meeting the key word criteria. After copies of the abstracts were obtained and reviewed, the pertinent articles were copied for detailed study.
APPENDIX B

SPATIAL FREQUENCY CONCEPTS

The physical significance of a spatial frequency can best be analyzed by considering a plane wave propagating at angles $\alpha$, $\beta$, and $\gamma$ relative to the x, y, and z axes respectively as shown in Figure B-1.

In Figure B-1, M₁ and M₂ are portions of planes of constant phase moving away from the origin at O. Lines OP₁ and OP₂ are perpendicular to the planes M₁ and M₂. As planes of constant phase, M₁ and M₂ are separated by a wavelength $\lambda$. The angles of propagation must satisfy the relationship

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad (B.1)$$

The intersection of the plane one wavelength from the origin with each of the coordinate axes is shown in Figure B-2. The intersection of the plane with the x and y axes defines the spatial wavelength in the x and y directions which is the inverse of the spatial frequency in the x and y directions ($f_x$ and $f_y$, respectively). Note that the spatial frequency has units of cycles/length and represents the number of $360^\circ$ phase changes in the optical signal per unit length in the x or y direction. Since the distance OP is
Figure B-1. Plane waves propagating at angles $\alpha$, $\beta$, and $\gamma$ relative to the $x$, $y$, and $z$ axes respectively.
Figure B-2. Plane wave at a distance $\lambda$ from the coordinate system origin intersecting the coordinate axes
the wavelength $\lambda$ in Figure B-2

$$\cos \alpha = \lambda f_x \quad (B.2)$$
$$\cos \beta = \lambda f_y \quad (B.3)$$

Thus from equation B.1, B.2, and B.3

$$\left(\lambda f_x\right)^2 + \left(\lambda f_y\right)^2 + \cos^2 \gamma = 1$$

$$\left(\lambda f_x\right)^2 + \left(\lambda f_y\right)^2 = 1 - \cos^2 \gamma$$

$$\lambda^2 (f_x^2 + f_y^2) = \sin^2 \gamma$$

$$\sin \gamma = \lambda \sqrt{f_x^2 + f_y^2} \quad (B.4)$$

From B.4, knowing the spatial frequency components in the $xy$ plane, uniquely determines the direction of propagation relative to the $z$ axis.

If line $OP$ is now extended to a point $P_3$ whose $z$ coordinate is $f$ as shown in Figure B.3

Then

$$\sin \gamma = \sqrt{x^2 + y^2} / \sqrt{f^2 + x^2 + y^2}$$

$$\sin \gamma = \sqrt{(x/f)^2 + (y/f)^2} / \sqrt{1 + (x/f)^2 + (y/f)^2}$$
Figure B-3. $P_3(x, y, f)$
Now for small angles ($\gamma < 15^\circ$)

$$\sin \gamma \approx \tan \gamma = \frac{\sqrt{x^2 + y^2}}{f}$$

$$\sin \gamma \approx \sqrt{(x/f)^2 + (y/f)^2} = (1/f)\sqrt{x^2 + y^2} \quad (B.5)$$

Combining B.4 and B.5 yields

$$(1/f)\sqrt{x^2 + y^2} = \lambda \sqrt{f_x^2 + f_y^2}$$

Since $f_x$ and $f_y$ depend only on the x and y components respectively

$$f_x = \left[\frac{x}{(\lambda f)}\right] \quad f_y = \left[\frac{y}{(\lambda f)}\right] \quad (B.6)$$

For larger angles since

$$\sin \gamma = \tan \gamma \cos \gamma$$

from B.4 and B.5

$$\lambda \sqrt{f_x^2 + f_y^2} = (1/f)\sqrt{x^2 + y^2} \cos \gamma$$

and

$$f_x = \left[\frac{x}{(\lambda f)}\right] \cos \gamma \quad f_y = \left[\frac{(y/(\lambda f))}{(\lambda f)}\right] \cos \gamma$$
Goodman (1968) provides a detailed analysis of the Fourier Transforming properties of lenses from its roots in diffraction theory.

Without reproducing the entire derivation, the optical amplitude $U(x_f, y_f)$ in one plane can be found from the optical amplitude $U(x_2, y_2)$ in a second parallel plane a distance $z$ from the first plane as shown in Figure C-1 from the formula

$$U(x_f, y_f) = \left[ \exp(ikz)/(j\lambda z) \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x_2, y_2) \exp\left[ j\left(\frac{k}{2z}\right) \left( (x_f - x_2)^2 + (y_f - y_2)^2 \right) \right] \, dx_2 \, dy_2$$

Where

- $\lambda =$ optical signal wavelength
- $k = 2\pi/\lambda$ the wave number
Figure C-1. Two planes \((x_2, y_2)\) and \((x_f, y_f)\) separated by a distance \(z\)
This formula assumes

a) that only a finite region around the origin of either plane is of interest and that the maximum linear dimension of this region of interest is much smaller than the distance \( z \) between the planes

b) the distance \( d \) between any two points on the two planes of interest can be approximated by the first two terms of the binomial expansion yielding

\[
d = \sqrt{z^2 + (x_f - x_2)^2 + (y_f - y_2)^2}
\]

\[
= z \sqrt{1 + [(x_f - x_2)/z]^2 + [(y_f - y_2)/z]^2}
\]

\[
= z[1 + (1/2)[(x_f - x_2)/z]^2 + (1/2)[(y_f - y_2)/z]^2]
\]

First the ability of a lens to Fourier Transform the optical field incident on its aperture will be examined. A lens has the property that a plane wave perpendicular to its axis will have phase delays which are a function of the lens thickness. For the lens of Figure C-2 with a thickness at \((x_2, y_2)\) given by \( \Delta (x_2, y_2) \), the total phase delay experienced by the plane wave at \( U_1(x_2, y_2) \) as it travels to \( U_1'(x_2, y_2) \) is given by
Figure C-2. Lens configuration
\[ U_1'(x_2, y_2) = U_1(x_2, y_2) \left[ k \eta \Delta (x_2, y_2) + k [\Delta_0 - \Delta (x_2, y_2)] \right] \]

Where \( \Delta_0 \) = maximum lens thickness
\( \eta \) = the refractive index of the lens
\( k \) = the wavenumber \((2\pi/\lambda)\)
\( k \eta \Delta (x_2, y_2) \) = the phase delay introduced by the lens
\( k[\Delta_0 - \Delta (x_2, y_2)] \) = the phase delay introduced by the non lens space between \( U_1 \) and \( U_1' \)

Now the thickness of the lens at any point \((x_2, y_2)\) can be solved in terms of the radius of curvature of each half of the lens. Again referring to Figure C-2, when only the area close to the lens axis is considered, the binomial expansion may be used to approximate

\[ \Delta (x_2, y_2) = \Delta_0 - \left[ \frac{(x_2^2 + y_2^2)}{2} \right] \left[ \frac{1}{R_1} - \frac{1}{R_2} \right] \]

Where
\( \Delta_0 \) = maximum lens thickness
\( R_1 \) = radius of curvature of one lens half
\( R_2 \) = radius of curvature of second lens half
The focal length \((f)\) of a lens is defined by

\[
\frac{1}{f} = (n - 1)[(1/R_1) - (1/R_2)]
\]  
(C.4)

Where

\(n\) = refractive index of the lens
\(R_1\) = radius of curvature of one lens half
\(R_2\) = radius of curvature of second lens half

Now if C.2 is represented in terms of a multiplicative exponential, then

\[
U_1'(x_2, y_2) = U_1(x_2, y_2) 
\exp[jk n \Delta(x_2, y_2) + jk(\Delta_o - \Delta(x_2, y_2))] 
= U_1(x_2, y_2) \exp(jk \Delta_o) \exp[jk(n - 1) \Delta(x_2, y_2)]
\]  
(C.5)

Substituting C.3 into C.5 yields

\[
U_1'(x_2, y_2) = U_1(x_2, y_2) \exp(jk n \Delta_o) 
\exp[-jk(n - 1)(x_2^2 + y_2^2)[(1/R_1) - (1/R_2)]/2]
\]  
(C.6)
Substituting C.4 into C.6 yields

\[ U_1'(x_2,y_2) = U_1(x,y)\exp(jk \Delta_0)\exp[-j(k/2f)(x_2^2 + y_2^2)] \]

(C.7)

If the lens is placed in the plane of \((x_2,y_2)\) in Figure C-1, then the field amplitude distribution at the focal plane \((x_f,y_f)\) at a distance \(f\) from the lens is found from equation C.1 as follows

\[
U(x_f,y_f) = \left[ \exp(jkf)/(j\lambda f) \right] \iint_{-\infty}^{\infty} U_1'(x_2,y_2) \exp\left[j(k/2f)[(x_f-x_2)^2+(y_f-y_2)^2]\right]dx_2dy_2
\]

= \left[ \exp(jkf)/(j\lambda f) \right] \iint_{-\infty}^{\infty} U_1(x_2,y_2)\exp(jk \Delta_0) \exp[-j(k/2f)(x_2^2 + y_2^2)] \exp[j(k/2f)[(x_f-x_2)^2+(y_f-y_2)^2]]dx_2dy_2
\[ U(x_f, y_f) = \exp(\frac{j k f}{\lambda f}) \exp(\frac{j k}{\lambda} \Delta_0) \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1(x_2, y_2) \exp[-j(k/2f)(x_2^2+y_2^2)] \]

\[ \exp[j(k/2f)(x_f^2+y_f^2)] \exp[-j(k/2f)(2x_f x_2 + 2y_f y_2)] \]

\[ \exp[j(k/2f)(x_2^2+y_2^2)] dx_2 dy_2 \]

\[ = \exp(\frac{j k f}{\lambda f}) \exp(\frac{j k}{\lambda} \Delta_0) \exp[j(k/2f)(x_f^2+y_f^2)] \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1(x_2, y_2) \exp[-j(k/f)(x_f x_2 + y_f y_2)] dx_2 dy_2 \]

Since \( k \) is the wave number \( k = \frac{2 \pi}{\lambda} \) and

\[ U(x_f, y_f) = \exp(\frac{j k f}{\lambda f}) \exp(\frac{j k}{\lambda} \Delta_0) \exp[j(k/2f)(x_f^2+y_f^2)] \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1(x_2, y_2) \exp[-j2 \pi [(x_f/(\lambda f)) x_2 + (y_f/(\lambda f)) y_2]] dx_2 dy_2 \]

\[ (C.8) \]

Thus the field distribution at the focal plane is proportional to the two dimensional Fourier Transform of the field incident on the lens.
In reality, the lens is never infinite and the light incident at \((x_2, y_2)\) will pass through some aperture before being focused at \((x_f, y_f)\). If the aperture is defined by a pupil function at \((x_2, y_2)\) such that

\[
P(x_2, y_2) = \begin{cases} 
0 & \text{outside the aperture} \\
1 & \text{inside the aperture} 
\end{cases}
\]

Then

\[
U(x_f, y_f) = \exp(jkf)/(j\lambda f) \exp(jk \eta \Delta_0) \\
\exp[j(k/2f)(x_f^2+y_f^2)] \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1(x_2, y_2)P(x_2, y_2) \\
\exp[-j2 \pi [(x_f/(\lambda f))x_2 + (y_f/(\lambda f))y_2]] \, dx_2 \, dy_2
\]

(C.9)

Now the response of the lens system to a point source will be found. Suppose the field at \(U_1(x_2, y_2)\) is derived from some point source of amplitude \(A_p\) at \((x, y)\) in a plane at distance \(d_0\) from \((x_2, y_2)\) as shown in Figure C-3. Since the source at \((x, y)\) is a point source of amplitude \(A_p\) the field distribution at \(U_1(x_2, y_2)\) may be found from (C.1) to yield
Figure C-3. Point source at a distance $d_0$ from a lens of focal length $f$
\[ U_1(x_2, y_2) = [\exp(jkd_0)/(j \lambda d_0)] \]
\[
\iint_{-\infty}^{\infty} A_p \delta(x, y) \exp[j(k/(2d_0))((x_2-x)^2+(y_2-y)^2)] \, dx \, dy
\]
\[
= [\exp(jkd_0)/(j \lambda d_0)] A_p \exp[j(k/(2d_0))(x_2^2+y_2^2)]
\]
\[ (C.10) \]

Substituting C.10 into C.9 yields

\[ U(x_f, y_f) = [\exp(jkf)/(j \lambda f)] \exp(jk \Delta_0) \]
\[ \exp[j(k/2f)(x_f^2+y_f^2)] \]
\[
\iint_{-\infty}^{\infty} P(x_2, y_2) A_p \left[ \exp(jkd_0)/(j \lambda d_0) \right] 
\exp[j(k/(2d_0))(x_2^2+y_2^2)] 
\exp[-j2 \pi \left( (x_f/(\lambda f))x_2 + (y_f/(\lambda f))y_2 \right)] \, dx_2 \, dy_2
\]
\[
= [\exp(jkf)/(j \lambda f)] \exp(jk \Delta_0) \]
\[ \exp[j(k/2f)(x_f^2+y_f^2)] \]
\[ F \left\{ P(x_2, y_2) \right\} \ast F \left\{ A_p \left[ \exp(jkd_0)/(j \lambda d_0) \right] \right\} 
\exp[j(k/(2d_0))(x_2^2+y_2^2)]
\]
\[ (C.11) \]

Now the Fourier Transform of the last term of C.11 will be found and substituted back into C.11 to solve for the field distribution at the focal plane due to a point source signal.
\[
F \left\{ A_p \left[ \exp( j k d_0 ) / ( j \lambda d_0 ) \right] \exp[ j (k/(2d_0))(x_2^2+y_2^2)] \right\}
\]

\[
= A_p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \exp( j k d_0 ) / ( j \lambda d_0 ) \right] \exp[ j (k/(2d_0))(x_2^2+y_2^2)]
\exp[-j2 \pi [(x_f/(\lambda f))x_2+(y_f/(\lambda f))y_2]] \, dx_2 \, dy_2
\]

Defining
\[
x_f/(\lambda f) = f_x \quad y_f/(\lambda f) = f_y
\]

And noting
\[
(2 \pi f_x)/(k/2d_0) = (4 \pi d_0 f_x)/(2 \pi / \lambda) = 2 \lambda d_0 f_x
\]

Then
\[
F \left\{ A_p \left[ \exp( j k d_0 ) / ( j \lambda d_0 ) \right] \exp[ j (k/2d_0)(x_2^2+y_2^2)] \right\}
\]

\[
= A_p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \exp( j k d_0 ) / ( j \lambda d_0 ) \right]
\exp[ j (k/2d_0)(x_2^2+y_2^2+2 \lambda d_0 f_x x_2+2 \lambda d_0 f_y y_2)]
\, dx_2 \, dy_2
\]
Completing the square

\[
F \left\{ A_p \left[ \exp( jk d_o )/( j \lambda d_o ) \right] \exp[ j(k/2d_o)(x_2^2+y_2^2) ] \right\}
\]

\[
= A_p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \exp( jk d_o )/( j \lambda d_o ) \right] \exp[ j(k/2d_o)(x_2^2+y_2^2) ] dx_2 dy_2
\]

\[
= A_p \exp[ -j \pi \lambda d_o (f_x^2+f_y^2) ] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \exp( jk d_o )/( j \lambda d_o ) \right] \exp[ j(k/2d_o)(x_2^2+y_2^2) ] dx_2 dy_2
\]

let \( p = x_2 + \lambda d_o f_x \)

\( dx_2 = dp \)

\( q = y_2 + \lambda d_o f_y \)

\( dy_2 = dq \)

\[
F \left\{ A_p \left[ \exp( jk d_o )/( j \lambda d_o ) \right] \exp[ j(k/2d_o)(x_2^2+y_2^2) ] \right\}
\]

\[
= A_p \exp[ -j \pi \lambda d_o (f_x^2+f_y^2) ] \exp(jk d_o ) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ 1/( j \lambda d_o ) \right] \exp[ j( \pi / \lambda d_o )(p^2+q^2) ] dp dq
\]
\[ F \left\{ A_p \left[ \exp(jk\sigma_0) \right] / (j \lambda \sigma_0) \exp[j(k/2\sigma_0)(x_2^2+y_2^2)] \right\} \]
\[ = A_p \exp[-j \pi \lambda \sigma_0(f_x^2+f_y^2)] \exp(jk\sigma_0) \]
\[ 2 \int_0^\infty 2 \int_0^\infty [1/(j \lambda \sigma_0)] \exp[j \pi / \lambda \sigma_0] (p^2+q^2)] dp dq \]

let \( m = \pi p^2 / (\lambda \sigma_0) \)   \( n = \pi q^2 / (\lambda \sigma_0) \)
\[ dp = (\lambda \sigma_0 / \pi)^{1/2} (1/2)^{m-1/2} \]
\[ dq = (\lambda \sigma_0 / \pi)^{1/2} (1/2)^{n-1/2} \]

Then
\[ F \left\{ A_p \left[ \exp(jk\sigma_0) \right] / (j \lambda \sigma_0) \exp[j(k/2\sigma_0)(x_2^2+y_2^2)] \right\} \]
\[ = A_p \exp[-j \pi \lambda \sigma_0(f_x^2+f_y^2)] \exp(jk\sigma_0) \]
\[ (1/j \pi) \int_0^\infty \int_0^\infty \exp(jm) \exp(jn) (mn)^{-1/2} dmdn \]
\[ = A_p \exp[-j \pi \lambda \sigma_0(f_x^2+f_y^2)] \exp(jk\sigma_0) [1/(j \pi)] \]
\[ \int_0^\infty (m^{-1/2}\cos m + jm^{-1/2}\sin m) dm \]
\[ \int_0^\infty (n^{-1/2}\cos n + j n^{-1/2}\sin n) dn \]
From Gradshteyn and Ryzhik (1980) equation 3.757

\[ \int_0^\infty x^{-1/2} \sin x \, dx = \int_0^\infty x^{-1/2} \cos x \, dx = \frac{\pi}{2} \]

And

\[
\begin{align*}
F \left\{ A_p \left[ \text{exp} \left( jk d_o \right) / (j \lambda d_o) \right] \text{exp} \left[ j \left( k / 2 d_o \right) \left( x_f^2 + y_f^2 \right) \right] \right\} \\
&= A_p \text{exp} \left[ - j \pi \lambda d_o (f_x^2 + f_y^2) \right] \text{exp} \left( jk d_o \right) \left[ 1 / (j \pi) \right] \\
&\quad \left[ (\pi/2) + j(\pi/2) \right] \left[ (\pi/2) + j(\pi/2) \right] \\
&= A_p \text{exp} \left[ - j \pi \lambda d_o (f_x^2 + f_y^2) \right] \text{exp} \left( jk d_o \right) \quad \text{(C.12)}
\end{align*}
\]

Substituting C.12 into C.11 yields

\[ U(x_f, y_f) = \left[ \text{exp} \left( jk f \right) / (j \lambda f) \right] \text{exp} \left( jk \eta \Delta_o \right) \text{exp} \left[ j \left( k / 2 f \right) \left( x_f^2 + y_f^2 \right) \right] \]

\[ F \left\{ P(x_2, y_2) \right\} * A_p \text{exp} \left[ - j \pi \lambda d_o (f_x^2 + f_y^2) \right] \text{exp} \left( jk d_o \right) \]

Again noting

\[ f_x = x_f / (\lambda f) \quad f_y = y_f / (\lambda f) \]

And

\[ \pi \lambda d_o f_x^2 = \pi \lambda d_o (x_f / (\lambda f))^2 = (k/2f)(d_o/f)x_f^2 \]
\[ \pi \lambda d_o f_y^2 = \pi \lambda d_o (y_f / (\lambda f))^2 = (k/2f)(d_o/f)y_f^2 \]
Results in

\[ U(x_f,y_f) = \left[ \exp(jk)/(j \lambda f) \right] \exp(jk \eta \Delta_0) \]
\[ \exp[j(k/2f)(x_f^2+y_f^2)(1-(d_0/f))] \]
\[ \exp(jkd_0)A_pF \{ P(x_2,y_2) \} \]  

(C.13)

The optical intensity at the focal plane is proportional to the square of the optical field amplitude, and the complex exponentials represent phase terms whose squared values are constant after time averaging. As a result

\[ I(x_f,y_f) \propto [A_p/(\lambda f)]^2 \left| F \left\{ P(x_2,y_2) \right\} \right|^2 \]  

(C.14)

Thus, a point source of amplitude \( A_p \) will yield a spatial frequency distribution at the lens focal plane which is a function of the Fourier Transform of the pupil function.

Now the Fourier Transform for a circular pupil function will be found and substituted into C.14 to find the intensity distribution at the focal plane due to a point source signal for a lens with a circular aperture.

If the lens aperture is circular

\[ P(x_2,y_2) = \text{circ}\{r/(D/2)\} = \begin{cases} 
1 & \text{for } r \leq (D/2) \\
0 & \text{for } r > (D/2) 
\end{cases} \]
Where

\( r = \text{distance from the center of the aperture} \)
\( D = \text{diameter of the aperture} \)

To find \( F \{ \text{circ}[r/(D/2)] \} \)

\[
F \{ \text{circ}[r/(D/2)] \} = F \{ \text{circ}[\sqrt{x_2+y_2^2}/(D/2)] \}
\]

\[
= \iint_{-\infty}^{\infty} \text{circ}[\sqrt{x_2+y_2^2}/(D/2)]
\]

\[
\exp[-j2\pi(f_x x+f_y y)]dxdy
\]

Where

\[
r = \sqrt{x_2+y_2^2}
\]
\[
f = \tan^{-1}(y/x)
\]
\[
s = \sqrt{f_x^2+f_y^2}
\]
\[
\theta_i = \tan^{-1}(f_y/f_x)
\]
\[
x = r \cos \theta_f
\]
\[
y = r \sin \theta_f
\]
\[
f_x = s \cos \theta_i
\]
\[
f_y = s \sin \theta_i
\]

Then

\[
F \{ \text{circ}[r/(D/2)] \} = \iint_{0}^{2\pi} \iint_{0}^{\infty} \text{circ}[r/(D/2)]
\]

\[
\exp[-j2\pi rs(\cos \theta_f \cos \theta_i + \sin \theta_f \sin \theta_i)]r \ dr \ d \theta_f
\]
\[ F \left\{ \text{circ}\left(\frac{r}{D/2}\right) \right\} = \int_0^\infty r \text{circ}\left(\frac{r}{(D/2)}\right) \]

\[ \int_0^{2\pi} \exp\left\{-j2\pi rs \cos(\theta_f - \theta_i)\right\} d\theta_f \, dr \]

The second integral is the Bessel function of the first kind of order zero is given by

\[ J_0(a) = \frac{1}{(2\pi)} \int_0^{2\pi} \exp\left\{-ja\cos(\theta_f - \theta_i)\right\} d\theta_f \]

Then

\[ F \left\{ \text{circ}\left(\frac{r}{D/2}\right) \right\} = 2\pi \int_0^\infty r \text{circ}\left(\frac{r}{D/2}\right) J_0(2\pi rs) \, dr \]

\[ = 2 \int_0^{D/2} r J_0(2\pi rs) \, dr \]

Let

\[ 2\pi rs = \rho \quad r = \frac{\rho}{2\pi s} \]

\[ dr = \frac{d\rho}{2\pi s} \]
Then

\[ F \left\{ \text{circ} \left\{ r/(D/2) \right\} \right\} = 2\pi \int_0^{(D/2)(2\pi s)} \frac{p}{(2\pi s)} J_0(p) \left\{ \frac{dp}{2\pi s} \right\} \]

\[ = \left[ \frac{1}{(2\pi s^2)} \right] \int_0^{\pi D s} pJ_0(p) \, dp \]

From

\[ \int_0^x bJ_0(b) \, db = xJ_1(x) \]

\[ F \left\{ \text{circ} \left\{ r/(D/2) \right\} \right\} = \left[ \frac{1}{(2\pi s^2)} \right] \pi Ds J_1(\pi Ds) \]

\[ = \frac{(D/2) J_1(\pi Ds)}{s} \]

\[ F \left\{ \text{circ} \left\{ r/(D/2) \right\} \right\} = (D/2)^2 \pi \left[ 2J_1(\pi Ds)/(\pi Ds) \right] \]

\( (C.15) \)

Substituting C.15 and C.14 yields

\[ I(x_f, y_f) \propto \left[ A_p/(\lambda f) \right]^2 (D/2)^4 \pi \left[ 2J_1(\pi Ds)/(\pi Ds) \right]^2 \]

\( (C.16) \)
C.16 gives the intensity distribution due to a point source at the focal plane of a lens with a circular aperture, and is commonly known as the Airy disc, after G.B. Airy who first derived it.

Values of the Bessel function for the Airy disc have been tabulated in Table C1. Using these values, the cross section of the Airy disc is shown graphically in Figure C-4.

TABLE C1
VALUES OF THE BESSEL FUNCTION FOR AN AIRY DISC

<table>
<thead>
<tr>
<th>Ds</th>
<th>( [2J_1(\pi Ds)/(\pi Ds)]^2 )</th>
<th>significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1</td>
<td>central maximum</td>
</tr>
<tr>
<td>1.220</td>
<td>0</td>
<td>first zero</td>
</tr>
<tr>
<td>1.635</td>
<td>0.0175</td>
<td>second maximum</td>
</tr>
<tr>
<td>2.233</td>
<td>0</td>
<td>second zero</td>
</tr>
<tr>
<td>2.679</td>
<td>0.0042</td>
<td>third maximum</td>
</tr>
<tr>
<td>3.238</td>
<td>0</td>
<td>third zero</td>
</tr>
<tr>
<td>3.699</td>
<td>0.0016</td>
<td>fourth maximum</td>
</tr>
</tbody>
</table>
Figure C-4. Plot of \( [2J_1(\pi Ds)/(\pi Ds)]^2 \) for varying values of (Ds)
APPENDIX D

OPTICAL BACKGROUND NOISE CONSIDERATIONS

An infrared detector's sensitivity is usually characterized either by its spectral noise equivalent power or spectral D-star ($D^*$) which is the spectral detectivity normalized for detector area and band width dependance (Wolfe, 1965). These are defined as follows

Spectral Noise Equivalent Power ($\text{NEP}_\lambda$)

That value of monochromatic incident rms signal power of wavelength $\lambda$ required to produce an rms signal to rms noise ratio of unity. The chopping frequency, the electrical bandwidth used in the measurement, and the detector area should be specified. Units: Watts

Spectral Detectivity ($D_\lambda$)

The reciprocal of spectral noise equivalent power. The chopping frequency, the electrical bandwidth used in the measurement, and the detector sensitive area should be specified. Units: Watt$^{-1}$
Spectral D-Star (D*)

A normalization of spectral detectivity to take into account the area and electrical bandwidth dependence of the detector. Units: cm Hz$^{1/2}$ Watt$^{-1}$

Presently available infrared semiconductor detectors have typical spectral D-stars of $10^9$ to $10^{11}$ which convert to noise equivalent powers of $10^{-12}$ to $10^{-14}$ watts (Laser Focus Buyers' Guide, 1982).

Typical optical backgrounds in the infrared region due to clear sky, sunlit clouds, and terrain (Wolfe, 1965) are between 100 and 1000 micro watts cm$^{-2}$ ster$^{-1}$. Even for very low backgrounds, narrow detector fields of view ($10^{-8}$ ster) and small aperture areas (1 cm$^2$) the optical background noise at the detector is

$$(100 \mu \text{watts cm}^{-2} \text{ster}^{-1})(1 \text{cm}^2)(10^{-8} \text{ster}) (1 \text{watt}/10^6 \mu \text{watt}) = 10^{-12} \text{watts}$$

The field of view and aperture radius corresponding to these conditions are as follows
Field of view

\[ 10^{-8} \text{ ster} = \text{area subtended by a solid angle/distance}^2 \text{ for a conical solid with an apex angle of } (L/R). \]

\[ 10^{-8} \text{ ster} = \pi \left[ \frac{1}{2} (L/R) R \right]^2 / R^2 \]

\[ 10^{-8} \text{ ster} = \left( \frac{\pi}{4} \right) (L/R)^2 \]

\[ (L/R)^2 = \frac{4}{\pi} 10^{-8} \]

\[ (L/R) = 1.13 \times 10^{-4} \text{ radians} \]

\[ = .0065 \text{ degrees} \]

Aperture radius

\[ 1 \text{ cm}^2 = \pi r^2 \]

\[ r = \left( \frac{1 \text{ cm}^2}{\pi} \right)^{1/2} \]

\[ r = .56 \text{ cm} \]

Since most optical systems of practical seeker use would have larger detector fields of view or aperture areas it is reasonable to assume the optical background noise at the detector will be the predominant detector noise source.
APPENDIX E

\[ 2\pi \int_0^\infty \int_0^\infty [2J_1(ar)/(ar)]^2 r \, dr \, d\theta \]

In order to solve for the total signal power in any Airy disc, the Bessel function must be integrated.

From equation 6.574.2 (Gradshteyn and Ryzhik, 1980)

\[ \int_0^\infty J_v(at)J_u(at)t^{-b}dt = a^{b-1} \Gamma(b) \Gamma[(v+u-b+1)/2]/[2^b \Gamma[(-v+u+b+1)/2] \Gamma[(v+u+b+1)/2] \Gamma[v-u+b+1)/2]] \]

Provided

\( \text{Re}(v+u+1) > \text{Re} b , \ a > 0 \)

Now

\( v = u = 1, \ b = 1 \)

\[ \int_0^\infty [J_1(at)]^2/t \, dt = \Gamma(1) \Gamma(1)/[2 \Gamma(1) \Gamma(2) \Gamma(1)] \]
From equation 8.338.1 (Gradshteyn and Ryzhik, 1980)

\[ \Gamma(1) = \Gamma(2) = 1 \]

And

\[ \int_0^\infty \frac{[J_1(at)]^2}{t} dt = \frac{1}{2} \]

For the integrated Airy disc intensity function over an infinite area

\[
\begin{aligned}
&\int_0^{2\pi} \int_0^\infty \left[ 2J_1(ar)/(ar) \right]^2 r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^\infty \left( 4a^{-2}[J_1(ar)]^2/r \right) dr \, d\theta \\
&= 4a^{-2} \int_0^{2\pi} d\theta \int_0^\infty ([J_1(ar)]^2/r) \, dr \\
&= 4(2\pi)a^{-2}(1/2) \\
&= \pi (2/a)^2
\end{aligned}
\]
APPENDIX F

DERIVATION OF ERROR PROBABILITIES AND OPTIMUM THRESHOLD

To evaluate the probability of making an error given that signal is present, the probability density function with signal included must be integrated below the threshold, T.

Thus

\[ P(\text{error/signal}) = \int_0^T \left( \frac{v}{n^2} \right) \exp\left[\frac{-\left(\frac{A^2+v^2}{2n^2}\right)}{}\right] \, dv \]

\[ = \frac{1}{n^2} \exp\left[\frac{-A^2}{2n^2}\right] \int_0^T v \exp\left[\frac{-v^2}{2n^2}\right] I_0\left(\frac{Av}{n^2}\right) \, dv \]

let \( v = mT \) for \( v = 0, m = 0 \)

\( dv = Tdm \) for \( v = T, m = 1 \)
Then

\[ P(\text{error/signal}) = \frac{1}{n} \left( \frac{1}{n^2} \right) \exp\left[ -\frac{A^2}{2n^2} \right] \int_0^1 \exp\left[ -m^2 T^2/(2n^2) \right] \]

\[ I_0(AmT/n^2) T \, dT \]

\[ = \left( \frac{T^2}{n^2} \right) \exp\left[ -\frac{A^2}{2n^2} \right] \int_0^1 \exp\left[ -m^2 T^2/(2n^2) \right] I_0(AmT/n^2) \, dm \]

For Gradshteyn and Ryzhik (1980), equation 8.447.1

\[ I_0(AmT/n^2) = \sum_{j=0}^{\infty} \left[ \frac{AmT/(2n^2)}{j!} \right]^2 (2j+1)/(j!)^2 \]

So

\[ P(\text{error/signal}) = \left( \frac{T^2}{n^2} \right) \exp\left[ -\frac{A^2}{2n^2} \right] \]

\[ \int_0^1 \exp\left[ -m^2 T^2/(2n^2) \right] \, dm \]

\[ \sum_{j=0}^{\infty} \left[ \frac{AmT/(2n^2)}{j!} \right]^2 (2j+1)/(j!)^2 \, dm \]
\[ P(\text{error/signal}) = \left( \frac{T^2}{n} \right) \exp \left[ -A^2/(2n^2) \right] \]

\[
\sum_{j=0}^{\infty} \left[ \frac{A}{(2n^2)} \right]^{2j} \left[ \frac{1}{(j!)^2} \right]
\]

\[
\int_{m=0}^{m=1} \int_{m}^{m+1} \exp\left[ -m^2 T^2 / (2n^2) \right] m^j \, dm
\]

\[= \left( \frac{T^2}{n} \right) \exp \left[ -A^2/(2n^2) \right] \]

\[
\sum_{j=0}^{\infty} \left[ \frac{A}{(2n^2)} \right]^{2j} \left[ \frac{1}{(j!)^2} \right]
\]

\[
\int_{m=0}^{m=1} \int_{m}^{m+1} \exp\left[ -m^2 T^2 / (2n^2) \right] m^{j+1} \, dm
\]

Let \[ m^2 = t \quad \text{for} \quad m = 0, \quad t = 0 \]

\[ m = t^{1/2} \quad \text{for} \quad m = 1, \quad t = 1 \]

\[ dm = (1/2)t^{-1/2} \, dt \]
Then

\[ P(\text{error/signal}) = \left(\frac{T^2}{n^2}\right) \exp\left[-\frac{A^2}{(2n^2)}\right] \]

\[ = \sum_{j=0}^{\infty} \left[\frac{AT}{(2n^2)}\right]^{2j}/(j!)^2 \int_0^1 t(j+1/2) \exp\left[-\frac{tT^2}{(2n^2)}\right](1/2)t^{-1/2}dt \]

\[ = \left(\frac{T^2}{n^2}\right) \exp\left[-\frac{A^2}{(2n^2)}\right] \]

\[ \sum_{j=0}^{\infty} \left[\frac{AT}{(2n^2)}\right]^{2j}/(j!)^2 \]

\[ (1/2) \int_0^1 t^j \exp \left[-\frac{tT^2}{(2n^2)}\right]dt \]

From the Chemical Rubber Company Handbook (1970), equation 668

\[ \int_0^1 t^j \exp \left[-\left(\frac{T^2}{(2n^2)}\right)t\right] dt = (j!)/\left[\frac{T^2}{(2n^2)}\right]^{j+1} \]

\[ \left\{1- \exp \left[-\left(\frac{T^2}{(2n^2)}\right)\right]\right\} \sum_{i=0}^{j} \left[\frac{T^2}{(2n^2)}\right]^{i/(i!)}\} \]
Then

\[ P(\text{error/signal}) = (T^2/n^2) \exp[-A^2/(2n^2)] \sum_{j=0}^{\infty} \frac{[AT/(2n^2)]^j}{(j!)^2} \]

\[ = (1/2)(j!)/[T^2/(2n^2)] j+1 \left( 1 - \exp[-T^2/(2n^2)] \right) \]

\[ \sum_{j=0}^{\infty} \frac{[T^2/(2n^2)]^j}{(j!)^2} \]

\[ = \exp[-A^2/(2n^2)] \sum_{j=0}^{\infty} \frac{[A^2/(2n^2)]^j}{(j!)} \]

\[ \left\{ 1 - \exp[-T^2/(2n^2)] \right\} \sum_{i=0}^{\infty} \frac{[T^2/(2n^2)]^i}{(i!)} \]

\[ = \exp[-A^2/(2n^2)] \left\{ \sum_{j=0}^{\infty} \frac{[A^2/(2n^2)]^j}{(j!)} \right\} \]

\[ - \exp[-T^2/(2n^2)] \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{[A^2/(2n^2)]^j}{(j!)\cdot[i!]} \}

\[ [T^2/(2n^2)]^i/(i!) \right\} \] (F.1)
Since

\[ \sum_{j=0}^{\infty} \frac{[A^2/(2n^2)]^j}{j!} = \exp \left[ \frac{A^2}{2n^2} \right] \]  \hspace{1cm} (F.2) 

And letting \( j = m + i \)

\[ \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left[ (A^2/(2n^2))^j / (j!) \right] \left[ (T^2/(2n^2))^i / (i!) \right] \]

\[ \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \left[ (A^2/(2n^2))^m \right] \left[ (A^2/(2n^2))^i \right] \left[ (T^2/(2n^2))^i \right] \]

\[ \left[ ((m+i)!) / (i!) \right] \]

\[ \sum_{m=0}^{\infty} \left( A^2/(2n^2) \right)^m \sum_{i=0}^{\infty} \left[ AT/(2n^2) \right]^i \left[ ((m+i)!) / (i!) \right] \]
\[
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{(A^2/(2n^2))^{j}}{(j!)} \right] \left[ \frac{(T^2/(2n^2))^{i}}{(i!)} \right]
\]

\[
= \sum_{m=0}^{\infty} (A/T)^m \left[ \frac{AT/(2n^2)}{m} \right] \sum_{i=0}^{\infty} \left[ \frac{AT/(2n^2)}{2i} \right] \frac{1}{[(m+i)!(i!)^{2}]}
\]

\text{(F.3)}

From Gradshteyn and Ryzhik (1980), equation 8.445

\[
I_v(z) = \sum_{k=0}^{\infty} \frac{1}{(k! \Gamma(v+k+1))} (z/2)^{v+2k}
\]

\[
= (z/2)^v \sum_{k=0}^{\infty} (z/2)^{2k} \left[ \left( \frac{1}{(k!)((k+v)!)} \right) \right]
\]

\text{(F.4)}
Substituting F.4 into F.3

\[
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left( \frac{[A^2/(2n^2)]^j}{(j!)} \right) \left( \frac{T^2/(2n^2)]^i}{(i!)} \right)
\]

\[
= \sum_{m=0}^{\infty} (A/T)^m I_m(AT/n^2)
\]

(F.5)

Substituting (F.2) and (F.5) into (F.1)

\[
P(\text{error/signal}) = 1 - \exp \left\{ -\frac{(A^2 + T^2)}{(2n^2)} \right\} \sum_{m=0}^{\infty} \frac{(A/T)^m I_m(AT/n^2)}{(A/T)^m}
\]

(F.6)

Similarly, the probability of making an error given that noise is present is

\[
P(\text{error/noise}) = \int_T^{\infty} \frac{v^2}{n^2} \exp \left\{ -\frac{v^2}{(2n^2)} \right\} dv
\]
\[ P(\text{error/noise}) = -\exp\left[-\frac{v^2}{2n^2}\right] \bigg|_T \]

\[ P(\text{error/noise}) = \exp\left[-\frac{T^2}{2n^2}\right] \quad (F.7) \]

Now that expressions for the probability of error given signal or noise (F.6 and F.7) have been derived in terms of the threshold, the threshold for minimum error will be derived.

The optimum threshold is the one which will make the total probability for error minimum. Letting

\[ M = \text{probability of signal: } P(\text{signal}) \]
\[ Q = \text{probability of noise: } P(\text{noise}) \]

Then the total probability of error is

\[
P(\text{error}) = M \left\{ 1 - \exp \left[ -\frac{(A^2 + T^2)}{2n^2} \right] \right. \\
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{[A^2/(2n^2)]^j}{(j!)} \right] \left[ \frac{T^2/(2n^2)]^i}{(i!)} \right] \\
+ Q \exp \left[ -\frac{T^2}{2n^2} \right] \right\} \quad (F.8)
\]
To find the conditions for minimum error probability, differentiate with respect to $T$ and set the derivative equal to zero.

\[
\frac{d}{dT} P(\text{error}) = 0
\]

\[
0 = -M \left( -\frac{T}{n^2} \right) \exp\left[ -\frac{(A^2 + T^2)}{2n^2} \right] \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{A^2}{(2n^2)} \right]^j \frac{1}{j!} \left[ \frac{T^2}{(2n^2)} \right]^i \frac{1}{i!}
\]

\[
- M \exp \left[ -\frac{(A^2 + T^2)}{2n^2} \right] \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{A^2}{(2n^2)} \right]^j \frac{1}{j!} \left[ \frac{T^2}{(2n^2)} \right]^i \frac{1}{i!}
\]

\[
+ Q \left( -\frac{T}{n^2} \right) \exp \left[ -\frac{T^2}{2n^2} \right]
\]
Dividing both sides by \( \exp \left[ -\frac{T^2}{2n^2} \right] \frac{(QT/n^2)}{(QT/n^2)} \) yields

\[
0 = -1 + M \left( \frac{T}{n^2} \right) \left( \frac{n^2}{QT} \right) \exp \left[ -\frac{A^2}{2n^2} \right] \\
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{A^2}{2n^2} \right]^{j/(j!)\left[ \frac{T^2}{2n^2} \right]^{i/(i!)}\left[ \frac{1}{2n^2} \right]}^{j/(j!)} \\
- M \left( \frac{n^2}{QT} \right) \exp \left[ -\frac{A^2}{2n^2} \right] \\
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{A^2}{2n^2} \right]^{j/(j!)\left[ \frac{T^2}{2n^2} \right]^{i/(i!)}\left[ \frac{1}{2n^2} \right]}^{j/(j!)} \\
\left[ 2iT^{2i-1} \left[ \frac{1}{2n^2} \right]^{i/(i!)} \right]
\]

\[
0 = -1 + \left( \frac{M}{Q} \right) \exp \left[ -\frac{A^2}{2n^2} \right] \\
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{A^2}{2n^2} \right]^{j/(j!)\left[ \frac{T^2}{2n^2} \right]^{i/(i!)}\left[ \frac{1}{2n^2} \right]}^{j/(j!)} \\
- \left( \frac{M}{Q} \right) \left( \frac{2n^2}{T^2} \right) \exp \left[ -\frac{A^2}{2n^2} \right] \\
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{A^2}{2n^2} \right]^{j/(j!)\left[ \frac{T^2}{2n^2} \right]^{i/(i!)}\left[ \frac{1}{2n^2} \right]}^{j/(j!)} \\
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{A^2}{2n^2} \right]^{j/(j!)\left[ \frac{T^2}{2n^2} \right]^{i/(i!)}\left[ \frac{1}{2n^2} \right]}^{j/(j!)} \\
\left[ 2iT^{2i-1} \left[ \frac{1}{2n^2} \right]^{i/(i!)} \right] \\
1 = \left( \frac{M}{Q} \right) \exp \left[ -\frac{A^2}{2n^2} \right] \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{A^2}{2n^2} \right]^{j/(j!)\left[ \frac{T^2}{2n^2} \right]^{i/(i!)}\left[ \frac{1}{2n^2} \right]}^{j/(j!)} \\
\left[ 2iT^{2i-1} \left[ \frac{1}{2n^2} \right]^{i/(i!)} \right]^{1-i(2n^2/T^2)}
Multiplying both sides by \((Q/M) \exp[A^2/(2n^2)]\)

\[
(Q/M) \exp \left[ \frac{A^2}{2n^2} \right] = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \frac{[A^2/(2n^2)]^j}{(j!)^i} \right] [T^2/(2n^2)]^i
\]

\[
1 - i(2n^2/T^2)
\]

(F.9)

Expanding the right-hand side of F.9 for various values of \(j\)

\[
\begin{array}{c|c}
j & \text{Right-hand side expansion} \\
\hline
0 & 1 \\
1 & [A^2/(2n^2)][1+[T^2/(2n^2)][1-(2n^2/T^2)]] \\
& = [A^2/(2n^2)][1+[T^2/(2n^2)]-1] \\
& = (A^2/2n^2)(T^2/2n^2) \\
2 & \left[ \left[ \frac{A^2}{2n^2} \right]^2 / 2 \right] + [T^2/(2n^2)] - 1 + \left[ \left[ \frac{T^2}{2n^2} \right]^2 / 2 \right] \\
& \left[ 1-2(2n^2/T^2) \right] \\
& = \left[ \left[ \frac{A^2}{2n^2} \right]^2 / 2 \right] \left[ \left[ \frac{T^2}{2n^2} \right] + \left[ \frac{T^2}{2n^2} \right]^2 / 2 \right] - \\
& \left[ \frac{T^2}{2n^2} \right] \\
& = \left[ \left[ \frac{A^2}{2n^2} \right]^2 / 2 \right] \left[ \left[ \frac{T^2}{2n^2} \right]^2 / 2 \right]
\end{array}
\]
From this trend the right hand side of F.9 becomes

\[
\sum_{j=0}^{\infty} \sum_{i=0}^{j} \left[ \frac{A^2}{(2n^2)} \right]^j\frac{j!}{(j!)} \left[ \frac{T^2}{(2n^2)} \right]^i\frac{i!}{(i!)} \left[1-i(2n^2/T^2)\right]
\]

\[
= \sum_{j=0}^{\infty} \left[ \frac{A^2}{(2n^2)} \right]^j\frac{j!}{(j!)} \left[ \frac{T^2}{(2n^2)} \right]^j\frac{j!}{(j!)}
\]

\[
= \sum_{j=0}^{\infty} \left[ \frac{AT}{(2n^2)} \right]^{2j}\frac{(j!)^2}{(j!)}
\]

\[
= I_0(\frac{AT}{n^2}) \quad (F.10)
\]
The optimum threshold is then defined by substituting F.10 into F.9 to yield

\[ I_0(\frac{AT}{n^2}) = \frac{(Q/M) \exp[A^2/(2n^2)]}{(2n^2)} \]  \hspace{2cm} (F.11)

When the probability of signal and noise are equal this reduces to

\[ I_0(\frac{AT}{n^2}) = \exp \left[ \frac{A^2}{(2n^2)} \right] \] \hspace{2cm} (F.12)

Equations F.11 and F.12 give an expression for the optimum threshold \( T \) in terms of the signal to noise power ratio \((\frac{A^2}{n^2})\).

Attempts to substitute F.11 back into F.8 to find the detectivity at the optimum threshold have been unsuccessful in eliminating the threshold as a variable.
APPENDIX G

COMPUTER CALCULATION OF OPTIMUM THRESHOLD AND DETECTIVITY

To evaluate optimum threshold and detectivity, two computer programs were written in basic language and run on a TRS 80 model II computer.

The program of Table G1 finds the optimum threshold \( T \) as a function of the signal amplitude \( A \). The result is the optimum threshold to signal amplitude ratio \( m = T/A \).

The program first finds \( \exp \left[ \frac{A^2}{2n^2} \right] \) from the signal to noise ratio, then finds two values for \( m \) such that \( I_0(mA^2/n^2) \) for the two values fall above and below \( \exp \left[ \frac{A^2}{2n^2} \right] \). By successively trying values for \( m \) which fall between these two values, the upper and lower bounds are brought closer together. For verification, the output data of Table G2 included the probability of signal, signal to noise ratio, optimum threshold, and the number terms used to calculate \( I_0 \left( \frac{mA^2}{n^2} \right) \) using the infinite series expression

\[
I_0 \left( \frac{mA^2}{n^2} \right) = \sum_{j=0}^{\infty} \left[ \frac{mA^2}{2n^2} \right]^j \frac{2^j}{(j!)^2}
\]
TABLE G1

COMPUTER PROGRAM FOR CALCULATING OPTIMUM THRESHOLD IN TERMS OF THRESHOLD TO SIGNAL RATIO, GIVEN SIGNAL TO NOISE RATIO AND SIGNAL PROBABILITY

100 CLS: CLEAR
105 A$ = "###.### ↑↑↓ ↑↓↑↑↓↑ ♫.###.### ↑↑↓↑↓↑
     ###.### ↑↑↓↑ ♫.###.### ↑↑↓↑↓↑"
110 DEFDBL V: DEFINT I, K: DEFSTR Z
120 Z1 = "OPTIMUM DETECTOR THRESHOLD CALCULATION"
130 Z2 = "P(SIG) P(NOISE)/P(SIG) SNR OPTIMUM T/A
     I0(EST)/I0(CALC) INFINITE TERMS"
140 PRINT Z1
142 LPRINT Z1
144 LPRINT
150 LPRINT Z2: LPRINT
160 INPUT "HOW MANY DIFFERENT SIGNAL PROBABILITIES DO YOU WISH?"; K1
170 INPUT "HOW MANY DIFFERENT SNR'S DO YOU WISH?"; K2
180 DIM V1(K1 - 1), V2(K2 - 1)
190 PRINT "ENTER THE PROBABILITIES OF RECEIVING A SIGNAL
     (1ST, 2ND, ETC.)"
200 FOR I1 = 0 TO K1 - 1: INPUT V1(I1): NEXT I1
210 PRINT "ENTER THE SNR'S OF INTEREST (1ST, 2ND, ETC.)"
220 FOR I2 = 0 TO K2 - 1: INPUT V2(I2): NEXT I2
230 FOR I1 = 0 TO K1 - 1: FOR I2 = 0 TO K2 - 1
TABLE G1 - Continued

240 IF V1(I1) <= 0 THEN LPRINT "P(SIG)<0 NOT ALLOWED": LPRINT: NEXT I1

250 IF V1(I1) > 1 THEN LPRINT "P(SIG)>1 NOT ALLOWED": LPRINT: NEXT I1

260 IF V2(I2) <= 0 THEN LPRINT "SNR<=0 NOT ALLOWED": LPRINT: NEXT I2

270 VM = 1: VL = -1: VH = 0

280 VR = ((1-V1(I1))/V1(I1))*EXP(V2(I2)/2)

290 GOSUB 600

300 IF VN<VR THEN VL=VM ELSE 320

310 IF VH=0 THEN VM=2*VM: GOSUB 600: GOTO 300 ELSE 330

320 VM=VM: IF VL<0 THEN VM=VH/2: GOSUB 600: GOTO 300 ELSE 330

330 VM=(VL + VH)/2

335 IF ABS ((VN/VR) - 1) > .1 THEN GOSUB 600: GOTO 350

340 IF ABS ((VL/VH) - 1) < 1D-5 THEN 500 ELSE GOSUB 600

350 IF VN<VR THEN VL=VM ELSE VH=VM

360 GOTO 330

500 LPRINT: LPRINT USING A$: V1(I1); (1-V1(I1))/V1(I1); V2(I2); VM; VN/VR; VE

520 NEXT I2: NEXT I1: GOTO 130

600 VN=1: VE=1: VT=1: VD=(VM*V2(I2)/2)^2

610 VT=VT*VD/(VE^2)

620 VN=VN+VT
TABLE G1 - Continued

630 IF VN>VR THEN RETURN
632 IF VR < 1D7 THEN 640
634 IF VE < 25 THEN 650
640 IF ABS (VT/VR)<1D-7 THEN RETURN
645 IF VE>1D3 THEN 670
650 VE=VE+1 : GOTO 610
670 LPRINT "M="; VM; "DOES NOT CONVERGE"; "P(SIG)=";
       V1(I1);"SNR="; V2(I2)
680 GOTO 520
<table>
<thead>
<tr>
<th>SNR (A²/n²)</th>
<th>P(Signal) = .001</th>
<th>P(Signal) = .010</th>
<th>P(Signal) = .100</th>
<th>P(Signal) = .500</th>
</tr>
</thead>
<tbody>
<tr>
<td>.001</td>
<td>8904.5</td>
<td>6423.2</td>
<td>3735.2</td>
<td>44.72</td>
</tr>
<tr>
<td>.002</td>
<td>4452.5</td>
<td>3211.9</td>
<td>1867.9</td>
<td>31.63</td>
</tr>
<tr>
<td>.005</td>
<td>1781.3</td>
<td>1285.1</td>
<td>747.51</td>
<td>20.01</td>
</tr>
<tr>
<td>.010</td>
<td>890.93</td>
<td>642.81</td>
<td>374.05</td>
<td>14.15</td>
</tr>
<tr>
<td>.020</td>
<td>445.73</td>
<td>321.68</td>
<td>187.32</td>
<td>10.01</td>
</tr>
<tr>
<td>.050</td>
<td>178.61</td>
<td>129.00</td>
<td>75.28</td>
<td>6.34</td>
</tr>
<tr>
<td>.001</td>
<td>89.57</td>
<td>64.77</td>
<td>37.93</td>
<td>4.50</td>
</tr>
<tr>
<td>.200</td>
<td>45.05</td>
<td>32.66</td>
<td>19.26</td>
<td>3.20</td>
</tr>
<tr>
<td>.500</td>
<td>18.34</td>
<td>13.39</td>
<td>8.05</td>
<td>2.06</td>
</tr>
<tr>
<td>1.000</td>
<td>9.43</td>
<td>6.96</td>
<td>4.31</td>
<td>1.50</td>
</tr>
<tr>
<td>2.000</td>
<td>4.98</td>
<td>3.75</td>
<td>2.44</td>
<td>1.12</td>
</tr>
<tr>
<td>5.000</td>
<td>2.31</td>
<td>1.82</td>
<td>1.31</td>
<td>.82</td>
</tr>
<tr>
<td>10.000</td>
<td>1.41</td>
<td>1.17</td>
<td>.92</td>
<td>.69</td>
</tr>
<tr>
<td>20.000</td>
<td>.96</td>
<td>.85</td>
<td>.72</td>
<td>.61</td>
</tr>
<tr>
<td>50.000</td>
<td>.69</td>
<td>.64</td>
<td>.60</td>
<td>.55</td>
</tr>
<tr>
<td>100.000</td>
<td>.60</td>
<td>.58</td>
<td>.55</td>
<td>.53</td>
</tr>
</tbody>
</table>
Table G2 lists output values of the optimum threshold to signal ratio for various values of SNR and P(signal). The results are plotted in Figure G-1 showing the optimum threshold as a function of signal to noise ratio. The graph shows that for low SNR's and signal probabilities the optimum threshold is orders of magnitude above the signal amplitude which minimizes the false alarm probability at little cost to detectivity since there is a low probability of signal presence.

The actual variation of threshold voltage required for optimum detectivity is shown in Figure G-2 where the detector output voltage has been normalized to 1 for a noise only input. The noise has been held constant and increases in detector output voltage are assumed to be due to additional signal only. The signal to noise ratio corresponding to the total detector voltage is shown on a parallel scale.

The program of Figure G-3 finds the detectivity as a function of threshold and signal to noise ratio. The program first finds $I_0(m\text{AT}/n^2)$ and $I_1(m\text{AT}/n^2)$ from which the higher order modified Bessel functions are calculated using the recurrence relationship

$$I_{p+1}(z) = I_{p-1}(z) + \frac{2p}{z} I_p(z)$$
Figure G-1. Optimum threshold in terms of the threshold to signal amplitude ratio ($T/A$) as a function of signal to noise ratio ($A^2/n^2$) and signal probability
Figure G-2. Threshold voltage as a function of detector voltage (due to signal plus noise) assuming the noise power only produces a detector voltage of 1
COMPUTER PROGRAM FOR CALCULATING DETECTIVITY GIVEN THRESHOLD (IN TERMS OF T/A), SIGNAL TO NOISE RATIO, AND SIGNAL PROBABILITY

100  CLS: CLEAR
110  DEFSTR Z: DEFDBL V: DEFINT I, K
120  A$="##.###tttt  ###.####  ##.####tttt  ##.l### tttt 0
130  Z1 = "DETECTIVITY CALCULATION"
140  Z2 = "P(SIG)   SNR    T/A    P(E/S)
      P(E/N) DETECTIVITY TERMS I0(ARG) I0"
150  PRINT Z1: LPRINT TAB(20) Z1: LPRINT: LPRINT Z2
160  INPUT "HOW MANY DIFFERENT SIGNAL PROBABILITIES DO YOU WISH TO ENTER?"; K1
170  INPUT "HOW MANY DIFFERENT SNR'S DO YOU WISH TO ENTER?"; K2
180  INPUT "HOW MANY DIFFERENT THRESHOLDS DO YOU WISH TO ENTER?"; K3
190  DIM V1(K1-1), V2(K2-1), V3(K3-1)
200  PRINT "ENTER THE PROBABILITIES OF RECEIVING A SIGNAL (1ST, 2ND, ETC.)"
210  FOR I1=0 TO K1-1: INPUT V1(I1): NEXT I1
220  PRINT "ENTER THE SNR'S OF INTEREST"
230  FOR I2=0 TO K2-1: INPUT V2(I2): NEXT I2
240  PRINT "ENTER THE THRESHOLDS OF INTEREST IN TERMS OF M=T/A"
TABLE G3 - Continued

250 FOR I3=0 TO K3-1: INPUT V3(I3): NEXT I3
260 FOR I1=0 TO K1-1: LPRINT: FOR I2=0 TO K2-1: LPRINT: FOR I3=0 TO K3-1
270 VL=V3(I3)*V2(I2)/2
280 IF VL>84 THEN 500
290 VP=EXP(-((V3(I3)+2)+1)*V2(I2)/2)
300 V0=1: VT=1: VE=1: V4=VL: VX=1: VY=1
310 VT=VT*(VL/VE)^2
320 V0=V0+VT
330 VB=VT*VL/(VE+1)
340 V4=V4+VB
350 VR=V0
360 IF ABS(ABS(V0/VX)-1)<1D-24 AND ABS(ABS(V4/VY)-1)<1D-24 THEN 380
370 VE=VE+1: VX=V0: VY=V4: GOTO 310
380 V8=(V0+(1/V3(I3))*V4)*VP
390 VC=1: VZ=1
400 V5=V0-(VC/VL)*V4
405 IF V5<0 THEN VC=VC-1: V5=V4: GOTO 450
410 V8=V8+VP*((1/V3(I3))+ (VC+1))*V5
420 IF VZ=0 THEN IF VZ+V8=0 THEN 450 ELSE 440
430 IF ABS(ABS(V8/VZ)-1)<1D-4 THEN 450
440 V0=V4: V4=V5: VC=VC+1: VZ=V8: GOTO 400
450 VS=1-V8
TABLE G3 - Continued

460 \( VQ = \exp(-(V3(I3)^2 + 2)V2(I2)/2) \)

470 \( VD = 1 - (V5\times V1(I1)) + (VQ\times (1 - V1(I1))) \)

480 LPRINT: LPRINT USING A$;
   V1(I1); V2(I2); V3(I3); VS; VQ; VD; VE; 2*VL; VR; (VC + 1);
   V5; VP*((1/V3(I3))t (VC + 1))*V5

490 NEXT 13: NEXT 12: NEXT I1: GOTO 100

500 LPRINT: LPRINT "SNR/THRESHOLD COMBINATION TOO HIGH"
   SNR="; V2(I2); "M=T/A="; V3(I3)

510 GOTO 490
The Bessel functions are then multiplied by $\exp \left[-\frac{(A^2+T^2)}{(2n^2)}\right]$, the appropriate power of $(A/T)$ and added to find

$$\exp \left[-\frac{(A^2+T^2)}{(2n^2)}\right] \sum_{j=0}^{\infty} \frac{1}{(A/T)^j} I_j(\frac{AT}{n^2})$$

The probability of error given signal and noise is then calculated and used to find the detectivity. Included in the printout are the calculated values of the $I_0$ and the last modified Bessel function of the series summation. These values were compared with Abramowitz (1965) to ensure the calculations were reasonably accurate. The accumulation of error began to affect the calculations for extremely high SNR's ($\geq 50$).

The results of the calculations are tabulated in Table G4. Figure G-3 graphically displays the detectivity as a function of threshold in terms of $(T/A)$. Figure G-4 shows the detectivity at optimum threshold as a function of SNR. Detectivities of 0.6, 0.7, 0.8, 0.9 and 0.95 correspond to SNR's of approximately 1.3, 3.1, 5.7, 10.4 and 15.3, respectively.
TABLE G4

COMPUTER OUTPUT DATA FOR DETECTIVITY CALCULATION,
P(SIGNAL) = 0.5

<table>
<thead>
<tr>
<th>THRESHOLD (T/A)</th>
<th>DETECTIVITY</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SNR 0.5</td>
</tr>
<tr>
<td>.1</td>
<td>.50</td>
</tr>
<tr>
<td>.2</td>
<td>.50</td>
</tr>
<tr>
<td>.3</td>
<td>.50</td>
</tr>
<tr>
<td>.4</td>
<td>.50</td>
</tr>
<tr>
<td>.5</td>
<td>.51</td>
</tr>
<tr>
<td>.6</td>
<td>.51</td>
</tr>
<tr>
<td>.7</td>
<td>.51</td>
</tr>
<tr>
<td>.8</td>
<td>.52</td>
</tr>
<tr>
<td>.9</td>
<td>.52</td>
</tr>
<tr>
<td>1.0</td>
<td>.52</td>
</tr>
<tr>
<td>1.5</td>
<td>.54</td>
</tr>
<tr>
<td>2.0</td>
<td>.54</td>
</tr>
<tr>
<td>2.5</td>
<td>.54</td>
</tr>
<tr>
<td>3.0</td>
<td>.53</td>
</tr>
<tr>
<td>3.5</td>
<td>.52</td>
</tr>
<tr>
<td>4.0</td>
<td>.51</td>
</tr>
<tr>
<td>5.0</td>
<td>.50</td>
</tr>
<tr>
<td>10.0</td>
<td>.50</td>
</tr>
</tbody>
</table>
Figure G-3. Detectivity as a function of threshold \((T)\) for various SNR's \(P\) \(\text{signal} = 0.5\)
Figure G-4. Detectivity at optimum threshold as a function of SNR, $P(\text{signal}) = 0.5$
APPENDIX H

CALCULATION OF THE FRACTION OF TOTAL POINT SOURCE SIGNAL POWER REACHING A DETECTOR AS A FUNCTION OF THE DISTANCE FROM THE AIRY DISC CENTROID TO THE DETECTOR EDGE

The total signal power reaching the image plane is given by equation 1.2 as

\[
\text{Total Signal Power} = \left[ \frac{A_p}{(\lambda f)} \right]^2 C^2 \frac{(D/2)^4 \pi^2}{2\pi \int_0^\infty \int_0^\infty [2J_1(\frac{\pi Dr}{(\lambda f)}) / \left[ \frac{\pi Dr}{(\lambda f)} \right]^2] r \ dr \ d\theta}
\]

In order to evaluate the detectivity as the point source image approaches the detector boundary, the fraction of the Airy disc power that remains on the detector as the boundary is approached must be calculated.

Since

\[
[J_1(\frac{\pi Dr}{(\lambda f)}) / \left[ \frac{\pi Dr}{(\lambda f)} \right]^2]
\]

is circularly symmetric and

\[
r = \sqrt{x^2 + y^2}
\]
Then

\[
\int_0^\infty \int_0^\infty [2J_1(\frac{\pi Dr}{\lambda f})] [\frac{\pi Dr}{\lambda f}]^2 \, r \, dr \, d\theta
\]

\[
= \int_{-\infty}^\infty \int_{-\infty}^\infty [2J_1(\frac{\pi D\sqrt{x^2 + y^2}}{\lambda f})] \left[\frac{\pi D\sqrt{x^2 + y^2}}{\lambda f}\right]^2 \, dx \, dy
\]

The fraction of the signal power contained in the Airy disc up to a line at \( x = B \) is given by

\[
\frac{\%\text{Power}}{100} = \frac{\int_0^B \int_0^\infty [2J_1(\frac{\pi D\sqrt{x^2+y^2}}{\lambda f})] [\frac{\pi D\sqrt{x^2+y^2}}{\lambda f}]^2 \, dy \, dx}{\int_0^\infty \int_0^\infty [2J_1(\frac{\pi Dr}{\lambda f})] [\frac{\pi Dr}{\lambda f}]^2 \, r \, dr \, d\theta}
\]

Since the denominator's integral is given by equation E.1 as

\[
\pi \left[\frac{2\lambda f}{(\pi D)}\right]^2
\]
The fraction becomes

$$\frac{\% \text{Power}}{100} = \left[ \frac{\pi D/(2\lambda f)}{\pi} \right] \int_{-\infty}^{B} \int_{-\infty}^{\infty} [2J_1(\pi D\sqrt{x^2+y^2}/(\lambda f))]^2 \, dy \, dx$$

Noting that the integral on $y$ yields

$$\int_{-\infty}^{\infty} \left[ 2J_1(\pi D\sqrt{x^2+y^2}/(\lambda f))/[\pi D\sqrt{x^2+y^2}/(\lambda f)] \right]^2 \, dy$$

$$\int_{-\infty}^{\infty}$$

$$= 2 \int_{0}^{\infty} \left[ 2J_1(\pi D\sqrt{x^2+y^2}/(\lambda f))/[\pi D\sqrt{x^2+y^2}/(\lambda f)] \right]^2 \, dy$$

$$= 2 \int_{0}^{\infty} 4 \left[ J_1(\pi D\sqrt{x^2+y^2}/(\lambda f)) \right]^2 (\lambda f/(\pi D))^2 (x^2+y^2)^{-1} \, dy$$

Let

$$\pi Dx/(\lambda f) = p \quad \pi Dy/(\lambda f) = q$$

$$x^2 = [\lambda f/(\pi D)]^2 p^2 \quad y^2 = [\lambda f/(\pi D)]^2 q^2$$

$$dy = [\lambda f/(\pi D)] dq$$
Then
\[
\int_{-\infty}^{\infty} \left[ 2J_1\left(\frac{D}{\pi} \sqrt{x^2+y^2} / (\lambda f)\right) / \left[ \pi D \sqrt{x^2+y^2} / (\lambda f) \right] \right]^2 dy
\]

\[
= 8 \int_{0}^{\infty} [J_1(\sqrt{p^2+q^2})]^2 \lambda f / (\pi D) \left[ \pi D / (\lambda f) \right]^2 (p^2+q^2)^{-1} \lambda f / (\pi D) dq
\]

\[
= 8 [\lambda f / (\pi D)] \int_{0}^{\infty} [J_1(\sqrt{p^2+q^2})]^2 (p^2+q^2)^{-1} dq
\]

From Gradshteyn And Ryzhik (1980) equation 6.596.13

\[
\int_{0}^{\infty} \left[ J_v^2(\frac{a^2+x^2}{(a^2+x^2)^v}) \right] x^{2v-2} dx =
\]

\[
[\Gamma (v-(1/2)) / (2a^v+1(\sqrt{\pi}))] H_v(2a)
\]

For

\[ \text{Re} v > 1/2 \]

\[ H_v(2a) \text{ is the Sturve function} \]
For \( v = 1, a = p, x = q \), the integral on \( y \) in H.l becomes

\[
8\left(\frac{\lambda f}{\pi D}\right) \int_0^\infty \left[J_1\left(\sqrt{p^2+q^2}\right)\right]^2 (p^2+q^2)^{-1} \, dq
\]

\[
= 8\left(\frac{\lambda f}{\pi D}\right)\left(\frac{\Gamma(1/2)}{(2p^2\sqrt{\pi})}\right) H_1(2p)
\]

Since

\[
\Gamma(1/2) = \sqrt{\pi}
\]

And recalling \( p = \left[\pi D/(\lambda f)\right] x \)

\[
8\left(\frac{\lambda f}{\pi D}\right)(1/2)\left(\frac{\pi D/(\lambda f)}{2}\right)^{-2} x^{-2} H_1\left[2\pi D/(\lambda f)\right]
\]

\[
= 4\left(\frac{\lambda f}{\pi D}\right)^3 x^{-2} H_1\left[2\pi D/(\lambda f)\right]
\]

(H.2)
Substituting \( H.2 \) into \( H.1 \) yields

\[
\%\text{Power} = \frac{\left[ \frac{\pi D/(2 \lambda f)}{\pi} \right]^2}{100} \int_{-\infty}^{B} 4\left[ \frac{\lambda f/(\pi D)}{\pi} \right]^3 x^{-2} H_1[2 \pi Dx/(\lambda f)] dx
\]

\[
= \left[ \frac{\pi D/(2 \lambda f)}{\pi} \right]^2 \int_{-\infty}^{\infty} x^{-2} H_1[2 \pi Dx/(\lambda f)] dx + \int_{0}^{B} x^{-2} H_1[2 \pi Dx/(\lambda f)] dx + \int_{0}^{B} x^{-2} H_1[2 \pi Dx/(\lambda f)] dx \}
\]

Since \( H_1(z) = H_1(-z) \)

\[
= \left[ \frac{\lambda f/(\pi D)}{\pi} \right] \int_{0}^{\infty} x^{-2} H_1[2 \pi Dx/(\lambda f)] dx + \int_{0}^{B} x^{-2} H_1[2 \pi Dx/(\lambda f)] dx \}
\]

Let

\[
2 \pi Dx/(\lambda f) = m, \text{ then } x^{-2} = [2 \pi D/(\lambda f)]^2 m^{-2} dx = \left[ \frac{\lambda f/(2 \pi D)}{\lambda f} \right] dm
\]
Then solving the first integral in H.3

\[
\int_0^\infty x^{-2} H_1[2 \pi D/(\lambda f)] \, dx = \int_0^\infty [2 \pi D/(\lambda f)]^{2m-2} H_1(m) \, \left[ \lambda f/(2 \pi D) \right] \, dm
\]

\[
= [2 \pi D/(\lambda f)] \int_0^\infty m^{-2} H_1(m) \, dm
\]

From Gradshteyn and Ryzhik (1980) equation 6.813.2

\[
\int_0^\infty x^{\nu-1} H_\nu(x) \, dx = (2^{\nu-1} \pi)/\Gamma(\nu+1) \quad \text{for } \Re\nu>-3/2
\]

let \( \nu = 1, \, x = m \)

Then the first integral in H.3 becomes

\[
\int_0^\infty x^{-2} H_1[2 \pi D/(\lambda f)] \, dx = [2 \pi D/(\lambda f)] (2^{-2} \pi) / \Gamma(2)
\]

\[= (\pi/2) \left[ \pi D/(\lambda f) \right] \quad (H.4)\]
Substituting H.4 into H.3 yields

\[ \% \text{Power} = 100 \left\{ \left[ \lambda f/(\pi D) \right] / \pi \right\} \left\{ (\pi/2) \left[ \pi D/(\lambda f) \right] + \right. \]

\[ \frac{B}{\pi} \int_{0}^{B} x^{-2} H_1 (2 \pi D/(\lambda f)) \, dx \}

\[ = (1/2) + \left[ \left[ \lambda f/(\pi D) \right] / \pi \right] \int_{0}^{B} x^{-2} H_1 (2 \pi D/(\lambda f)) \, dx \]

Now consider the integral in H.5

\[ \frac{B}{\pi} \int_{0}^{B} x^{-2} H_1 (2 \pi D/(\lambda f)) \, dx \]

From the series expansion of \( H_1 \)

\[ H_1 (z) = \sum_{m=0}^{\infty} (-1)^{m} (z/2)^{2m+2} / \left( \Gamma(m+3/2) \Gamma(m+5/2) \right) \]
Substituting the series expansion for $H_1$ into the integral of equation $H.5$ yields

$$
\int_{0}^{B} x^{-2} H_1 \left[ 2 \pi \frac{Dx}{(\lambda f)} \right] \ dx
$$

$$
= \int_{0}^{B} x^{-2} \sum_{m=0}^{\infty} (-1)^m \left[ 2 \pi \frac{Dx}{(2 \lambda f)} \right]^{2m+2} / \left[ \Gamma \left( m+\left( \frac{3}{2} \right) \right) \Gamma \left( m+\left( \frac{5}{2} \right) \right) \right] \ dx
$$

$$
= \sum_{m=0}^{\infty} (-1)^m \left[ \pi \frac{D}{(\lambda f)} \right]^{2m+2} / \left[ \Gamma \left( m+\left( \frac{3}{2} \right) \right) \Gamma \left( m+\left( \frac{5}{2} \right) \right) \right]
$$

$$
\int_{0}^{B} x^{-2} x^{2m+2} \ dx
$$

$$
= \sum_{m=0}^{\infty} (-1)^m \left[ \pi \frac{D}{(\lambda f)} \right]^{2m+2} / \left[ \Gamma \left( m+\left( \frac{3}{2} \right) \right) \Gamma \left( m+\left( \frac{5}{2} \right) \right) \right]
$$

$$
\left[ \frac{1}{(2m+1)} \right] x^{2m+1} \bigg|_{0}^{B}
$$
\[ \int_{\frac{B}{x}} x^{-2} \frac{H_1[2 \pi Dx/(\lambda f)]}{\pi} \, dx \]

\[ = \sum_{m=0}^{\infty} (-1)^m \left[ \frac{\pi D/(\lambda f)}{\Gamma(m+3/2) \Gamma(m+5/2)} \right]^{2m+2} \]

\[ \times \left[ \frac{1}{(2m+1)} \right] (B^{2m+1}) \]  

(H.6)

Substituting H.6 into H.5

\[ \%\text{Power} = \frac{1}{100} \left( \frac{1}{2} + \left[ \frac{\lambda f}{\pi D} \right] \sum_{m=0}^{\infty} (-1)^m \left[ \frac{\pi D/(\lambda f)}{\Gamma(m+3/2) \Gamma(m+5/2)} \right]^{2m+2} \right. \]

\[ \left. \times \frac{B^{2m+1}}{(2m+1) \Gamma(m+3/2) \Gamma(m+5/2)} \right] \]

\[ \%\text{Power} = \frac{1}{100} \left( \frac{1}{2} + \frac{1}{\pi} \sum_{m=0}^{\infty} (-1)^m \left[ \frac{\pi DB/(\lambda f)}{\Gamma(m+3/2) \Gamma(m+5/2)} \right]^{2m+1} \right. \]

\[ \left. \times \left[ (2m+1) \Gamma(m+3/2) \Gamma(m+5/2) \right] \right) \]  

(H.7)

A computer program (Table H1) was written and used to evaluate the series in H.7. The results are tabulated in Table H2 and plotted in Figure H-1.
TABLE H1

COMPUTER PROGRAM FOR CALCULATING THE FRACTION OF AIRY DISC POWER ON A DETECTOR WHEN THE AIRY DISC CENTROID IS A DISTANCE B (IN TERMS OF λ Ef/D) FROM THE EDGE OF THE DETECTOR.

100 CLS: CLEAR
110 DEFSTR Z: DEFDBL V: DEFI NT I, K
120 A$="##.### ##. #### tttt ##. #### tttt
    #.##### ↑↑↑↑ #.##### ↑↑↑↑ #.##### #
    #.##### ↑↑↑↑"
130 Z1="PERCENT ENERGY CALCULATION"
140 Z2="DIST(WLF/D) % SIG RECEIVED % SIG LOST TERMS LAST TERM"
150 PRINT Z1; LPRINT TAB(20) Z1: LPRINT: LPRINT Z2
160 INPUT "HOW MANY DIFFERENT DISTANCES DO YOU WISH TO ENTER?"; K1
170 DIM V1 (K1-1)
180 PRINT "ENTER THE DISTANCES OF INTEREST IN TERMS OF WLF*F/D"
190 FOR IL=0 TO K1-1: INPUT V1(IL): NEXT IL
200 FOR IL=0 TO K1-1
210 V2=3.1415927
220 VE=1: VU=V1(IL)/2: VN=(1/V2)*VU/(((1/2)^3)*(3/2))
230 VU=VU*((V2*V1(IL))/(((2*VE)-1)/2))↑2
240 VT=(1/V2)*VU*((-1)^VE)/(((2*VE)+1)/2)^3)
    *(((2*VE)+3)/2))
TABLE H1 - Continued

250 VN=VN+VT
260 IF ABS(VT)<1D-8 THEN 300
270 VE=VE+1: GOTO 230
300 LPRINT USING A$: VI(IL); ((1/2)+VN); ((1/2)-VN); VE; VT
310 NEXT IL: GOTO 100
### TABLE H2

**COMPUTER OUTPUT DATA FOR FRACTION OF AIRY DISC POWER ON A DETECTOR**

<table>
<thead>
<tr>
<th>Distance of Airy Disc Centroid from Detector Edge in units of (λf/D)</th>
<th>Fraction of the total Airy Disc Power Reaching the Detector</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Airy Disc Centroid Off Detector</td>
</tr>
<tr>
<td>.1</td>
<td>.42</td>
</tr>
<tr>
<td>.2</td>
<td>.34</td>
</tr>
<tr>
<td>.3</td>
<td>.26</td>
</tr>
<tr>
<td>.4</td>
<td>.20</td>
</tr>
<tr>
<td>.5</td>
<td>.15</td>
</tr>
<tr>
<td>.6</td>
<td>.12</td>
</tr>
<tr>
<td>.7</td>
<td>.09</td>
</tr>
<tr>
<td>.8</td>
<td>.08</td>
</tr>
<tr>
<td>.9</td>
<td>.07</td>
</tr>
<tr>
<td>1.0</td>
<td>.06</td>
</tr>
<tr>
<td>1.5</td>
<td>.04</td>
</tr>
<tr>
<td>2.0</td>
<td>.03</td>
</tr>
<tr>
<td>2.5</td>
<td>.03</td>
</tr>
<tr>
<td>3.0</td>
<td>.02</td>
</tr>
<tr>
<td>3.5</td>
<td>.02</td>
</tr>
<tr>
<td>3.6</td>
<td>.01</td>
</tr>
</tbody>
</table>
Figure H-1. Fraction of the total point source signal power reaching a detector as a function of the distance (in terms of $\lambda f/D$) from the Airy disc centroid to the detector edge.
As the plot in Figure H-1 shows, only 5% of the signal energy in the Airy disc falls across a line $1.4(\lambda f/D)$ from its centroid and over 80% of the signal energy remains on the detector until the Airy disc centroid is within $.4(\lambda f/D)$ of the detector's edge. At the opposite extreme, if the Airy disc is at least $3.7\lambda f/D$ from the detector edge virtually no signal power falls across the detector boundary.

In order to get a closed form solution consider

$$
\int_{0}^{B} x^{-2} H_{1}[2\pi Dx/(\lambda f)] \, dx
$$

Let $x = Bm$

$dx = Bdm$

for $x = 0, m = 0$

for $x = B, m = 1$
Then

\[
B \int_{0}^{1} x^{-2} H_1[2 \pi Dx/(\lambda f)] \, dx
\]

\[
= \int_{0}^{1} (Bm)^{-2} H_1[2 \pi DBm/(\lambda f)] \, Bdm
\]

\[
= B^{-1} \int_{0}^{1} m^{-2} H_1[2 \pi DBm/(\lambda f)] \, dm
\]

Let \( b = 2 \pi DB/(\lambda f) \)

\[
= B^{-1} \int_{0}^{1} m^{-2} H_1(bm) \, dm
\]

Let \( m = \sqrt{x} \)

\[
\frac{dm}{dx} = \frac{1}{2} x^{-1/2}
\]

\[
dm = (1/2) x^{-1/2} \, dx
\]
Then

\[
\int_0^1 x^{-2} H_1 [2 \pi Dx/(\lambda f)] \, dx
\]

\[
= B^{-1} \int_0^1 x^{-1} H_1 (b \sqrt{x}) \, (1/2) \, x^{-1/2} \, dx
\]

\[
= B^{-1} \int_0^1 x^{-3/2} H_1 (b \sqrt{x}) \, dx
\]

From Gradshteyn and Ryzhik (1980), equation 6.815.2

\[
\int_0^1 x^{d-(1/2)v-3/2} (1-x)^{u-1} H_v(a \sqrt{x}) \, dx
\]

\[
= [B(d,u)a^{v+1}/(2\sqrt{\pi} \Gamma(v+(3/2)))]
\]

\[
\text{2F}_3[1,d;3/2,v+(3/2),d+u;-(a^2/4)]
\]

[Re d>0, Re u>0], B(d,u) = the beta function

\[
\text{2F}_3 (a_1, a_2; b_1, b_2, b_3; z) = \text{the generalized hypergeometric series}
\]
For $v = 1$, $a = b$, $d = 1/2$, $u = 1$

\[
\int_0^1 x^{-2} H_1 \left[ 2 \pi \frac{Dx}{(\lambda f)} \right] dx = (1/2)B^{-1} \int_0^1 x^{-3/2} H_1 \left( b\sqrt{x} \right) dx
\]

\[
= (1/2)B^{-1}[B(1/2,1) b^2/
\left[ 2\sqrt{\pi} \Gamma \left( 5/2 \right) \right]]
\]

\[
2F_3[1,(1/2);(3/2),(5/2),(3/2);
-(b^2/4)]
\]

From Gradshteyn and Ryzhik (1980), equation 8.384.1

\[
B(x,y) = \Gamma (x) \Gamma (y)/\Gamma (x+y)
\]

So

\[
\int_0^1 x^{-2} H_1 \left[ 2 \pi \frac{Dx}{(\lambda f)} \right] dx =
\]

\[
= (1/2)B^{-1}[ \Gamma (1/2) \Gamma (1)/ \Gamma (3/2)]
\]

\[
\left[ b^2/[2\sqrt{\pi} \left( 3/2 \right)(1/2) \Gamma (1/2)] \right]
\]

\[
2F_3[1,(1/2);(3/2),(5/2),(3/2);(-b^2/4)]
\]

\[
= B^{-1}(2/3 \pi )b^2
\]

\[
2F_3[1,(1/2);(3/2),(5/2),(3/2);-(b^2/4)]
\]
Where $b = (B2 \pi D/(\lambda f))$

Thus

$$B \int x^{-2} H_1[2 \pi Dx/(\lambda f)] \, dx$$

$$= \left[\frac{2}{(3 \pi)}\right] [B2 \pi D/(\lambda f)]^2 B^{-1}$$

$$2F_3[1, (1/2); (3/2), (5/2), (3/2); - [\pi DB/(\lambda f)]^2]$$

And substituting into H.5

$$\%\text{Power} = \frac{(1/2) + \left[\frac{\lambda f/(\pi D)}{\pi} \right] [2/(3 \pi)] B^{-1} [B2 \pi D/(\lambda f)]^2}{100}$$

$$2F_3[1, (1/2); (3/2), (5/2), (3/2); - [\pi DB/(\lambda f)]^2]$$

$$= (1/2) + \left[\frac{8}{(3 \pi)}\right] [BD/(\lambda f)]$$

$$2F_3[1, (1/2); (3/2), (5/2), (3/2); - [\pi DB/(\lambda f)]^2]$$

(Equation H.8)

Equation H.8 is the closed form equivalent of equation H.7
APPENDIX I

DETERMINATION OF THE FIELD OF VIEW CONTAINING COHERENT OPTICAL NOISE FOR TWO APERTURES

To determine the field of view containing coherent noise for two apertures separated by a distance \((D/N_g)\) first find the correlation between points in the field of view.

Following the development of Fowles (1968) as illustrated in Figure I-1, the correlation of the two receiving points \((P_1\) and \(P_2\)) is given by

\[ \gamma_{12}(\tau) = \frac{<E_1(t)E_2^*(t+\tau)>}{\sqrt{I_1 I_2}} \]

Where
- \(E_1\) = electric field strength at point 1
- \(E_2\) = electric field strength at point 2
- \(I_1 = <|E_1|^2>\)
- \(I_2 = <|E_2|^2>\)
- \(\tau\) = variable time delay
- \(E^*\) = the complex conjugate of \(E\)
- \(<|E|^2>\) = the time average of the magnitude of the square of the electric field

If \(S_A\) and \(S_B\) are two mutually incoherent point sources then

\[ E_1 = E_{1a} + E_{1b} \]
\[ E_2 = E_{2a} + E_{2b} \]
Figure I-1. Illustration for computing the angular field of view of coherent noise
Where

\[ E_{1a} = \text{contribution from } S_A \text{ to the electric field strength at point 1} \]

\[ E_{1b} = \text{contribution from } S_B \text{ from the electric field strength at point 1} \]

\[ E_{2a} = \text{contribution from } S_A \text{ to the electric field strength at point 2} \]

\[ E_{2b} = \text{contribution from } S_B \text{ to the electric field strength at point 2} \]

Then

\[ \gamma_{12}(\tau) = \frac{\left| E_{1a}(t) + E_{1b}(t) \right| \left| E_{2a}^*(t+\tau) + E_{2b}^*(t+\tau) \right|}{\sqrt{I_1 I_2}} \] (I.1)

Assume the detector system has a narrow band filter so only one optical frequency is of interest and that the noise power is uniformly distributed through the field of view. With noise uniformly distributed through the field of view the coherence may be analyzed for points in a plane at various distances \( R \) from the plane of points \( P_1 \) and \( P_2 \). The only difference between the planes will be the optical attenuation of the noise intensity which will be the same for all the noise point sources in the plane. Since the noise is uniformly distributed, and the noise point sources are mutually incoherent they may be represented by
\[ \begin{align*}
S_A &= N_0 \exp[-i\omega t] \exp[i \phi_a(t)] \\
S_B &= N_0 \exp[-i\omega t] \exp[i \phi_b(t)]
\end{align*} \]

After being attenuated over a distance \( R \) these become

\[ \begin{align*}
E_{1a}(t) &= N_R \exp[-i\omega t] \exp[i \phi_a(t)] \\
E_{2a}(t) &= N_R \exp[-i\omega(t+(r_{2a}-r_{1a})/C)] \\
& \quad \exp[i \phi_a(t+(r_{2a}-r_{1a})/C)] \\
E_{1b}(t) &= N_R \exp[-i\omega t] \exp[i \phi_b(t)] \\
E_{2b}(t) &= N_R \exp[-i\omega(t+(r_{2b}-r_{1b})/C)] \\
& \quad \exp [i \phi_b(t+(r_{2b}-r_{1b})/C)]
\end{align*} \]  

(1.2)

Where

\begin{align*}
\exp(i\omega t) &= \text{sinusoidally varying component of optical noise} \\
\exp[i \phi(t)] &= \text{randomly varying phase of optical noise} \\
r_{2a}-r_{1a} &= \text{path length difference between points 1 and 2 from source A} \\
r_{2b}-r_{1b} &= \text{path length difference between point 1 and 2 from the source b} \\
C &= \text{speed of light}
\end{align*}
The time it will take the optical signal reaching point 2 to travel to point 1 be

\[(P_2-P_1)/C\]

Where \(C\) is the speed of light.

Since we are interested in how the correlation changes as the separation between \(P_2\) and \(P_1\) changes let

\[\tau = (P_2-P_1)/C\]  \hspace{1cm} (I.3)

Now

\[\gamma_{12}(\tau) = E_{1a}(t)E_{2a}^*(t+\tau)+E_{1b}(t)E_{2b}^*(t+\tau) + E_{1a}(t)E_{2b}^*(t+\tau)+E_{1b}(t)E_{2a}^*(t+\tau)\]  \hspace{1cm} (I.4)

The last two terms are zero since \(S_A\) and \(S_B\) are mutually incoherent and therefore uncorrelated. Substituting I.2 into I.4 yields

\[\gamma_{12}(\tau) = \langle NR\exp[-i\omega t]\exp[i\phi_a(t)]\rangle NR\exp[i\omega (t+(r_{2a}-r_{1a})/C+\tau)]\]

\[\exp[-i\phi_a(t+(r_{2a}-r_{1a})/C+\tau)]/\sqrt{I_1I_2}\]

\[+\langle NR\exp[-i\omega t]\exp[i\phi_b(t)]\rangle NR\exp[i\omega (t+(r_{2b}-r_{1b})/C+\tau)]\]

\[\exp[-i\phi_b(t+(r_{2b}-r_{1b})/C+\tau)]/\sqrt{I_1I_2}\]
\[ \gamma_{12}(\tau) = <N_R^2 \exp[iw((r_{2a}-r_{1a})/C+\tau)] \]
\[ \cdot \exp[i(\phi_a(t)-\phi_a(t+(r_{2a}-r_{1a})/C+\tau))] > <\sqrt{I_1 I_2} \]
\[ + <N_R^2 \exp[iw((r_{2b}-r_{1b})/C+\tau)] \]
\[ \cdot \exp[i(\phi_b(t)-\phi_b(t+(r_{2b}-r_{1b})/C+\tau))] > <\sqrt{I_1 I_2} \]  
\( (I.5) \)

Now let

\[ \exp[i(\phi_a(t)-\phi_a(t+(r_{2a}-r_{1a})/C+\tau))] = \exp[i \Delta_a(t)] \]
\[ \exp[i(\phi_b(t)-\phi_b(t+(r_{2b}-r_{1b})/C+\tau))] = \exp[i \Delta_b(t)] \]

Where \( \Delta_a \) and \( \Delta_b \) are random phase differences.

The sources \( S_A \) and \( S_B \) will have some characteristic coherence time \( \tau_0 \) during which the optical signal originating from the source will have a constant phase. After this time (\( \tau_0 \)) the phase will change abruptly and randomly to a new phase. The time average of the variable terms in the numerators of I.5 is then found from
\[ \frac{1}{\tau_0} \int_0^{\tau_0} \exp[i(\phi_a(t) - \phi_a(t + (r_2a-r_1a)/C + \tau))] \]

\[ = \left( \frac{1}{\tau_0} \right) \int_0^{\tau_0} dt \]

\[ + \left( \frac{1}{\tau_0} \right) \int_{\tau_0 - [(r_2a-r_1a)/C + \tau]}^{\tau_0} \exp[i \Delta a(t)] dt \]

Where the first integral represents the time period of the light traveling from point A over the same path length to points 1 and 2 (and therefore arriving at the same time resulting in no phase difference) and the second integral represents the time required to travel the additional path length from point 1 to point 2 (during which a random phase change may occur).

Since the time average of \( \exp[i \Delta a(t)] \) is zero the integral will yield

\[ \frac{1}{\tau_0} \int_0^{\tau_0} \exp[\phi_a(t) - \phi_a(t + (r_2a-r_1a)/C + \tau))] dt \]

\[ = \tau_0 - [(r_2a-r_1a)/C + \tau)]/ \tau_0 \] (I.6)
Similarly

\[ \frac{1}{\tau_0} \int_0^{\tau_0} \exp[\phi \, b(t) - \phi \, b(t+(r_2 b-r_1 b)/C+\tau)] \, dt \]

\[ = \tau_0 - [(r_2 b-r_1 b)/C+\tau]/\tau_0 \quad \text{(I.7)} \]

Now

\[ \sqrt{I_1 I_2} = \sqrt{\langle |E_1|^2 \rangle \langle |E_2|^2 \rangle} \]

\[ = \sqrt{\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle} \]

\[ = \sqrt{\langle (E_{1a} + E_{1b})(E_{1a}^* + E_{1b}^*) \rangle} \]

\[ \sqrt{\langle (E_{2a} + E_{2b})(E_{2a}^* + E_{2b}^*) \rangle} \]

\[ = \sqrt{\langle (E_{1a} E_{1a}^* + E_{1a} E_{1b}^* + E_{1a}^* E_{1b} + E_{1b} E_{1b}^*) \rangle} \]

\[ \sqrt{\langle (E_{2a} E_{2a}^* + E_{2a} E_{2b}^* + E_{2a}^* E_{2b} + E_{2b} E_{2b}^*) \rangle} \]
Substituting from I.2

\[ \sqrt{I_1I_2} = \sqrt{<NR^2 + NR^2 \exp[i \phi_a(t)] \exp[-i \phi_b(t)]} \]

\[ + NR^2 \exp[-i \phi_a(t)] \exp[i \phi_b(t)] + NR^2 > \]

\[ \sqrt{<NR^2 + NR^2 \ exp(i \phi_a(t)) \exp(-i \ \phi_b(t))} \]

\[ NR^2 \exp[-i \ \phi_a(t)] \exp[i \ \phi_b(t)] + NR^2 > \]

\[ \sqrt{I_1I_2} = NR^2 <2 + \exp[i(\phi_a(t) - \phi_b(t))] \]

\[ + \exp[i(\phi_b(t) - \phi_a(t))] > \]

\[ \sqrt{I_1I_2} = NR^2 <2 + 2 \cos [\phi_a(t) - \phi_b(t)] > \]

The right hand side time averages to \(2NR^2\), so

\[ \sqrt{I_1I_2} = 2NR^2 \] (I.8)

Simplifying I.5 by substituting in I.6, I.7, and I.8

\[ \gamma_{12}(\tau) = NR^2 [\exp[iw((r_{2a} - r_{1a})/C + \tau)]/(2NR^2)] \]

\[ [[(\tau_0 - (r_{2a} - r_{1a})/C + \tau)]/\tau_0] \]

\[ + NR^2 [\exp[iw((r_{2b} - r_{1b})/C + \tau)]/(2NR^2)] \]

\[ [[(\tau_0 - (r_{2b} - r_{1b})/C + \tau)]/\tau_0] \]
Let

\[
\frac{(r_{2a} - r_{1a})/C}{\tau} = \tau_a \\
\frac{(r_{2b} - r_{1b})/C}{\tau} = \tau_b 
\]

\[
\gamma_{12}(\tau) = (1/2)\exp(iw\tau_a)[(\tau_o - \tau_a)/\tau_o] + (1/2)\exp(iw\tau_b)
\]

\[
[[(\tau_o - \tau_b)/\tau_o]
\]

From Figure I-1 \( \tau_b = \tau_a \)

Since \( r_{1a} \approx r_{1b} \approx r_{2a} \approx r_{2b} \approx R \)

Then

\[
|\gamma_{12}(\tau)|^2 = (1/2)^2\left|[(\tau_o - \tau_a)/\tau_o]\exp(iw\tau_a) + [(\tau_o + \tau_b)/\tau_o]\exp(iw\tau_b)\right|^2
\]

\[
= (1/2)^2\left[((\tau_o - \tau_a)/\tau_o)^2 + ((\tau_o - \tau_b)/\tau_o)^2 + ((\tau_o - \tau_a)/\tau_o)((\tau_o - \tau_b)/\tau_o) \right.
\]

\[
\left. (\exp(iw(\tau_b - \tau_a)) + \exp(iw(\tau_b - \tau_a)))\right]
\]

\[
\approx (1/2)^2((\tau_o - \tau_a)/\tau_o)((\tau_o - \tau_b)/\tau_o)
\]

\[
[2 + 2\cos[w(\tau_b - \tau_a)]
\]

\[
|\gamma_{12}(\tau)|^2 \approx (1/2)(1 - \tau_a/\tau_o)(1 - \tau_b/\tau_o)[1 + \cos[w(\tau_b - \tau_a)]]
\]
Now the coherence will be zero (and therefore the noise at points 1 and 2 completely uncorrelated) when

\[
\cos \left( (\tau b - \tau a) \right) = -1 \text{ or } w(\tau b - \tau a) = \pm \pi \quad (I.9)
\]

From Figure I-1 assume \( r_{2b} = r_{1b} \)

Then

\[
\tau b - \tau a = (r_{1a} - r_{2a}) / C
\]

\[
= (1/C)(r_{1a} - r_{2a})
\]

\[
\tau b - \tau a = (1/C) \left[ \sqrt{R^2 + [L_c + (D/2N_S)]^2} - \sqrt{R^2 + [L_c - (D/2N_S)]^2} \right]
\]

\[
= (R/C) \left[ \sqrt{1 + ([L_c + (D/2N_S)]/R)^2} - \sqrt{1 + ([L_c - (D/2N_S)]/R)^2} \right]
\]

\[
= (R/C) \left[ 1 + (1/2)([L_c + (D/2N_S)]/R)^2 \right]
\]

\[
-1 - (1/2)([L_c + (D/2N_S)]/R)^2
\]

\[
= (1/2)(R/C)(1/R^2) [L_c^2 + 2(L_cD/2N_S) + (D/2N_S)^2]
\]

\[
-L_c^2 + 2(L_cD/2N_S) - (D/2N_S)^2
\]
\[ \tau_b - \tau_a = \frac{1}{2}(R/C)(1/R^2)(4L_cD/2N_S) \]

\[ = \frac{L_c}{R}(D/N_S)(1/C) \]  

(I.10)

Substituting I.10 into I.9:

\[ w(\tau_b - \tau_a) = \left(2 \pi C/\lambda\right)\left(\frac{L_c}{R}(D/N_S)(1/C) = \pm \pi \right. \]

\[ (L_c/R) = \pm \left[ \lambda/(D/N_S) \right](1/2) \]

Since \(L_c/R\) is the angular separation between two sources which are coherent, to account for the full angular spread for coherence the positive and negative directions must be added and the coherence angle \(\Theta_c\) will be given by

\[ \Theta_c = \frac{L_c}{R} = \frac{\lambda}{(D/N_S)} \]  

(I.11)

Equation I.11 gives the angle of the field of view which contains coherent noise.
BIBLIOGRAPHY


