Modal Analysis by State Space Approach in Frequency Domain

1983

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MODAL ANALYSIS BY STATE
SPACE APPROACH IN FREQUENCY DOMAIN

BY

FELICIANO FEIJO
B.S.M.E., Ricardo Palma University, 1973

RESEARCH REPORT

Submitted in partial fulfillment of the requirements
for the degree of Master of Science
in the Graduate Studies Program of the College of Engineering
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1983
ABSTRACT

This report tests a new procedure to identify the eigenvalue problem matrix "A" by utilizing the frequency response of a system in state-space for a given sinusoidal forcing input.

Two sample identification tests are performed and a detailed explanation of both cases is presented. The method proved to be correct and the results obtained were accurate in both cases. A random error was then added to both sample tests in order to simulate real measuring conditions. Results indicate that it is more difficult to identify the "A" matrix of an undamped system. The error in the system must be very small for this type of system to be identified. However, when the identification is repeated several times and then averaged, the identification becomes more accurate.
ACKNOWLEDGEMENTS

The author wishes to express his gratitude and special thanks to Dr. S. M. Metwalli for his guidance and support throughout this research report.

Thanks are also extended to Dr. D. Jenkins and Dr. H. Klee for their helpful comments, review of the report and for serving on my committee.

Last but not least, my sincere appreciation and thanks to Mrs. Sharon Darling for her expert accomplishment of typing this report.
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<td>A</td>
<td>System's matrix in state space equation</td>
</tr>
<tr>
<td>B</td>
<td>Matrix with the added random error</td>
</tr>
<tr>
<td>b₁</td>
<td>Forcing input (\sin wt)</td>
</tr>
<tr>
<td>b₂</td>
<td>Forcing input (\cos wt)</td>
</tr>
<tr>
<td>C</td>
<td>Damping matrix</td>
</tr>
<tr>
<td>cᵢⱼ</td>
<td>Element of matrix C</td>
</tr>
<tr>
<td>D</td>
<td>Diagonal matrix</td>
</tr>
<tr>
<td>E</td>
<td>Error matrix</td>
</tr>
<tr>
<td>eᵢⱼ</td>
<td>Element of matrix E</td>
</tr>
<tr>
<td>F*</td>
<td>Dissipation function</td>
</tr>
<tr>
<td>G</td>
<td>Gyroscopic matrix</td>
</tr>
<tr>
<td>gᵢⱼ</td>
<td>Element of matrix G</td>
</tr>
<tr>
<td>H</td>
<td>Circulatory forces matrix</td>
</tr>
<tr>
<td>hᵢⱼ</td>
<td>Element of matrix H</td>
</tr>
<tr>
<td>I</td>
<td>Identity matrix</td>
</tr>
<tr>
<td>i</td>
<td>Integer number</td>
</tr>
<tr>
<td>K*</td>
<td>Stiffness matrix in 2n-dimensional state space representation</td>
</tr>
<tr>
<td>K</td>
<td>Stiffness matrix</td>
</tr>
<tr>
<td>kᵢⱼ</td>
<td>Element of matrix K</td>
</tr>
<tr>
<td>L</td>
<td>The Lagrangian</td>
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NOMENCLATURE (Continued)

\[ M^* \] Mass matrix in 2n-dimensional state space representation

\[ M \] Mass matrix

\[ m_{ij} \] Element of matrix M

\[ n \] Dimension of a vector matrix

\[ Q \] Generalized forces

\[ q \] Generalized coordinate

\[ r \] Number of constraints on the original system

\[ T \] Kinetic energy of the system

\[ t \] Time

\[ U \] Input vector

\[ u \] r-dimensional input vector

\[ V \] Potential energy

\[ X \] Right eigenvectors matrix

\[ x \] Element of the right eigenvectors matrix, state vector

\[ Y \] Left eigenvectors matrix

\[ y \] Element of the left eigenvectors matrix

\[ Z \] m-dimensional output

\[ \alpha \] Real part of complex eigenvalue

\[ \beta \] Imaginary part of complex eigenvalue

\[ \varepsilon \] A given number

\[ \lambda \] Eigenvalues of the system's matrix

\[ \rho \] Eigenvalues of the error matrix
Phase angle
Phase of the state variable
Frequency
CHAPTER I
INTRODUCTION

Modal analysis can be defined as the theoretical or experimental analysis of the structure's dynamic characteristics of a mechanical system in terms of its modal parameters; that is, finding the eigenvalues and eigenvectors of the equations of motion which define the mechanical system. A review of the past, present and future of modal analysis technology will improve our knowledge on different methods and techniques that are being used for estimating, testing, and solving vibration problems.

Past Modal Analysis Technology

The development of modal testing techniques, which took place mainly after World War II, was made principally in the aircraft and space vehicle fields (Budd 1969). Before 1960, frequency response determination for modal analysis was limited to a discrete sine testing method, and mode shape was determined by a simple sine dwell method. Sand patterns and other similar testing techniques were also used to determine nodal (zero amplitude) patterns. During the early 1960's, the most important development was the tracking filter that allowed the use of the mechanical impedance measuring analyzer, mostly known as the transfer function analyzer. The transfer function analyzer measured frequency response by using a sinusoidal
excitation signal to excite the mechanical system. This process made it possible to get frequency response plots, and from the poles, natural frequencies, and damping could be estimated. Another improvement was the development of a co-quad meter which allowed the real and imaginary components of the frequency response to be measured directly. Instead of using the total amplitude for the mode shape values, the quadrature response could be used to separate closely coupled modes.

Experimental modal analysis became divided into two fields: (1) the multi-input sine dwell (MISD), and (2) the single input frequency response (SIFRQ).

Multi-input Sine Dwell Method

The multi-input sine dwell test was the most popular modal testing method. Several classical papers were published which discuss the calculation of force ratios among shakers, like C.C. Kennedy (1947), R.C. Lewis (1950), G.W. Asher (1958) and R.E.D. Bishop (1963). In this method, a sine dwell is performed to determine the frequencies of the system poles. A number of exciters are connected to the structure and tuned to one of the pole frequencies. The force amplitude and phase are adjusted such that only one mode is excited by this combined forcing function. There are several checks which are performed to determine if the mode is properly tuned. Tuning each mode of vibration, which included positioning the shakers, adjusting the frequency and adjusting the forcing vector was the difficult procedure with this method. Only one frequency
could be measured at a time, so it was required to have a large number of response transducers in parallel to minimize the test time.

Single-input Frequency Response Method

For the single-input frequency response method it was necessary to measure a large number of frequency responses. This was a very slow process using existing analog equipment at that time. But, in the late sixties, the minicomputer systems and the implementation of the fast Fourier transform algorithm in these computers made it possible to develop a Fourier analysis system which could be used to measure frequency response in a small fraction of the time required with the analog equipment. J.W. Cooley and J.W. Tukey (1965) developed the Fast Fourier Transform procedure which made all that possible and did some studies on complex Fourier series machine calculations.

Testing Methods

Many new excitation and testing methods were developed: impact testing, random, pseudo-random, etc., as a result of these new Fourier techniques. During the early seventies, a great deal of research was done to develop these testing methods to define their limitations and application areas. The main applications were made in trouble-shooting vibration problems and verifying finite element results. For the trouble-shooting applications it was not necessary to
measure exact estimates of the modal vectors, since in most of the trouble-shooting applications these estimates were not used to build analytical models for finite element modeling.

However, it was difficult to compare experimental results with finite element results. Finite element analysis does not consider damping in most cases and damping is always present in experimental results.

Analytical Methods

Analytical methods were improved by the emergence of the finite element method, which was in turn made attractive by Fourier analysis and the digital computer. This method for dynamics got started in the early sixties with lumped mass finite element models of structural systems. The analysts were developing their modeling expertise: What kind of structures could be modeled and what elements? How should various boundary conditions be approximated, etc.? Methods for building analytical models by merging substructures based upon finite elements and experimental results were started. The results obtained during the sixties were limited by the ability to extract good modal data bases from measurements and by inadequate methods for including effects of damping and non-linearities. The seventies brought new reduction and solution algorithms, much better pre- and post-processing of the data, very good computer graphics, more powerful and cheaper computers, all of which greatly reduced the cost of constructing a finite element model.
Present Modal Analysis Technology

Experimentally, the present technology is a continuation of the technologies developed during the sixties but much more refined. With the introduction of low cost minicomputers, and the development of software programs for modal testing, for example General Radio Time/Data Division (1977) and Hewlett-Packard (1974), more tests are done using digital equipment. The key to conducting the modal test using a digital signal processing technique is the measurement of the frequency response function between the input forces and the responses at different locations of the structure. Modal parameters for defining the dynamic characteristics of the structure are then extracted from these measured functions. Substantial literature has dealt with this approach, for example, M. Richardson and R. Potter (1974); K.A. Ramsey (1975, 1976); A. Klosterman and R. Zimmerman (1975); D. Brown, R. Alleman, R. Zimmerman and M. Mergeay (1979). Single excitation frequency response methods have probably been developed nearly to their limit. The multi-shaker sine dwell method, as in R.R. Craig and Y.W.T. Su (1974), is also reaching its practical limits. Multi-input methods are currently being developed which are combining the best features of both methods, also new non-traditional methods have been developed (Brown 1982).
Testing Methods

Single-Input Frequency Response Function (SIFRF)

Multidegrees of freedom parameter estimation methods were developed which allowed a single frequency response measurement to be curve-fit reliably with minimum error. For trouble-shooting applications or frequency regions of low modal density, the single point excitation method has become very popular. In fact, more than ninety-five percent of the systems in service use this technology. We can find an example of this technique in Potter (1979).

Multi-Input Sine Dwell (MISD)

The MISD method has been limited to large aerospace companies. This method requires considerable investment in excitation and transducer systems. An approach to this method is studied by T. Comstock, J. Niebe, G. Wylie and F.H. Chu (1979). The method is only practical if a large number of transducers can be permanently mounted to the test structure, since only one mode can be determined at a time and it is necessary to tune each mode. Then, the testing time can become prohibitive.

Multi-Input Frequency Response (MIFRF)

The MIFRF method is an attempt at combining the advantages of the SIFRF method and the MISD method. The main advantage of the SIFRF method is that the complete dynamic response for a frequency band is determined between the input and output points while the
major advantage of the MISD method is the excellent modal separation which is obtained. Also with the SIFRF method there is less chance of missing a mode, but a greater chance of measuring a contaminated mode. The data must be measured simultaneously with MIFRF method in order to separate the modes. The use of this procedure for accurate measurements of the frequency response function is explained by R.J. Allemang, R.W. Rost, and D.L. Brown (1982).

Ibrahim Time Domain (ITD)

The Ibrahim method is one of the new methods which uses free-decay responses in time domain to compute the modal parameters. It is not necessary to measure the system inputs, but impulse response functions can be used in the place of the free responses. The poles and modal vectors can be estimated from the free decays, but it is not possible to estimate the modal scale factors (modal mass or stiffness) without measuring the input force (unit impulse functions). The procedure is to measure the free decays at various points on the structure. A recurrence matrix is created from the free decays; the eigenvalues of this matrix are the poles of the system, the eigenvectors are the measured modal vectors. This method of identification of parameters is studied in S.R. Ibrahim and E.C. Mikulick (1973, 1976).
**Autoregressive Moving Average (ARMA)**

This method computes the modal characteristics by assuming that the response data is caused by a white random noise input to the system. The technique computes the best statistical model of the system in terms of its poles (from the autoregressive part), and zeros (from the moving average part), as well as statistical confidence factors on the parameters. Since in the general case the inputs are not measured, the modal vectors are determined by referencing each response function to a single response to provide relative magnitude and phase information. A method to estimate modal parameters using this technique has been developed by David H. Friedman (1982).

**Direct Parameter Identification (DPI)**

In this method, mass, stiffness and damping matrices are computed directly from the measured force input and response measurements taken from the test article. The system's modal parameters are computed directly from the mass, stiffness, and damping matrices by an eigenvalue decomposition method. This method does not require any restrictions on the relationships between input forces and can compute repeated roots, which none of the above methods do directly. An application of this procedure can be seen in Mauri Maattanen (1982).
**Poly Reference (PR)**

The poly reference method is a technique which uses measured unit impulse responses taken from a number of simultaneous exciter positions. It is a multi-dimensional complex exponential algorithm method. The complex exponential is a standard, multi-degree of freedom parameter estimation algorithm which is currently used to curve fit single input measurements. The method generates a companion matrix whose roots are the poles of the systems. Since it measures data from several exciter positions, it can be used to solve for repeated roots or in the ordinary measurement case pseudo-repeated roots. Pseudo-repeated roots and poles so closely coupled that they cannot, in a measurement sense, be distinguished from a true repeated root. An application of this method is shown by Havard Vold and G. Thomas Rocklin (1982).

**Modal Assurance Criterion (MAC)**

This technique measures the modal vector directly from the responses of a system to a series of randomly applied inputs. The procedure is to excite the system from a randomly distributed number of input points. The responses are measured at a minimum of two points on the structure per measurement cycle. For each input, the residues of both the reference and roving responses are estimated for each pole. Once the residues are estimated, a linear regression analysis is performed between the residues measured at the reference point and the roving point. The constant of proportionality between
these two data sets is an estimate of the modal vector at the roving point. This method does not directly estimate poles (frequencies and damping) only modal vectors. An application of this technique has been done by G.D. Carbon, D.L. Brown and R.J. Allemang (1982).

Measurement Equipment

The measurement equipment that is currently used can be divided into three major categories: (a) Small dedicated two-channel Fourier analyzer systems that are dedicated systems which are not programmable and are very useful for the trouble-shooting type of problem. They, in general, use SIFRF method excitation and simple parameter estimation algorithms. (b) Larger multi-channel Fourier analyzer systems that are programmable general purpose computer systems with Fourier analysis software. These systems use the newer parameter estimation programs. The excitation signal used is of the random type, which tends to average out non-linear distortion errors so the measurements are more compatible with the parameter estimation algorithms. (c) Data acquisition systems are used to gather data for input into a more powerful computer system. This is done with some of the newer techniques which do not use frequency response information. Some studies on measurements have been done by K.A. Ramsey (1975, 1976) and R.W. Potter (1979).

Analytical Methods

The knowledge in analytical methods has been and is continuing to advance very rapidly. The finite element theory has been
extensively developed with a large number of new programs for pre-
and post-processing of data files, large new data base programs inter-
faced with drafting and solids generation programs used in the
computer-aided manufacture areas. Finite element programs have been
developed which run on typical desk top computers and small minicom-
puter systems for use in limited applications or by users with infre-
quent requirements. There are commercial programs available for
taking modal measurements or data bases, and building an analytic
model of the system which can be used to predict modifications.
Some of these analytical methods are explained in the following papers:
R.G. Stroud, C.J. Bonner and G.J. Chambers (1978); and C.V. Stahle
(1978).

Future Application of Modal Analysis

In the testing area, systems with a large number of input chan-
nels should be developed. The cost of transducers and data acqui-
sition should be reduced so industry, in general, can afford the sys-
tems. The parameter estimation will use the complete data base in
one pass to get the best modal model based upon this complete data
set.

New transducers or procedures should be developed to measure mo-
ments and rotations which are not presently measured, but which are
very important in building an analytical model based upon experimental
data. Less expensive, more powerful and portable field test units
should be developed for the trouble-shooting applications. Damping and non-linearities should be incorporated into the analytical models.

In the analytical area, the methods presented here and other advances in the area will, hopefully, help in making modal analysis more accurate, easier to perform and simpler to modify.
CHAPTER II

ANALYTICAL MODAL ANALYSIS

Analytical Determination of Modal Data

Methods of analytical determination of modal parameters are most readily examined in terms of the various steps of the analysis. The major steps in the analysis are the following:

1. Modeling of the system
2. Formulation of equations of motion
3. Solution for modal data

The derivation of the mathematical model of the system and the associated choice of coordinate system are linked with the formulation of the equations of motion. In all practical cases, at least parts of the system have distributed masses, and so the number of degrees of freedom required to represent exactly its dynamic motion is infinite. The reduction of a real structure to systems of finite degrees of freedom represents one of the basic simplifications in the modeling process. This can be done by such techniques as lumping of the masses (springs), by application of finite element techniques, or by a Ritz (1911) approach. The application of these techniques results finally in the equations of motion which reduce to a matrix eigenvalue problem by removal of the time factor. The methods by which these equations are solved are independent of the
derivation of the mathematical model and the formulation of the equations of motion.

**Modeling of the System**

The mathematical model is the most important factor in obtaining satisfactory modal definition for the system. If the model is of poor quality, mathematical rigor in the solution of the equations of motions will not improve results. The following basic factors are given careful consideration in the synthesis of the mathematical model:

1. Stiffness distribution
2. Mass distribution
3. Boundary conditions

Neglect of any of these considerations may result in a model that is not dynamically similar to the actual system.

**Stiffness Distribution**

The definition of the stiffness distribution is the most difficult step in the synthesis of a mathematical system dynamic model. For systems having continuously distributed properties, a model exhibiting those same properties must be utilized (Hurty and Rubinstein 1964). Models of this type are used with displacement functions to obtain modal parameters.

Most structures are complex and contain discontinuities in stiffness. For these structures, mathematical models are used which are composed of independently modeled structural components (finite
elements) joined at coordinate points through common displacement and force components. These are defined as finite element models.

In the analysis of systems with many components and large differences in stiffness parameters, the modal parameters of the components are used to synthesize the overall structural characteristics (Hurty 1965). Some advantages of using this technique are that it solves lower-order eigenvalue problems and generates the stiffness and mass parameters for smaller substructures.

Mass Distribution

Mass distribution depends on the physical system under consideration and the method of analysis. For a system with a not uniform mass distribution, commonly handled by an analysis based on continuous-system equations, the mass distribution is readily defined by the actual structural distribution.

Several methods are used to define the mass distribution. These include the lumped-mass method, the consistent-mass method, and a number of approaches that use various velocity-interpolation functions to define a mass matrix (Bisplinghoff, Ashley, and Halfman 1955). The lumped-mass method distributes the element masses in concentrations located at the coordinate points in a physically reasonable manner which maintains the center of mass of the structure (Fowler 1959). This method of distribution is well suited for analysis of structures with concentrated masses. The disadvantage of the method is the relatively large number of coordinate points.
required for accurate analysis of systems with preponderantly distributed masses. The consistent-mass represents the mass in agreement with the actual distribution of mass in the structure (Archer 1963; Melosh and Lang 1965). The consistent-mass distribution technique gives more accurate results than the concentrated-mass technique for systems where the mass is largely distributed in the structure.

Boundary Conditions

Because natural modes of a structure are sensitive to boundary conditions, the same boundary conditions are imposed on the model as on the actual structure, insofar as feasible. Frequently, static tests must be performed to determine the influence coefficients defining the boundary conditions at an interstage connection with a supporting stage structure.

Formulation of Equations of Motion

The methods of formulating the equations of motion can be classified under the following categories:

1. Integral equation methods
2. Differential equation methods
3. Energy methods

Each method may include either the theoretically exact or approximate approach and can be used to handle both the distributed and discrete system models. In all but a few special cases, however, the mathematically exact solution to the equations of motion cannot be found.
and the analyst must resort to numerical techniques. Examples of these methods are found in Melosh and Lang (1965), Melosh (1963), and Meirovitch (1967).

Integral Equation Methods

Integral equation methods of formulating the equations of motion make use of an influence-coefficient function that defines the displacement of any point of a structure in terms of the applied load. The displacement under the inertial loads is obtained through integral equations which become the equations of motion. The advantage of this method is that it includes the boundary conditions in the basic equation of motion.

Differential Equation Methods

The natural modes and frequencies of a system can be determined through differential equations which relate the structure's distortions to the inertial forces on the structure. This is a classical approach, and modal data obtained by this approach are available for a large variety of simple configurations. However, it is difficult to apply this method to a complex structure.

Energy Methods

Energy methods for formulation of the equations of motion are based on energy principles of mechanics, such as conservation of energy, virtual work, Lagrange's equation and Hamilton's principle. In this approach, displacement functions that approximate the mode
shape are used to represent the system behavior. While the chosen functions are not theoretically restricted to those satisfying all the boundary conditions for the mode shapes, the accuracy of the solution depends strongly on how well the boundary conditions are satisfied. The advantage of energy methods lies in their versatility; they can be applied to any structural configuration and can approximate the system behavior to any desired degree of precision.

Solution for Modal Data

The equations of motion can be solved to obtain the modal data by several methods. Hand solution is usually limited to small order systems because of the overwhelming amount of numerical labor involved. The solution of large-order systems is generally achieved on electronic computers.

Exact solutions are available for linear differential equations with constant coefficients, such as equations used to characterize systems with both uniform mass and stiffness properties. Where these properties are not uniformly distributed, exact solutions are not always possible and approximate solutions are obtained. Studies on solution for modal data have been done by Hintz (1975), Caughey (1960), Hou (1969) and Meirovitch (1975).

System Equations of Motion

Mathematical models of dynamic systems can be divided into two large classes: discrete and distributed. The reference is to
the system parameters such as mass, damping and stiffness. Discrete systems are described by functions depending on time alone, and distributed systems are described by functions depending on time and space. The minimum number of dependent variables required to fully describe the motion of a discrete system is known as the number of degrees of freedom of the system. These variables are called coordinates. Defining the system Lagrangian as:

\[ L = T - V \]  

where:

\[ T = \text{kinetic energy of the system} \]
\[ V = \text{potential energy of the system} \]

The equations of motions take the form of Lagrange's equations (Meirovitch 1980):

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial q_i} = Q_i \]  

where:

\[ F = \text{dissipation function and } Q_i = \text{generalized forces} \]

The solution of equation 2 consists of the n generalized coordinates \( q_i(t) \) (\( i = 1, 2, \ldots, n \)). This solution can be interpreted geometrically by conceiving of an n-dimensional Euclidean space with \( q_i \) as axes, where the space is known as the configuration space. The solution represents a point in the configuration space defined by the tip of a vector \( \bar{q}(t) \) whose components are the generalized coordinates \( q_i(t) \).

The configuration space is not very convenient for a geometric representation of the motion. The main reason is that two dynamical
paths can intersect, so that a given point in the configuration space does not define the state of the system uniquely. If the generalized velocities $\dot{q}_i$ are used as a set of auxiliary variables, then the motion can be described in a $2n$-dimensional Euclidean space defined by $q_i$ and $\dot{q}_i$ and known as the state space. The set of $2n$ variables $q_i$ and $\dot{q}_i$ define a state vector and the tip of this vector traces a trajectory in the state space. The advantage of the representation in the state space is that two trajectories never intersect, so that a given point in the state space corresponds to a unique trajectory.

The interest of this study lies in small motions about equilibrium points, and motions confined to small neighborhoods of the equilibrium points. The assumption of small motions about an equilibrium point implies linearization of the equations of motion, so we can ignore terms of degree higher than two. From this assumption, Meirovitch (1980) defines the mass, stiffness, damping, gyroscopic and circulatory forces coefficients such that:

$$
M = [m_{ij}], \quad K = [k_{ij}], \quad C = [c_{ij}], \quad G = [g_{ij}], \quad H = [h_{ij}] \quad (3)
$$

where:

- $m_{ij}$ = mass coefficient
- $k_{ij}$ = stiffness coefficient
- $c_{ij}$ = damping coefficient
- $g_{ij}$ = gyroscopic coefficient
- $h_{ij}$ = circulatory forces coefficient
\( M = \text{mass matrix} \)
\( K = \text{stiffness matrix} \)
\( C = \text{damping matrix} \)
\( G = \text{gyroscopic matrix} \)
\( H = \text{circulatory forces matrix} \)

\( M, K \) and \( C \) are symmetric matrices, then we have:
\[
M = M^T, \quad K = K^T, \quad C = C^T \tag{4}
\]

The equation of motion becomes:
\[
\ddot{\vec{q}} + (G + C)\vec{q} + (K + H)\vec{q} = \vec{Q} \tag{5}
\]

where \( \vec{Q} \) is the generalized forcing vector.

**Forced Vibration and the Eigenvalue Problem**

A convenient way of deriving the solution of equation 5 is modal analysis, which requires the solution of the so-called eigenvalue problem for the system. Meirovitch (1980) derives the eigenvalue problem for various cases. Here we will be concerned with the two most frequent cases.

**Conservative Non-gyroscopic System**

In the absence of gyroscopic, damping, circulatory forces, we have \( G = C = H = 0 \). Then, equation 5 reduces to:
\[
M\ddot{\vec{q}}(t) + K\vec{q}(t) = \vec{Q} \tag{6}
\]

We can rewrite this equation as:
\[
\dot{\vec{x}}(t) + K\vec{x}(t) = \vec{U} \tag{7}
\]

Where \( \vec{x}(t) \) is the 2n-dimensional state vector.
\[ \dot{x}(t) = \begin{bmatrix} \dot{q}(t) \end{bmatrix}^T \begin{bmatrix} \ddot{q}(t) \end{bmatrix}^T \]

and

\[ M^* = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}, \quad K^* = \begin{bmatrix} 0 & K \\ -I & 0 \end{bmatrix} \]

2n x 2n are real matrices and where \( \overline{U} \) is a forcing vector.

\[ \overline{U} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

The final formulation is then given by:

\[ \dot{x} = Ax - M^*^{-1}\overline{U} \]

where

\[ A = -M^*^{-1}K^* = \begin{bmatrix} 0 & -M^{-1}K \\ I & 0 \end{bmatrix} \]

Assuming that \( M^* \) is not singular, \( A \) is an arbitrary real matrix.

Because \( A \) is real, if the eigenvalues \( \lambda \) are complex, then they must occur in pairs of complex conjugates and so must the eigenvectors \( x \).

**Damped Non-gyroscopic Systems**

If we let \( G \) and \( H \) be zero in equation 5, then the equation becomes:

\[ \dddot{M}q(t) + \ddot{C}q(t) + \dot{K}q(t) = \overline{Q} \]

It describes the forced vibration of a damped non-gyroscopic system, where all three matrices are real and symmetric. As it was done
before, equation 10 can be rewritten in the form of equation 7:
\[ \dot{\bar{x}}(t) + K\bar{x}(t) = \bar{U} \]

where

\( \bar{x}(t) \) is a 2n-dimensional state vector (equation 8) and

\[
M^* = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}, \quad K^* = \begin{bmatrix} C & K \\ -I & 0 \end{bmatrix}
\]  (11)

2n x 2n are real matrices.

Assuming that \( M^* \) is non-singular, the equation of motion becomes

\[
\dot{x} = Ax - M^*U
\]

where

\[
A = -M^*-1K^* = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I & 0 \end{bmatrix}
\]  (12)

Again, \( A \) is real with complex eigenvalues which occur in pairs of complex conjugates, and so also the eigenvectors \( \bar{x} \).

The Eigenvalue Problem

It can be easily seen (Meirovitch 1980) that the equation of the eigenvalue problem for damped non-gyroscopic systems can be written in the form:

\[ \lambda M^*\bar{x} + K^*\bar{x} = 0 \]  (13)

in which \( \bar{x} \) is a 2n-vector, where \( n \) represents the number of degrees of freedom of the system and:
\[ M^* = \begin{bmatrix} M & 0 \\ \hline -I & 0 \end{bmatrix} \quad K^* = \begin{bmatrix} C & K \\ \hline I & 0 \end{bmatrix} \]

2n x 2n are real symmetric matrices which are clearly not positive definite.

For distinct eigenvalues, the orthogonality relations are:

\[ x_i^T M^* x_j = 0, \quad \lambda_i \neq \lambda_j \quad i,j = 1, 2, \ldots, 2n \]  \hspace{1cm} (14)

\[ x_i^T K^* x_j = 0, \quad \lambda_i \neq \lambda_j \quad i,j = 1, 2, \ldots, 2n \]  \hspace{1cm} (15)

The eigenvectors can also be "M" normalized by setting

\[ x_i^T M^* x_i = 1 \quad i = 1, 2, \ldots, 2n \]  \hspace{1cm} (16)

So that if \( X = [x_1, x_2, \ldots, x_{2n}] \) represents the square matrix of the "M" normalized eigenvectors, then we can get:

\[ X^T M^* X = I \]  \hspace{1cm} (17)

and also

\[ -X^T K^* X = D \]  \hspace{1cm} (18)

where \( D \) is the diagonal matrix of the eigenvalues.

For the case where \( A \) (equation 12) is a real non-symmetric matrix, we define:

\[ \bar{A}^T y_i = \lambda y_i \]  \hspace{1cm} (19)

\[ A \bar{x}_i = \lambda \bar{x}_i \]  \hspace{1cm} (20)

where:

\( X = \) right eigenvector matrix; \( \bar{x}_i = \) element of matrix \( X \)
Y = left eigenvector matrix; \( \overline{y}_i \) = element of matrix Y

After minor manipulations, we find that:

\[
X^T Y = I
\]  

and

\[
Y^T A X = X^{-1} A X =
\]  

The two sets of eigenvectors \( \overline{x}_i \) and \( \overline{y}_i \) are then biorthogonal.

\[
\overline{y}_i^T \overline{x}_i = 1 \quad i = 1, 2, \ldots, 2n
\]

**Eigensolution Error**

If the matrix A is found with an error E such that

\[
B = A + E
\]

where:

- **B** eigenvalues are \( B_1, B_2, B_3, \ldots, B_n \)
- **A** eigenvalues are \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \)
- **E** eigenvalues are \( \rho_1, \rho_2, \rho_3, \ldots, \rho_n \)

We find that according to Weyl's Theorem (Meirovitch 1980):

\[
B_r - \lambda_r \leq ||E||
\]

where

\[
r = \text{number of constraints on the original system providing a characterization of the eigenvalues}
\]

where

\[
||E|| = ||B - A|| = \max. ( -\rho_1, \rho_n )
\]

where

\[
||E|| \leq \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |e_{ij}|^2 \right)^{\frac{1}{2}}
\]

or
Where the maximum is with respect to any row $i$ of $E$.

As an illustration, let us assume that all we know about $E$ is that it is a small matrix and that the magnitudes of its elements do not exceed a given number $\varepsilon$. Then:

$$|e_{ij}| \leq \varepsilon, \quad ||E|| \leq n\varepsilon$$

Using Weyl's Theorem, we get:

$$|B_r - \lambda_r| \leq n\varepsilon$$  \hspace{1cm} (28)

**Frequency Response in State Space**

Conceptually, this method is more involved than the time domain approach. It requires the transformation of measured data from the time domain to the frequency domain and hence, involves the use of digital processing techniques. The key in this method is the measurement of a response function between the Fourier transform of the response to the Fourier transform of the input force. If a Laplace transform is used to transform the test data instead of a Fourier transform, this function is called a transfer function. Modal parameters of the system are then determined from these measured frequency response functions using curve fitting techniques. The major steps involved in this method can be summarized in general as follows:

1. Input a measurable force. This force can be sinusoidal, random, pseudo-random, transient or impulse in nature.
2. Measure the response at a selected point (or points) on the system.

3. Fourier transform the input force and output response measurements.

4. Compute the frequency response functions.

5. Estimate the modal parameters from these functions.

Due to the fact that the internal state of a system is represented by a vector, it is to be expected that the state-space approach will be more mathematically complex than the classical scalar input-output frequency-response technique. Although no design techniques have yet emerged from this new point of view that were not available in the classical frequency-response approach, the insight we gain into the internal state of linear systems in sinusoidal input-output steady-state may eventually lead to new design and analysis approaches.

The following procedure tries to narrow the gap between the classical control viewpoint based on the frequency-response technique and the contemporary approach using the state-space vector methods. Y. Takahashi, M.J. Rabins and D.M. Auslander (1972) develop this procedure where they consider a linear lumped system described by the following state and output equations:

\[
\frac{d\bar{x}}{dt} = \bar{A}\bar{x} + \bar{B}\bar{u} \quad \text{or} \quad \frac{d\bar{x}}{dt} = \bar{A}\bar{x} + \bar{b}\bar{u}
\]

and

\[
\bar{Z} = \bar{C}\bar{x}
\]

where:
\[ x(t) = \text{an n-dimensional state} \]
\[ u = \text{r-dimensional input} \]
\[ Z = \text{m-dimensional output} \]

The system is characterized by an \( 2n \times 2n \) matrix \( A \), input matrix \( B \) which is \( 2n \times r \) and output matrix \( C \) which is \( m \times 2n \).

The derivative operator \( \frac{d}{dt} \) is replaced by \( j \omega \) to deal with sinusoidal steady-state (or frequency response), where \( \omega \) (rad/sec) is the angular frequency of the sinusoidal input and \( j = \sqrt{-1} \).

Finally, the following form for the \( i \)th state variable in sinusoidal steady-state is obtained:

\[
x_i(\omega) = \alpha_i \sin \omega t + \beta_i \cos \omega t
\]

with

\[
\alpha_i = A_i \cos \psi_i, \quad \beta_i = A_i \sin \psi_i
\]

where:

\( A_i = \text{amplitude} \)

\( \psi_i = \text{phase of the } i \text{th state space variable} \)

From equation 30, we get:

\[
\mathbf{x}(\omega) = [\alpha, \beta] \begin{bmatrix} \sin \omega t \\ \cos \omega t \end{bmatrix}
\]

where:

\[
\mathbf{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{2n} \end{bmatrix}, \quad \mathbf{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{2n} \end{bmatrix}
\]

The time derivative for \( x(\omega) \) is given by:
\[
\frac{d}{dt} \bar{x}(t) = [\bar{\alpha}, \bar{\beta}][-\omega \cos \omega t \quad -\omega \sin \omega t] = [-\omega \bar{\beta}, \omega \bar{\alpha}]\begin{bmatrix} \sin \omega t \\ \cos \omega t \end{bmatrix}
\]  

Finally, the values for \( \bar{\alpha} \) and \( \bar{\beta} \) are found to be (Takahashi, Rabins and Auslander 1972):

\[
\bar{\alpha} = (\omega I + A^2)^{-1} (-\bar{A} \bar{b}_1 + \omega \bar{b}_2)
\]

\[
\bar{\beta} = (\omega I + A^2)^{-1} (-\bar{A} \bar{b}_2 - \omega \bar{b}_1)
\]

where:
- \( \bar{\alpha} \) = real part of the solution \( \bar{x} \)
- \( \bar{\beta} \) = imaginary part of the solution \( \bar{x} \)
- \( \omega \) = circular frequency (rad/sec)
- \( A \) = eigenvalue problem matrix
- \( \bar{b}_1 \) = forcing input (\( \sin \omega t \)) vector
- \( \bar{b}_2 \) = forcing input (\( \cos \omega t \)) vector

The amplitude is given by:

\[
|x_i| \text{ Amplitude} = \sqrt{\alpha_i^2 + \beta_i^2}, \ i = 1, 2, \ldots, 2n
\]

The phase angle is obtained from:

\[
\phi = \tan^{-1} \frac{\alpha_i}{\beta_i}
\]

**Eigenvalue Problem Matrix Identification**

In frequency domain and for non-homogeneous equations, we start

the procedure of identifying the eigenvalue problem matrix with
equations 29 to 36. That process would describe relationships for sinusoidal steady-state.

Equations 35 and 36 were derived by generalizing a vector input, assuming:

$$\bar{u}(\omega) = [\bar{b}_1, \bar{b}_2][\sin \omega t]$$  \hfill (39)

where $\bar{b}_1$ and $\bar{b}_2$ are prescribed magnitudes with each as a 2n dimensional vector.

Substituting equations 32, 34 and 39 into equation 29, we find:

$$[-\omega \bar{\beta}, \omega \bar{\alpha}][\sin \omega t] = A[\bar{\alpha}, \bar{\beta}][\sin \omega t] + [\bar{b}_1, \bar{b}_2][\sin \omega t]$$  \hfill (40)

And from this relationship, we obtain:

$$[-\omega \bar{\beta}, \omega \bar{\alpha}] = A[\bar{\alpha}, \bar{\beta}] + [\bar{b}_1, \bar{b}_2]$$  \hfill (41)

or

$$-\omega \bar{\beta} = A\bar{\alpha} + \bar{b}_1$$  \hfill (42)

$$\omega \bar{\alpha} = A\bar{\beta} + \bar{b}_2$$  \hfill (43)

Metwalli (1982) shows that by measuring at 2n the $\omega_1$, we can get $\bar{\alpha}$, and $\bar{\beta}$ for any $\omega_1$, then arranging the different vectors in equation 43 such that:

$$A = [\omega_1 \bar{\alpha}_1 - \bar{b}_2, 1] \begin{bmatrix} \omega_2 \bar{\alpha}_2 - \bar{b}_2, 2 \end{bmatrix} \ldots [\omega_{2n} \bar{\alpha}_{2n} - \bar{b}_2, 2n] [\bar{\beta}_1 \bar{\beta}_2 \ldots \bar{\beta}_{2n}]^{-1}$$  \hfill (44)
Also from equation 42,

\[ A = \begin{bmatrix}
-w_1 \tilde{b}_1 & -w_2 \tilde{b}_1 & \cdots & -w_n \tilde{b}_1 \\
-w_1 \tilde{b}_2 & -w_2 \tilde{b}_2 & \cdots & -w_n \tilde{b}_2 \\
\vdots & \vdots & \ddots & \vdots \\
-w_1 \tilde{b}_{2n} & -w_2 \tilde{b}_{2n} & \cdots & -w_n \tilde{b}_{2n}
\end{bmatrix}
\begin{bmatrix}
\tilde{\alpha}_1 \\
\tilde{\alpha}_2 \\
\vdots \\
\tilde{\alpha}_{2n}
\end{bmatrix}
\] \hspace{1cm} (45)

The following step is to add a random error to the response so that to simulate real measuring conditions.

Each of the ith columns of the first RH matrix in equation 43 has been changed to:

\[ \text{ith column} = (w_1 \tilde{\alpha}_1 - b_{2,i})[1 + \%R.E.] \] \hspace{1cm} (46)

where:

\[ \text{R.E.} = \text{the percentage of random error added} \]

The computer which was used to run the identification program was a Tektronix 4050 Series. The random function used causes an internal pseudorandom number generator to output a random number from 0 to 1. The random number generator contains approximately 140 trillion numbers within the range 0 to 1. These numbers are linked together in a chain in such a way as to appear random when they are output from the generator. Each time the random function is executed, one number in the chain is retrieved. The next time the random function is executed, another number in the chain is obtained and so on. The probability distribution of these random numbers is uniform, which has been used as such for simplicity.
CHAPTER III
SAMPLE IDENTIFICATION TESTS

This chapter gives the sample calculation of two different cases. The first one is a damped non-gyroscopic system, and the other is a conservative non-gyroscopic system. In this section we will derive the eigenvalue problem for both cases, and then we will check the properties of their solutions. Next, we will get the graphical representation of frequency versus amplitude, and phase angle versus amplitude in the frequency domain. Finally, a state space identification in the frequency domain for non-homogeneous equations will be done with a unit step sinusoidal input, and the addition of a certain percentage of random error.

Damped Non-gyroscopic System

From Figure 1, we can get the kinetic and potential energy for the system:

Fig. 1. Damped non-gyroscopic system.
\[ T = \frac{1}{2} m_{11} \ddot{q}_1 + \frac{1}{2} m_{22} \ddot{q}_2 \]

\[ V = \frac{1}{2} k_1 q_1^2 + \frac{1}{2} k_2 (q_2 - q_1)^2 \]  

(47)

And the dissipation function:

\[ F^* = \frac{1}{2} c_1 \dot{q}_1^2 + \frac{1}{2} c_2 (q_2 - q_1)^2 + \frac{1}{2} c_3 \dot{q}_1^2 + \frac{1}{2} c_4 \dot{q}_2^2 \]  

(48)

The system Lagrangian is:

\[ L = T - V \]  

(49)

\[ L = \frac{1}{2} m_{11} \ddot{q}_1 + \frac{1}{2} m_{22} \ddot{q}_2 - \frac{1}{2} k_1 q_1^2 - \frac{1}{2} k_2 (q_2 - q_1)^2 \]  

(50)

Solving by Lagrange's equations we have:

\[ m_{11} \ddot{q}_1 + (c_1 + c_2 + c_3) \dot{q}_1 + (k_1 + k_2) q_1 - c_2 \dot{q}_2 - k_2 q_2 = 0 \]  

(51)

\[ m_{22} \ddot{q}_2 + (c_2 + c_4) \dot{q}_2 + k_2 q_2 - k_1 q_1 - c_2 \dot{q}_1 = 0 \]

In a matrix form:

\[
\begin{bmatrix}
  m_{11} & 0 \\
  0 & m_{22}
\end{bmatrix}
\begin{bmatrix}
  \ddot{q}_1 \\
  \ddot{q}_2
\end{bmatrix}
+
\begin{bmatrix}
  c_1 + c_2 + c_3 & -c_2 \\
  -c_2 & c_2 + c_4
\end{bmatrix}
\begin{bmatrix}
  \dot{q}_1 \\
  \dot{q}_2
\end{bmatrix}
+
\begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
  q_1 \\
  q_2
\end{bmatrix}
= 0
\]  

(52)

Assuming:
m_{11} = m_{22} = 1

c_1 = 2, c_2 = 2.8, c_3 = 0, c_4 = 0 \quad \text{(53)}
k_1 = 4, k_2 = 4

The equations of motion are then:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_1 \\
\ddot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
4.8 & -2.8 \\
-2.8 & 2.8
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
8 & -4 \\
-4 & 4
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

This is a damped non-gyroscopic system with a characteristic equation of motion (equation 10) as follows:

\[M\dddot{q} + C\dot{q} + Kq = 0\]

where:

\[M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4.8 & -2.8 \\ -2.8 & 2.8 \end{bmatrix}, \quad K = \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}\]

In state space representation is given by equation 11:

\[
M\dot{x} = \begin{bmatrix}
M & 0 \\
0 & I
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
K\ddot{x} = \begin{bmatrix}
C & K \\
-I & 0
\end{bmatrix}
= \begin{bmatrix}
4.8 & -2.8 & 8 & -4 \\
-2.8 & 2.8 & -4 & 4 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
\]
where the matrix $A$ of the eigenvalue problem is:

$$A = -M^{-1} K^* = \begin{bmatrix}
-4.8 & 2.8 & -8 & 4 \\
2.8 & -2.8 & 4 & -4 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}$$

Using the computer program EIGRF from International Mathematical and Statistical Libraries, Inc. (1975) (Appendix A) we get the eigenvalues and normalized eigenvectors that are listed in Appendix B, Table 1. These eigenvectors are called the right eigenvectors.

To check the accuracy of the eigenvalues, we calculate the determinant of $A$:

$$\det A = \begin{vmatrix}
-4.8 & 2.8 & -8 & 4 \\
2.8 & -2.8 & 4 & -4 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{vmatrix}$$

$$\det A = 16$$

This must be equal to the product of the eigenvalues:

$$\prod_{i=1}^{4} \lambda_i = (-4.4094)(-0.42305 + 1.1699i)(-0.42305 - 1.1699i)(-2.3455)$$

$$\prod_{i=1}^{4} \lambda_i \approx 16 \approx \det A$$

The trace of matrix $A$ must also be equal to the addition of all the eigenvalues.

$$\text{Trace } A = -4.8 - 2.8 = -7.6$$
\[ \sum_{i=1}^{4} \lambda_i = -4.4094 + (-0.42305 + 1.1699i) + (-0.42305 - 1.1699i) - 2.3455 \]

\[ \sum_{i=1}^{4} \lambda_i = -7.6 = \text{Trace } A \]

In order to check the orthogonality, we use equations 14 and 15. The results we already have indicate that the solution is correct.

System Response

Next, we obtain a graphical representation of frequency versus amplitude and frequency versus phase angle for a unit step sinusoidal input, using equations 37 and 38, for a frequency between 0 and 10 cycles/sec. These graphs can be seen in Figures 2 through 9.

We have used a computer program for frequency response that is shown in Appendix A. The input for the frequency response was equations 35 and 36:

\[ \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \omega_{\text{increment}} = 0.1 \text{ cycles/sec.} \]

\[ \omega_{\text{maximum}} = 10 \text{ cycles/sec.} \]

System Identification

The following step is to identify the matrix A (eigenvalue problem) for a given frequency range that usually contains all the set of eigenvalues of A. A unit step sinusoidal input is utilized
Damped non-gyroscopic system

Velocity $q_1$ versus $\omega$

Fig. 2. Damped non-gyroscopic system amplitude $I$ versus frequency (cycles/sec).
Fig. 3. Damped non-gyroscopic system amplitude $q_2$ versus frequency (cycles/sec).
Fig. 4. Damped non-gyroscopic system amplitude \( x_1 \) versus frequency (cycles/sec).
Fig. 5. Damped non-gyroscopic system amplitude $x_2$ versus frequency (cycles/sec).
Fig. 6. Damped non-gyroscopic system phase angle $\theta_1$ versus frequency (cycles/sec).
Damped non-gyroscopic system

Phase Angle \((q_2)\) versus \(\omega\)

Fig. 7. Damped non-gyroscopic system phase angle \(q_2\) versus frequency (cycles/sec).
Damped non-gyroscopic system

Phase Angle ($x_1$) versus $\omega$

Fig. 8. Damped non-gyroscopic system phase angle III versus frequency (cycles/sec).
Damped non-gyroscopic system

Phase angle \( x_2 \) versus \( \omega \)

Fig. 9. Damped non-gyroscopic system phase angle \( x_2 \) versus frequency (cycles/sec).
to excite the system, as indicated before. The input data for the computer program for identification (Appendix A) is:

\[ b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \omega_1 = 1.3 \text{ cycles/sec.} \]
\[ \omega_2 = 5.2 \text{ cycles/sec.} \]

where:

- \( \omega_1 \) = initial frequency
- \( \omega_2 \) = maximum frequency
- \( b_1 \) = unit input (sine)
- \( b_2 \) = unit input (cosine)

In addition, we have added a random error of 0.2%, 0.4%, 0.6%, 0.8% and 1.0% in every case, and we identified the matrix \( A \) for every percentage of random error. Then, we applied equation 25 that gives the error matrix:

\[ E = B - A \]

where:

- \( B \) = matrix with an assumed random error
- \( A \) = real system matrix without random error
- \( E \) = error matrix

The eigenvalues and eigenvectors of the error matrix are shown in Tables 2, 3, 4, 5 and 6 of Appendix B. We notice that the eigenvalues of the error matrix do not exceed 5% of the eigenvalues of the original matrix \( A \).
Finally, we averaged 100 times the identification of the matrix $A$ for a fixed random error (0.5%). We obtained a very good approximation to the original matrix $A$, i.e., $B$ after averaging 100 times is:

$$B = \begin{bmatrix} -4.797540 & 2.805658 & -8.023725 & 4.017355 \\ 2.800209 & -2.799554 & 3.998017 & -3.999138 \\ 0.998862 & -0.002508 & 0.010779 & -0.008241 \\ -0.002132 & 0.995271 & 0.020753 & -0.015628 \end{bmatrix} \approx A$$

The computer program for the frequency response was used again for the case when the system is critically damped. This case occurs when we have 95.266% of the original damping. We then have two equal real roots and one pair of complex conjugate roots that are shown in Table 7 of Appendix B. The frequency response versus amplitude and phase angle graphs are very similar to those of the original system. The plots can be seen in Figures 10 through 17.

The identification computer program was used again with a 0.5% of random error. It was performed 100 times and then averaged. The matrix that was obtained is practically the same as the original (with 95.266% of the initial damping), i.e., for

$$A = \begin{bmatrix} -4.572753 & 2.667439 & -8 & 4 \\ 2.667439 & -2.667439 & 4 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
Fig. 10. Critically damped system amplitude $q_1$ versus frequency (cycles/sec).
Fig. 11. Critically damped system amplitude $q_2$ versus frequency (cycles/sec).
Fig. 12. Critically damped system amplitude $x_1$ versus frequency (cycles/sec).
Fig. 13. Critically damped system amplitude $x_2$ versus frequency (cycles/sec).
Fig. 14. Critically damped system phase angle $\phi$ versus frequency (cycles/sec).
Fig. 15. Critically damped system phase angle $\phi_2$ versus frequency (cycles/sec).
Fig. 16. Critically damped system phase angle $\phi_1$ versus frequency (cycles/sec).
Fig. 17. Critically damped system phase angle $\phi_2$ versus frequency (cycles/sec).
Another interesting case which has been examined is the under-damped system with 10% of the original damping in the matrix $A$. The eigenvalues and eigenvectors were obtained, and we can see that the eigenvalues are two pairs of complex conjugate roots, as shown in Table 8 of Appendix B.

The graphs obtained for amplitude and phase angle versus frequency show an increase in magnitude for amplitude and a decrease in the values of the phase angle with respect to the original matrix $A$. It can be seen in Figures 18 through 25.

The identification process was performed 100 times with 0.5% of random error and the matrix obtained was similar to the original (with 10% of the initial damping), i.e., for

$$ A = \begin{bmatrix}
-0.48 & 0.28 & -8 & 4 \\
0.28 & -0.28 & 4 & -4 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix} , $$

$$ B = \begin{bmatrix}
-0.480265 & 0.278332 & -7.993251 & 3.999450 \\
0.280299 & -0.279460 & 3.999220 & -4.000215 \\
1.000007 & -0.000675 & 0.001206 & -0.000794 \\
-0.000666 & 0.999987 & 0.002201 & -0.001730 \\
\end{bmatrix} \sim A \cdot 

Fig. 18. Underdamped system amplitude $q_1$ versus frequency (cycles/sec).
Fig. 19. Underdamped system amplitude $q_2$ versus frequency (cycles/sec).
Fig. 20. Underdamped system amplitude \( x_1 \) versus frequency (cycles/sec).
Fig. 21. Underdamped system amplitude $x_2$ versus frequency (cycles/sec).
Fig. 22. Underdamped system phase angle $q_1$ versus frequency (cycles/sec).
Underdamped system

Phase angle $\varphi_2$ versus $\omega$

Fig. 23. Underdamped system phase angle $\varphi_2$ versus frequency (cycles/sec).
Underdamped system
Phase angle ($\phi_1$) versus $\omega$

Fig. 24. Underdamped system phase angle $\phi_1$ versus frequency (cycles/sec).

Underdamped system
Phase angle ($\phi_1$) versus $\omega$

Fig. 24. Underdamped system phase angle $\phi_1$ versus frequency (cycles/sec).
Fig. 25. Underdamped system phase angle $\phi_2$ versus frequency (cycles/sec).
Conservative Non-gyroscopic System

A number of Cam follower systems that have been used for automotive and aircraft piston engines have a complex follower system involving two levers for motion amplification. In this example, we will consider a double lever Cam mechanism with a four degree of freedom dynamic system model. A study about modeling this system has been done by Delbert Tesar and Gary K. Matthew (1976), and the equations of motion and detailed modeling procedures are also explained by Barkan (1953).

The details of the double lever Cam follower system and system model are shown in Figure 26. The differential equations of motion of this system are written in terms of displacement variables $q_1$, lumped masses $m_i$, and linear spring stiffness $K_i$.

Damping coefficients are neglected by Tesar and Matthew (1976) for simplicity. The potential and kinetic energy for the system are:

$$T = \frac{1}{2} m_1 \ddot{q}_1^2 + \frac{1}{2} m_2 \ddot{q}_2^2 + \frac{1}{2} m_3 \ddot{q}_3^2 + \frac{1}{2} m_4 \ddot{q}_4^2$$

$$V = \frac{1}{2} k_1 (q_1 - q_0)^2 + \frac{1}{2} k_2 (q_2 - q_1)^2 + \frac{1}{2} k_3 (q_3 - q_2)^2 + \frac{1}{4} k_4 (q_4 - q_3)^2 + \frac{1}{2} k_5 (q_5 - q_4)^2 = 0$$

Solving by Lagrange's equation and setting $q_5 = 0$, and $q_0 = y$, we get:

$$m_1 \ddot{q}_1 + (k_1 + k_2)q_1 - k_2 q_2 = k_1 y$$

$$m_2 \ddot{q}_2 - k_2 q_1 + (k_2 + k_3)q_2 - k_3 q_3 = 0$$

(55)
Fig. 26. Conservative non-gyroscopic system double lever cam follower system and model (Young and Shoup 1982).
In a matrix form:

\[
\begin{bmatrix}
 m_1 & 0 & 0 & 0 \\
 0 & m_2 & 0 & 0 \\
 0 & 0 & m_3 & 0 \\
 0 & 0 & 0 & m_4 \\
\end{bmatrix}
\begin{bmatrix}
 \ddot{q}_1 \\
 \ddot{q}_2 \\
 \ddot{q}_3 \\
 \ddot{q}_4 \\
\end{bmatrix}
+ \begin{bmatrix}
 k_1 + k_2 & -k_2 & 0 & 0 \\
 -k_2 & k_2 + k_3 & 0 & 0 \\
 0 & 0 & k_3 + k_4 & -k_4 \\
 0 & 0 & -k_4 & k_4 + k_5 \\
\end{bmatrix}
\begin{bmatrix}
 q_1 \\
 q_2 \\
 q_3 \\
 q_4 \\
\end{bmatrix}
= \begin{bmatrix}
 k_1 y \\
 0 \\
 0 \\
 0 \\
\end{bmatrix}
\]

Next, we consider the data of a sample calculation given by S.S.D. Young and T.E. Shoup (1982) where they use the following values:

\[
m_1 = 1.075 \text{ lb-s}^2/\text{in} \\
m_2 = 3.25 \text{ lb-s}^2/\text{in} \\
m_3 = 0.95 \text{ lb-s}^2/\text{in} \\
m_4 = 2.14 \text{ lb-s}^2/\text{in} \\
k_1 = 56 \times 10^4 \text{ lb/in} \\
k_2 = 19 \times 10^4 \text{ lb/in} \\
k_3 = 7 \times 10^4 \text{ lb/in} \\
k_4 = 31 \times 10^4 \text{ lb/in} \\
\]
are:

\[ k_5 = 11 \times 10^4 \text{ lb/in} \]

\[ y = 1 \text{ in} \]

Replacing these values in equation 56, the equations of motion are:

\[
\begin{bmatrix}
1.075 & 0 & 0 & 0 \\
0 & 3.25 & 0 & 0 \\
0 & 0 & 0.95 & 0 \\
0 & 0 & 0 & 2.14
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\dot{q}_4
\end{bmatrix}
+
\begin{bmatrix}
75 & -19 & 0 & 0 \\
-19 & 26 & -7 & 0 \\
0 & -7 & 38 & -31 \\
0 & 0 & -31 & 42
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
= \begin{bmatrix}
56 \times 10^4 \\
0 \\
0 \\
0
\end{bmatrix}
\]

This is a conservative non-gyroscopic system with a characteristic equation of motion as follows:

\[ M\ddot{q} + Kq = 0 \]

where:

\[
M = \begin{bmatrix}
1.075 & 0 & 0 & 0 \\
0 & 3.25 & 0 & 0 \\
0 & 0 & 0.95 & 0 \\
0 & 0 & 0 & 2.14
\end{bmatrix}
\]

\[
K = \begin{bmatrix}
75 & -19 & 0 & 0 \\
-19 & 26 & -7 & 0 \\
0 & -7 & 38 & -31 \\
0 & 0 & -31 & 42
\end{bmatrix}
\times 10^4
\]

In state space representation, the system matrix (equation 8) is given by:
The matrix $A$ of the eigenvalue problem is then (see equation 9):

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & -697,674 & 176,744 & 0 & 0 \\ 0 & 0 & 0 & 0 & 58,461 & -80,000 & 21,538 & 0 \\ 0 & 0 & 0 & 0 & 0 & 73,684 & -400,000 & 326,315 \\ 1 & -1.776 \times 10^{-15} & 2.664 \times 10^{-15} & 1.776 \times 10^{-15} & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.065 \times 10^{-14} & -7.105 \times 10^{-15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.329 \times 10^{-15} & -2.842 \times 10^{-14} & 1 & 0 \\ -3.552 \times 10^{-15} & 7.105 \times 10^{-15} & 1 & 0 & 0 & 0 & 0 & 0 \\ -3.552 \times 10^{-15} & 5.329 \times 10^{-15} & -2.842 \times 10^{-14} & 1 & 0 & 0 & 0 & 0 \\ \end{bmatrix}$$
With the computer program EIGRF from International Mathematical and Statistical Libraries, Inc. (1975) (Appendix A), we obtained the eigenvalues and normalized eigenvectors that are listed in Table 9 of Appendix B.

The eigenvectors were taken to be $M$ orthogonal so that:

$$\phi_i^T M \phi_i = 1 \quad i = 1, \ldots, n$$

and

$$\phi_i^T K \phi_i = \omega_i^2 \quad i = 1, \ldots, n$$

In state space,

$$\phi_i^T M^* \phi_i = 1 \quad i = 1, \ldots, 2n$$

$$\phi_i^T K^* \phi_i = \omega_i \quad i = 1, \ldots, 2n$$

and these relations are checked and found to be correct.

System Response

The computer program for frequency response was used to get the graphs of frequency versus amplitude and phase angle that can be seen in Figures 27 through 42.

The input data was the matrix $A$ (equation 61) and the input is defined by:
Conservative non-gyroscopic system

Velocity $q_1$ versus $\omega$

Fig. 27. Conservative non-gyroscopic system: amplitude $I$ versus frequency (cycles/sec).
Fig. 28. Conservative non-gyroscopic system amplitude $q_2$ versus frequency (cycles/sec).
Fig. 29. Conservative non-gyroscopic system amplitude $q_3$ versus frequency (cycles/sec).
Conservative non-gyroscopic system

Velocity $q_4$ versus $\omega$

Fig. 30. Conservative non-gyroscopic system amplitude $q_4$ versus frequency (cycles/sec).
Fig. 31. Conservative non-gyroscopic system amplitude $x_1$ versus frequency (cycles/sec).
Conservative non-gyroscopic system

Displacement $x_2$ versus $\omega$

Fig. 32. Conservative non-gyroscopic system amplitude VI versus frequency (cycles/sec).
Fig. 33. Conservative non-gyroscopic system amplitude $x_3$ versus frequency (cycles/sec).
Conservative non-gyroscopic system

Displacement $x_4$ versus $\omega$

Fig. 34. Conservative non-gyroscopic system amplitude VIII versus frequency (cycles/sec).
Conservative non-gyroscopic system

Phase angle \( q_1 \) versus \( \omega \)

Fig. 35. Conservative non-gyroscopic system phase angle \( q_1 \) versus frequency (cycles/sec).
Fig. 36. Conservative non-gyroscopic system phase angle $q_2$ versus frequency (cycles/sec).
Conservative non-gyroscopic system

Phase angle \( q_3 \) versus \( \omega \)

Fig. 37. Conservative non-gyrosopic system phase angle III versus frequency (cycles/sec).
Conservative non-gyroscopic system

Fig. 38. Conservative non-gyroscopic system phase angle \( \phi_4 \) versus frequency (cycles/sec).
Fig. 39. Conservative non-gyroscopic system phase angle $\phi_1$ versus frequency (cycles/sec).
Conservative non-gyroscopic system
Phase angle \((x_2)\) versus \(\omega\)

Fig. 40. Conservative non-gyroscopic system phase angle \(x_2\) versus frequency (cycles/sec).
Fig. 41. Conservative non-gyroscopic system phase angle $x_3$ versus $\omega$. 

Conservative non-gyroscopic system

Phase angle $(x_3)$ versus $\omega$
Conservative non-gyroscopic system phase angle \( (x_4) \) versus \( \omega \).

Fig. 42. Conservative non-gyroscopic system phase angle \( x_4 \) versus frequency (cycles/sec).
As can be noted, we used a unit step sinusoidal input and a range of frequencies that includes the values of the eigenvalues. We can see in the graphs of frequency versus amplitude that the four peaks of the response curves correspond exactly to the values of the eigenvalues. In the graphs of frequency versus phase angle, it is easily seen that the sudden changes in slope take place for the frequencies of the eigenvalues.

System Identification

The computer program for matrix identification was used in order to get the original matrix A of this example. A $0.5/1 \times 10^4$ or $0.005\%$ random error was added to the frequency response solution, but the original matrix was not obtained.

Finally, different percentages of random error were added to the solution. The following is a list of the percentages considered:

- $0$ or $0\%$ R.E.
- $0.5/1 \times 10^4$ or $0.005\%$ R.E.
- $0.5/1 \times 10^5$ or $0.0005\%$ R.E.
- $0.5/1 \times 10^6$ or $0.00005\%$ R.E.
- $0.5/1 \times 10^7$ or $0.000005\%$ R.E.
- $0.5/1 \times 10^8$ or $0.0000005\%$ R.E.
- $0.5/1 \times 10^{10}$ or $0.000000005\%$ R.E.

\[
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix},
\]

$\omega_{\text{increment}} = 10 \text{ cycles/sec.}$

$\omega_{\text{maximum}} = 1,000 \text{ cycles/sec.}$
Only in the last case, \(0.5/1 \times 10^{10}\), the identified matrix is exactly the same as the original. For the other cases between \(0.5/1 \times 10^{4}\) and \(0.5/1 \times 10^{8}\), a graph was plotted to observe the behavior of both the real and complex part of the eigenvalues. This is shown in Figure 43.

The input used in these previous cases was:

\[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} \quad \bar{b}_1 = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} \quad \bar{b}_2 = \begin{bmatrix}
\omega_1 = 120 \text{ cycles/sec.} \\
\omega_2 = 960 \text{ cycles/sec.} \\
\end{bmatrix}
\]

It can be noticed that the input is a unit sine and unit cosine, because if we input only sine or cosine, we will have a singular matrix during the manipulation process.

As in the case of the damped non-gyroscopic system, the identification computer program was used 100 times and averaged for the added \(0.5/1 \times 10^{10}\) random error. The matrix obtained was very similar to the original matrix A, i.e. (see equation 61):

\[
B = \begin{bmatrix}
-2.214 \times 10^{-7} & 2.609 \times 10^{-5} & -3.256 \times 10^{-4} & 3.927 \times 10^{-4} & -697,673.999 & 176,743.970 & 0.251523 & -0.259750 \\
-2.211 \times 10^{-7} & 2.318 \times 10^{-5} & -2.813 \times 10^{-4} & 3.365 \times 10^{-4} & 58,461.000 & -80,000.025 & 21,538.216 & -0.222124 \\
3.776 \times 10^{-8} & -3.882 \times 10^{-6} & 4.630 \times 10^{-5} & -5.454 \times 10^{-5} & -1.368 \times 10^{-4} & 73,684.006 & -400,000.035 & 326,315.036 \\
7.813 \times 10^{-9} & -7.755 \times 10^{-7} & 9.114 \times 10^{-6} & -1.076 \times 10^{-5} & -2.769 \times 10^{-5} & 8.416 \times 10^{-4} & 144,858.992 & -196,260.992 \\
1.000 & -9.571 \times 10^{-9} & 7.214 \times 10^{-8} & -6.556 \times 10^{-8} & -4.623 \times 10^{-7} & 8.107 \times 10^{-6} & -5.310 \times 10^{-5} & 5.009 \times 10^{-5} \\
-5.171 \times 10^{-11} & 1.000 & -4.176 \times 10^{-8} & 4.700 \times 10^{-8} & 1.596 \times 10^{-7} & -4.009 \times 10^{-6} & 3.178 \times 10^{-5} & -3.206 \times 10^{-5} \\
-4.366 \times 10^{-12} & 2.317 \times 10^{-10} & 0.999 & 1.119 \times 10^{-9} & 1.160 \times 10^{-8} & -1.812 \times 10^{-7} & 1.131 \times 10^{-6} & -9.743 \times 10^{-7} \\
-4.172 \times 10^{-11} & 4.379 \times 10^{-9} & -5.613 \times 10^{-8} & 1.000 & 1.566 \times 10^{-7} & -5.091 \times 10^{-6} & 4.329 \times 10^{-5} & -4.453 \times 10^{-5}
\end{bmatrix}
\]
Fig. 43. Matrix error eigenvalues versus random error.
In addition to that, the error matrix E was calculated for every case, and the eigenvalues and eigenvectors obtained are depicted in Tables 10 through 14 of Appendix B. It can be noticed that the complex part of the eigenvalues of the error matrix is very small compared with the eigenvalues of the original matrix A. The eigenvalues of the real part are larger compared with the original eigenvalues, and for a $0.5/1 \times 10^4$ random error there is one eigenvalue which is very large (493.6) compared with the values of the original matrix A.

**Discussion of Results**

The simulated system response for the damped non-gyroscopic system shows that the system is stable. It can be seen that the peak occurs for the frequency corresponding to the value of the imaginary part of the two complex conjugate roots, in the frequency versus amplitude graph of the original system. For the critically damped case, we obtain practically the same results for amplitude and phase angle because the original system is only approximately 5% overdamped compared with the critically damped.

Reducing the damping to a 10% of the initial value caused an increase in amplitude up to 400% of the unit sinusoidal input to the system initially.

Also for the underdamped case the peaks occur at the frequency that corresponds to the imaginary part of the four complex conjugate roots of the system.
The error eigenvalues obtained for the added random errors (0.2% to 1.0%) are not bigger than 5% of the original eigenvalues. The identification of the system matrix was performed one hundred times and averaged with a 0.5% added random error for all the previous cases. The matrices obtained through the new identification process are very good approximations to the original system matrix. For the case of the conservative non-gyroscopic system, the response shows the peaks occurring, as expected, at a frequency corresponding to the complex part of the eight complex conjugate eigenvalues. But for the phase angle, the graphs show sudden changes in slope because there is no damping in the system.

In order to identify the system matrix, several percentages of random error were assumed. Only good results are obtained for very low percentages of error. The identified system matrix was obtained, but for a $0.5/10^{10}$ random error. The identification results are improved by performing the identification process 100 times and then averaging it.
CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

The purpose of this work was to present a method to determine the modal parameters of a system. This method was verified by finding the response of two different sample cases and the identification tests were then performed for both of them.

The first case was a damped non-gyroscopic system. The modeling of this system, setting of the equations of motion, obtaining the modal data, and identifying the matrix of the system were easily performed. The results obtained for identification were accurate. The input used for the identification process was a unit sinusoidal force. The process could also be done with a unit cosinusoidal force. Both inputs can be used at the same time, with same results obtained. For the case of measurement error, results indicate that averaging can improve the accuracy of the identification process.

The second case of a conservative non-gyroscopic system was also modeled and solved as in the first case with the same method. The identification of the system matrix was then obtained in this case with both unit sinusoidal and cosinusoidal forces as inputs. If we use only a unit sinusoidal or cosinusoidal force as an input, we will find singularities during the manipulation process.
that will not permit us to get a solution. This is caused because of the lack of damping in this particular case. The error added in order to simulate real measuring conditions had to be much smaller than in the first case in order to get the original matrix A. Therefore, measurements for such systems should be very accurate. Based on the methods studied in this report, it is recommended that future research is needed in the following directions.

1. Use different sets of frequencies to perform the identification of the system matrix, and perform an averaging process over these sets.

2. Experimentally confirm the method to identify the system matrix from the real and complex part of the frequency response.
APPENDICES
APPENDIX A

COMPUTER PROGRAMS FOR DATA ANALYSIS
PROGRAM 1

COMPUTER PROGRAM FOR EIGENVALUES AND NORMALIZED EIGENVECTORS

0001 INTEGRAL X1, X2, X3, IRC
0002 REAL A(AA), X1(X1), X2(X2), X3(X3), X4(X4), X5(X5)
0003 COMPLEX X1, X2, X3, IRC
0004 EQUIVALENCE (W(1), R(1)), (Z(1), R(1))
0005 DATA A/7
0006 N=4
0007 IA=4
0008 I=4
0009 IJ=2
0010 CALL EIGENV (A, N, IA, X1, R(1), Z(1), IRC)
0011 WRITE (*, 10) IFER, W(1)
0012 20 FORMAT (1X, 10D12.12, 1X, 12D12.12, 1X, 12D12.12)
0013 WRITE (*, 10)
0014 10 FORMAT (1X, 'EIGENVALUES', 1X)
0015 C, 50 = 1
0016 CC=50
0017 30 FORMAT (1X, 'EIGENVECTORS', 1X)
0018 50 CONTINUE
0019 C, 51
0020 FORMAT (1X, 'EIGENVECTORS', 1X)
0021 CO 9 = 1
0022 WRITE (*, 40) Z(K), J=1, N
0023 40 FORMAT (1X, 12D12.12, 1X, 12D12.12, 1X, 12D12.12)
0024 C, 5 CONTINUE
0025 CO 60 = 1
0026 CO 70 = 1
0027 CO 90 = 1
0028 CO 71 = 1
0029 CO 72 = 1
0030 CO 73 = 1
0031 CO 74 = 1
0032 CO 75 = 1
0033 80 FORMAT (1X, 'NORMALIZED EIGENVECTORS', 1X)
0034 CO 90 = 1
0035 WRITE (*, 80) Z(K), J=1, N
0036 CONTINUE
0037 STOP
0038 END
PROGRAM 2

COMPUTER PROGRAM FOR FREQUENCY RESPONSE

100 INIT
110 PAGE
120 REM: ****** -.---.-. - ******
130 REM: ****** -.---.-. - ******
140 PRINT "L ENTER THE TYPE OF DATA G_"
150 REM: ****** -.---.-. - ******
160 REM: ****** -.---.-. - ******
170 PRINT "J_ K1=1 FOR DATA FILE G_"
180 PRINT "J_ K1=2 FOR FUNCTIONS G_"
190 PRINT "J_ K1=3 FOR DIRECT INPUT G_"
200 PRINT "J_ ENTER THE TYPE OF DATA ,K1>G_"
210 INPUT K1
220 REM: ****** -.---.-. - ******
230 REM: ****** -.---.-. - ******
240 REM: ****** -.---.-. - ******
250 PRINT "L ENTER THE TYPE OF OUTPUT DO YOU WANT? "
260 REM
270 REM: ****** -.---.-. - ******
280 PRINT "J_ K2=1 FOR TABLES G_"
290 PRINT "J_ K2=2 FOR GRAPH G_"
300 PRINT "J_ K2=3 FOR OUTPUT FILE G_"
310 PRINT "J_ ENTER THE TYPE OF OUTPUT ,K2>G_"
320 INPUT K2
330 IF K1=1 THEN 680
340 IF K1=2 THEN 870
350 IF K1=3 THEN 360
360 PRINT "J_ DATA FROM DIRECT INPUT "
370 PRINT "L ENTER THE DIMENSION SIZE N >G_"
380 INPUT N
390 DIM A(N,N), A2(N,N), C(N,N), D1(N,1), D2(N,1), U1(N,1), U2(N,1)
400 DIM F(N,1), B(N,1)
410 PRINT "ENTER THE VALUES OF MATRIX A_6": SPACE(10)
420 FOR I=1 TO N
430 FOR J=1 TO N
440 PRINT USING 460: "ENTER A(',I,",",J,') = '?
450 INPUT A(I,J)
460 IMAGE SA.FD, 1A, FD.1A.S
470 NEXT J
480 NEXT I
490 PRINT "ENTER THE VALUES OF UL J": SPACE(10)
500 FOR I=1 TO N
510 PRINT USING 510: "UL(',I,') = '?
520 INPUT U1(I,1)
530 IMAGE 3A, FD.1A.S
540 NEXT I
550 PRINT "ENTER THE VALUES OF U2 J": SPACE(10)
560 FOR I=1 TO N
570 PRINT USING 590: "U2(',I,') = '?
580 INPUT U2(I,1)
590 IMAGE 3A, FD.1A.S
600 NEXT I
610 PRINT "ENTER THE OMEGA INCREMENT": SPACE(10)
620 PRINT "OMEGA = '?
630 INPUT WL
640 PRINT "ENTER THE FINN OMEGA, OMEGA MAX": SPACE(10)
650 PRINT "OMEGA MAX = '?
PROGRAM 2 (Continued)

660 INPUT W2
670 GO TO 910
680 PRINT "J_ DATA FROM DATA FILE G_"
690 CALL "OPEN", "FEIJO14 LFN: 1 TACCESS: FULL"
700 READ #1, N
710 DIM A(N,N), A2(N,N), C(N,N), D1(N,1), D2(N,1), U1(N,1), U2(N,1)
720 DIM F(N,1), B(N,1)
730 FOR I=1 TO N
740 FOR J=1 TO N
750 READ #1, A(I,J)
760 NEXT J
770 NEXT I
780 FOR I=1 TO N
790 READ #1, U1(I,1)
800 NEXT I
810 FOR I=1 TO N
820 READ #1, U2(I,1)
830 NEXT I
840 READ #1, W1
850 READ #1, W2
860 GO TO 910
870 REM FUNCTIONS FOR INPUTS
880 REM 
890 PRINT "J_ CLOSED FORM FUNCTIONS G_"
900 REM: GET YOUR DATA FROM FUNCTIONS
910 PRINT "J_ THE MAIN PROGRAM J_G_"
920 M=W2/W1
930 DIM E(M,N), H(M,N), W(M)
940 FOR L=1 TO M
950 W(L)=W1*L
960 A2=A2*W1
970 FOR I=1 TO N
980 A2(I,1)=A2(I,1)+W(L)*W(L)
990 NEXT I
1000 C=INV(A2)
1010 D1=0
1020 D2=0
1030 D1=A MPY U1
1040 D2=A MPY U2
1050 FOR I=1 TO N
1060 D1(I,1)=-D1(I,1)+W(L)*U2(I,1)
1070 D2(I,1)=-D2(I,1)-W(L)*U1(I,1)
1080 NEXT I
1090 F=0
1100 B=0
1110 F=C MPY D1
1120 B=C MPY D2
1130 FOR I=1 TO N
1140 E(L, I)=SQR(F(I,1)*F(I,1)+B(I,1)*B(I,1))
1150 SET DEGREES
1152 IF F(I,1)=0 THEN 1162
1155 IF B(I,1)=0 THEN 1166
1160 H(L, I)=ATN(B(I,1)/F(I,1))
1161 GO TO 1170
1162 H(L, I)=90
1163 IF B(I,1)>0 THEN 1200
1164 H(L, I)=-90
1165 GO TO 1200
1166 H(L, I)=0
1167 IF F(I,1)>0 THEN 1200
1168 H(L, I)=-180
1169 GO TO 1200
1170 IF B(I,1)<0 AND F(I,1)<0 THEN 1190
1171 IF B(I,1)<0 AND F(I,1)<0 THEN 1190
1180 GO TO 1200
1190 H(L, I)=H(L, I)-180
1200 NEXT I
1210 NEXT L
1220 PAGE
1230 PRINT "OMEGA   ":
1240 FOR I=1 TO N
1250 PRINT " ;";"A";I;": "
1260 NEXT I
1270 PRINT "J_J_"
1280 FOR I=1 TO M
1290 PRINT USING 1295: W(I)
1295 IMAGE 5D, X, S
1300 FOR J=1 TO N
1310 PRINT USING 1315: E(I, J)
1315 IMAGE 2E, X, S
1320 NEXT J
1330 PRINT
1340 NEXT I
1350 PRINT "J_J_"
1360 PRINT "PRESS (RETURN) TO CONTINUE"
1370 INPUT A$ 
1380 PAGE
1390 PRINT "OMEGA " ;
1400 FOR I=1 TO N
1410 PRINT " "; "FAY"; I; " ";
1420 NEXT I
1430 PRINT "J_J_"
1440 FOR I=1 TO M
1450 PRINT USING 1295: W(I)
1460 FOR J=1 TO N
1470 PRINT USING 1475: H(I, J)
1475 IMAGE 4D, 2D, X, S
1480 NEXT J
1490 PRINT
1500 NEXT I
1510 FOR J=1 TO N
1520 PRINT "J_J_"
1530 PRINT "PRESS (RETURN) TO CONTINUE"
1540 INPUT A$
1550 PAGE
1560 VIEWPORT 10.120.20.90
1570 WINDOW 0.1000.0.100
1580 AXIS 50.5
1590 MOVE 0.0
PROGRAM 2 (Continued)

1600 FOR I=1 TO M
1610 DRAW W(I),H(I,J)
1620 NEXT I
1630 HOME
1640 PRINT "I_ AMPLITUDE A";J;" VERSUS OMEGA ";
1650 PRINT @32,21:0.4
1660 PRINT " PRESS [RETURN] TO CONTINUE ";
1670 INPUT A$
1680 NEXT J
1690 FOR J=1 TO N
1700 PRINT "J_J_"
1710 INPUT A$
1720 PAGE
1730 VIEWPORT 10,120,20,90
1740 WINDOW 50,1000,-270,90
1750 AXIS 50,10
1760 MOVE 0,0
1770 FOR I=1 TO M
1780 DRAW W(I),H(I,J)
1790 NEXT I
1800 HOME
1810 PRINT "I_ PHASE ANGLE FAY(";J;") VERSUS OMEGA ";
1820 PRINT @32,21:0.4
1830 PRINT " PRESS [RETURN] TO CONTINUE ";
1840 INPUT A$
1850 NEXT J
1860 IF K2=3 THEN 1960
1870 REM: FILE SUBROUTINE FOR OUTPUT
1880 CALL "OPEN","FILE01" LFN:2 TACCESS:FULL
1890 WRITE #2,1:N,M
1900 FOR I=1 TO M
1910 WRITE #2,W(I)
1920 FOR J=1 TO N
1930 WRITE #2,E(I,J),H(I,J)
1940 NEXT J
1940 END
PROGRAM 3

COMPUTER PROGRAM FOR SYSTEM  MATRIX IDENTIFICATION

100 INIT
110 PAGE
120 PRINT "THE TYPE OF DATA"
130 PRINT "K1=1 FOR DIRECT INPUT"
140 PRINT "K1=2 FOR FUNCTIONS"
150 PRINT "K1=3 FOR DATA FILE"
160 PRINT "ENTER THE TYPE OF DATA K1"
170 INPUT K1
180 PRINT "TYPE OF OUTPUT REQUIRED"
190 PRINT "K2=1 FOR TABLES"
200 PRINT "K2=2 FOR GRAPHS"
210 PRINT "K2=3 FOR OUTPUT FILE"
220 PRINT "ENTER THE TYPE OF OUTPUT REQUIRED K2"
230 INPUT K2
240 IF K1=1 THEN 270
250 IF K1=2 THEN 780
260 IF K1=3 THEN 590
270 PRINT "DATA FROM DIRECT INPUT"
280 PRINT "ENTER THE DIMENSION SIZE N"
290 INPUT N
300 DIM A(N,N), A2(N,N), C(N,N), D1(N,1), D2(N,1), U1(N,1), U2(N,1)
310 DIM F(N,1), B(N,1)
320 PRINT "ENTER THE MATRIX A"
330 FOR I=1 TO N
340 FOR J=1 TO N
350 PRINT USING 370: "ENTER A(I,J)"
360 INPUT A(I,J)
370 NEXT J
380 NEXT I
390 PRINT "ENTER THE MATRIX U1"
400 FOR I=1 TO N
410 FOR J=1 TO N
420 PRINT USING 440: "U1(I,J)"
430 INPUT U1(I,J)
440 NEXT J
450 NEXT I
460 PRINT "ENTER THE MATRIX U2"
470 FOR I=1 TO N
480 PRINT USING 500: "U2(I,J)"
490 INPUT U2(I,J)
PROGRAM 3 (Continued)

500 IMAGE 3A: FD. 5A: S
510 NEXT 1
520 PRINT "ENTER THE OMEGA INCREMENT";
530 INPUT M
540 PRINT "ENTER THE FINAL OMEGA, OMEGA MAX";
550 PRINT M
560 GO TO 820
570 INPUT N
580 PRINT "ENTER DATA FROM DATA FILE";
590 PRINT N
600 CALL "OPEN", FILE: 1, ACCESS: FULL"
610 READ 11, L
620 DIM A(L, L), A2(L, L), C(L, L), D1(L, L), D2(L, L), U1(L, L), U2(L, L)
630 DIM F(L, L), G(L, L)
640 FOR J = 1 TO L
650 FOR J = 1 TO L
660 READ 11, A(J, J)
670 NEXT J
680 NEXT L
690 FOR I = 1 TO L
700 READ 11, U1(I, I)
710 NEXT I
720 FOR I = 1 TO L
730 READ 11, U2(I, I)
740 NEXT I
750 FOR I = 1 TO L
760 READ 11, W1(I, I)
770 NEXT I
780 REM: FUNCTIONS FOR INPUTS
790 REM: "CLOSED FORM FUNCTIONS"
800 PRINT "THE PROGRAM"
810 PRINT "FUNCTIONS"
820 PRINT "F(J, I) = 0"
830 FOR L = 1 TO 100
840 FOR I = 1 TO M
850 FOR J = 1 TO M
860 A2(J, I) = A(J, I)
870 NEXT J
880 NEXT I
890 NEXT L
PROGRAM 3 (Continued)

880 FOR I = 1 TO N
910 NEXT I
920 D1 = 0
930 D2 = 0
940 Dt = A t1 PY Ul
950 D22 = A l' IPY U2
960 FOR I = 1 TO N
970 Dt(I, I) = Dt(I, I) - W <L> • U2(I, I)
980 D2(I, I) = D2(I, I) - W <L> • U2(I, I)
990 NEXT I
1000 F = O
1010 B = O
1020 F = C f'PV D1
1030 B = C MPV D2
1040 ZS = RND(-1)
1050 FOR I = 1 TO N
1060 E(I, L) = F(I, I) • W <L> - U2(I, I) • (1 + RND(I) - 0, 5)/100)
1070 H(I, L) = E(I, I)
1080 NEXT I
1090 NEXT I
1100 A1 = INV<A>
1110 A4 = E f'IPY A3
1120 T1 = T1 + A4
1130 PRINT T1 ... 
1140 NEXT T
1150 T1 = T1 / 100
1160 PRINT 'L THE AVERAGED MATRIX=J: T1
1170 STOP
1190 END
<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>$w_1 = -0.440590 \times 10^{-1}$</td>
<td>0.0</td>
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<tr>
<td>$w_2 = -0.719380 \times 10^{-1}$</td>
<td>0.0</td>
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<tr>
<td>$w_3 = -0.116190 \times 10^{-1}$</td>
<td>0.0</td>
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<tr>
<td>$w_4 = -0.234460 \times 10^{-1}$</td>
<td>0.0</td>
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</thead>
<tbody>
<tr>
<td>$-0.779390 \times 10^{-1}$</td>
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<tr>
<td>$0.586220 \times 10^{-1}$</td>
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<tr>
<td>$0.176750 \times 10^{-1}$</td>
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<tr>
<td>$-0.132950 \times 10^{-1}$</td>
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</tr>
</tbody>
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<table>
<thead>
<tr>
<th>NORMALIZED EIGENVECTORS</th>
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<th></th>
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<tbody>
<tr>
<td>$-0.779380 \times 10^{-1}$</td>
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<tr>
<td>$0.586220 \times 10^{-1}$</td>
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<tr>
<td>$0.176750 \times 10^{-1}$</td>
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</table>
TABLE 2
DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.2% RANDOM ERROR

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 ) = 0.492950</td>
<td>( 0.1660370 )</td>
<td>( 0.727220 )</td>
<td>( 0.209370 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \omega_2 ) = 0.492950</td>
<td>( 0.1660370 )</td>
<td>( 0.727220 )</td>
<td>( 0.209370 )</td>
<td>0.596060</td>
<td>0.162350</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EIGENVECTORS</th>
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<td>( 0.727220 )</td>
<td>( 0.209370 )</td>
<td>( 0.596060 )</td>
<td>0.162350</td>
</tr>
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<td>( \omega_2 ) = 0.492950</td>
<td>( 0.1660370 )</td>
<td>( 0.727220 )</td>
<td>( 0.209370 )</td>
<td>( 0.596060 )</td>
<td>0.162350</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NORMALIZED EIGENVECTORS</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 ) = 0.492950</td>
<td>( 0.1660370 )</td>
<td>( 0.727220 )</td>
<td>( 0.209370 )</td>
<td>( 0.596060 )</td>
<td>0.162350</td>
</tr>
<tr>
<td>( \omega_2 ) = 0.492950</td>
<td>( 0.1660370 )</td>
<td>( 0.727220 )</td>
<td>( 0.209370 )</td>
<td>( 0.596060 )</td>
<td>0.162350</td>
</tr>
</tbody>
</table>
### TABLE 3

**DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.4% RANDOM ERROR**

**EIGENVALUES**

| \( w(1) \) | 0.241270 - 0.6 | 0.4 |
|\( w(2) \) | 0.416310 - 0.6 | 0.791250 - 0.2 |
|\( w(3) \) | 0.419610 - 0.6 | -0.741250 - 0.2 |
|\( w(4) \) | -0.255870 - 0.6 | 0.9 |

**EIGENVECTORS**

| \( \lambda_1 \) | 0.579520 0.0 |
| \( \lambda_2 \) | -0.599490 0.0 |
| \( \lambda_3 \) | -0.423140 0.0 |
| \( \lambda_4 \) | -0.354570 0.0 |

| \( 0.596420 0.0 \) | 1.0 | \( 0.596420 0.0 \) | \( 0.404460 0.0 \) | \( 0.404460 0.0 \) | \( 0.721360 0.1 \) | \( 0.0 \) |
| \( -0.251390 0.0 \) | 1.0 | \( -0.251390 0.0 \) | \( -0.867960 0.0 \) | \( -0.251390 0.0 \) | \( 0.679600 0.0 \) | \( 0.472290 0.0 \) | \( 0.0 \) |
| \( -0.246750 0.0 \) | 1.0 | \( -0.246750 0.0 \) | \( 0.574510 0.0 \) | \( -0.246750 0.0 \) | \( -0.574510 0.0 \) | \( -0.206170 0.0 \) | \( 0.0 \) |
| \( -0.254210 0.0 \) | 1.0 | \( -0.254210 0.0 \) | \( 0.102720 0.0 \) | \( -0.254210 0.0 \) | \( -0.102720 0.0 \) | \( -0.468730 0.0 \) | \( 0.0 \) |

**NORMALIZED EIGENVECTORS**

| \( \lambda_1 \) | 0.596420 0.0 |
| \( \lambda_2 \) | -0.599490 0.0 |
| \( \lambda_3 \) | -0.423140 0.0 |
| \( \lambda_4 \) | -0.354570 0.0 |

| \( 0.596420 0.0 \) | 1.0 | \( 0.596420 0.0 \) | \( 0.204390 0.0 \) | \( 0.204390 0.0 \) | \( 0.208390 0.0 \) | \( 0.812570 0.0 \) | \( 0.0 \) |
| \( -0.596190 0.0 \) | 1.0 | \( -0.596190 0.0 \) | \( -0.262570 0.0 \) | \( -0.262570 0.0 \) | \( 0.762570 0.0 \) | \( 0.532010 0.0 \) | \( 0.0 \) |
| \( -0.613950 0.0 \) | 1.0 | \( -0.613950 0.0 \) | \( 0.905300 -0.01 \) | \( -0.613950 0.0 \) | \( -0.905300 -0.01 \) | \( 0.232170 0.0 \) | \( 0.0 \) |
| \( -0.641240 0.0 \) | 1.0 | \( -0.641240 0.0 \) | \( 0.129380 0.0 \) | \( -0.641240 0.0 \) | \( -0.129380 0.0 \) | \( -0.199500 0.0 \) | \( 0.0 \) |
TABLE 4
DAMPED NON-GYROSCOPIC SYSTEM – ERROR MATRIX EIGENVALUES AND EIGENVECTORS – 0.6% RANDOM ERROR

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 ) = -0.1249670 00 0.0</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_2 ) = 0.3642370 01 0.0</td>
<td>1</td>
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<tr>
<td>( \lambda_3 ) = 0.2612490 02 0.0</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_4 ) = -0.111260 01 0.0</td>
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</tbody>
</table>

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<thead>
<tr>
<th>EIGENVECTORS</th>
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<tbody>
<tr>
<td>-0.166670 01 0.0</td>
<td>-0.228180 01 0.0</td>
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<tr>
<td>-0.290070 00 0.0</td>
<td>0.115940 01 0.0</td>
</tr>
<tr>
<td>-0.293600 00 0.0</td>
<td>0.899310 00 0.0</td>
</tr>
<tr>
<td>-0.904880 00 0.0</td>
<td>0.933870 00 0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NORMALIZED EIGENVECTORS</th>
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<tbody>
<tr>
<td>-0.858730 00 0.0</td>
<td>-0.705340 00 0.0</td>
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<tr>
<td>-0.149450 00 0.0</td>
<td>0.440410 00 0.0</td>
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<tr>
<td>-0.151270 00 0.0</td>
<td>0.313540 00 0.0</td>
</tr>
<tr>
<td>-0.466220 00 0.0</td>
<td>0.335239 00 0.0</td>
</tr>
</tbody>
</table>
### TABLE 5

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.8% RANDOM ERROR

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>EIGENVECTORS</th>
<th>NORMALIZED EIGENVECTORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \begin{bmatrix} 0.8% \text{ RANDOM ERROR} \end{bmatrix} )</td>
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</tbody>
</table>
**TABLE 6**

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 1.0% RANDOM ERROR

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>0.190490-01</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>0.680480-02</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>0.179490-01</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>0.195420-01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EIGENVECTORS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{v}_1 )</td>
<td>0.115420 01 0.0</td>
</tr>
<tr>
<td>( \mathbf{v}_2 )</td>
<td>0.214970 01 -0.324960 01 0.214970 01 0.324960 01 0.649000 00 0.0</td>
</tr>
<tr>
<td>( \mathbf{v}_3 )</td>
<td>0.230910 01 0.268720 01 0.230910 01 -0.268720 01 -0.280500 01 0.0</td>
</tr>
<tr>
<td>( \mathbf{v}_4 )</td>
<td>0.273540 00 0.102790 01 0.273540 00 -0.102790 01 0.254480 01 0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NORMALIZED EIGENVECTORS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{u}_1 )</td>
<td>0.756630 00 0.0</td>
</tr>
<tr>
<td>( \mathbf{u}_2 )</td>
<td>0.115390 01 0.537620 00 0.115390 01 -0.537620 00 -0.872190 00 -0.360700 00 0.727460 00 0.0</td>
</tr>
<tr>
<td>( \mathbf{u}_3 )</td>
<td>0.730840 00 0.872190 00 -0.730840 00 -0.872190 00 -0.497430 00 0.0</td>
</tr>
<tr>
<td>( \mathbf{u}_4 )</td>
<td>0.265080-01 0.138060 00 0.318920 00 -0.318920 00 0.138060 00 0.458380 00 0.0</td>
</tr>
<tr>
<td>( \mathbf{u}_5 )</td>
<td>0.391710 00 0.360700 00 -0.169450 00 -0.360700 00 0.169450 00 0.727460 00 0.0</td>
</tr>
</tbody>
</table>
TABLE 7
CRITICALLY DAMPED NON-CYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>EIGENVECTORS</th>
<th>NORMALIZED EIGENVECTORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 = -0.4031 )</td>
<td>( \begin{bmatrix} 0.702280 &amp; -0.286050 &amp; 0.702280 &amp; 0.702280 &amp; 0.702280 \ 0.184230 &amp; -0.537070 &amp; 0.184230 &amp; 0.184230 &amp; 0.184230 \ -0.490740 &amp; -0.597140 &amp; -0.490740 &amp; -0.490740 &amp; -0.490740 \ -0.680370 &amp; -0.629400 &amp; -0.680370 &amp; -0.680370 &amp; -0.680370 \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0.659100 &amp; -0.188060 &amp; 0.659100 &amp; 0.659100 &amp; 0.659100 \ 0.989250 &amp; -0.385210 &amp; 0.989250 &amp; 0.989250 &amp; 0.989250 \ -0.314930 &amp; -0.452410 &amp; -0.314930 &amp; -0.314930 &amp; -0.314930 \ -0.547200 &amp; -0.653490 &amp; -0.547200 &amp; -0.547200 &amp; -0.547200 \end{bmatrix} )</td>
</tr>
<tr>
<td>( \lambda_2 = -0.43120 )</td>
<td>( \begin{bmatrix} 0.286050 &amp; 0.702280 &amp; 0.286050 &amp; 0.286050 &amp; 0.286050 \ 0.537070 &amp; 0.184230 &amp; 0.537070 &amp; 0.537070 &amp; 0.537070 \ 0.490740 &amp; 0.597140 &amp; 0.490740 &amp; 0.490740 &amp; 0.490740 \ 0.680370 &amp; 0.629400 &amp; 0.680370 &amp; 0.680370 &amp; 0.680370 \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0.184440 &amp; -0.547200 &amp; 0.184440 &amp; 0.184440 &amp; 0.184440 \ 0.593940 &amp; -0.385210 &amp; 0.593940 &amp; 0.593940 &amp; 0.593940 \ -0.314930 &amp; -0.452410 &amp; -0.314930 &amp; -0.314930 &amp; -0.314930 \ -0.653490 &amp; -0.747200 &amp; -0.653490 &amp; -0.653490 &amp; -0.653490 \end{bmatrix} )</td>
</tr>
<tr>
<td>( \lambda_3 = -0.3219 )</td>
<td>( \begin{bmatrix} 0.104230 &amp; -0.286050 &amp; 0.102280 &amp; 0.102280 &amp; 0.102280 \ 0.593940 &amp; 0.184440 &amp; 0.593940 &amp; 0.593940 &amp; 0.593940 \ 0.232270 &amp; 0.593940 &amp; 0.232270 &amp; 0.232270 &amp; 0.232270 \ 0.753270 &amp; 0.184440 &amp; 0.753270 &amp; 0.753270 &amp; 0.753270 \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0.439710 &amp; -0.184440 &amp; 0.439710 &amp; 0.439710 &amp; 0.439710 \ 0.989250 &amp; -0.547200 &amp; 0.989250 &amp; 0.989250 &amp; 0.989250 \ -0.547200 &amp; -0.989250 &amp; -0.547200 &amp; -0.989250 &amp; -0.989250 \ -0.184440 &amp; -0.439710 &amp; -0.184440 &amp; -0.439710 &amp; -0.439710 \end{bmatrix} )</td>
</tr>
<tr>
<td>( \lambda_4 = -0.3214 )</td>
<td>( \begin{bmatrix} 0.0 &amp; 0.702280 &amp; 0.0 &amp; 0.702280 &amp; 0.0 \ 0.0 &amp; 0.184230 &amp; 0.0 &amp; 0.184230 &amp; 0.0 \ 0.0 &amp; 0.490740 &amp; 0.0 &amp; 0.490740 &amp; 0.0 \ 0.0 &amp; 0.680370 &amp; 0.0 &amp; 0.680370 &amp; 0.0 \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0.0 &amp; 0.104230 &amp; 0.0 &amp; 0.104230 &amp; 0.0 \ 0.0 &amp; 0.439710 &amp; 0.0 &amp; 0.439710 &amp; 0.0 \ 0.0 &amp; 0.989250 &amp; 0.0 &amp; 0.989250 &amp; 0.0 \ 0.0 &amp; 0.439710 &amp; 0.0 &amp; 0.439710 &amp; 0.0 \end{bmatrix} )</td>
</tr>
</tbody>
</table>
# Table 8

Underdamped Non-Gyroscopic System Eigenvalues and Eigenvectors

<table>
<thead>
<tr>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.135650$</td>
</tr>
<tr>
<td>$0.927510$</td>
</tr>
<tr>
<td>$-0.383750$</td>
</tr>
<tr>
<td>$0.247090$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.153690$</td>
</tr>
<tr>
<td>$0.901210$</td>
</tr>
<tr>
<td>$-0.383750$</td>
</tr>
<tr>
<td>$0.247090$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Normalized Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.892460$</td>
</tr>
<tr>
<td>$0.553490$</td>
</tr>
<tr>
<td>$0.339590$</td>
</tr>
<tr>
<td>$0.161760$</td>
</tr>
</tbody>
</table>
Damped non-gyroscopic system

Velocity $q_1$ versus $\omega$

Fig. 2. Damped non-gyroscopic system amplitude $I$ versus frequency (cycles/sec).
Fig. 3. Damped non-gyroscopic system amplitude $q_2$ versus frequency (cycles/sec).
Fig. 4. Damped non-gyroscopic system amplitude $x_1$ versus frequency (cycles/sec).
Fig. 5. Damped non-gyroscopic system amplitude $x_2$ versus frequency (cycles/sec).
Fig. 6. Damped non-gyroscopic system phase angle $\varphi_1$ versus frequency (cycles/sec).
FIG. 7. Damped non-gyroscopic system phase angle $\theta$ versus frequency (cycles/sec).
Fig. 8. Damped non-gyroscopic system phase angle (\(x_1\)) versus frequency (cycles/sec).
Fig. 9. Damped non-gyroscopic system phase angle $\phi_2$ versus frequency (cycles/sec).
Critically damped system

Velocity $q_1$ versus $\omega$.

Fig. 10. Critically damped system amplitude $I$ versus frequency (cycles/sec).
Fig. 11. Critically damped system amplitude $q_2$ versus frequency (cycles/sec).
Fig. 12. Critically damped system amplitude $x_1$ versus frequency (cycles/sec).
Fig. 13. Critically damped system amplitude $x_2$ versus frequency (cycles/sec).
Fig. 14. Critically damped system phase angle $\theta_1$ versus frequency (cycles/sec).
Fig. 15. Critically damped system phase angle $\phi_2$ versus frequency (cycles/sec).
Critically damped system

Phase angle ($x_1$) versus $\omega$

Fig. 16. Critically damped system phase angle III versus frequency (cycles/sec).
Fig. 17. Critically damped system phase angle $\phi_2$ versus frequency (cycles/sec).

Critically damped system

Phase angle $\phi_2$ versus $\omega$

PHASE ANGLE ($\phi_2$)

0 90 180 270

-90 -180 -270

0 1 2 3 4 5 6 7 8 9 10

$\omega$
Underdamped system

Velocity $q_1$ versus $\omega$

Fig. 18. Underdamped system amplitude $q_1$ versus frequency (cycles/sec).
Fig. 19. Underdamped system velocity $q_2$ versus frequency (cycles/sec).
Underdamped system
Displacement $x_1$ versus $\omega$

Fig. 20. Underdamped system amplitude $x_1$ versus frequency (cycles/sec).
Fig. 21. Underdamped system amplitude $x_2$ versus frequency (cycles/sec).
Fig. 22. Underdamped system phase angle $\phi_1$ versus frequency (cycles/sec).
Fig. 23. Underdamped system phase angle $\theta_2$ versus frequency (cycles/sec).
Fig. 24. Underdamped system phase angle $\angle x_1$ versus frequency (cycles/sec).
Fig. 25. Underdamped system phase angle vs. frequency (cycles/sec).
Fig. 27. Conservative non-gyroscopic system - amplitude $q_1$ versus frequency (cycles/sec).
Conservative non-gyroscopic system
Velocity $q_2$ versus $\omega$

Fig. 28. Conservative non-gyroscopic system amplitude II versus frequency (cycles/sec).
Fig. 29. Conservative non-gyroscopic system amplitude $q_3$ versus frequency (cycles/sec).
Conservative non-gyroscopic system

Velocity $q_4$ versus $\omega$

Fig. 30. Conservative non-gyroscopic system amplitude IV versus frequency (cycles/sec).
Conservative non-gyroscopic system

Displacement $x_1$ versus $\omega$

Fig. 31. Conservative non-gyroscopic system amplitude $V$ versus frequency (cycles/sec).
Conservative non-gyroscopic system

Displacement $x_2$ versus $\omega$

Fig. 32. Conservative non-gyroscopic system amplitude VI versus frequency (cycles/sec).
Conservative non-gyroscopic system

Displacement $x_3$ versus $\omega$

Fig. 33. Conservative non-gyroscopic system amplitude VII versus frequency (cycles/sec).
Conservative non-gyroscopic system

Displacement $x_4$ versus $\omega$

Fig. 34. Conservative non-gyroscopic system amplitude VIII versus frequency (cycles/sec).
Fig. 35. Conservative non-gyroscopic system phase angle $q_1$ versus frequency (cycles/sec).
Conservative non-gyroscopic system

Phase angle ($q_2$) versus $\omega$

Fig. 36. Conservative non-gyroscopic system phase angle $q_2$ versus frequency (cycles/sec).
Conservative non-gyroscopic system

Phase angle \( \dot{q}_3 \) versus \( \omega \)

Fig. 37. Conservative non-gyroscopic system phase angle III versus frequency (cycles/sec).
Conservative non-gyroscopic system

Phase angle \( \phi_4 \) versus \( \omega \)

Fig. 38. Conservative non-gyroscopic system phase angle IV versus frequency (cycles/sec).
Conservative non-gyroscopic system

Phase angle ($\phi_1$) versus $\omega$

Fig. 39. Conservative non-gyroscopic system phase angle $\phi_1$ versus frequency (cycles/sec).
Fig. 40. Conservative non-gyroscopic system phase angle $\theta$ versus frequency (cycles/sec).
**Conservative non-gyroscopic system**

Phase angle \((x_3)\) versus \(\omega\)

---

**Fig. 41.** Conservative non-gyroscopic system phase angle VII versus frequency (cycles/sec).
Fig. 42. Conservative non-gyroscopic system phase angle $x_4$ versus frequency (cycles/sec).
PROGRAM 1

COMPUTER PROGRAM FOR EIGENVALUES AND NORMALIZED EIGENVECTORS

```
0001 INTEGER IA, IJCR, ICRA
0002 REAL A(N, N), IX(N), RZ(N), Z(J, I)
0003 COMPLEX*16 W(N), Sigma(N), V(N, N), Z(N, N)
0004 EQUIVALENCE (W(1) = RZ(1)), (Z(1, 1) = RZ(1))
0005 DATA I/0/
0006 K = 0
0007 IA = 0
0008 I = 0
0009 J = 0
0010 CALL EIGRF (A, N, IA, IJCR, ICRA, RZ, Z, WX, IFR)
0011 WRITE (*, 20) IER, WX(1)
0012 20 FORMAT (1X, 'EIGENVALUES')
0013 WRITE (*, 10)
0014 FORMAT (1X, '*12.6', 12X, '*12.5', 12X, '*12.2')
0015 CONTINUE
0016 FORMAT (1X, 'EIGENVECTORS')
0017 DO 30 K = 1, N
0018 WRITE (*, 51) I, I2, I3, I4
0019 51 FORMAT (1X, 'E12.2', 12X, 'E12.5', 12X, 'E12.2')
0020 CONTINUE
0021 FORMAT (1X, 'NORMALIZE EIGENVECTORS')
0022 WRITE (*, 20) IER, WX(1)
0023 20 FORMAT (1X, 'EIGENVALUES')
0024 CONTINUE
0025 FORMAT (1X, 'NORMALIZE EIGENVECTORS')
0026 WRITE (*, 10)
0027 FORMAT (1X, 'EIGENVALUES')
0028 CONTINUE
0029 FORMAT (1X, 'EIGENVALUES')
0030 CONTINUE
0031 WRITE (*, 20) IER, WX(1)
0032 20 FORMAT (1X, 'EIGENVALUES')
0033 CONTINUE
0034 WRITE (*, 10)
0035 CONTINUE
0036 STOP
0037 END
```
PROGRAM 2

COMPUTER PROGRAM FOR FREQUENCY RESPONSE

100 INIT
110 PAGE
120 REM:  ***** -.-.-.-.-.-  *****
130 REM:
140 PRINT "_ ENTER THE TYPE OF DATA G_
150 REM:
160 REM:  ***** -.-.-.-.-  *****
170 PRINT "_ K1=1 FOR DATA FILE G_
180 PRINT "_ K1=2 FOR FUNCTIONS G_
190 PRINT "_ K1=3 FOR DIRECT INPUT G_
200 PRINT "_ ENTER THE TYPE OF DATA ,K1>G_
210 INPUT K1
220 REM:
230 REM:  ***** -.-.-.-.-  *****
240 REM:
250 PRINT "_ WHAT TYPE OF OUTPUT DO YOU WANT?  "
260 REM:
270 REM:  ***** -.-.-.-.-  *****
280 PRINT "_ K2=1 FOR TABLES G_
290 PRINT "_ K2=2 FOR GRAPH G_
300 PRINT "_ K2=3 FOR OUTPUT FILE G_
310 PRINT "_ ENTER THE TYPE OF OUTPUT ,K2>G_
320 INPUT K2
330 IF K1=1 THEN 680
340 IF K1=2 THEN 870
350 IF K1=3 THEN 360
360 PRINT "_ DATA FROM DIRECT INPUT  
370 PRINT "_ ENTER THE DIMENSION SIZE N G_

PROGRAM 2 (Continued)

380 INPUT N
390 DIM A(N,N),A2(N,N),C(N,N),D1(N,1),D2(N,1),U1(N,1),U2(N,1)
400 DIM F(N,1),B(N,1)
410 PRINT "J_ENTER THE VALUES OF MATRIX A G_"
420 FOR I=1 TO N
430 FOR J=1 TO N
440 PRINT USING 460:"ENTER A(",I,"","J,"') G_ "
450 INPUT A(I,J)
460 IMAGE SA.FD,1A,FD.6A.S
470 NEXT J
480 NEXT I
490 PRINT "J_ENTER THE VALUES OF U1 J_G_"
500 FOR I=1 TO N
510 PRINT USING 530:"U1(",I,"') G_ "
520 INPUT U1(I,1)
530 IMAGE SA.FD,6A.S
540 NEXT I
550 PRINT "J_ENTER THE VALUES OF U2 J_G_"
560 FOR I=1 TO N
570 PRINT USING 590:"U2(",I,"') G_ "
580 INPUT U2(I,1)
590 IMAGE SA.FD,6A.S
600 NEXT I
610 PRINT "J_ENTER THE OMEGA INCREMENT"
620 PRINT "J_G_ OMEGA=";
630 INPUT W1
640 PRINT "J_ENTER THE FINAL OMEGA, OMEGA MAX"
650 PRINT "J_G_ OMAX=";
PROGRAM 2 (Continued)

660 INPUT W2
670 GO TO 910
680 PRINT "J_ DATA FROM DATA FILE G_
690 CALL "OPEN", "FEIJ014 LFN: I TACCESS: FULL"
700 READ #1, I
710 DIM A(N, N), A2(N, N), C(N, N), D1(N, 1), D2(N, 1), U1(N, 1), U2(N, 1)
720 DIM F(N, 1), B(N, 1)
730 FOR I=1 TO N
740 FOR J=1 TO N
750 READ #1: A(I, J)
760 NEXT J
770 NEXT I
780 FOR I=1 TO N
790 READ #1: U1(I, I)
800 NEXT I
810 FOR I=1 TO N
820 READ #1: U2(I, I)
830 NEXT I
840 READ #1: W1
850 READ #1: W2
860 GO TO 910
870 REM FUNCTIONS FOR INPUTS
880 REM.
890 PRINT "J_CLO SED FORM FUNCTIONS G_
900 REM: GET YOUR DATA FROM FUNCTIONS
910 PRINT "J_ THE MAIN PROGRAM J_G_
920 M=W2/W1
930 DIM E(M, N), M(M, N), W(M)
940 FOR L=1 TO M
950 W(L)=W1*L
960 A2=A MPY A
970 FOR I=1 TO N
980 A2(I, I)=A2(I, I)+W(L)*W(L)
990 NEXT I
PROGRAM 2 (Continued)

1000 C=INV(A2)
1010 D1=0
1020 D2=0
1030 D1=A MPY U1
1040 D2=A MPY U2
1050 FOR I=1 TO N
1060 D1(I,1)=-D1(I,1)+W(L)*U2(I,1)
1070 D2(I,1)=-D2(I,1)-W(L)*U1(I,1)
1080 NEXT I
1090 F=0
1100 B=0
1110 F=C MPY D1
1120 B=C MPY D2
1130 FOR I=1 TO N
1140 E(L, I)=SQR(F(I, 1)*F(I, 1)+B(I, 1)*B(I, 1))
1150 SET DEGREES
1152 IF F(I, 1)=0 THEN 1162
1155 IF B(I, 1)=0 THEN 1166
1160 H(L, I)=ATN(B(I, 1)/F(I, 1))
1161 GO TO 1170
1162 H(L, I)=90
1163 IF B(I, 1)>0 THEN 1200
1164 H(L, I)=90
1165 GO TO 1200
1166 H(L, I)=0
1167 IF F(I, 1)>0 THEN 1200
1168 H(L, I)=-180
1169 GO TO 1200
1170 IF B(I, 1)<0 AND F(I, 1)<0 THEN 1190
1171 IF B(I, 1)>0 AND F(I, 1)<0 THEN 1190
1180 GO TO 1200
1190 H(L, I)=H(L, I)-180
1200 NEXT I
1210 NEXT L
1220 PAGE
1230 PRINT "OMEGA ":
1240 FOR I=1 TO N
1250 PRINT " ;"A";I," ;
1260 NEXT I
1270 PRINT "J_J_"
1280 FOR I=1 TO M
1290 PRINT USING 1295.W(I)
1295 IMAGE 5D.X.S
1300 FOR J=1 TO N
1310 PRINT USING 1315.E(I,J)
1315 IMAGE 2E.X.S
1320 NEXT J
1330 PRINT
1340 NEXT I
1350 PRINT ".J_."
1360 PRINT "PRESS (RETURN) TO CONTINUE":
1370 INPUT A$,
1380 PAGE
1390 PRINT "OMEGA ":
1400 FOR I=1 TO N
1410 PRINT ".I:.FAYI;"I";I"
1420 NEXT I
1430 PRINT ".J_."
1440 FOR I=1 TO M
1450 PRINT USING 1295.W(I)
1460 FOR J=1 TO N
1470 PRINT USING 1475.E(I,J)
1475 IMAGE 4D.2D.X.S
1480 NEXT J
1490 PRINT
1500 NEXT I
1510 FOR J=1 TO N
1520 PRINT ".J_."
1530 PRINT " PRESS (RETURN) TO CONTINUE",
1540 INPUT A$,
1550 PAGE
1560 VIEWPORT 10.120.20.90
1570 WINDOW 0.1000.0.100
1580 AXIS 50.5
1590 MOVE 0.0
PROGRAM 2 (Continued)

1600 FOR I=1 TO M
1610 DRAW W(I),E(I,J)
1620 NEXT I
1630 HOME
1640 PRINT "I_ AMPLITUTE A";J;" VERSUS OMEGA ";
1650 PRINT @32:21:0.4
1660 PRINT " PRESS [RETURN] TO CONTINUE ";
1670 INPUT A$
1680 NEXT J
1690 FOR J=1 TO N
1700 PRINT "J_J_"
1710 INPUT A$
1720 PAGE
1730 VIEWPORT 10,120,20,90
1740 WINDOW 50,1000,-270,90
1750 AXIS 50,10
1760 MOVE 0,0
1770 FOR I=1 TO M
1780 DRAW W(I),H(I,J)
1790 NEXT I
1800 HOME
1810 PRINT "I_ PHASE ANGLE FAY(";j;")VERSUS OMEGA";
1820 PRINT @32:21:0.4
1830 PRINT " PRESS [RETURN] TO CONTINUE ";
1840 INPUT A$
1850 NEXT J
1860 IF K2<3 THEN 1960
1870 REM: FILE SUBROUTINE FOR OUTPUT
1880 CALL "OPEN","FEU015 LFN:2 TACCESS:FULL"
1890 WRITE #2,1:N,M
1900 FOR I=1 TO M
1910 WRITE #2,W(I)
1920 FOR J=1 TO N
1930 WRITE #2,E(I,J),H(I,J)
1940 NEXT J
1950 END
PROGRAM 3

COMPUTER PROGRAM FOR SYSTEM MATRIX IDENTIFICATION

300 INIT
310 PAGE
320 PRINT "THE TYPE OF DATA"
330 PRINT "K1=1 FOR DIRECT INPUT"
340 PRINT "K1=2 FOR FUNCTIONS"
350 PRINT "K1=3 FOR DATA FILE"
360 PRINT "ENTER THE TYPE OF DATA K1;"
370 INPUT K1
380 PRINT "TYPE OF OUTPUT REQUIRED"
390 PRINT "K2=1 FOR TABLES"
400 PRINT "K2=2 FOR GRAPHS"
410 PRINT "K2=3 FOR OUTPUT FILE"
420 PRINT "ENTER THE TYPE OF OUTPUT REQUIRED K2;"
430 INPUT K2
440 IF K1=1 THEN 270
450 IF K1=2 THEN 780
460 IF K1=3 THEN 590
470 PRINT "DATA FROM DIRECT INPUT"
480 PRINT "ENTER THE DIMENSION SIZE N;"
490 INPUT N
500 DIM A(N,N), A2(N,N), C(N,N), D1(N,1), D2(N,1), U1(N,1), U2(N,1)
510 DIM F(N,1), B(N,1)
520 PRINT "ENTER THE MATRIX A"
530 FOR I=1 TO N
540 FOR J=1 TO N
550 PRINT USING 370: "ENTER A[I,J];"
560 INPUT A(I,J)
570 NEXT J
580 NEXT I
590 PRINT "ENTER THE MATRIX U1"
600 FOR I=1 TO N
610 PRINT USING 440: "U1[1,I];"
620 INPUT U1(I,1)
630 NEXT I
640 PRINT "ENTER THE MATRIX U2"
650 FOR I=1 TO N
660 PRINT USING 500: "U2[I,1];"
670 INPUT U2(I,1)
PROGRAM 3 (Continued)

500 IMAGE 3A, FD, 5A, S
510 NEXT I
520 PRINT "ENTER THE OMEGA INCREMENT"
530 PRINT "OMEGA";
540 INPUT W1
550 PRINT "ENTER THE FINAL OMEGA, OMEGA MAX"
560 PRINT "OMEGAMAX=";
570 INPUT W2
580 GO TO 820
590 PRINT "DATA FROM DATA FILE"
600 CALL "OPEN", "FEJO14 LFN: I ACCESS: FULL"
610 READ #1, 1
620 DIM A(N, N), A2(N, N), C(N, N), D1(N, 1), D2(N, 1), U1(N, 1), U2(N, 1)
630 DIM F(N, 1), B(N, 1)
640 FOR I = 1 TO N
650 FOR J = 1 TO N
660 READ #1: A(I, J)
670 NEXT J
680 NEXT I
690 FOR I = 1 TO N
700 READ #1: U1(I, 1)
710 NEXT I
720 FOR I = 1 TO N
730 READ #1: U2(I, 1)
740 NEXT I
750 READ #1: W1
760 READ #1: W2
770 GO TO 820
780 REM: FUNCTIONS FOR INPUTS
790 REM:
800 PRINT "CLOSED FORM FUNCTIONS"
810 REM: GET YOUR DATA FROM FUNCTIONS
820 PRINT "THE PROGRAM"
830 M=W2/W1
840 DIM E(N, M), H(N, M), W(M), A3(N, M), A4(N, M), T1(N, M)
844 T1=0
846 FOR T = 1 TO 100
850 FOR L = 1 TO M
860 W(L)=W1*L
870 A2= A MPY A
PROGRAM 3 (Continued)

880 FOR I=1 TO N
900 NEXT I
910 C=INV(A2)
920 D1=0
930 D2=0
940 D1=A MPY U1
950 D2=A MPY U2
960 FOR I=1 TO N
970 D1(I, I)=D1(I, I)+W(L)*U2(I, 1)
980 D2(I, I)=D2(I, I)-W(L)*U1(I, 1)
990 NEXT I

1000 F=0
1010 B=0
1020 F=C MPY D1
1030 B=C MPY D2
1040 Z5=RND(-1)
1050 FOR I=1 TO N
1060 E(I, L)=(F(I, 1)+W(L)+U2(I, 1)+1+RND(1)−0.5)/100
1070 H(I, L)=Z(I, 1)
1090 NEXT I
1100 NEXT L
1110 A3=INV(H)
1120 A4=E MPY A3
1130 TI=TI+A4
1135 PRINT "T=" ... ";
1140 NEXT T
1150 TI=TI/100
1160 PRINT "L THE AVERAGED MATRIX=:");TI
1170 STOP
1180 END
| EIGENVALUES | | EIGENVECTORS | | NORMALIZED EIGENVECTORS |
|------------|------------|----------------|-------------------------|
| \( w_1 = -0.4409 \) | \( 0.90 \) | \( -0.7793 \) \( 0.0 \) | \( 0.1727 \) \( 0.0 \) |
| \( w_2 = -0.2344 \) | \( 0.0 \) | \( 0.5862 \) \( 0.0 \) | \( 0.5862 \) \( 0.0 \) |
| \( w_3 = -0.1169 \) | \( 0.1169 \) \( 0.0 \) | \( 0.1767 \) \( 0.0 \) | \( 0.1767 \) \( 0.0 \) |
| \( w_4 = -0.2344 \) | \( 0.0 \) | \( 0.1329 \) \( 0.0 \) | \( 0.1329 \) \( 0.0 \) |

**TABLE 1**

**DAMPED NON-GYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS**

| \( w_1 \) | \( 0.1223 \) \( 0.0 \) | \( 0.4903 \) \( 0.0 \) | \( -0.5504 \) \( 0.0 \) |
| \( w_2 \) | \( 0.2575 \) \( 0.0 \) | \( 0.7417 \) \( 0.0 \) | \( 0.3184 \) \( 0.0 \) |
| \( w_3 \) | \( 0.3407 \) \( 0.0 \) | \( -0.2278 \) \( 0.0 \) | \( 0.3184 \) \( 0.0 \) |
| \( w_4 \) | \( 0.4903 \) \( 0.0 \) | \( -0.3974 \) \( 0.0 \) | \( -0.1329 \) \( 0.0 \) |

| \( w_1 \) | \( -0.4950 \) \( 0.0 \) | \( -0.4950 \) \( 0.0 \) | \( -0.1862 \) \( 0.0 \) |
| \( w_2 \) | \( -0.2575 \) \( 0.0 \) | \( -0.7417 \) \( 0.0 \) | \( -0.4404 \) \( 0.0 \) |
| \( w_3 \) | \( 0.3407 \) \( 0.0 \) | \( 0.2278 \) \( 0.0 \) | \( -0.1329 \) \( 0.0 \) |
| \( w_4 \) | \( -0.4903 \) \( 0.0 \) | \( 0.3974 \) \( 0.0 \) | \( 0.1684 \) \( 0.0 \) |
TABLE 2
DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.2% RANDOM ERROR

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>EIGENVECTORS</th>
<th>NORMALIZED EIGENVECTORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(.w_1)</td>
<td>(w_1 = 0.92144\cdot 10^{-1})</td>
<td>(0.502950)</td>
</tr>
<tr>
<td>(w_2)</td>
<td>(w_2 = 0.47386\cdot 10^{-1})</td>
<td>(0.697320)</td>
</tr>
<tr>
<td>(w_3)</td>
<td>(w_3 = 0.16847\cdot 10^{-2})</td>
<td>(0.614950)</td>
</tr>
<tr>
<td>(w_4)</td>
<td>(w_4 = 0.10084\cdot 10^{-2})</td>
<td>(0.209370)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(-0.2% )</th>
<th>(-0.2% )</th>
<th>(-0.2% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.502950)</td>
<td>(0.697320)</td>
<td>(0.502950)</td>
</tr>
<tr>
<td>(-0.16847\cdot 10^{-1})</td>
<td>(-0.614950)</td>
<td>(-0.16847\cdot 10^{-2})</td>
</tr>
<tr>
<td>(-0.20937\cdot 10^{-2})</td>
<td>(-0.209370)</td>
<td>(-0.20937\cdot 10^{-1})</td>
</tr>
</tbody>
</table>

| \(0.502950\) | \(0.697320\) | \(0.502950\) |
| \(-0.697320\) | \(-0.502950\) | \(-0.697320\) |
| \(-0.209370\) | \(-0.209370\) | \(-0.209370\) |

| \(0.502950\) | \(0.697320\) | \(0.502950\) |
| \(-0.697320\) | \(-0.502950\) | \(-0.697320\) |
| \(-0.209370\) | \(-0.209370\) | \(-0.209370\) |
### TABLE 3

**DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.4% RANDOM ERROR**

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>0.4%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w(1) = 0.261270 - 0.0 )</td>
<td>0.4</td>
</tr>
<tr>
<td>( w(2) = 0.415600 - 0.0 )</td>
<td>0.781250 - 0.2</td>
</tr>
<tr>
<td>( w(3) = 0.415600 - 0.0 )</td>
<td>0.781250 - 0.2</td>
</tr>
<tr>
<td>( w(4) = -0.255370 - 0.0 )</td>
<td>0.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EIGENVECTORS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.579520 00 0.2</td>
<td></td>
</tr>
<tr>
<td>-0.599490 00 0.2</td>
<td></td>
</tr>
<tr>
<td>-0.423140 00 0.2</td>
<td></td>
</tr>
<tr>
<td>-0.354570 00 0.2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NORMALIZED EIGENVECTORS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.579520 00 0.2</td>
<td></td>
</tr>
<tr>
<td>-0.599490 00 0.2</td>
<td></td>
</tr>
<tr>
<td>-0.423140 00 0.2</td>
<td></td>
</tr>
<tr>
<td>-0.354570 00 0.2</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE 4

**DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.6% RANDOM ERROR**

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>W(1) = -0.174034D+00</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>W(2)</td>
<td>0.364967D-01</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>W(3)</td>
<td>0.251843D-02</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>W(4)</td>
<td>-0.111260D-01</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EIGENVECTORS</th>
<th>-0.166567D 01 0.0</th>
<th>I -0.228140 01 0.0</th>
<th>I 0.195120 01 0.0</th>
<th>I 0.526730 00 0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.290070 00 0.0</td>
<td>I 0.115940 01 0.0</td>
<td>I 0.351260 01 0.0</td>
<td>I -0.426320 00 0.0</td>
</tr>
<tr>
<td></td>
<td>-0.293600 00 0.0</td>
<td>I 0.899310 09 0.0</td>
<td>I 0.179750 01 0.0</td>
<td>I 0.193840 01 0.0</td>
</tr>
<tr>
<td></td>
<td>-0.904800 00 0.0</td>
<td>I 0.933070 00 0.0</td>
<td>I -0.106310 01 0.0</td>
<td>I 0.282460 01 0.0</td>
</tr>
</tbody>
</table>

| NORMALIZED EIGENVECTORS | -0.850730 00 0.0 | I -0.705340 00 0.0 | I 0.439880 00 0.0 | I 0.150850 00 0.0 |
|                         | -0.149450 00 0.0 | I 0.404100 00 0.0 | I 0.775680 00 0.0 | I -0.122980 00 0.0 |
|                         | -0.151270 00 0.0 | I 0.313640 00 0.0 | I 0.396940 00 0.0 | I 0.555570 00 0.0 |
|                         | -0.464220 00 0.0 | I 0.352330 00 0.0 | I 0.234750 00 0.0 | I 0.808850 00 0.0 |
TABLE 5

DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 0.8% RANDOM ERROR

### Eigenvalues

<table>
<thead>
<tr>
<th>Mode</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvector</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.118850</td>
<td>0.137060 01 0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.137060</td>
<td>-0.118850 01 0.0</td>
</tr>
<tr>
<td>3</td>
<td>-0.455050</td>
<td>0.118850 01 0.0</td>
</tr>
<tr>
<td>4</td>
<td>0.382990</td>
<td>-0.455050 01 0.0</td>
</tr>
</tbody>
</table>

### Eigenvectors

<table>
<thead>
<tr>
<th>Mode</th>
<th>Eigenvector</th>
<th>Normalized Eigenvector</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.142760</td>
<td>0.134570 01 0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.134570</td>
<td>-0.142760 01 0.0</td>
</tr>
<tr>
<td>3</td>
<td>-0.390620</td>
<td>0.142760 01 0.0</td>
</tr>
<tr>
<td>4</td>
<td>0.382990</td>
<td>-0.390620 01 0.0</td>
</tr>
</tbody>
</table>
TABLE 6
DAMPED NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS - 1.0% RANDOM ERROR

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
</tr>
</thead>
<tbody>
<tr>
<td>W( 1 )</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
</tr>
<tr>
<td>W( 2 )</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
</tr>
<tr>
<td>W( 3 )</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
<td>0.6021890-00</td>
</tr>
<tr>
<td>W( 4 )</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
<td>0.1021890-00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EIGENVECTORS</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
<th>0.1021890-00</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NORMALIZED EIGENVECTORS</th>
<th>0.756630 00</th>
<th>0.756630 00</th>
<th>0.756630 00</th>
<th>0.756630 00</th>
<th>0.756630 00</th>
<th>0.756630 00</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.214970 01</td>
<td>-0.324960 01</td>
<td>0.649060 00</td>
<td>0.0</td>
</tr>
</tbody>
</table>


### Table 7

Critically Damped Non-Gyroscopic System Eigenvalues and Eigenvectors

<table>
<thead>
<tr>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1 = -0.4031$</td>
</tr>
<tr>
<td>$\lambda_2 = -0.0312$</td>
</tr>
<tr>
<td>$\lambda_3 = -0.3219$</td>
</tr>
<tr>
<td>$\lambda_4 = -0.3214$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1 = (0.1022, 0.2860, 0.1022, -0.7532)$</td>
</tr>
<tr>
<td>$v_2 = (0.1042, 0.5310, -0.1042, 0.5901)$</td>
</tr>
<tr>
<td>$v_3 = (0.6803, -0.6529, -0.6803, 0.1857)$</td>
</tr>
<tr>
<td>$v_4 = (0.6591, -0.4524, 0.6591, -0.7478)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Normalized Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{v}_1 = (0.6591, 0.6591, 0.6591, 0.6591)$</td>
</tr>
<tr>
<td>$\tilde{v}_2 = (0.9892, 0.9892, 0.9892, 0.9892)$</td>
</tr>
<tr>
<td>$\tilde{v}_3 = (0.3149, 0.3149, 0.3149, 0.3149)$</td>
</tr>
<tr>
<td>$\tilde{v}_4 = (0.5472, 0.5472, 0.5472, 0.5472)$</td>
</tr>
</tbody>
</table>
TABLE 8
UNDERDAMPED NON-GYROSCOPIC SYSTEM EIGENVALUES AND EIGENVECTORS

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>EIGENVECTORS</th>
<th>NORMALIZED EIGENVECTORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 \approx -0.337560 )</td>
<td>( \cos \theta_1 \approx 0.321820 )</td>
<td>( -0.892460 )</td>
</tr>
<tr>
<td>( \lambda_2 \approx 0.337560 )</td>
<td>( \cos \theta_2 \approx -0.321820 )</td>
<td>( 0.539590 )</td>
</tr>
<tr>
<td>( \lambda_3 \approx -0.424160 )</td>
<td>( \cos \theta_3 \approx 0.133540 )</td>
<td>( 0.161760 )</td>
</tr>
<tr>
<td>( \lambda_4 \approx -0.424160 )</td>
<td>( \cos \theta_4 \approx -0.133540 )</td>
<td>( -0.115700 )</td>
</tr>
</tbody>
</table>
### Table 9

**Conservative Non-Gyroscopic System Eigenvalues and Eigenvectors**

#### Eigenvalues

<table>
<thead>
<tr>
<th>mt</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wt</td>
<td>$-0.101190-13$</td>
<td>$-0.485000-03$</td>
<td>$-0.485000-03$</td>
<td>$-0.611010-13$</td>
<td>$-0.485000-03$</td>
<td>$-0.611010-13$</td>
<td>$-0.485000-03$</td>
<td>$-0.611010-13$</td>
</tr>
</tbody>
</table>

#### Eigenvectors

<table>
<thead>
<tr>
<th>mt</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wt</td>
<td>$-0.291140-02$</td>
<td>$-0.240410-12$</td>
<td>$-0.240410-12$</td>
<td>$-0.240410-12$</td>
<td>$-0.240410-12$</td>
<td>$-0.240410-12$</td>
<td>$-0.240410-12$</td>
<td>$-0.240410-12$</td>
</tr>
</tbody>
</table>

#### Normalized Eigenvectors

<table>
<thead>
<tr>
<th>mt</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wt</td>
<td>$-0.339990-16$</td>
<td>$-0.305800-16$</td>
<td>$-0.399490-16$</td>
<td>$-0.256900-03$</td>
<td>$-0.463370-13$</td>
<td>$-0.463370-13$</td>
<td>$-0.463370-13$</td>
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### Table 10

**CONSERVATIVE NON-GYROSCOPIC SYSTEM – ERROR MATRIX EIGENVALUES AND EIGENVECTORS WITH 0.005% RANDOM ERROR**

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#### Normalized Eigenvectors

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### Table 11

**Conservative Non-Gyroscopic System - Error Matrix Eigenvalues and Eigenvectors with 0.0005% Random Error**

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<th>Eigenvalues</th>
<th>Eigenvalues</th>
<th>Normalized Eigenvectors</th>
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<tbody>
<tr>
<td><em>w</em>&lt;sub&gt;1&lt;/sub&gt;</td>
<td>0.15540(0.6)</td>
<td>0.193710(0.1)</td>
</tr>
<tr>
<td><em>w</em>&lt;sub&gt;2&lt;/sub&gt;</td>
<td>0.15540(0.6)</td>
<td>0.193710(0.1)</td>
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<tr>
<td><em>w</em>&lt;sub&gt;3&lt;/sub&gt;</td>
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<td>0.193710(0.1)</td>
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<tr>
<td><em>w</em>&lt;sub&gt;4&lt;/sub&gt;</td>
<td>0.15540(0.6)</td>
<td>0.193710(0.1)</td>
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<tr>
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<td>0.15540(0.6)</td>
<td>0.193710(0.1)</td>
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<tr>
<td><em>w</em>&lt;sub&gt;6&lt;/sub&gt;</td>
<td>0.15540(0.6)</td>
<td>0.193710(0.1)</td>
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<tr>
<td><em>w</em>&lt;sub&gt;7&lt;/sub&gt;</td>
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<td>0.193710(0.1)</td>
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<tr>
<td><em>w</em>&lt;sub&gt;8&lt;/sub&gt;</td>
<td>0.15540(0.6)</td>
<td>0.193710(0.1)</td>
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</table>

The table includes eigenvalues and eigenvectors with normalized values for the conservative non-gyroscopic system. The errors are presented with 0.0005% random error.
### TABLE 12

**CONSERVATIVE NON-GYROSCOPIC SYSTEM – ERROR MATRIX EIGENVALUES AND EIGENVECTORS WITH 0.00005% RANDOM ERROR**

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<tr>
<td>( u ) ( i ) = 0.113740 ( i ) 0.2</td>
<td>0.0</td>
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<tr>
<td>( v ) ( i ) = 0.104460 ( i ) 0.2</td>
<td>0.0</td>
</tr>
<tr>
<td>( w ) ( i ) = 0.1958 -0.1</td>
<td>0.4</td>
</tr>
<tr>
<td>( x ) ( i ) = 0.012570 ( i ) -0.1</td>
<td>0.4</td>
</tr>
<tr>
<td>( y ) ( i ) = 0.649310 ( i ) -0.1</td>
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<tr>
<td>( z ) ( i ) = 0.256940 ( i ) -0.1</td>
<td>0.4</td>
</tr>
<tr>
<td>( \mu ) ( i ) = 0.165400 ( i ) -0.1</td>
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<table>
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<tr>
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<td>[ 0.113740 \quad 0.2 ] ( \begin{bmatrix} b \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ 0.256940 \quad 0.0 ] ( \begin{bmatrix} \end{bmatrix} ]</td>
<td>[ 0.104460 \quad 0.2 ] ( \begin{bmatrix} c \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ 0.1958 \quad -0.1 ] ( \begin{bmatrix} d \end{bmatrix} ]</td>
<td>[ 0.012570 \quad -0.1 ] ( \begin{bmatrix} e \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ 0.649310 \quad -0.1 ] ( \begin{bmatrix} f \end{bmatrix} ]</td>
<td>[ 0.256940 \quad -0.1 ] ( \begin{bmatrix} g \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ 0.165400 \quad -0.1 ] ( \begin{bmatrix} h \end{bmatrix} ]</td>
<td>[ 0.113740 \quad -0.2 ] ( \begin{bmatrix} i \end{bmatrix} ]</td>
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<table>
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<tr>
<td>[ 0.391200 \quad 0.1 ] ( \begin{bmatrix} j \end{bmatrix} ]</td>
<td>[ 0.495270 \quad 0.1 ] ( \begin{bmatrix} k \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ 0.411960 \quad 0.1 ] ( \begin{bmatrix} l \end{bmatrix} ]</td>
<td>[ 0.629300 \quad 0.1 ] ( \begin{bmatrix} m \end{bmatrix} ]</td>
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<tr>
<td>[ 0.842520 \quad 0.1 ] ( \begin{bmatrix} n \end{bmatrix} ]</td>
<td>[ 0.312360 \quad 0.1 ] ( \begin{bmatrix} o \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ 0.792200 \quad 0.1 ] ( \begin{bmatrix} p \end{bmatrix} ]</td>
<td>[ 0.765940 \quad 0.1 ] ( \begin{bmatrix} q \end{bmatrix} ]</td>
</tr>
<tr>
<td>[ 0.945640 \quad 0.1 ] ( \begin{bmatrix} r \end{bmatrix} ]</td>
<td>[ 0.424840 \quad 0.1 ] ( \begin{bmatrix} s \end{bmatrix} ]</td>
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**II**
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TABLE 13
CONSERVATIVE NON-GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS
WITH 0.000005% RANDOM ERROR

118
TABLE 14

CONSERVATIVE NON–GYROSCOPIC SYSTEM - ERROR MATRIX EIGENVALUES AND EIGENVECTORS
WITH 0.0000005% RANDOM ERROR

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