Classroom Mathematical Practices In A Preservice Elementary Mathematics Education Course Using An Instructional Sequence Related

2006

Janet Andreasen

University of Central Florida

Find similar works at: http://stars.library.ucf.edu/etd

University of Central Florida Libraries http://library.ucf.edu

Part of the Science and Mathematics Education Commons

STARS Citation


http://stars.library.ucf.edu/etd/741

This Doctoral Dissertation (Open Access) is brought to you for free and open access by STARS. It has been accepted for inclusion in Electronic Theses and Dissertations by an authorized administrator of STARS. For more information, please contact lee.dotson@ucf.edu.
© 2006 Janet B. Andreasen
ABSTRACT

This qualitative study documents a classroom teaching experiment in a semester-long undergraduate mathematics education course for 16 prospective elementary school teachers. The purpose of this study was to investigate how social aspects of the classroom environment facilitated the collective mathematical learning of place value and whole number operations by preservice elementary school teachers. Design-based research methodology was used for formulating the study. A hypothetical learning trajectory and instructional sequence related to place value and operations were created and refined in the two semesters prior to this study. The instructional sequence was in its third iteration for this study.

The developmental levels that children progress through in learning place value and operations were used in identifying the learning trajectory and supporting tasks in which the preservice teachers were asked to engage. A large portion of the instructional sequence involved a setting of base eight instead of base ten. The sequence returned to base ten in order to discuss whole number operations and alternative strategies for operations in an effort to further develop the preservice teachers’ conceptual understandings of place value and operations and to examine children’s thinking strategies.

Data were collected through video-taped recordings of class sessions, audio-taped recordings of table discussions and research team meetings, field notes, and journals written by the research team. Sixteen preservice teachers participated in the study which lasted over 5 class sessions of 3 hours and 10 minutes each.

The emergent perspective which attempts to coordinate the individual learning and the social aspects of the classroom that support collective learning was used as an interpretive lens for data collection and analysis. The social aspects along with some aspects of individual student
understandings together give an indication of collective mathematical understandings of the students as a whole group.

Social norms established were: a) the expectation of providing explanations and justifications for solutions and solution methods, b) making sense of each other’s solutions and c) asking questions of classmates or the instructor. Sociomathematical norms that were valued but not fully established were: a) criteria for different solutions and solution methods and b) criteria for what constituted a good explanation. Data analysis for the establishment of classroom mathematical practices was conducted using Toulmin’s argumentation model (Toulmin, 1969). A three phase approach described by Rasmussen and Stephan (in press) was used in determining what constituted a classroom mathematical practice. The classroom mathematical practices that facilitated student learning in this study were: a) unitizing, b) flexibly representing numbers, and c) reasoning about operations. This study led to the refinement of the hypothetical learning trajectory and further progress in defining an instructional theory of how preservice teachers may come to understand place value and whole number operations.
This is dedicated to my wonderful husband, Robbie, and my two children, Zachary and Sarah.

Your support through this process has been priceless to me. I love you.
ACKNOWLEDGMENTS

Where do I start in acknowledging those who have helped me through this process. First, thank you to my committee members, Dr. Juli K. Dixon, Dr. Laura Blasi, Dr. Michele Gill, Dr. Enrique Ortiz, and Dr. Michelle Stephan for their feedback and support. Thank you to my fellow doctoral students who helped with this project, Debbie Wheeldon and George Roy. Without your support and help, this project never would have happened. Lastly, but certainly not least, thank you to my friends and family who were so supportive of my efforts. Zachary and Sarah, mommy can come play with you now.
# TABLE OF CONTENTS

LIST OF FIGURES ....................................................................................................................... xi

LIST OF TABLES ........................................................................................................................ xii

LIST OF ACRONYMS/ABBREVIATIONS .............................................................................. xiii

DEFINITION OF TERMS .......................................................................................................... xiv

CHAPTER ONE: INTRODUCTION ............................................................................................. 1

  Statement of the Problem .................................................................................................... 3

  Significance of Study .......................................................................................................... 7

  Research Focus ................................................................................................................... 8

  Conclusion ......................................................................................................................... 10

CHAPTER TWO: LITERATURE REVIEW ............................................................................... 12

  Children’s Understanding of Place Value ......................................................................... 13

  Whole Number Operations ............................................................................................... 21
    Addition and Subtraction ................................................................................................ 21
    Multiplication and Division ............................................................................................ 25

  Teaching Place Value and Operations ............................................................................. 32

  Preservice and Inservice Teachers’ Number Sense ......................................................... 39

  Hypothetical Learning Trajectory ................................................................................... 45

  Conclusion ......................................................................................................................... 47

CHAPTER THREE: METHODOLOGY ..................................................................................... 50

  Design-based Research .................................................................................................... 51

  Classroom Teaching Experiment ..................................................................................... 53

  Theoretical Framework ..................................................................................................... 55
LIST OF FIGURES

Figure 1: Toulmin's argumentation scheme for 2 x 3 ............................................................... 83
Figure 2: Three groups of six with 10 frames ........................................................................... 92
Figure 3: The Dot Problem ....................................................................................................... 93
Figure 4: Candy Problem ......................................................................................................... 99
Figure 5: Traditional Regrouping Algorithm for 51 - 23 ......................................................... 99
Figure 6: Double 10 for 6 plus 3 ............................................................................................. 109
Figure 7: Double 10 Frame Example ...................................................................................... 117
Figure 8: A representation of Katrina's drawing of rolls of candy ......................................... 125
Figure 9: Sample Inventory Form .......................................................................................... 139
Figure 10: Equal Compensation Algorithm (base ten) .......................................................... 148
Figure 11: Partial Sums Algorithm ......................................................................................... 178
Figure 12: Column Addition .................................................................................................... 178
Figure 13: Partial Differences Algorithm ................................................................................ 179
Figure 14: Equal Compensation Algorithm ........................................................................... 179
Figure 15: Partial Products Algorithm .................................................................................... 180
Figure 16: Partial Differences Algorithm ................................................................................ 180
LIST OF TABLES

Table 1: Addition Strategies ......................................................................................................... 23
Table 2: Subtraction Strategies ..................................................................................................... 24
Table 3: Multiplication Strategies................................................................................................. 31
Table 4: Division Strategies .......................................................................................................... 31
Table 5: Emergent Perspective Interpretive Framework (Cobb & Yackel, 1996) ....................... 56
Table 6: Social aspects for this classroom teaching experiment................................................... 58
Table 7: Heuristics of RME .......................................................................................................... 65
Table 8: Initial Hypothetical Learning Trajectory for this Teaching Experiment ....................... 70
Table 9: Instructional Sequence for Place Value and Operations................................................. 73
Table 10: Phases in documenting collective activity (Rasmussen & Stephan, in press) ............... 83
Table 11: Social and Sociomathematical Norms for this Classroom ......................................... 89
Table 12: Initial Hypothetical Learning Trajectory .................................................................... 162
Table 13: Revised Hypothetical Learning Trajectory for Place Value and Operations .......... 166
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Full Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>CGI</td>
<td>Cognitively Guided Instruction</td>
</tr>
<tr>
<td>HLT</td>
<td>Hypothetical Learning Trajectory</td>
</tr>
<tr>
<td>IRB</td>
<td>Institutional Review Board</td>
</tr>
<tr>
<td>NCATE</td>
<td>National Council for Accreditation of Teacher Education</td>
</tr>
<tr>
<td>NCTM</td>
<td>National Council of Teachers of Mathematics</td>
</tr>
<tr>
<td>PUFM</td>
<td>Profound Understanding of Fundamental Mathematics</td>
</tr>
<tr>
<td>RME</td>
<td>Realistic Mathematics Education</td>
</tr>
</tbody>
</table>
## DEFINITION OF TERMS

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classroom Mathematical Practice</td>
<td>The ways of reasoning, explaining, and justifying related to content-specific mathematical ideas that are taken-as-shared by the classroom community.</td>
</tr>
<tr>
<td>Classroom Teaching Experiment</td>
<td>A sequence of teaching episodes which include a teacher, one or more students, at least one witness of the teaching episode, and a method for recording the teaching episode with the goal of constructing models of students’ mathematics that can be useful in teaching other students and in learning how students learn and understand mathematics.</td>
</tr>
<tr>
<td>Design-based Research</td>
<td>A qualitative research methodology which provides a basis for testing learning theories in the actual classroom with real situations in a cyclical process of testing, modifying, retesting, and remodifying.</td>
</tr>
<tr>
<td>Hypothetical Learning Trajectory</td>
<td>A framework of instructional tasks with expectations for how the class will engage in thinking and learning as they participate in the instruction; provides the basis for the decisions that are made with respect to instructional tasks and plans for instruction; includes three aspects: the learning goals, associated instructional tasks, and projected student learning outcomes.</td>
</tr>
<tr>
<td>Instructional Sequence</td>
<td>Sequence of instructional tasks which support the learning goals of the hypothetical learning trajectory.</td>
</tr>
<tr>
<td>Mathematics for Understanding</td>
<td>The ability to apply mathematical understandings to new situations; Conceptual understandings of mathematics.</td>
</tr>
<tr>
<td>Number Sense</td>
<td>The ability to decompose numbers naturally, use particular numbers like 100 or 1/2 as referents, use the relationships among arithmetic operations to solve problems, understand the base-ten number system, estimate, make sense of numbers, and recognize the relative and absolute magnitude of numbers.</td>
</tr>
<tr>
<td>Social Norm</td>
<td>Taken-as-shared ways of participating in the classroom community</td>
</tr>
<tr>
<td>Sociomathematical Norm</td>
<td>Criteria for what counts as a different, unique, or sophisticated mathematical solution as well as what counts as an acceptable mathematical explanation and justification</td>
</tr>
</tbody>
</table>
CHAPTER ONE: INTRODUCTION

With increased pressure to demonstrate student achievement through high stakes testing, many schools are evaluating teachers based on the achievement levels of their students. Studies have shown that the content knowledge of the teacher has a positive impact on the achievement levels of his/her students (Goldhaber & Anthony, 2003; Sanders & Rivers, 1996; Wright, Horn, & Sanders, 1997). Colleges of education are also under increased pressure to demonstrate production of teachers who have the knowledge, skills, and dispositions to be effective educators (NCATE, 2002). In order to teach mathematics for understanding, elementary school teachers need to have mathematical understandings of their own beyond the procedural methods taught in many schools today (Ball, 1990a; Shulman, 1987). Preservice elementary education programs need to include courses which seek to improve the prospective teachers’ own mathematical understanding prior to entering the classroom.

Although there is a need for improvement in nearly all areas of mathematics, place value and whole number operations was chosen as the focus for this study. A thorough understanding of place value has implications for understandings of operations with whole numbers, decimals, percents, and fractions. The National Council of Teachers of Mathematics (NCTM), in its publication Principles and Standards for School Mathematics (2000), stated that the central idea behind the content standard Number and Operations is number sense, described as “the ability to decompose numbers naturally, use particular numbers like 100 or 1/2 as referents, use the relationships among arithmetic operations to solve problems, understand the base-ten number system, estimate, make sense of numbers, and recognize the relative and absolute magnitude of numbers” (p. 32). It advocates that all students in prekindergarten through grade 12 should have
an understanding of number including multiple representations such as with objects and numerals, or on number lines. Place value and operations are crucial aspects of the development of number sense. The foundation for this understanding must begin in elementary school. If teachers are entering the profession without a conceptual understanding of place value and operations, it is unlikely they will provide learning experiences for their students which lead to the type of understanding the NCTM advocates (Ma, 1999).

This study was a design-based research study set in a classroom teaching experiment with the purpose of documenting the social aspects of the classroom that supported concepts of place value and operations. The data presented here were collected during the third iteration of the instructional sequence. The study was conducted at a major research university in the southeastern United States in a semester-long mathematics education course for prospective elementary school teachers over five class sessions that were each 3 hours and 10 minutes long. Sixteen students agreed to participate in the study. This study sought to add to the limited research related to documenting preservice teachers’ understandings of place value and whole number operations in the social context of the classroom (McClain, 2003; Speiser & Walter, 2000). While adult’s conceptions of place value and operations has not been fully analyzed, children’s understanding of place value and operations has been widely studied (Baroody, 1990; Cobb & Wheatley, 1988; Fuson, 1990; Hiebert & Wearne, 1992; Ross, 1986; Steffe, 1983). “Evidence indicates that U.S. children do not learn place-value concepts or multidigit addition and subtraction adequately and even many children who calculate correctly show little understanding of the procedures they are using” (Fuson, 1990, p. 273). Place value and operations are foundational to students’ understandings of mathematics throughout their schooling. The concepts related to place value and operations have an impact on children in
numerous areas throughout their mathematical development. By improving preservice teachers’ mathematical knowledge and understanding related to place value and operations, they will learn the concepts themselves and be better prepared to improve the learning of their students in these areas.

**Statement of the Problem**

The National Council for Accreditation of Teacher Education (NCATE) has developed standards for preservice teachers and teacher education programs. Standard one states that “Candidates preparing to work in schools as teachers or other professional school personnel [must] know and demonstrate the content, pedagogical, and professional knowledge, skills, and dispositions necessary to help all students learn” (NCATE, 2002, p. 10). To meet the content portion of this standard, many elementary education programs require preservice teachers to take some form of mathematics content courses through their undergraduate elementary education programs. Many state universities require some level of mathematics as part of the general education requirements all students must meet regardless of major. However, the mathematics instruction received in these courses is often no different from the procedural teaching present in many K-12 classrooms as noted by the Conference Board of the Mathematical Sciences in its publication *The Mathematical Education of Teachers*.

It seems, then, that we are caught in a vicious cycle: poor K-12 mathematics instruction produces ill-prepared college students, and undergraduate education often does little to correct the problem. Indeed, some universities mandate next to no mathematics coursework for the prospective elementary teacher. However, simply increasing the number of required credit hours is no solution—courses that allow students to get by using the same
stratagems that got them through K-12 just perpetuate the problem” (2001, p. 55-56)

We cannot assume that teachers are adequately prepared to teach mathematics for understanding simply because they have passed their required mathematics courses. “Relying on what prospective teachers have learned in their … mathematics classes is unlikely to provide adequate subject matter preparation for teaching mathematics for understanding” (Ball, 1990b, p. 142). The mathematics education of prospective teachers should include experiences in which they are challenged to become problem solvers and asked to explain and justify their answers (Conference Board of the Mathematical Sciences, 2001).

Shulman (1987) stated that “teaching necessarily begins with a teacher’s understanding of what is to be learned and how it is to be taught” (p. 7). Since the late 1980s, efforts have been made to define the mathematical knowledge required for teaching mathematics at various levels. Shulman (1986) coined the term pedagogical content knowledge to describe the special kind of subject matter knowledge needed to be an effective teacher. His distinction between general knowledge of the content and pedagogical content knowledge – knowledge of student misconceptions, typical errors, various representations effective in teaching, and familiarity with topics children find interesting or difficult – contributed to attempts to determine qualities and resources necessary for effective teaching. Further research into the field of mathematics with regards to pedagogical content knowledge has shown that what teachers of mathematics needed to understand about the mathematics being taught in their classrooms was substantially different from the mathematics the everyday person needed to understand (Ball, 1990a, 1991; Leinhardt & Smith, 1985).
Ball (1990a) was able to describe differences between teachers’ knowledge and ability to solve problems involving division by a fraction and their ability to represent the same problems accurately for children. This began to mark a distinction between content knowledge in general and specialized mathematical knowledge for teaching. Work by Ma (1999) added to this distinction with her attempts to define profound understanding of fundamental mathematics (PUFM) by comparing teachers in China and teachers in the United States.

With respect to pedagogical content knowledge, NCATE believes the target goal for all preservice teachers is that “teacher candidates reflect a thorough understanding of pedagogical content knowledge delineated in professional, state, and institutional standards. They have in-depth understanding of the subject matter that they plan to teach, allowing them to provide multiple explanations and instructional strategies so that all students learn. They present the content to students in challenging, clear, and compelling ways and integrate technology appropriately” (NCATE, 2002, p. 15). The structure of many elementary education programs makes the development of pedagogical content knowledge difficult to accomplish. Preservice teachers are often lacking in their own understanding of school mathematics, making pedagogical content knowledge all the more difficult to develop. Ball (1990b) found that “teacher candidates demonstrated that they wanted to give the pupils what they considered to be meaningful answers, but often they could not do so because their subject matter knowledge … was insufficient to act on that commitment” (p. 142). In order to improve their ability to provide multiple explanations and instructional strategies, their own knowledge and understanding must be improved first (Shulman, 1987).

As a result of the push for higher student achievement in school mathematics and the emphasis on knowledge, skills, and dispositions advocated by NCATE, teacher-educators are
focusing on developing programs which attempt to strengthen teacher content knowledge and pedagogical content knowledge related to school mathematics at the preservice and inservice levels in an effort to improve instruction in the classroom (Carpenter, Fennema, & Franke, 1996; Koency & Swanson, 2000; McClain, 2003; Steele, 1994). Carpenter, Fennema, and Franke (1996) found that the program Cognitively Guided Instruction (CGI), which focused on understanding children’s thinking, helped teachers to develop a broader knowledge base for mathematics. CGI focused directly on helping teachers to understand children’s thinking by helping them formulate models of children’s thinking in specific content areas, including whole number operations, fractions, geometry, and measurement (Carpenter et al., 1996). The work of CGI is focused primarily on inservice teachers; however, preservice teachers can benefit from similar programs.

An influencing factor in selecting preservice teachers for this study was the procedural focus common in K-12 schools in the United States (Conference Board of the Mathematical Sciences, 2001; U.S. Department of Education, 2003). Most preservice teachers are products of the U.S. educational system. With its emphasis on procedures rather than conceptual understandings, preservice teachers are entering their teacher preparation courses without the knowledge of school mathematics necessary to teach for understanding. By shaping preservice teachers’ conceptual understandings of place value and operations, they can begin their teaching careers with some basis of pedagogical content knowledge to build upon. Additionally, a course focused on elementary school mathematics rather than methods for teaching elementary school mathematics was selected as the setting for this study in order to develop the preservice teachers’ own mathematical understandings of place value and operations before they were exposed to
techniques for teaching mathematics in the elementary school which rely in many ways on conceptual understandings of the mathematics to be taught.

**Significance of Study**

The purpose of this study was to document the social aspects of the classroom and the ways in which preservice elementary school teachers’ collective development of concepts related to place value and whole number operations was facilitated by these social aspects. Using a design-based research methodology (Stephan & Cobb, 2003), the ways in which the social aspects of the classroom, including a) social norms, b) sociomathematical norms, and c) classroom mathematical practices supported preservice teachers’ development of an understanding of place value and operations, were investigated. An instructional sequence related to place value and operations was developed and implemented in an undergraduate elementary mathematics education course, with particular attention paid to the classroom mathematical practices established throughout the instructional sequence. With an understanding of the processes through which children progress in learning place value and operations (Cobb & Wheatley, 1988; Ross, 1986; Steffe, 1983) and through which preservice teachers seem to progress in learning alternative bases (McClain, 2003), a hypothetical learning trajectory (HLT), which combined learning goals, instructional tasks, and expected collective student outcomes, was developed and refined through the cyclical process of design-based research (Gravemeijer, 2004; Gravemeijer, Bowers, & Stephan, 2003; Simon, 1995). A sequence of instructional tasks, called an instructional sequence, related to place value and operations was based on this HLT and included the use of base eight rather than base ten for supporting preservice teachers’
development of place value concepts (McClain, 2003; Speiser & Walter, 2000; Zazkis & Khoury, 1993). The instructional sequence and hypothetical learning trajectory for this study were in their third iteration. Student thinking and misconceptions were examined as they developed alternative strategies for whole number operations. The instructional sequence included an examination of children’s thinking strategies related to whole number operations. The sequence was designed in an attempt to allow preservice teachers to better anticipate difficulties their students may encounter in learning mathematics by experiencing similar difficulties and struggles as children, contributing to the preservice teachers’ acquisition of mathematical content knowledge as well as mathematics knowledge for teaching (Hill, Ball, & Schilling, 2004).

Through the cyclical process of design-based research, the instructional sequence along with the HLT can be formulated into an instructional theory of how preservice teachers understand place value and operations and how their understanding developed through their experiences with the instructional sequence (Gravemeijer, 2004). This instructional theory provides a framework for teacher educators to develop place value and operations HLTs for their own students. The instructional theory provides a guide for suggested tasks, as well as other means of support, which teacher educators can implement in their own classrooms to further increase and improve the learning of preservice teachers related to place value and operations.

**Research Focus**

This research study focuses on the ways in which the social aspects of the classroom facilitated student learning. The social aspects of the classroom included: a) social norms, b)
sociomathematical norms, and c) classroom mathematical practices. These were measured by examining the video-taped class sessions as well as research team meeting transcripts and field notes and identifying where norms were being established as well as what mathematical ideas became taken-as-shared. Toulmin’s argumentation scheme (1969) was used as a basis for determining classroom mathematical practices through an analysis of taken-as-shared knowledge using student discourse, namely the ways in which students explained and justified their solutions and solution processes through whole class discussions. The research questions were:

1. In what ways did the social context of the classroom including a) social norms, b) sociomathematical norms, and c) classroom mathematical practices facilitate preservice elementary school teachers’ development of place value and operation concepts?

2. In what ways did the instructional sequence and subsequent revision facilitate the collective development of place value and operation concepts with preservice elementary school teachers?

The social and sociomathematical norms were determined by examining the transcripts episode-by-episode to determine when norms were being established (Cobb & Whitenack, 1996; Glaser & Strauss, 1967), and classroom mathematical practices were analyzed using Toulmin’s (1969) argumentation scheme related to students explaining their solutions and justifying their solution processes. The classroom mathematical practices that were established documented the ways in which the prospective elementary school teachers developed concepts of place value and operations. Through a design-based research methodology, the instructional sequence and hypothetical learning trajectory were revised throughout the implementation of this third iteration and the ways in which those changes facilitated student learning were also examined as
documented by the video-taped class sessions, researcher’s field notes, and audio-taped team meetings. The classroom teaching experiment was conducted with 16 preservice teachers over the course of a six-week summer semester. The research team consisted of six people: the instructor for the course, two doctoral students at the dissertation stage, two doctoral students at the pre-dissertation stage, and one research mentor with extensive experience in conducting classroom teaching experiments. The data were collected over five class sessions of 3 hours and 10 minutes each and consisted of video-taped recordings of each class session, audio-taped recordings of student-to-student interactions and research team meetings, field notes, and individual student work. These data were collected to document collective student learning or the ways of reasoning mathematically that are taken-as-shared by the classroom community (Rasmussen & Stephan, in press), that took place through the implementation of the instructional sequence.

**Conclusion**

The topics of place value and operations were selected for this study because they are foundational for understanding most other areas of mathematics. The research base for children’s development and understanding of place value and operations is extensive; however, the research literature related to preservice teachers’ understanding of place value and operations is limited. Chapter two addresses the literature related to children’s development of understandings related to place value and operations as well as the relevant research related to preservice teachers’ acquisition and understanding of place value and operations. This literature forms the basis for
decisions made in creating, revising, and implementing the hypothetical learning trajectory (HLT) and the instructional sequence. Literature related to HLTs is also presented here.

Chapter three discusses the methodology for this study. Included here is a discussion of aspects of design-based research and the classroom teaching experiment that are relevant for the study at hand as well as the conceptual framework based on the emergent perspective and Realistic Mathematics Education (RME). Pilot research related to the current study is discussed as it relates to the development of the instructional sequence. Data collection is detailed here with video-tapes of class sessions, field notes from the research team, and audio-tapes of research team meetings used as data for documenting collective student learning.

Chapter four provides the data analysis process and results of the study. Social and sociomathematical norms that were identified are presented. Classroom mathematical practices that developed throughout the implementation of the instructional sequence are discussed. The ways in which the instructional sequence and hypothetical learning trajectory were modified and the subsequent student learning was examined and reported.

Chapter five presents conclusions that can be drawn from the data analysis process, including changes that were made to the hypothetical learning trajectory as a result of this iteration of the instructional sequence. Implications for future research and the limitations of this study are discussed here as well.
CHAPTER TWO: LITERATURE REVIEW

The National Council of Teachers of Mathematics (NCTM) developed principles which should guide the teaching of school mathematics. In its publication *Principles and Standards for School Mathematics*, ten content and process standards were proposed, one of which was Number and Operation (NCTM, 2000). Although the content standards are consistent across all grade bands, NCTM recognizes that the emphasis of number and operation is highest in prekindergarten through grade 2 and next highest in grades 3 – 5. By high school, the emphasis on number and operation has diminished to facilitate the understandings of other content standards like Algebra and Geometry.

Since Number and Operation, which includes place value, is a priority for elementary school students, a vast amount of time and energy is spent in the elementary grades focusing on place value and operations. Therefore, a similar amount of time and energy should be spent in preservice preparation of teachers as it relates to Number and Operation (Menon, 2004). It is of vital importance that preservice teachers are adequately prepared for the challenges that await them in teaching elementary school, one of those challenges being teaching place value and operations.

This study seeks to add to the limited research related to the ways the social aspects of the classroom support preservice elementary school teachers’ collective learning of place value and operations. As part of their preparation to be elementary school teachers, preservice teachers need to experience for themselves the struggles and progressions children experience in learning mathematics (Conference Board of the Mathematical Sciences, 2001). Therefore, children’s development of understanding of place value and operations needs to be examined. This research
informs the hypothetical learning trajectory and the selection of appropriate tasks in which
preservice teachers engage in order to deepen their conceptual understandings of place value and
operations (McClain, 2003). Following this discussion, research regarding preservice and
inservice teachers’ understandings of place value and operations is discussed, developing the
need for this study. This chapter concludes with a discussion of hypothetical learning
trajectories, an important aspect of the implementation of this classroom teaching experiment.

**Children’s Understanding of Place Value**

Children’s understanding of place value and operations naturally begins with
understanding number. One important aspect in understanding number is conservation of
number. When given a collection of objects, children who grasp conservation of number will
immediately recognize that the same number of objects are present regardless of their
arrangement. Ross (1986) investigated conservation of number and identified phases of
development through which children progress. She found that children at the beginning level of
conservation of number were convinced that the total number of objects had changed when the
groupings were changed. As children progressed through understanding conservation of number, they then began to count to determine if the number of objects was the same. Finally, mastery occurred when the child responded that the quantity was the same without counting.

Conservation of number provides the foundation on which to build further number
concepts including concepts of ten (Dimitrovsky & Almy, 1975; Piaget, 1952). Before children
can begin to group objects and create efficient counting strategies, children must understand that
the number of objects present does not change with changes in how the objects are grouped.
Conservation of number is critical for the development of further mathematical ideas, including place value and operations. Children solve simple addition and subtraction problems by counting the number of objects in a collection, often as early as first or second grade (NCTM, 2000). The accuracy of that counting relies on conservation of number. The efficiency of that counting is dependent on the construction of numerical units. As children count more and more objects, they progress through developmental phases related to concepts of ten. Conservation of number provides the basis for recognizing that ten ones is the same whether it is separated out into ones or grouped together as one ten. Steffe (1983) proposed three levels of conceptual development in children’s construction of the concept of ten: 1) ten as a numerical composite, 2) ten as an abstract composite unit, and 3) ten as an iterable unit.

The first developmental level documented by Steffe (1983) was identification of ten as a numerical composite. At this level, children focus on the individual ones that make up the ten rather than on the ten itself as a single entity. Students at the level of ten as a numerical composite are characterized by the need to count single, discrete objects rather than units of ten. They are unable to construct ten as a unit of any kind. For example, students who are shown 43 beans grouped as 4 groups of 10 and 3 individual beans will count each bean individually to determine how many are in the collection. They do not recognize the unit of 10. They may call a strip of ten squares a ten, but do not recognize it as being composed of ten ones and simultaneously equal to one ten. They simply give it the name of a ten which signifies the result of counting ten ones. Ross (1986) identified similar levels of development in looking at children’s ability to efficiently determine the number of objects in a collection grouped by tens and ones. She identified that children were at one of three developmental levels with respect to ten as a numerical composite: a) the inability to determine how many were in the set, b) the
ability to quantify the number in the set, but only by counting by ones, and finally, c) the ability to quantify the number using a more efficient method like counting by tens and ones, the second level in Steffe’s analysis.

Cobb and Wheatley (1988) examined Steffe’s (1983) work and conducted another study of second grade children. Cobb and Wheatley (1988) interviewed 14 children to determine their levels of development of the concept of ten using various tasks including horizontal addition sentences, ten strips, and vertical addition problems, some of which included the same problems that were previously presented horizontally and others of which built on prior problems (i.e. 24 + 32 = ??, and then 24 + 33 = ??) (Cobb & Wheatley, 1988). The tens strip task involved showing students strips of ten dots and individual dots and asking the students how many dots were present. Then, more dots were revealed in different configurations of tens and ones and the students were asked again to determine how many dots were present.

Cobb and Wheatley (1988) identified similar levels of development to Steffe (1983), with children at level one, ten as a numerical composite, only being able to solve horizontal addition sentences by counting on by ones and often losing track of their counting. They did not count by tens. In the ten strips task, children at this level counted on by ones each time more strips were revealed. Some children at level one counted the ten strips and then counted on the units, but started over after each new piece of information was revealed (Cobb & Wheatley, 1988).

With the second concept, ten as an abstract composite unit, children begin to view composites of ten as single entities and as ten units simultaneously. Children who construct ten as an abstract composite will begin counting in the middle of a decade and coordinate counting by tens and counting by ones without difficulty. For example, one student was shown four strips of 10 squares and two individual squares. He was told that 25 squares were hidden and asked to
determine how many squares in all. In solving this problem, he pointed first to each visible ten strip and then to each visible square as he counted. He began counting, however, with 25 and counted “35, 45, 55, 65, 66, 67” (Steffe, Cobb, & von Glasersfeld, 1988). The significance of what this student did was that he began counting in the middle of the decade. He did not start with 20, he started with 25 and counted up from there. He also coordinated counting by tens and ones without difficulty. For these reasons, this student was believed to have developed ten as an abstract composite unit (Steffe et al., 1988). For these children, however, when counting by ten they are simply counting how many tens and do not recognize the concept of incrementing or decrementing by ten. For example, a child at this level may count 45, 55, 65, 66, 67, but may not recognize that from 45 to 55 is ten units. Instead, he may only see it as one more ten than before. Furthermore, these children are dependent upon representations of some kind to make units of ten and have difficulty with traditional paper-and-pencil algorithms. These children can construct a number with units of ten and units of one or as a single entity constructed of all units of one, but not both simultaneously (Steffe, 1983). These children may have difficulty understanding the traditional addition and subtraction algorithms which involve regrouping. These children are about to construct a number with tens and ones, but may have difficulty adding to or taking away from that number in instances where tens can be composed or decomposed.

Cobb and Wheatley (1988) found children who could construct ten as an abstract composite unit viewed horizontal addition sentences as vertical problems and approached them procedurally, but with errors. One child counted on by ones, but organized her counting into groups of ten. In the tens strips task, children at this level were characterized by separating tens and ones and were able to count on by tens and ones from any given number. Their success in solving these problems, however, was limited to having the representations of tens and ones in
front of them. They were unable to coordinate counting by tens and ones without suitable materials (Cobb & Wheatley, 1988).

Steffe’s (1983) third level, ten as an iterable unit, involves children’s abilities to solve tasks through counting by tens and ones without concrete materials. The children at this level are not dependent upon representations of tens and ones to create composite units. Counting by ten at this level includes incrementing or decrementing by ten. For instance, a child at this level who counts 39, 49, 59, … understands that from 39 to 49, there is one unit of ten and then to go to 59, that is another unit of ten which makes twenty. For example, given the problem: We have seventy-one. We take away a number and we are left with 39. How many did we take away? One student solved this by sequentially putting up three fingers on one hand and counting 61, 51, 41, and then putting up two fingers on the other hand and counting 40, 39. He then immediately wrote 32 for the solution to the problem. He viewed 32 as a single entity that could be simultaneously expressed as composite units of ten and units of one. He no longer needed the visual picture of objects to count and he iterated the units of ten without needing the concrete representations. Also, he recognized that from 71 to 61 was ten and from 61 to 51 gave a total of twenty, further substantiation that he had developed ten as an iterable unit (Cobb, 1991).

Children at this developmental level gave evidence they were attempting to create or had already created their own algorithms for solving addition problems. For example, one child justified her answer of 61 to the problem 37 plus 24 by saying “I knew there were 50 [i.e. 30 + 20] you make 60 and you have one left over” (Cobb & Wheatley, 1988, p. 12). Cobb and Wheatley (1988) called this level of development ten as an abstract collectible unit since she created tens and ones, but did not recognize that she was adding on 24. She started from the beginning and made tens and then added on the ones that were left. Another student added on to
the first number to make tens and then adding on the ones. For the problem $28 + 13$, he first added 2 to 28 to get 30, then added 10 more for 40, and then recognized there was only one left of the 13 and added that to get 41 (Cobb & Wheatley, 1988).

In addition to the three levels Steffe (1983) and Cobb and Wheatley (1988) identified and examined, Ross (1986) also examined other developmental milestones children reach in learning number and place value which are consistent with the developmental progressions identified by Steffe (1983) and Cobb and Wheatley (1988). These milestones included the ability of children to give meaning to, and not just recognition of, the individual digits in a two-digit number and the ability to compose and decompose numbers using base ten blocks. With respect to giving meaning to the individual digits in a two-digit number, Ross found that students at level one understood that the number represented the whole quantity, but gave no meaning to the individual digits. As students’ place value understanding grew into level two, the children began to invent meanings for the individual digits which were unrelated to the notion of tens and ones. Then, the children began to understand that the individual digits represented tens and ones, ten as a numerical composite, but failed to see that the sum of the parts must equal the whole. In the fourth and final level, the children understood that the individual digits in a number represented partitioning into tens and ones and that the sum of the tens and ones must equal the whole number. This corresponds to the developmental level of ten as an abstract composite identified by Steffe (1983).

Ross (1986) also examined children’s ability to compose and decompose two-digit numbers using base ten blocks. She initially asked them to represent a given number with no more than nine unit blocks. Children at the lowest level were unsuccessful at solving the problem. Children performing at the next level began by trying to represent the number with only
unit blocks. They discovered there were not enough unit blocks, so they used rods of ten. This relates to Steffe’s (1983) developmental level of ten as an abstract composite unit. These children were able to see the number as composed of unit block or rods of ten with blocks, but not necessarily both simultaneously. For children at the highest level on this skill the problem was viewed as routine and the children quickly chose the blocks needed to complete the problem. These children have constructed ten as an iterable unit (Steffe, 1983).

Those students who were successful with the previous task were then asked to represent the same number using more than nine unit blocks. The students at the beginning level, although successful with the prior task, were unsuccessful at this task. They were unable to be convinced that the number could be represented in any other way, giving more confirmation that these children viewed ten as an abstract composite unit (Steffe, 1983). The second level showed students were again successful only after an initial unsuccessful attempt to use all unit blocks. Some of these students required a prompt from the interviewer to see that other pieces could be used. Again, those students at mastery level for this task viewed the problem as routine. Many students at the mastery level transformed the initial collection of ten-blocks and unit-blocks into other representations, many by using a trade one ten-block for ten unit-blocks approach, a common approach taken with elementary school students. This trading relies on the developmental progressions previously identified by Steffe (1983) and Cobb and Wheatley (1988).

Researchers have shown that children progress through various developmental phases in learning number and place value (Cobb & Wheatley, 1988; Ross, 1986; Steffe, 1983). These include conservation of number as well as viewing ten as a numerical composite, an abstract composite unit, and an iterable unit or abstract collectable unit (Cobb & Wheatley, 1988; Steffe,
An understanding of these developmental phases is important for examining preservice teachers’ understandings of place value in an alternative base and for developing instructional tasks and goals for preservice elementary school teachers (McClain, 2003). Preservice teachers’ mathematical understandings of place value and operations, however, are scarce in the literature.

In examining preservice teachers’ development of place value understandings in pilot research for this study, it was found that using the instructional sequence and hypothetical learning trajectory developed for this study, the preservice teachers tended to progress through similar levels of development as children. Although most preservice teachers begin with an understanding of conservation of number, when exposed to an alternative base, in this case base eight, the preservice teachers tended to progress through similar levels of development. The hypothetical learning trajectory and instructional sequence used in this study intended to provide tasks and experiences for preservice teachers to develop base eight understandings of “ten” in similar ways as children have been found to develop. The developmental levels children tend to progress through is then vitally important for encouraging similar developmental levels in preservice teachers.

As children begin to develop more sophisticated understandings of number and place value, whole number operations can be introduced. Although most textbooks address whole number operations after place value has been mastered, some researchers advocate introducing multidigit addition and subtraction alongside place value instruction (Fuson, 1990; Fuson et al., 1997). In either case, an understanding of how children learn whole number operations is needed in order to understand the developmental progressions through which preservice teachers may be expected to progress.
Whole Number Operations

For all operations, most children begin with a need for modeling situations including using manipulatives or drawings to represent scenarios in story problems or analyzing story problems in a context the student can understand (Fuson, 1990). Modeling often initially begins with counting objects represented by manipulatives or drawings in order to find solutions eventually leading to recognizing operations for various problem situations. Children may consistently count by ones in these situations, giving indication that they have not yet fully developed place value grouping concepts (Cobb & Wheatley, 1988; Steffe, 1983). Other children who have a firm understanding of the base ten place value system may model directly, but make groups of tens throughout the problem solving process to make counting more efficient (Cobb & Wheatley, 1988; Steffe, 1983). Children’s understandings of whole number operations, then, is important in developing the hypothetical learning trajectory and instructional sequence for preservice teachers, if they are expected to progress through similar development progressions. A discussion of addition and subtraction is included here and is followed by a discussion of multiplication and division.

Addition and Subtraction

Children’s initial encounters with addition and subtraction occur mainly with single digit addends. Children can see the addends and then count to find the sum, initially counting all the objects and then counting on from one addend and then from the larger addend (Fuson, 1992). Sometimes, children will be able to count on when the second addend is one or two, but not be able to count on five or more. For subtraction, children may model the scenario presented in a
word problem exactly. This leads to counting on strategies with some method for keeping track of how much was counted on. These counting strategies are often abstract representations of the direct-modeling strategies already used (Carpenter et al., 1996). Counting strategies are more efficient than direct-modeling, and require a greater conceptual understanding of number than direct modeling with manipulatives. Some counting strategies require concepts of ten as an abstract composite unit and/or ten as an iterable unit (Steffe, 1983). Children then progress to being able to complete more sophisticated addition and subtraction problems by counting on from the larger addend or counting back from the sum as their conceptual understanding of addition and subtraction develops (Fuson, 1992). Their procedural understandings, however, are not separated from their conceptual development. Children need to have developed concepts of units of ten (Cobb & Wheatley, 1988; Steffe, 1983) to some degree before addition and subtraction strategies begin to be meaningful. Children also might begin to use addition and subtraction facts with compensation strategies. For example, a student might use the fact that 6 + 6 is 12 to find 6 + 7 by compensating (Carpenter et al., 1996).

When beginning multidigit addition and subtraction, many errors children make indicate that they treat multidigit numbers as single-digit numbers placed side by side (Fuson, 1992; Fuson et al., 1997). This is a place value misconception that leads to problems with multidigit operations as well as a misconception regarding units within multidigit numbers. The student does not recognize that 42 is made up of 4 tens and 2 ones, but treats the 4 and the 2 as equal units and is, therefore, unable to recognize that 42 is composed of 4 iterable units of ten and two ones (Steffe, 1983). Even with some place value understanding, children often initially do not see that it is problematic to arrive at different answers to the same problem, especially when problems are presented vertically rather than horizontally (Cobb & Wheatley, 1988; Fuson,
1992), giving indication that these children may not developed the concept of conservation of number in that they find different answers to the same problem.

Strategies children used initially in multidigit addition and subtraction problems were similar to those used for single-digit problems – counting all the objects, counting on from one addend, separating into tens and ones, and counting up or counting down to find the difference. These procedures quickly lead to errors due to the larger numbers involved and children begin to develop other strategies for adding and subtracting multidigit numbers, leading to the development of invented strategies, many of which are based on conceptual understandings of place value.

Children’s invented strategies for solving multidigit addition and subtraction problems are often more flexible than traditional algorithms, often beginning on the left hand side instead of the right hand side, and viewing the numerals as entities, not individual digits. By allowing children to develop their own strategies, understanding of multidigit addition and subtraction is deepened and can later be applied to understanding the traditional algorithms (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Carroll & Porter, 1997). Invented strategies for addition and subtraction include adding tens then ones, moving some to make tens, counting up, taking away tens then ones, taking extra and adding back, and compensation strategies (Schifter, Bastable, & Russell, 1999). These strategies are elaborated upon in tables 1 and 2.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Example (48 + 25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adding tens then ones</td>
<td>40 + 20 = 60; 8 + 5 = 13; 60 + 13 = 73</td>
</tr>
<tr>
<td>Moving some to make tens</td>
<td>48 + 2 = 50; 50 + 23 = 73</td>
</tr>
<tr>
<td>Counting up</td>
<td>48, 58, 68, 69, 70, 71, 72, 73</td>
</tr>
</tbody>
</table>
Table 2: Subtraction Strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Example (74 – 58)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taking away tens then ones</td>
<td>74 – 50 = 24; 24 – 8 = 16</td>
</tr>
<tr>
<td>Taking extra and adding back</td>
<td>74 – 60 = 14; 14 + 2 = 16</td>
</tr>
<tr>
<td>Compensation strategies</td>
<td>74 – 58 = 76 – 60 = 16</td>
</tr>
</tbody>
</table>

These strategies are often built upon place value understandings. For example, in the adding tens then ones strategy, children must see that 48 is not 4 and 8, but is 40 and 8. They have developed both conservation of number and ten as an abstract and/or iterable unit (Cobb & Wheatley, 1988; Steffe, 1983) and applied their knowledge to adding multidigit numbers. Similarly, in the counting up strategy, children have developed the concept of ten as an abstract composite or iterable unit and can count by tens starting with any number and can coordinate 10 units as 1 ten (Cobb & Wheatley, 1988; Steffe, 1983). Children’s invented strategies often can lead to the introduction of the more traditional algorithms in ways in which students develop a conceptual, rather than purely procedural, understanding of the addition and subtraction algorithms (Carpenter et al., 1998; Carroll & Porter, 1997). The researchers hoped to see the preservice teachers invent strategies in base eight and to see if they were similar to young children’s constructions through the hypothetical learning trajectory and instructional sequence developed for this study. One aspect of developing a profound understanding of multidigit addition and subtraction was examining alternative thinking strategies and allowing the freedom to develop strategies that made sense to the preservice teachers. This was also true of multiplication and division strategies.
Multiplication and Division

When children initially encounter multiplication and division problems they begin by modeling the actions in the problem directly similar to addition and subtraction problems (Baek, 1998; Carpenter et al., 1996). They build on what they already know about addition and subtraction in solving situational problems that are multiplicative (Steffe, 1994). One of the most common models used for multiplication is that of groups of objects in which one factor, the multiplier, represents how many groups, the other factor, the multiplicand, represents the number in each group, and the product represents the total number of objects. An alternative, and often used, model for multiplication is that of arrays or areas in which the number of rows is one factor, the number in each row is the other factor, with the area being the product (Simon & Blume, 1994).

Initial attempts at solving multiplication problems often rely on concepts of number related to addition and subtraction and counting sequences. Children will often count up repeatedly, but use something other than one as their unit, what Steffe (1994) refers to as composite units, or a unit that is itself composed of units. Baek (1998) refers to this as complete number strategies, or adding the multiplicand but not partitioning the multiplier or multiplicand in any way. For example, they might find 4 x 7 by counting up by seven, four times. They may keep track of how many times they have counted by seven with their fingers or other objects. They view seven as a composite unit and repeat that unit four times (Steffe, 1994). Similarly, they may find 6 x 12 by counting up by 12, six times. They do not partition the 12 to make the counting easier.
Multiplicative reasoning requires using two different ranks, the number of groups and the number in each group. When children begin to develop multiplicative reasoning skills, these units of different rank begin to be coordinated. For example, one child, who was beginning to coordinate these units of different rank, was asked how many orange pieces of paper it would take to cover a red piece of paper if two orange pieces covered a blue piece of paper and six blue pieces covered a red piece. The child tapped six of her fingers each two times and kept track of how many she tapped, but counted in pairs – “1, 2, …, 3, 4, …, 5, 6, …, 7, 8, …, 9, 10, …, 11, 12” (Steffe, 1994). As children continue to develop multiplicative reasoning concepts, they become more efficient and create iterable units. The development of multiplicative reasoning concepts also leads to more efficient and conceptually-based strategies for multiplying, including partitioning strategies and compensation strategies (Baek, 1998).

As children’s strategies develop into more efficient ways of multiplying, they begin to develop partitioning strategies (Baek, 1998). In these strategies, the multiplier is often partitioned into smaller numbers to make the multiplication easier. Other times, the multiplicand and sometimes both the multiplier and multiplicand are partitioned into smaller numbers. Some children will partition into tens and ones while others may partition into other combinations of numbers. For example, in solving the problem to determine the total number of apples if there are 15 boxes with 177 apples in each box, one upper-elementary student used the fact that 15 was 5 x 3 to make the multiplication of 15 x 177 easier. She made 15 x 177 into (5 x 3) x 177, or 5 x (3 x 177). She combined the three 177s first by partitioning 177 into 100, 70, and 7 and adding each one three times. This gave her 531. Then she found five groups of 531 by first finding four groups of 531 by doubling 531 twice and then adding another 531 to get 2655. Her strategy for solving 15 x 177 was far from the traditional algorithm commonly associated with these
problems, and although it was complex, relied predominately on partitioning the 15 into 5 x 3. She used additive properties associated with multiplication to find 3 x 177, but used multiplicative properties to find 5 x 531 by breaking 5 into 4 and 1 and doubling to find 4 x 531 (Baek, 1998).

Students also were found to partition into tens and ones, providing even more efficient ways of multiplying. One student solved the problem 26 times 39 by saying 20 x 30, 20 x 9, 6 x 30, and 6 x 9. This partitioning into tens and ones corresponds nicely with the traditional algorithm and with the alternative algorithm “partial products”. As students develop strategies for partitioning into tens and ones, they develop more sophisticated ways of solving multiplication problems. These often include recognizing that 5 is half of 10 and that 20 is double 10. Students might find 5 times 25 by first finding 10 times 25 and then dividing by two (Baek, 1998).

As children’s partitioning strategies develop into more sophisticated ways of multiplying, compensation strategies begin to be developed as well. Compensation strategies recognize the mathematical ability to manipulate the original problem to make the process easier. For example, when multiplying 5 x 250, students who have developed compensation strategies might say that 5 is half of 10, so to make the 5 a 10, I have to divide 250 by 2. Therefore, 5 x 250 is the same as 10 x 125, which is 1250. They recognize the relationship between the multiplier and the multiplicand and use those relationships to make the problem easier to solve. Compensation strategies can also be used in ways that only change the multiplier or the multiplicand, often when one is close to a multiple of ten. They may solve a problem like 17 x 70 by first finding 20 x 70 and then taking away 3 x 70 (Baek, 1998). These types of multiplication strategies provide
Strategies students use in solving multiplication and division problems often neglect the context of the problem. Instead, students report strategies of looking at the numbers to determine the operation to use, try all the operations until the most reasonable answer is found, or look for key words or phrases to determine which operation to use (Greer, 1992). Although this is seen with addition and subtraction problems as well (Cobb, 1991), the issues involving the coordination of units of different rank make this all the more common with multiplication and division problems. The context of the problem and modeling it to determine the operation are often left out of solving multiplication and division problems. This situation must be addressed for students to be successful at the elementary level and the preservice educational level and is accomplished in this study by using contextually-based problems to begin the examination of aspects of both place value and whole number operations.

Development of division concepts is directly related to multiplicative reasoning concepts and builds on multiplication in ways similar to addition and subtraction. Direct modeling with division problems leads to the recognition that there are two basic types of division problems: sharing (or partitive) division and measurement (or quotitive) division (Carpenter et al., 1996; Greer, 1992). Children see these as different, despite both of them being division problems. “Although adults may recognize both [sharing and measurement] problems as division problems, young children initially think of them in terms of the actions or relationships portrayed in the problems” (Carpenter et al., 1996, p. 8). This distinction is critical for preservice teachers to recognize and understand in seeking to develop both their own mathematical knowledge and their pedagogical content knowledge. If preservice teachers do not recognize this critical
difference in division understanding, they are less likely to provide their students with experiences dealing with both types of division problems, leading to potential misconceptions on the parts of their students. The instructional sequence involved in this study began with experiences that were experientially real to students, so many initial experiences with both place value and operations involved context-based problem situations. Direct modeling was critical and expected from the preservice teachers and led to discussions of alternative strategies for multiplication and division problems, many of which may be encountered by the preservice teachers in their own classrooms with their own students.

Sharing division, also called partitive division, is the most commonly recognized model for division. Sharing division can be modeled like dealing cards or sharing cookies in which the total number and number of groups is known and the number in each group is unknown. Children will often model these problems directly by counting out objects and then dealing them out (Carpenter et al., 1996). Measurement division, also called quotitive division, is used less frequently and involves knowing the total number and the number in each group. In this case, the number of groups is the unknown. Research has shown that sharing division is most commonly taught and measurement division, although helpful for development of division by fractions, is less common in schools (Graeber, Tirosh, & Glover, 1986; Simon, 1993). The emphasis on sharing division over measurement division also leads to the misconceptions that multiplication makes bigger and division makes smaller (Greer, 1992; Tirosh & Graeber, 1989) as well as the misconception that the quotient must be less than the dividend (Tirosh & Graeber, 1990). In whole number division, the quotient is smaller than the total number of objects. When dividing with fractions, however, the quotient is often larger than the total number of objects with which the student started. For example, in the situation of having 4 cookies and wanting to give each
friend one-half of a cookie \((4 \div 1/2)\), 8 friends can share the cookies, so the quotient, 8, is larger than the dividend, 4. This is counter-intuitive to many students who have not encountered an adequate number of contextual division problems. Children who have been exposed almost exclusively to sharing division problems often find it difficult to understand how the quotient can be larger than the dividend. Even if the teacher does not make that conclusion for the students, if they are not given experiences with both types of division problems or given contextual problems to make sense of, children often make the conclusion for themselves that division makes smaller and multiplication makes bigger.

The issues involved in teaching multiplication and division often correspond to those involved in teaching addition and subtraction, especially with respect to place value. Children often begin solving multiplication and division problems by applying what they know about addition and subtraction. They count how many objects, but can now use strategies they are familiar with from addition and subtraction. Although many of the strategies for developing understanding of multiplication and division are similar to addition and subtraction, there are issues specific to multiplication and division learning. One issue that is involved in multiplication and division and not in addition and subtraction is that of referents. Addition and subtraction involve the same referents throughout. One cannot add apples and oranges and get pears. One must add apples and apples to get apples. However, in multiplication and division, the referents involved in each aspect of the problem are different (Greer, 1992). In sharing and measurement division problems, the referent of the solution changes depending on the type of division problem. In sharing division, the solution is the number in each group while in measurement division, the solution is the number of groups. This issue makes conceptually understanding and interpreting multiplication and division problems all the more challenging.
Preservice teachers need to understand these aspects of multiplication and division in order to provide their students with conceptually and contextually-based learning experiences.

Similar to addition and subtraction, direct modeling strategies are eventually replaced with counting strategies, addition or subtraction strategies, or using derived number facts, often involving doubling. These may include repeated addition and subtraction, skip counting, and using number facts to determine a reasonable answer (See Tables 3 and 4). Similar strategies were found to occur with the preservice teachers in the pilot study for this research and the instructional sequence and hypothetical learning trajectory included a focus on the preservice teachers’ development of invented strategies for all whole number operations.

### Table 3: Multiplication Strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Example (4 x 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repeated addition</td>
<td>6 + 6 = 12; 12 + 6 = 18; 18 + 6 = 24</td>
</tr>
<tr>
<td>Skip counting for multiplication</td>
<td>6, 12, 18, 24 (extends one finger for each time he counts to keep track of the number of groups)</td>
</tr>
<tr>
<td>Number facts</td>
<td>6 + 6 = 12; 12 + 12 = 24</td>
</tr>
</tbody>
</table>

### Table 4: Division Strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Example (32 ÷ 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repeated subtraction</td>
<td>32 – 4 = 28; 28 – 4 = 24; 24 – 4 = 20; 20 – 4 = 16; 16 – 4 = 12; 12 – 4 = 8; 8 – 4 = 4; 4 – 4 = 0; that 4 is subtracted 8 times.</td>
</tr>
<tr>
<td>Skip counting for division</td>
<td>4, 8, 12, 16, 20, 24, 28, 32 (extends one finger for each time he counts, then counts how many fingers)</td>
</tr>
<tr>
<td>Number facts</td>
<td>5 fours is 20. I need 12 more, so that is 3 more fours, so 8.</td>
</tr>
</tbody>
</table>
The ways in which children learn and understand multiplication and division of whole numbers is critical for preservice teachers to experience and understand. Alternative and invented strategies for whole number operations are important for preservice teachers to have experience developing for themselves in order to foster that same development in their own students. The hypothetical learning trajectory and instructional sequence for this study rely on contextually-based learning experiences as well as the preservice teachers’ development of concepts of place value and whole number operations in ways similar to those that children experience. An understanding of how children learn and understand whole number operations is then critical for developing and refining the hypothetical learning trajectory and instructional sequence for this study. Additionally, an understanding of how place value and operations is often taught in schools is important for understanding how the preservice teachers may have been taught previously and what awaits them as they enter the profession of education. Knowledge of the ways in which place value and operations are currently taught can be informative in examining what preservice teachers need to experience and understand prior to entering the classroom in order for them to become effective teachers.

**Teaching Place Value and Operations**

The ways in which children learn place value and operations have been studied in many contexts. Research suggests implications for how place value and operations can be taught within the elementary school classroom as well as approaches that can be taken with preservice elementary school teachers (Bowers, Cobb, & McClain, 1999; Cobb & Wheatley, 1988; Fuson, 1992; Fuson et al., 1997; McClain, 2003; McClain, Cobb, & Bowers, 1998; Ross, 1986; Steffe,
1983). The textbook approach of emphasizing assignment of values to digits based on their position does not advocate a conceptual understanding of units of ten and often leads to misconceptions (Ross, 1986). Students are encouraged to memorize the place value based on the column rather than conceptually understanding ten as an iterable unit (Baroody, 1990; Hiebert & Wearne, 1992; Steffe, 1983; Uprichard & Collura, 1977). Other approaches have been shown to be more effective in developing children’s conceptual understandings of place value and operations (Baroody, 1990; Bowers, 1996; Bowers et al., 1999; Hiebert & Wearne, 1992; McClain et al., 1998) and deserve examination in developing the hypothetical learning trajectory and instructional sequence for this study.

Most textbooks present place value apart from operations and spend a considerable amount of time developing place value concepts before moving on to operations (Fuson, 1990). Although some aspects of place value are necessary to understand operations, particularly multidigit operations, some researchers advocate integrating teaching of place value concepts with the teaching of multidigit addition and subtraction with and without regrouping (Baroody, 1990; Fuson, 1990). On the one hand, “understanding operations on multidigit numbers requires understanding how to compose and decompose multidigit numbers into these multiunits in order to carry out the various operations” (Fuson, 1990, p. 273). On the other hand, “early textbook emphasis on two-digit problems with no trades may support typical errors like writing both numbers in a given column and subtracting the smaller number from the larger number in a column…only trading requires relating adjacent columns to each other” (Fuson, 1990, p. 276) which requires an understanding of place value relationships. These may also be compounded by students not developing the concept of ten as an iterable unit, leading to misconceptions regarding the structure of multidigit numbers (Cobb & Wheatley, 1988; Steffe, 1983).
The traditional textbook approach to teaching place value and operations procedurally has been shown to be problematic for student learning (Hiebert & Wearne, 1992). Hiebert and Wearne (1992) investigated differences in learning place value and addition and subtraction without regrouping based on instructional design differences. They compared a traditional, textbook-based approach in which instruction followed the textbook closely including extensive practice in specific algorithms to an alternative approach which relied on building connections between different models and allowed the students develop their own algorithms. They found that students in the alternative instruction classrooms performed better on items related to place value and operations. When items were analyzed by topic, they found that students in the traditional textbook instruction classroom did not have higher proficiency on written procedures despite increased practice through the instruction they received. Additionally, students in the alternative instruction classroom were able to adapt their understanding of addition and subtraction without regrouping to be able to solve problems involving regrouping (Hiebert & Wearne, 1992). The alternative approach to teaching place value and operations provided students with the tools they needed to increase their own understanding and apply their knowledge outside the areas specifically addressed in the classroom. This approach is similar to that advocated in this study – giving students experiences that help them develop conceptual understandings of place value and operations and building on place value understandings to assist in understanding whole number operations.

Uprichard and Collura (1977) found that seven- and eight-year-olds who performed at the lowest level on a pretest regarding number, place value, and addition and subtraction computations performed higher on a post-test after receiving instruction that emphasized mathematical structure. The students in the experimental group of their study were exposed to
experiences which facilitated their development of grouping by tens through concrete, representational, and abstract levels. Students in the control group received instruction using games involving only drill and practice. The students in the experimental group performed significantly better on the post-test in which all the items were abstract in nature and required either the reading or writing of numerals, despite the fact they were not given drill and practice type games. The conceptual understandings and development of concepts of ten provided a framework on which to build further conceptual understandings of operations.

These studies demonstrate that the traditional drill and practice, textbook-based approach to teaching place value and operations may be flawed. Klein, Beishuizen, and Treffers (1998) found that using an empty (unmarked) number line in teaching addition and subtraction up to 100 was advantageous over not only the traditional drill and practice approach, but also over approaches using base-ten blocks or hundred grids. The empty number line provided the students with ways to reason flexibly as marks for specific numbers were not already present on the number line. The students were able to determine what numbers to place where and how to solve the problem at hand. Although base-ten blocks were useful for place value as well as composition and decomposition of numbers and the hundred grids simulated sequential counting by tens, they found that the empty number line aided in the development of informal strategies. Students who used the empty number line were “cognitively involved in their actions. In contrast, students who use[d] materials such as [base-ten] blocks or the hundred [grid] sometimes tend[ed] to depend primarily on visualization, which result[ed] in a passive ‘reading off’ behavior rather than cognitive involvement in the actions undertaken” (Klein et al., 1998, p. 447). Alternative approaches including allowing students to develop their own invented strategies for operations and making connections between representations of numbers and
operations take advantage of the levels of development through which children progress in learning number and operation and provide for a deeper conceptual understanding of place value and operations than the traditional textbook approaches, leading to improved learning in other areas of mathematics, including other aspects of number and operation. These alternative approaches are similar to those used in the instructional sequence for this study. Making connections between areas of mathematics and inventing strategies for whole number operations was critical in the development of preservice teachers’ understandings of place value and operations.

Bowers (1996) investigated children’s development of place value and multidigit addition and subtraction strategies using an instructional sequence set in a candy factory (see also Bowers et al., 1999; Yackel & Karnes, 1992, 1993). The candy factory scenario provided a basis for developing place value concepts in that 10 pieces of candy made a roll and 10 rolls of candy made a box. Children were given experiences that facilitated their development of concepts related to place value and operations and the instructional sequence used by Bowers (1996) formed the starting point for the instructional sequence used with preservice teachers in this study. Bowers (1996) found that the children who experienced the candy factory instructional sequence demonstrated improved understanding of paper-and-pencil algorithms and developed increasingly sophisticated place value concepts. The students developed concepts of place value related to composing and decomposing numbers and concepts of addition and subtraction with multidigit numbers both in context and out of context (Bowers, 1996; Bowers et al., 1999; McClain et al., 1998; Yackel & Karnes, 1992, 1993).

The implications of how children learn place value and whole number operations extend into all of the Number and Operations standard (NCTM, 2000). Students with a conceptual
understanding of place value and operations with whole numbers are less likely to make mistakes with procedures for operating on decimals while students with only a procedural understanding of place value are more likely to make mistakes with procedures for operating on decimals (Wearne, Hiebert, & Campbell, 1994). For example, a student using a place value explanation for addition of multidigit numbers, explaining that one should add ones with ones, tens with tens, hundreds with hundreds, and so forth, would be more likely to naturally line up the decimal point when learning addition and subtraction of decimals. Another student with a procedural understanding of lining up the numbers on the right hand side when adding and subtracting multidigit numbers would be more likely to line up the numbers on the right hand side, irrespective of where the decimal point occurred in the number. The way in which students understand place value and operation with whole numbers is directly related to their ability to apply their mathematical understandings to operations with decimals (Wearne & Hiebert, 1986; Wearne et al., 1994). Clearly the way in which place value and operations is taught has implications for future understandings of concepts related to number and operations and for ways in which preservice teachers need to understand place value and operations themselves to avoid the common misconceptions and mistakes prevalent in schools today.

Although researchers advocate the use of multiple representations and building connections between mathematical topics and representations (NCTM, 2000), there is conflicting research regarding the use of multiple concrete representations for learning place value and operation concepts. Edge and Ashlock (1983) found that using multiple representations of three digit numbers – using coffee stirrers, base-ten blocks, and colored chips – did not produce a difference in achievement measures as compared to only using one of the three representations. Hiebert and Wearne (1992) examined aspects of instructional practice that were different
between classes which used a traditional, textbook approach versus alternative approaches and found that the alternative instruction classroom used fewer kinds of hands-on materials but used them more often than the textbook instruction classroom. Additionally, the teachers in the textbook driven class demonstrated the use of hands-on materials more often while the children in the alternative approach classroom used the materials more often than the teacher. Results indicated that the alternative type of instruction was beneficial for children’s understanding of place value and operations and students made more connections between their current understandings and other areas of place value and operations. For instance, children were able to solve addition and subtraction problems that involved regrouping based on their understandings of addition and subtraction without regrouping. The underlying conceptual understandings of place value and operations had been established more effectively through the alternative instruction route and their use of fewer representations and more time with each type of hands-on material. In developing and refining the hypothetical learning trajectory and instructional sequence for this study, the aspects of effective instruction and use of hands-on materials for teaching place value and operations were considered and used in developing and making revisions to the sequence.

With an understanding of how children learn number and numeration concepts as well as place value concepts and operations and the belief that preservice teachers can progress through similar levels of development, teacher-educators can be better prepared to develop instructional strategies that facilitate development of profound understandings of place value and operations in preservice teachers. Many studies have shown that this transition and connection of representations can facilitate deeper understanding of mathematics in general and place value and operations in particular (Hiebert & Wearne, 1992; Jones, Thornton, & Zoest, 1992;
Underhill, 1983). These transitions and connections are important for preservice teachers to develop and understand in order to deepen their mathematical content knowledge and their pedagogical content knowledge related to place value and operations. Research has shown that preservice and inservice teachers are lacking in their understandings related to place value and operations. Solutions to these problems, however, are lacking in the current research.

**Preservice and Inservice Teachers’ Number Sense**

Although there is a vast amount of research related to children’s understanding and learning of place value and operations, research on preservice teachers’ understandings of place value and operations has been scarce. Related research on inservice teachers’ understandings includes the work of Cognitively Guided Instruction (CGI) and other teacher professional development programs (see for example Carpenter et al., 1996; Fennema et al., 1996). Although preservice teachers can benefit from the findings of CGI and other teacher professional development programs, the knowledge base for inservice teachers can be vastly different from preservice teachers.

In studies conducted related to the teacher development program Cognitively Guided Instruction (CGI), Fennema et al. (1996) found that when teachers participated in a CGI program, there were fundamental changes in both the beliefs and instruction of the teachers. These included changes from demonstrating mathematical procedures to children to helping children build on their own mathematical thinking through problem solving situations and encouraging the communication of mathematical thinking. They found that changes in instruction were directly related to increases in student achievement. These inservice teachers
had experiences to draw from and were able to apply their knowledge directly to the classroom. Preservice teachers do not have that opportunity. They must take what they know and visualize how it could be used in the classroom. Without the opportunities to develop their own understandings of mathematics, they are unlikely to be able to provide the mathematical experiences and understandings current reform movements desire (Broadbent, 2004; Ma, 1999). The only experiences they have to build upon are their own experiences as students in a classroom. By providing experiences which rely upon reform-based instruction, preservice teachers can be better prepared to be elementary school teachers. A goal of this instructional sequence was to provide the types of experiences necessary to deepen preservice teachers’ understandings of place value and operations so as to improve their own knowledge of mathematics in order for them to be better prepared to be elementary school teachers. A research focus was on documenting the ways the instructional sequence and corresponding hypothetical learning trajectory supported the development of collective mathematical understandings of preservice elementary school teachers.

The current state of preservice teachers’ mathematical knowledge has been shown to be lacking. Menon (2004) conducted a study which examined preservice teachers in elementary mathematics methods courses, all of whom had taken the mandatory mathematics content courses required by their university which included algebra and geometry. In spite of that fact, 85% of them still cited a lack of confidence in teaching middle school mathematics, citing a feeling of insecurity with their own ability to successfully perform middle school mathematics content. A ten item test on number sense was given to the preservice teachers on the first day of class which included questions related to the students’ ability to make mathematical judgments as well as the ability to develop useful and efficient strategies for numerical situations. The
students were asked not only to determine the correct answer, but provide an explanation in terms of a correct solution. Questions were scored for correct answer, correct solution, both, or neither. Menon found that a majority of preservice teachers studied were able to make appropriate judgments about relevance and sufficiency of given information; they were able to determine if an adequate amount of information was provided to determine a solution to the problem. This aspect of number sense was quite strong among the preservice teachers studied. However, in the second component, using efficient strategies for numerical situations, 91% of students gave a correct answer, but only 41% provided correct explanations which demonstrated the use of effective and efficient strategies related to the problem at hand. When given an addition problem and asked to compute a different, but related, addition or subtraction problem (i.e. $45 + 32 = 77$, what is $46 + 32 = ?$), 75% of students used actual computations to arrive at their answer and did not use any relationship with the information given. Menon (2004) determined that the preservice teachers had an over-reliance on algorithms and used inefficient strategies for managing numerical situations. He concluded “if these future K – 8 mathematics teachers seem to rely on learned procedures, without the profound understanding of fundamental mathematics suggested by Ma (1999), as shown by some of their explanations to the number sense test items, how well equipped will they be to teach conceptually?” (p. 57).

In a teaching experiment conducted by McClain (2003), preservice teachers were given tasks situated in a candy factory in which eight candies made a roll and eight rolls made a box. This scenario paralleled a teaching experiment with third graders in which the same scenario was used, but with groupings of tens and provided the basis for the instructional sequence used in this study. She found that the preservice teachers constructed in many ways the learning paths of the third graders when supported by similar instructional tasks. The teachers began by estimating the
number of candies in a bag and then transitioned to being able to count collections by boxes, rolls, and pieces. This led to the preservice teachers’ understanding of a roll being a single entity and eight pieces at the same time, or what Cobb and Wheatley (1988) referred to as viewing ten as a composite unit. The sequence proceeded through reorganizing boxes, rolls, and pieces by packing and unpacking and led eventually to developing efficient strategies for adding and subtracting with three digit numbers including subtracting with regrouping across zeros. McClain observed that the preservice teachers were better able to understand the conventional base ten algorithms for addition and subtraction as a result of their experiences in the course.

In another study, Simon (1993) found that preservice teachers lacked a conceptual understanding of division. When given a task of writing three different story problems which would give three different answers to the same division problem (i.e. 51 divided by 4 gives 12 3/4, 13, or 12), a majority of students were unable to connect the context of the problem to the answer. They seemed to believe that the answer was dependent upon the directions with respect to rounding rather than the context of the story. Simon also verified earlier studies (Graeber et al., 1986) which found that preservice teachers tended to think only of sharing models of division and neglected measurement models.

These studies highlighted here demonstrate that preservice teachers lack a full understanding of number. Menon (2004) gives some suggestions for ways to begin to assist preservice teachers in developing this deep conceptual understanding of mathematics. He advocates for mathematics educators to make preservice teachers aware of “the disadvantages of learning mathematics procedurally and the empowering nature of making sense of mathematics” (p. 58). He also concludes that to have a long-term impact on preservice teachers that lasts throughout their teaching careers, programmatic and systemic change has to be implemented.
within teacher education programs to focus more on increasing the pedagogical content knowledge of preservice teachers. This may include examining the mathematics content required of preservice teachers and the manner in which it is taught as well as re-examining the perceived division between content and methods implied by the common practice of content courses being taught in the mathematics department and pedagogy being taught by the education department (Menon, 2004).

With Menon’s (2004) conclusions that preservice teachers are lacking in content knowledge and pedagogical content knowledge related to place value and operations, preservice teachers need high quality instruction related to these areas. McClain’s (2003) study and preliminary research related to this study, has indicated that preservice teachers can progress in learning from counting to computing in alternative bases to base ten. These developmental progressions can be informed by those that children progress through in learning counting and computing in base ten (Cobb & Wheatley, 1988; McClain, 2003; Ross, 1986; Steffe, 1983). This conceptual process is important for researchers to examine in developing and refining an instructional sequence for preservice teachers related to place value and operations. These findings should be used and investigated with regards to facilitating the preservice teachers’ understandings of number and operation and in developing mathematical knowledge for teaching (Hill et al., 2004).

In the study reported here, the instructional sequence began with tasks and learning trajectories similar to those McClain (2003) used in her study; revisions to the sequence were based upon the specific experiences of the research team involved in this study. Experiences in the first two semesters of implementing the instructional sequence provided data which indicated that the preservice teachers were able to examine and understand conceptual foundations for
place value and whole number operations by using base eight as an alternative to base ten, seemed to progress through similar development progressions in constructing “ten” as a unit, and developed alternative and invented strategies for whole number operations. The focus of this dissertation is to examine the social aspects of the classroom, namely the social and sociomathematical norms and the classroom mathematical practices, and how they supported learning in an effort to improve preservice teachers’ mathematical content knowledge and mathematical knowledge for teaching in ways Menon (2004) discussed and advocated. Additionally, this study focuses on documenting and examining the ways in which the instructional sequence was refined and how those revisions supported student learning. The ways in which the mathematical content was developed in this sequence and the social aspects of the classroom are not the most common in colleges today. This break from the procedural focus of mathematical teaching in colleges and K-12 schools throughout the country directly supports the conclusions Menon made in relation to improving preservice teacher education programs. The results of this study has implications for how preservice teachers are prepared to be in the elementary school classroom, an area that needs critical attention if children are to have highly qualified teachers in their classrooms. The research base surrounding children’s understandings of place value and operations as well as the ways in which these mathematical topics have historically been taught to both children and preservice teachers is informative for the creation of the hypothetical learning trajectory that forms a basis for instructional decisions made in this classroom teaching experiment.
Hypothetical Learning Trajectory

Design-based research uses the concept of a hypothetical learning trajectory (HLT) as a basis for the development of instructional sequences. Hypothetical learning trajectories (HLTs) were first introduced by Simon (1995) as a tool to plan for and describe the pedagogical thinking involved in teaching mathematics for understanding. Several interpretations of HLT exist in the body of research (Clements & Sarama, 2004; Gravemeijer, 2004; Gravemeijer et al., 2003; Simon, 1995; Simon & Tzur, 2004), but most agree that the HLT includes three aspects: 1) the learning goals, 2) the instructional sequence of tasks to support those learning goals, and 3) the developmental progressions of students expected as a result of the instructional sequence. The HLT is developed initially as a framework of instructional tasks with expectations for how the class will engage in thinking and learning as they participate in the instruction. The HLT provides the basis for the decisions that are made with respect to instructional tasks and plans for instruction. The learning goals associated with HLT aid in determining the instructional tasks which may support those goals. As the instructional sequence is implemented, however, the tasks are determined or modified on a class-by-class basis taking into consideration the learning that occurred during the preceding tasks and how it matched or did not match what was expected. At the completion of the teaching experiment, the instructional sequence at hand is then refined through a cyclical process of analyzing the tasks used, the teacher’s role, student learning individually and collectively, and the classroom environment to determine how the HLT was realized through the sequence and changes that need to be made to the HLT and instructional sequence for future iterations. The HLT is not intended to be a scripted lesson plan, but instead a
framework that teachers can use to adapt instructional sequences that fit their own classroom needs (Clements & Sarama, 2004; Simon, 1995; Simon & Tzur, 2004).

In the initial development of a HLT, developmental progressions of the population being studied are identified. The HLT for a given topic is built upon prior research which gives an indication of mathematical development for the specific domain being investigated. This aspect of the HLT distinguishes it from other instructional design models that break a goal into sub-skills based on the researcher’s thinking rather than the students’ developmental progressions (Clements & Sarama, 2004). Since research has shown that preservice teachers can progress through similar developmental progressions as children in developing understandings of place value and operations when supported with particular experiences and environments (McClain, 2003), this aspect of the HLT is met through the research team’s knowledge of children’s conceptual development and prior research in which researchers have developed an HLT for supporting children’s development of understandings of place value and operations. Children’s developmental progressions were used as a model for preservice teachers’ developmental progressions as related to place value and operations including unitizing ten, composing and decomposing numbers, flexible representations of numbers, and invented strategies for whole number operations. These formed a basis for the development of the HLT used in this study.

Research on the conceptual development of students is then used to inform the design of the instructional sequence in determining the manner and sequence in which tasks are given and the projected outcomes of the implementation of those tasks. Tasks are designed that include tools and actions that support the mathematical activity in which students are expected to engage throughout the sequence. The tasks are sequenced intentionally based on the developmental progression expected. Through the process of implementing the instructional sequence and
analyzing the student learning both individually and collectively, the tasks are modified and the sequence evaluated to determine any changes in the HLT that may have taken place as a result of the implementation of the sequence. The sequence may be altered for future use as a result of the impact seen and learning evidenced with the initial implementation. The HLT and instructional sequence are then implemented in this cyclical manner, complete with constant revision and review, until a local instructional theory is developed (Clements & Sarama, 2004; Gravemeijer, 2004; Simon & Tzur, 2004). The HLT can become a tool for curriculum development in that trajectories for various mathematical topics can be developed in this manner and tested in classroom using design-based research and classroom teaching experiments. As these trajectories and instructional sequences are refined, curriculum for mathematical teaching and learning can be established. This can take place both in K-12 classrooms by classroom teachers as well as at the post-secondary education level with mathematics courses or with teacher preparation courses, as is the case for this study.

The HLT specific to this study is presented in chapter 3 along with a discussion of Realistic Mathematics Education (RME), the instructional theory used for creating the learning trajectory and instructional sequence. The HLT provides the classroom mathematical practices that are expected to develop through the implementation of the instructional sequence (Cobb, Stephan, McClain, & Gravemeijer, 2001).

**Conclusion**

The need for an understanding of children’s development of concepts of number, place value, and operations is important for preservice teachers and teacher-educators in providing
learning experiences for understanding mathematics. Knowledge of the development of children’s concepts of number, place value, and whole number operations was critical to the development and refining of the hypothetical learning trajectory and instructional sequence for this study. It is believed that preservice teachers can construct number, place value, and operations concepts in ways similar to children when supported by particular instructional tasks and environments in the context of a base other than base ten. However, preservice teachers’ understanding of school mathematics has not been studied to great extents. Little research has documented how preservice teachers understand and make sense of mathematics for themselves and even fewer studies have examined the social aspects of the classroom at the university level. There is a vast research base for how children learn and understand place value and operations (see for example Cobb & Wheatley, 1988; Fuson, 1990; Ross, 1986; Steffe, 1983) and how the social nature of the classroom influences children’s development of mathematical understanding (for example, Cobb, 2003; Lo, Wheatley, & Smith, 1994), but the same type of research has not been widely conducted dealing with preservice elementary school teachers. With reform-based mathematics instruction advocating teaching mathematics for understanding, the knowledge of the teacher is of vital importance (Sanders & Rivers, 1996; Wright et al., 1997). Preservice teachers are a critical population to increase their knowledge of mathematics required for teaching. They are not classroom teachers yet and are able to begin their teaching careers with a deeper mathematical understanding in order to provide their students with a mathematics education that is conceptually driven and reform-based (Hill et al., 2004; Ma, 1999; NCTM, 2000).

Although the individual learning of the preservice teacher is important, the focus of this study was on the social aspects of the classroom and how they support collective student
learning. These social aspects are critical for preservice teachers to experience and to be encouraged to develop in their own students. This study has implications for how preservice teachers are taught mathematics and what classroom environments may have a positive influence on collective mathematics understandings.
CHAPTER THREE: METHODOLOGY

Using a design-based research methodology, this study used qualitative research methods to document a classroom teaching experiment that was conducted in a semester-long undergraduate mathematics education course for 16 prospective elementary school teachers. The emergent perspective, which attempts to coordinate the individual learning and the social aspects of the classroom that support collective learning, was used as an interpretive lens for data collection and analysis. Data collection consisted of video and/or audio-taped recordings of the five, three hour and ten minute class sessions, small group interactions, and research team meetings; field notes; research team journals; and individual student work. The focus of this research study was on documenting ways in which the social aspects of the classroom, namely a) social norms, b) sociomathematical norms, and c) classroom mathematical practices facilitated collective student learning of place value and whole number operations. Collective student learning can be defined as the ways in which reasoning in the classroom community becomes taken-as-shared through patterns of interaction (Rasmussen & Stephan, in press). Argumentation analysis (Toulmin, 1969) was used as a basis for determining when mathematical ideas became taken-as-shared and analysis was conducted using the emergent perspective (Cobb & Yackel, 1996). Using a design-based research methodology, the instructional sequence and hypothetical learning trajectory were revised throughout the classroom teaching experiment documented here. The ways in which those revisions further facilitated collective student learning were also examined. These research foci were documented primarily by video-taped class sessions, researcher’s field notes, audio-taped research team meetings, and student work. The research team consisted of six people: the instructor for the course, two doctoral students at the
This chapter begins with a discussion of design-based research and the classroom teaching experiment and its influence on educational research. A discussion of the conceptual framework which includes the emergent perspective and Realistic Mathematics Education (RME) is then included. The hypothetical learning trajectory (HLT) specific for this study is discussed, followed by a description of the participants for this study and results of pilot research conducted in prior semesters. Finally, procedures for data collection and analysis including the methods used for documenting collective activity using Toulmin’s (1969) argumentation scheme are presented.

**Design-based Research**

Design-based research, also called design experiment, began as an effort to apply experimental methodology conducted in science and invention to education (Brown, 1992). The process of designing, testing, and refining an invention was applied to educational practice in the application of design-based research to the classroom. In traditional educational research, in order to determine what strategies or interventions are effective in teaching, research is often conducted in made-up situations in laboratories. In contrast, design-based research provides a basis for testing learning theories in the actual classroom with real situations in a cyclical process of testing, modifying, retesting, and remodifying (Gorard, Roberts, & Taylor, 2004). Design-based research takes both what seems to be ineffective and what was shown to be effective and uses it as a starting point for the next phase of revisions. Lessons learned from the ineffective
intervention are used to make the intervention better and more effective. The purpose of design-based research is not to demonstrate that the initial instructional sequence works; rather, the purpose is to test and refine the initial intervention based on current and past analyses of classroom activities and events in an effort to develop an instructional theory related to the topics at hand (Cobb, 2003).

Design-based research can be conducted in a variety of settings including one-on-one teaching experiments, classroom teaching experiments, preservice teacher development experiments, inservice teacher development studies, and school and school district restructuring experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Despite the differing settings, five features are present in all these situations: a) the purpose is to develop a class of theories about the process of learning and the means to support that learning, b) the methodology is highly interventionist with the intent of investigating the possibilities for educational improvement by bringing about new forms of learning in order to study them, c) conditions for developing theories are created and then tested, d) the design is iterative in nature, and e) the theories developed during the process are concerned with domain-specific learning processes (Cobb et al., 2003). In order for design-based research to contribute to reform in mathematics education, Cobb (2003) cites three criteria that must be present. “The results from the analyses should feed back to improve the instructional designs, the methodology should permit documentation of the collective mathematical learning of the classroom community over extended periods of time spanned by design experiments, and the analysis should permit documentation of the developing mathematical reasoning of individual students as they participate in communal classroom processes” (p. 11). In this study, the results will feed back to improve instructional designs with revisions made to both the HLT and the instructional
sequence as a result of this study. The methodology of design-based research is designed to permit documenting collective mathematical learning, and the first research focus is supported by this goal. The third goal according to Cobb will be met with future research related to this study.

**Classroom Teaching Experiment**

The instructional sequence was implemented in a semester-long preservice elementary mathematics education classroom as a classroom teaching experiment. The classroom teaching experiment began as a research methodology for mathematics education in the early 1970s. The need for models that accounted for the progress students made as a result of mathematical communication as well as the gap between teaching and research provided the impetus for the use and acceptance of the classroom teaching experiment. Prior to 1970, most educational research was conducted as experimental treatment and control groups where the effect of one treatment was compared to the effect of no treatment at all to specify differences between them and to prescribe what works better for educating children. This design inhibited investigations of students’ making sense of mathematics (Steffe & Thompson, 2000). As more and more researchers have sought to make sense of student thinking, classroom teaching experiments have become an accepted methodology for mathematics education research. “Whereas the clinical interview is aimed at understanding students’ current knowledge, the teaching experiment is directed toward understanding the progress students make over extended periods” (Steffe & Thompson, 2000, p. 274).
The design of a classroom teaching experiment allows the researchers to experience firsthand the ways in which students learn and reason about mathematics. “Students’ mathematics is indicated by what they say and do as they engage in mathematical activity, and a basic goal of the researchers in a teaching experiment is to construct models of students’ mathematics” (Steffe & Thompson, 2000, p. 269). The teaching experiment is designed as a sequence of teaching episodes which include a teacher, one or more students, at least one witness of the teaching episode, and a method for recording the teaching episode. The sequence of teaching episodes can span anywhere from several class sessions to entire courses. The goal is to construct models of students’ mathematics that can be useful in teaching other students and in learning how students learn and understand mathematics. This falls in line with the cyclical nature of design-based research in which the results of one cycle of hypothesizing what students may learn, testing what they did learn, and reconstructing hypotheses of what students may learn and how they may learn provide the beginning of the next cycle.

The classroom teaching experiment for this study was implemented with 16 prospective elementary school teachers. The primary goal of this analysis was to examine prospective teachers’ collective learning and reasoning related to place value and operations (Steffe & Thompson, 2000). The teaching experiment involved a sequence of teaching episodes which were implemented by an instructor with the students in the class. Five independent observers were involved in the teaching experiment and a method of recording what transpired during the teaching, in this case, videotape, audiotape, and field notes was used (Steffe & Thompson, 2000). The focus of this teaching experiment was on eliciting and understanding student’s thinking related to place value and whole number operations and determining collective mathematical understanding of place value and operations. This happened within the confines of a semester-
long preservice elementary mathematics education course on elementary school mathematics. The ways in which the social environment of the classroom supported student learning was a research focus.

The social aspects of the classroom including collective argumentation were examined as they facilitated the refinement and implementation of the instructional sequence. Additionally, the ways in which the instructional sequence and hypothetical learning trajectory facilitated collective student learning were examined. Analysis of these means of support provides a rich description of the mathematical practices present in the classroom and how they support mathematical learning of the students.

**Theoretical Framework**

The theoretical framework provides the basis for decisions made in data collection and provides the lens through which the analysis of data and conclusions can be drawn. In this study, the design of the instructional sequence and hypothetical learning trajectory (HLT) relied on the instructional theory of Realistic Mathematics Education (RME) discussed later in this chapter. The analysis of the classroom environment and its support of student learning was conducted through the lens of the emergent perspective (Cobb, 2000; Cobb & Yackel, 1996; Stephan & Cobb, 2003). The emergent perspective involves coordinating the social and the individual perspectives on mathematical learning; learning is both a psychological process on the part of the individual learner and a social process on the part of the group or classroom environment, with neither taking precedence over the other (Stephan, 2003). “A basic assumption of the emergent perspective is…that neither individual students’ activities nor classroom mathematical practices
can be accounted for adequately except in relation to the other” (Cobb, 2000, p. 310). The social aspects of the emergent perspective are also consistent with Blumer’s (1969) Symbolic Interactionism. The emergent perspective enables student mathematical learning to be situated within the social context of the classroom (Cobb, 2003). The individual and the classroom community cannot be separated and “the existence of one depends on the existence of the other” (Stephan, 2003, p. 28). The social perspective and psychological perspective are equally important in organizing analysis of collective mathematical learning (see Table 5).

Table 5: Emergent Perspective Interpretive Framework (Cobb & Yackel, 1996)

<table>
<thead>
<tr>
<th>Social Perspective</th>
<th>Psychological Perspective</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classroom social norms</td>
<td>Beliefs about one’s own role, others’ roles, and the general nature of mathematical activity in school</td>
</tr>
<tr>
<td>Sociomathematical norms</td>
<td>Mathematical beliefs and values</td>
</tr>
<tr>
<td>Classroom mathematical practices</td>
<td>Mathematical conceptions and activity</td>
</tr>
</tbody>
</table>

This study examined the ways in which the social aspects of the classroom facilitated individual learning. The social perspective entails an examination of the social norms, sociomathematical norms, and mathematical practices of the classroom in looking at collective mathematical understanding of the students in the classroom. The classroom social norms refer to the taken-as-shared ways of participating in the classroom community. Social norms include that students develop meaningful solutions to problems, explain and justify solutions and solution processes, attempt to make sense of other student’s solutions, and ask questions or raise challenges when there are misunderstandings or disagreements (Yackel, 2001). In this classroom, the norms of explaining and justifying, making sense of solutions, and asking questions were
valued and the instructor attempted to facilitate their development. The classroom
sociomathematical norms involve criteria for what counts as a different, unique, or sophisticated
mathematical solution as well as what counts as an acceptable mathematical explanation and
justification (Yackel, 2001). In this classroom, the sociomathematical norms that were valued
included the development of criteria for what counted as a different solution and an acceptable
mathematical explanation. Again, the development of these sociomathematical norms was
facilitated by the instructor for the course. Social and sociomathematical norms continued to be
established throughout the entire course and were not content-specific. The classroom
mathematical practices, however, refer to the content-specific mathematical ideas that are taken-as-shared for the classroom community. These were established throughout the implementation
of the instructional sequence and related directly to the mathematics at hand, namely place value
and whole number operations. Although the social and sociomathematical norms were
established throughout the entire classroom teaching experiment which included topics outside
of place value and whole number operations, the classroom mathematical practices were specific
to the five class sessions which dealt with the topics of place value and operations. The
individual students’ reasoning as presented in whole class discussions is viewed as participation
in the classroom community through these norms.

In contrast, the psychological aspect of the emergent perspective focuses on the specific
individual’s reasoning and the student’s particular ways of interacting with the classroom
community. The social and sociomathematical norms are important to examine before examining
the individual’s reasoning. The ways in which the individual student interacts with other students
and how that supports the individual learning is related to the social and sociomathematical
norms that are established in the classroom. In that way, the social norms, sociomathematical
norms, and classroom mathematical practices need to be determined prior to examining the individual student’s learning. The social and individual aspects of the emergent perspective correspond together, in that when examining the social aspects including social norms, sociomathematical norms, and classroom mathematical practices, the individual student’s learning contributes to the social development of taken-as-shared mathematical ideas. Similarly, when examining the individual student’s understandings of mathematical ideas that were accepted by the classroom, the social aspects help to define that individual student’s participation in the classroom. The social and the individual aspects must be coordinated for a full analysis of the classroom community and its facilitation of individual student and collective mathematical understandings (Cobb et al., 2001; Cobb & Yackel, 1996). The two perspectives support each other and cannot be separated; however, the analysis process in examining the social aspects of the classroom differs from the analysis process used in examining the aspects of individual student learning. For that reason, this study only examines the social aspects of the classroom. Although the emphasis of this study is on examining the social aspects of the classroom and their support of collective student learning, the individual learning that took place in the classroom was of vital importance and was examined through participation in whole class discussions which hinge on individual student reasoning and understanding. The social aspects of the emergent perspective including a) social norms, b) sociomathematical norms, and c) classroom mathematical practices are elaborated upon here and shown in table 6.

Table 6: Social aspects for this classroom teaching experiment

<table>
<thead>
<tr>
<th>Social Perspective</th>
<th>Valued in this Study</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classroom social norms</td>
<td>• Explaining and justifying</td>
</tr>
<tr>
<td></td>
<td>• Making sense of other student’s solutions</td>
</tr>
<tr>
<td></td>
<td>• Asking questions of each other when there is a misunderstanding</td>
</tr>
</tbody>
</table>
Social and Sociomathematical Norms

Social and sociomathematical norms were established throughout the course of the teaching experiment. These norms were negotiated between the students and the teacher and constituted what was acceptable participation socially and mathematically (Cobb, 2000; Stephan & Whitenack, 2003). The norms were crucial to the development of the preservice teacher’s learning in this teaching experiment. The establishment of social and sociomathematical norms began prior to the implementation of the place value and operations sequence and was continually refined throughout the sequence. The norms established assisted in examining the depth of discussions students engaged in as well as the manners in which they communicated with each other and with the instructor.

Social Norms

Social norms that have been identified in mathematical classrooms include the expectation that students develop meaningful solutions to problems, explain and justify their thinking and solutions, listen to and attempt to makes sense of other student’s solutions to problems, and ask questions or raise challenges if there are misunderstandings or disagreements (Yackel, 2001). Social norms that were valued to become taken-as-shared in this classroom included the need to explain and justify solutions and attempt to make sense of other student’s
solution strategies. Questions were expected to be asked when students were not clear as to another student’s mathematical thinking or the justification of their choices. These norms must be established by the teacher and the students cooperatively for them to be successful. Although the teacher is an authority figure in the classroom, from the emergent perspective, the teacher can only initiate and guide the process of renegotiating the social norms. The teacher cannot demand that specific norms be established; it is also dependent upon the students in the classroom (Cobb, 2000). The norms were negotiated through whole class discussions as well as small group interactions that took place in the classroom.

**Sociomathematical Norms**

Sociomathematical norms, like social norms, were established interactively with the instructor and students in the course. Sociomathematical norms that were valued included the criteria for what counts as a “different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation and justification” (Cobb, 2000, p. 323). These criteria were established most often in the course of whole class discussions, but also can be established through small group interactions and individual work including homework. For example, the establishment of criteria for what counts as a different mathematical solution was facilitated by the teacher asking the students if there was another solution that anyone found. As students present alternative solutions, they come to an understanding of what counts as different through a discussion of accepting and declining solutions as different and examining what made them different. This process works to establish in the students’ and the teacher’s minds what constitutes a different mathematical solution.
(Cobb, 2000; Stephan & Whitenack, 2003). These types of norms are not easily established, and most likely take shape throughout the course of the teaching experiment through classroom discussions focused on different solutions and efficient strategies. The focus of the classroom teaching experiment was on explaining and justifying solutions more than alternative strategies; however, alternative strategies were examined throughout the teaching experiment. These helped formulate the norms of what counted as different solutions and a good explanation.

The social norms and sociomathematical norms often coordinate together to establish classroom mathematical activity. For example, the social norm of making sense of solutions relates to the sociomathematical norm of presenting different solution methods. The establishment of the criteria for what counts as a different mathematical solution is supported by the social norm of making sense of other student’s solutions and explanations. By students presenting alternative solutions and other students making sense of their solution methods and explanations, the classroom community develops mathematical understandings. These mathematical understandings are supported by both the social norm of making sense of other student’s solutions and the sociomathematical norm of developing the criteria for what counts as a different mathematical solution. Social norms and sociomathematical norms coordinate together to support the development of taken-as-shared classroom mathematical practices and understandings, the third of the social aspects of the emergent perspective.

**Classroom Mathematical Practices**

Classroom mathematical practices are those ways of reasoning, explaining, and justifying that are taken-as-shared by the mathematical community of the classroom (Stephan & Cobb,
They evolve as the instructor and students discuss problems and solutions. “Classroom mathematical practices are … localized to the classroom and are established jointly by the students and the teacher through discussion; they emerge from the classroom rather than come in from the outside” (Stephan & Cobb, 2003, p. 41-42). Although related to social and sociomathematical norms, classroom mathematical practices differ from the social and sociomathematical norms in that they are content-specific rather than general ways of participating in the classroom. The social norm of explaining and justifying solutions and the sociomathematical norm of developing different solution methods for the same problem are generalized to the mathematics classroom. The classroom mathematical practice is specific to the mathematical topic at hand. For example, a classroom mathematical practice may be that measuring the length of an object can be found by repeatedly iterating a measuring device like a ruler. This is specific to the mathematics being discussed while the social norm of explaining and justifying solutions or the sociomathematical norm of providing a good explanation can be applied to the measurement situation, but also to discussions of place value, operations, fractions, or any other mathematical topic. The social norms and sociomathematical norms support the development of classroom mathematical practices, but are not specific to the mathematical ideas. The norms of the classroom transcend the mathematics being discussed, but provide support for the social interactions of the classroom.

Although the focus of the classroom mathematical practices is on collective learning, individual student learning is also important to examine with respect to the mathematics at hand. The documentation of the ways in which individual students participated in and contributed to the development of classroom mathematical practices is important in the analysis. The individual student’s interpretations of the mathematics relates to the classroom mathematical practices in
that the individual’s learning occurs as they contribute to the development of the mathematical practices. In discussing the taken-as-shared mathematical ideas, the individual student’s learning and contribution to the classroom discussion is inherent in the discussion of the establishment of taken-as-shared mathematical ideas that lead to classroom mathematical practices. In that way, the individual side is included in the discussion of the classroom mathematical practices. Further analysis should be completed to document more thoroughly the individual student learning, but that is outside the scope of this study.

**Participants**

Participants in this study were undergraduate students enrolled in a semester-long 2000-level elementary mathematics education course through the college of education at a major university located in the southeastern United States. They were predominately preservice teachers in their sophomore or junior years of college, although the course was not limited to education majors. The course met three times a week for three hours and ten minutes each session during the summer for six weeks. This was a four credit, elementary mathematics education content course. The classroom was organized with students seated at tables of four to five students each. Students were given time on most tasks to work individually and with their small groups before a whole class discussion was begun. Many of the students in this study were working in the daytime and participating in this class in the evenings. There were 16 preservice teachers that agreed to participate in the study. Five students were males and 11 students were females. Three students were minority students and the remaining 13 students were Caucasian. An emphasis of this course was on the preservice teachers’ conceptual understandings of place
value and operations. The research team included six people: the instructor for the course, two doctoral students at the dissertation stage, two doctoral students at the predissertation stage, and one researcher with extensive experience in conducting classroom teaching experiments.

**Pilot Research**

In the fall prior to this study, an instructional sequence related to place value and operations was developed collaboratively by the research team and the first cycle of the sequence was implemented. The instructional sequence included tasks in which the preservice teachers were required to think and reason entirely in base 8 instead of base 10 as well as tasks which paralleled student activities for operations including discussions of alternative algorithms, children’s thinking, and error patterns (Ashlock, 2002; McClain, 2003; Yackel & Bowers, 1997). The doctoral students and research mentor were observers in the course and any changes in the instructional sequence that needed to be made during the class were approached collaboratively by the research team (Cobb, 2000; Simon, 2000). Throughout the implementation of this first cycle of the instructional sequence, revisions were made based on student-to-student interactions, instructor-to-student interactions, and researcher-to-student interactions observed. Student work was examined in analyzing adjustments that needed to be made to the sequence and the HLT. Field notes, video and audio tapes, and journals were kept by the research team to document changes made and future changes that may be necessary as well as impressions of successfulness of the implementation of the sequence. After the sequence was completed in the fall, additional revisions were made to the sequence for implementation in the spring (Cobb, 2000; Simon, 2000). The sequence was then implemented again in the spring by the instructor for the course.
Although the research team did not meet on a regular basis during this iteration, decisions about adjustments to the HLT or tasks supporting the learning goals were made collaboratively with members of the research team. Student interactions during the spring semester were similar to those which occurred in the prior fall semester. The instructional sequence and HLT was in its third iteration for this study, meeting the criteria Cobb (2003) cited that “the methodology should permit documentation of the collective mathematical learning of the classroom community over extended periods of time spanned by design experiments” (p. 11)

Realistic Mathematics Education

The instructional theory used in developing the hypothetical learning trajectory and instructional sequence for this study was Realistic Mathematics Education (RME) which was developed at the Freudenthal Institute (Cobb, 2000; Gravemeijer, 2004). The instructional theory of RME forms the basis for decisions made in developing and refining the instructional sequence related to place value and operations used in this study. There are three design heuristics that are key principles of RME: 1) the reinvention principle, 2) didactical phenomenology, and 3) emergent modeling (Gravemeijer, 2004; Whitenack & Knipping, 2002) (see Table 7).

<table>
<thead>
<tr>
<th>Reinvention Principle</th>
<th>Didactical Phenomenology</th>
<th>Emergent Modeling</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Children’s historical development of mathematical concepts</td>
<td>• Particular sequence of activities that support student learning</td>
<td>• Tools that are used by students for reasoning mathematically</td>
</tr>
<tr>
<td>• Experientially real</td>
<td></td>
<td>• How students model problems</td>
</tr>
</tbody>
</table>

The reinvention principle and didactical phenomenology are closely related and often difficult to separate. The reinvention principle focuses on developing a learning trajectory which
takes into account the ways children have historically developed mathematical understandings. The reinvention principle of RME provides the justification for examining how children learn place value and operations for this study. The preservice teachers are expected to progress through similar developmental progressions, so the historical nature of how children learn place value and operations is important for the reinvention principle of RME. The didactical phenomenology heuristic is similar to the reinvention principle, but focuses more so on identifying particular activities to sequence in order to support the students’ mathematical development, leading to the instructional sequence of tasks and activities. One focus of this study was on documenting the ways in which the revisions to that instructional sequence supported student learning, which was directly related to this didactical phenomenology heuristic (Gravemeijer, 2004; Whitenack & Knipping, 2002).

The heart of the didactical phenomenology heuristic is that the starting point of instructional sequences must be experientially real to the students. They should be able to engage immediately in the mathematical activity of the sequence. The starting points are not required to be realistic situations for the students to actually participate in, but must be feasible and believable by students (Nicol & Crespo, 2005). Nicol and Crespo (2005), for example, explored place value using ancient Mayan numeration systems. This context was not directly realistic for students, but they could believe the situation could happen. Nicol and Crespo found that the preservice teachers were actively engaged in the task in spite of the lack of immediate application to their day-to-day lives as evidenced by the questions they posed to each other and to the instructor, the ways they approached the task, and their persistence through attempts to understand the scenario at hand.
According to RME, starting points of the realistic situations should be justified by the potential learning outcomes of the sequence. RME also argues that the instructional sequences should include activities in which students “create and elaborate symbolic models of their informal mathematical activities. This modeling activity might entail making drawings, diagrams, or tables, or it could entail developing informal notations or using conventional mathematical notations” (Cobb, 2000, p. 319).

In the instructional sequence used here, students engaged in mathematical activity through the use of base eight. Students were asked to live in base eight and think in base eight. A scenario of a candy factory in 8-world was used throughout the learning activities. Although living in 8-world was not immediately relevant to the students, the scenario of operating a candy factory was experientially real to students. They could imagine that in this world, there would be a candy factory and could believe the situations in which they were asked to engage. The candy factory packaging situations were justified by the potential learning outcomes of unitizing, composing and decomposing numbers, and developing strategies for operating with whole numbers. The candy factory scenario provided imagery that the students continued to use throughout the instructional sequence whether in the context of the candy factory or not. By placing the students in 8-world, they were able to experience learning place value and operations in much the same way as young children do while learning base ten. Preservice teachers often come into the classroom with at least a procedural understanding of place value and operations in base ten. This can inhibit their understandings of children’s thinking in that their mathematics is compressed. “Though each of us once inhabited the mathematical world of the young child, that world is lost to most of us” (Conference Board of the Mathematical Sciences, 2001, p. 56). Ball and Bass (2000) argue that teachers must be able to “work backwards from mature and
compressed understanding of the content to unpack its constituent elements” (p. 98) In order to accomplish this with preservice teachers, the scenario of living in 8-world was used. The scenario used in this experiment differed from McClain (2003) and was first developed by a team of researchers at Purdue University Calumet. McClain (2003) chose to have her students reason in base eight, but use base ten language. In that way, one box was 64 pieces and one roll was 8 pieces. The numbers were still written in base ten. This was done because she felt that many of her students were reasoning in base ten and translating to base eight. This was not seen in the first two iterations of this study, and the notation was left in base eight. In addition, the transition back to base ten and the examination of alternative strategies for whole number operations was included in this instructional sequence, but was excluded from McClain’s sequence.

The third heuristic of RME, emergent models, examines how students model problems and how the models are related to their mathematical understandings. Students initially use models that are informal and relate to the task at hand. Through the course of instruction, however, these models become more and more sophisticated and begin to represent more formal ways of mathematical representing, interpreting, and reasoning (Whitenack & Knipping, 2002). Gravemeijer (2004) described four levels of activity with relation to models. The first level, activity in the task setting, is one in which the model is inside the situation itself. Gravemeijer refers to this as out-of-school settings. The student is reasoning informally based on models they may understand from elsewhere. As model use progresses, students develop and reason with models that are based in a specific problem context, referential activity. Students in the candy shop scenario, for example, may use a model of boxes, rolls, and pieces for place value because the task gives that model naturally. Over time, however, the model is transitioned to a general
activity level, in which students use the model to reason about problems independent of the context of the problem. In the candy shop scenario, for example, students may use the boxes, rolls, and pieces language in a problem involving packaging sticks of gum into groups of varying sizes. The context of the problem does not necessarily lend itself to the boxes, rolls, and pieces language, but the model is extended to a more general function. The model becomes common language and imagery that students use in solving problems. Finally, at the formal level, students reason with mathematical ideas that are no longer framed by the model. Students in the candy shop scenario, for example, may begin to talk about addition and subtraction strategies that no longer use boxes, rolls, and pieces as a model. They reason mathematically without the need for the model (Gravemeijer, 2004; Whitenack & Knipping, 2002).

**Hypothetical Learning Trajectory**

Through the process of the pilot study, a hypothetical learning trajectory (HLT) was developed and refined as part of the initial instructional sequence (see Table 8). The HLT was based upon prior research involving how children learn place value and operations and a HLT from the Purdue Calumet Group (Yackel & Karnes, 1992, 1993, Stephan, personal communication) and was modified based on the results of the first two implementations of the instructional sequence. The research team received the initial HLT for the first iteration through contacts from the Purdue Calumet Group and it underwent two cycles of modifications before this iteration. This type of research was new to the research team, so the first two iterations also included the team working together in an effort to modify the curriculum for this elementary mathematics content course. The social aspects of the theoretical framework created the need to
teach in ways that were different from those used in the past. This shift took the first two iterations to work out so that by the third iteration, the teaching methods were no longer as much of a focus.

Table 8: Initial Hypothetical Learning Trajectory for this Teaching Experiment

<table>
<thead>
<tr>
<th>HLT Phase</th>
<th>Learning Goal</th>
<th>Supporting Tasks for Instructional Sequence</th>
<th>Supporting Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase One</td>
<td>Counting objects within base eight and unitizing</td>
<td>Counting objects; Skip counting; Dot Frames; Empty Number Line</td>
<td>Snap Cubes; 10 frames; Number Line</td>
</tr>
<tr>
<td>Phase Two</td>
<td>Flexible representations of numbers in base eight</td>
<td>Candy Factory Scenario involving: Estimating; Packing and Unpacking Candy; Inventory Forms</td>
<td>Pictorial representations of boxes, rolls, and pieces; inventory forms</td>
</tr>
<tr>
<td>Phase Three</td>
<td>Operations within base eight</td>
<td>Candy Factory Transactions; Inventory Forms; Dot Frames; Dot Arrays</td>
<td>Pictorial representations of boxes, rolls, and pieces; inventory forms; 10 frames; Snap Cubes</td>
</tr>
<tr>
<td>Phase Four</td>
<td>Base ten representations and alternative strategies for whole number operations</td>
<td>Base-ten Blocks; Arrays; Number Lines</td>
<td>Base-ten blocks; Number Line; Dot Arrays</td>
</tr>
</tbody>
</table>

The first phase of the learning trajectory included the goal of having students count objects within the base eight system. This phase also included the development of efficient counting strategies in base eight supported by tasks involving the blank number line. These included the expected stages of grouping by “10”s but verifying by counting by ones and then grouping by “10”s and then counting by “10”s. Contrary to what McClain (2003) did with preservice teachers, this was done in the context of base eight, so eight pieces was written as 10. In an effort to not confuse 10 pieces in base eight with ten, a new name, in this case “one-ee-zero” was given to 10 pieces in base eight. Subscript notation was not used since everything was
in base eight. Likewise, 100 pieces was given the name “one-hundree pieces.” This terminology was first developed by the Purdue University Calumet mathematics education group (Stephan, personal communication). The goal was that by the end of phase one, students would understand not only the semantics of the base eight system, but also have developed unitizing and efficient counting strategies built upon place value understanding and reasoning.

After this first phase of counting and working with numbers in base eight, the next phase included developing flexible representations for numbers in base eight. This was implemented through the context of a candy factory in which 10 pieces made a roll and 10 rolls made a box. The Candy Factory sequence for base eight was initially developed by the Purdue Calumet group and used research with third grade children as the starting point for the sequence (Bowers, 1996; Bowers et al., 1999; Cobb & Yackel, 1996; McClain et al., 1998; Yackel & Bowers, 1997). Although McClain (2003) conducted a Candy Factory sequence with preservice teachers, she chose to keep the context in base ten instead of base eight. This was based on prior attempts to use the Candy Factory scenario with base eight notation where she found that the preservice teachers attempted to connect the base eight notation with base ten and did not explore the underlying issues of place value. They reasoned in base ten and then transferred to base eight. In the pilot for this study, students initially tried to reason between base ten and base eight, but transitioned quickly to thinking in terms of base eight only. The students were able to explore the issue of place value involved in base eight in ways they could not have done otherwise, so it was decided to keep the notation in base eight.

The supporting instructional tasks in phase two provided students with the experiences necessary to understand place value and flexibly represent numbers in base eight. These tasks attempted to allow the preservice teachers the experiences necessary in order to further develop a
concept of “one-ee-zero”, or “ten” in base eight, and compose and decompose numbers in order to develop flexible representations of numbers. These goals were based on children’s development of a concept of ten (Cobb & Wheatley, 1988; Steffe, 1983) and provided experiences to give the students common imagery with which to discuss the mathematics.

The third phase of the HLT dealt with operations in base eight. By the end of phase three, the classroom community was expected to understand the meanings of the operations and develop, explain, and justify alternative strategies for all four operations in similar ways as children develop operational fluency in base ten. Instructional tasks which supported learning of operations in base eight included solving addition and subtraction problems in the context of the Candy Factory pictorially and abstractly, modeling multiplication and division problems in various contexts, and using an area model for multiplication (see Appendix A).

For the final phase of the HLT, students were asked to transition back into base ten and asked to apply their knowledge of and reasoning about place value and operations to understanding and developing strategies for whole number operations in base ten. Alternative algorithms were discussed as they naturally appeared in student solutions. These alternatives included partial sums, column addition, partial differences, equal compensation, partial products, and partial quotients (see Appendix B). Students were asked to reason and explain why the algorithms worked. Problems were posed in which students were required to identify conceptual errors in student work and provide suggestions for correcting the conceptual errors. Many of these scenarios were modeled after Ma’s (1999) work and others were modeled after errors students had made in initial attempts at solving problems. This phase of the HLT focused on developing the prospective teachers’ specialized content knowledge for teaching while the first three phases focused on general content knowledge.
The learning goals of the HLT correspond to the classroom mathematical practices which were expected to be established (Cobb et al., 2001). The tools and imagery which were used to support the development of these mathematical ideas was expected to support the student learning for each phase of the HLT. Thus, the HLT provided a basis with which to begin the data analysis process.

**Instructional Sequence**

The classroom teaching experiment in this study began with one and a half class sessions dealing with general problem solving. The goal of these tasks was to begin to establish the social and sociomathematical norms of the classroom. The mathematics involved in these problem solving situations was not the focus; instead, the focus was on discussing the mathematics to facilitate the establishment of classroom norms. The instructional sequence dealing with place value and operations began with the second half of the second class session (see Table 9).

**Table 9: Instructional Sequence for Place Value and Operations**

<table>
<thead>
<tr>
<th>Day</th>
<th>Learning Goals</th>
<th>Tasks</th>
<th>Supporting Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>• Counting in base eight</td>
<td>• Counting objects</td>
<td>• Blank Number Line</td>
</tr>
<tr>
<td></td>
<td>• Skip Counting Forward and Backward</td>
<td>• Blank Number Line</td>
<td>• Snap Cubes</td>
</tr>
<tr>
<td></td>
<td>• Blank Number Line</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Two</td>
<td>• Counting in base eight and unitizing</td>
<td>• 10 Frames</td>
<td>• 10 Frames</td>
</tr>
<tr>
<td></td>
<td>• Flexibly representing numbers in base eight</td>
<td>• Estimating with the Candy Shop</td>
<td>• Pictorial representations of boxes, rolls, and pieces</td>
</tr>
<tr>
<td></td>
<td>• Operations in base eight</td>
<td>• Candy Shop Configurations</td>
<td>• Inventory Forms</td>
</tr>
<tr>
<td></td>
<td>• Candy Shop Transactions Pictorially and Abstractly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Three</td>
<td>• Operations in base eight</td>
<td>• Explanation example (See Appendix C)</td>
<td>• Pictorial representations of boxes, rolls, and</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Multiplication with 10</td>
<td></td>
</tr>
</tbody>
</table>
The instructional sequence began with the learning goal of counting efficiently in base eight, corresponding to the ways in which children learn place value as discussed in chapter two. Tasks involved skip counting forwards and backwards with the intent of developing initial understandings of the number words and sequences in base eight. Situations were given in which it was expected that students would be required to develop three digit numbers in base eight. Following the counting situations, students were introduced to the blank number line as a tool for solving problems involving addition and subtraction situations, with the goal of developing more efficient counting strategies and further developing unitizing. In the next class session, 10 frames were introduced as a tool for counting objects with the goal of facilitating the students’ development of taken-as-shared concepts of 10. These activities completed phase one of the hypothetical learning trajectory whose learning goal was counting objects in base eight and unitizing. This learning goal supported the classroom mathematical practice of unitizing, the goal
of which was to develop taken-as-shared ways of reasoning about counting and unitizing in base eight. This phase was concluded as the tasks which were aimed directly at facilitating development of unitizing 10 were completed, although the development of the classroom mathematical practice of unitizing continued throughout the remainder of the instructional sequence.

The second day of the instructional sequence continued into phase two of the hypothetical learning trajectory with the learning goal of flexibly representing numbers in base eight, the second classroom mathematical practice. The tasks which supported learning in this phase began with introducing the candy factory scenario which continued to support the classroom mathematical practice of unitizing, but also gave imagery students used to support taken-as-shared reasoning about representations of numbers. Students were asked to estimate the number of rolls of candy in a bag of individual candies. Tasks followed in which students were given pictorial configurations of boxes, rolls, and pieces and were asked to find other configurations to represent the same number of candies, further developing taken-as-shared ways of reasoning about flexibly representing numbers and unitizing. In a subsequent task, students were shown pictorial configurations of rolls and pieces with an overhead projector. The configurations were only shown for 2-3 seconds and students were asked to mentally determine how many candies were shown. These tasks facilitated the continued development of unitizing and the development of flexible representations of numbers by composing and decomposing. This concluded phase two of the hypothetical learning trajectory, as the tasks which supported students development of taken-as-shared ways of flexibly representing numbers was concluded and it was determined that the classroom mathematical practice of flexibly representing numbers had begun to be established, although operations in base eight continued to facilitate the
development of flexibility in representing numbers. This is consistent with the recommendations by Fuson (1992) that place value and operations be considered simultaneously. The second day of the instructional sequence continued with addition and subtraction in the candy factory scenario with transactions both pictorially and abstractly with an inventory form, a type of place value chart, giving the students further tools with which to build taken-as-shared understandings about place value.

Day three of the instructional sequence continued with further operations within base eight with multiplication and division. The tool of the 10 frames was again revisited with groups of 10 frames as a model for multiplication. Contextually-based problems were also given involving multiplication with a machine that packaged sticks of gum and bags of candy. As students’ development of understandings related to multiplication progressed, division situations were also introduced. The division situations were given to include both partitive and quotitive division problems and the difference in modeling those problems was discussed. The learning goal of this sequence of activities was to develop multiplicative reasoning and operational fluency with respect to multiplication and division in base eight. Tasks for multiplication sought to build upon each other to develop multiplicative strategies and relating multiplication to division.

The next class session concluded discussion of division in base eight and transitioned the students back into base ten in order to examine alternative strategies for operations in base ten. Day four concluded phase three of the hypothetical learning trajectory and began phase four with the introduction of base ten strategies. Students were introduced back into base ten by examining children’s strategies for subtraction. This led to discussion of alternative strategies for addition and subtraction through the use of base ten blocks. The final class session, day five, concluded
alternative strategies for subtraction by examining compensation strategies and developed
alternative strategies for multiplication and division by using base ten blocks and dot arrays. The
goals associated with this phase of the HLT included operational fluency and understanding
alternative strategies for operations in base ten. This class session concluded the instructional
sequence for place value and operations.

**Ethical Considerations**

Prior to the pilot research, approval was given through the University’s Institutional
Review Board (IRB). This approval was continued through the data collection for this study. See
Appendix D for the IRB Approval form. All data were collected with consent of the participants,
all of whom were adults and provided informed consent. Two students declined to participate in
the study, and although their audio could be heard as part of the classroom discussions, they were
not captured on video tape. The recording was halted when either of these students presented a
solution strategy at the front of the classroom. Their work was not examined as it related to this
study and their contributions to the classroom discussions were not examined as part of the data
analysis process. The other sixteen students agreed to participate in the study and provided
informed consent via letter (See Appendix E). All names that are included in this study are
pseudonyms to protect student confidentiality.

The research team consisted of six members: 1) the instructor for the course, 2) two
doctoral students at the dissertation stage, 3) two doctoral students at the pre-dissertation stage,
and 4) one researcher with extensive experience conducting classroom teaching experiments and
design-based research. The two doctoral students at the dissertation stage assisted each other in
examining the data by first conducting an initial analysis individually and then collaboratively working together to come to agreements regarding data analysis. Decisions about the instructional sequence and hypothetical learning trajectory were made with input from all members of the research team so as to avoid bias.

**Data Collection**

Beginning with the first class session in the summer, the instructor and participants in the course were videotaped each class session. Two video recorders were used – one focused on specific small group dynamics and whole class discussions and the other focused on small group dynamics and board work. Additionally, one group was audio taped to gather additional student-to-student interactions during the course of the instructional sequence. The groups which were chosen for focused audio and/or video taping were determined based on observations of social interactions and collaborative learning prior to the first day of the instructional sequence to determine the groups whose specific conversations would be useful for data collection.

The course began with one class focused on problem solving. Social and sociomathematical norms for the classroom began to be established in the first few classes of the semester and these videotaped recordings, along with observations noted in the class, assisted in identifying those norms (Cobb, 2000; Stephan & Whitenack, 2003). Norms which were established included asking questions of each other or the instructor when there was a misunderstanding and establishing what counted as an acceptable and different mathematical solution, explanation, and justification. These are elaborated upon in chapter 4. During each class session, the research team recorded field notes to further supplement the videotapes. At the
Conclusion of each class, each member of the research team was asked to journal to record individual perspectives of the sequence implementation, classroom dynamics, and individual student development as well as changes which may need to be made to the instructional sequence based on the actual learning trajectory on which students progressed.

The instructional sequence on place value and operations was then implemented in the course. Videotaping, audiotaping, and observations continued throughout the course and field notes were collected. As in the fall and spring, modifications and revisions were made throughout the sequence as it was implemented through discussion by the research team during and between class sessions. The research team met after each class session to discuss the day’s results and implementation of the sequence for the next session (Cobb, 2000; Simon, 2000). Team meetings were audio taped.

Data Analysis

Analysis examining social and sociomathematical norms was done through the use of the methodology proposed by Cobb and Whitenack (1996). This is consistent with Glaser and Strauss’ (1967) Constant Comparative Method. The video transcripts were first analyzed chronologically and instances which were believed to assist in establishing classroom social and sociomathematical norms were marked. These chronological instances were then taken as data themselves and instances which corresponded to different norms were notated. These instances were then synthesized to determine social and sociomathematical norms which were taken-as-shared by the classroom community. Discussions at research team meetings also included conjectures about the establishment of norms based on field notes and observations and were
used as support of the conjectures made in analyzing the video transcripts. Times in which norms were seemingly being established were marked and then later reviewed for instances of consistency between class sessions and/or inconsistencies that may indicate that a classroom norm was not taken-as-shared, as in a case where a student might violate what was thought to be a norm of the classroom and the students did not object. The process of identifying norms provided a framework for what was expected in the student-to-student interactions and student-to-instructor interactions throughout the course and specifically during the instructional sequence related to place value and operations. Although initial analysis for classroom norms was completed before the place value and operations instructional sequence began, norms continued to be established and refined through the implementation of the instructional sequence. This analysis was consistent with Cobb and Whitenack’s (1996) process which has been shown to be consistent with Glaser and Strauss’ (1967) constant comparative method (Cobb & Whitenack, 1996; McClain, 2003).

The classroom discourse during the instructional sequence was analyzed using Toulmin’s (1969) argumentation analysis to determine social factors which facilitated the classroom community’s learning of and reasoning about place value and operations. The analysis process proposed by Rasmussen and Stephan (in press) was used to document the collective mathematical activity occurring in the classroom. Collective activity can be defined as the ways in which mathematical ideas become established in a classroom through interactions. This collective activity examines the classroom as a whole and not the individual students per se. The focus of this study was on the taken-as-shared knowledge and practices that the collective students understand and use and gives little indication of the individual student understanding for each and every student. Although individual student learning is used as the support for the
development of taken-as-shared mathematical ideas and classroom mathematical practices, it
does not take into account each and every student in the classroom. The community as a whole is
the focus of the analysis.

Rasmussen and Stephan (in press) propose a three phase approach for documenting this
collective activity of the classroom (see Table 10). The first phase involves transcribing every
whole class discussion and notating claims made by either students or the instructor. The
transcripts are then analyzed to determine data, warrants, and backings which correspond with
the claims made. Phase two uses this argumentation analysis as data to identify taken-as-shared
mathematical ideas. Phase three uses these mathematical ideas to then identify the mathematical
activities that are associated with those taken-as-shared ideas. Toulmin’s argumentation scheme
was used for determining when ideas become taken-as-shared. Although various levels of
discussion and argumentation occur in the classroom, namely small group, individual students,
and whole class discussions, since collective activity focuses on the unit of the classroom and not
on the individual student understanding, only the whole class discussions that all students had the
opportunity to hear and participate in were used for determining classroom mathematical
practices.

**Toulmin’s Argumentation**

Toulmin (1969) created a model to describe the structure of arguments. These arguments
need not be solely involved in debate, but can be applied to classroom discussion as done here
and proposed by Rasmussen and Stephan (in press). Toulmin (1969) determined that an
argument consisted of four parts: the claim, the data, the warrant, and the backing. The claim is
that which a student determines to be true. The data are the evidence the student uses to explain why the claim is correct, typically the procedures or facts that lead to the student’s conclusion. A warrant is used to justify why the data leads to the claim, generally how the data supports the claim. If further challenges occur, then a backing comes into play which explains why the claim has merit. Students were expected to engage in classroom discussions that revolved around providing data and warrants for their claims. This was related to the social norms of explaining and justifying solutions and solution processes, i.e. providing data for a claim, and making sense of other student’s mathematical solutions, i.e. asking for and providing warrants and/or backings when needed. Although other forms of discourse analysis are prevalent (Phillips & Jorgensen, 2002), most of them fail to provide a basis for documenting collective activity. They focus on the individual student learning. Toulmin’s argumentation scheme was selected as the basis for data analysis for this study for documenting collective activity as supported by Rasmussen and Stephan (in press).

**Documenting Collective Activity**

Documenting the structure and function of arguments is difficult; however Rasmussen and Stephan (in press) propose “rules of thumb” that can be useful. Claims are the easiest to identify and consist of either an answer to a problem or a mathematical statement in which a student may be asked for further clarification. Data generally involve the mathematical relationships or methods that lead to the answer. Warrants often remain implied and are used to connect the implications of the data to the claims. Backings are typically found by asking why the claim should be accepted. Backings give validity to the claim. For example, a student may
claim that 2 times 3 is 6. The data that student may use to support that is 3 plus 3 is six. The warrant that links their data to their claim may be that 2 times 3 is 3 plus 3. The backing may be that 2 times 3 is 2 groups of 3 objects (see figure 1).

![Toulmin's argumentation scheme for 2 x 3](image)

Rasmussen and Stephan (in press) provide a three phase approach to analyzing the classroom interactions to determine classroom mathematical practices and taken-as-shared ideas (see Table 10).

Table 10: Phases in documenting collective activity (Rasmussen & Stephan, in press)

<table>
<thead>
<tr>
<th>Phase of Research</th>
<th>Activity</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase One</td>
<td>• Transcribe every whole class discussion</td>
<td>Argumentation Log</td>
</tr>
<tr>
<td></td>
<td>• Notate claims made by students or instructor</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Identify data and conclusions, as well as warrants and/or backings if present</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Compare argumentation schemes and come to agreement</td>
<td></td>
</tr>
<tr>
<td>Phase Two</td>
<td>• Use Argumentation Log as data</td>
<td>Mathematical Ideas Chart</td>
</tr>
<tr>
<td></td>
<td>• Identify taken-as-shared mathematical ideas</td>
<td></td>
</tr>
</tbody>
</table>
Phase Three

• Use Mathematical Ideas Charts to identify common mathematical activities associated with taken-as-shared mathematical ideas

Classroom Mathematical Practices

The first phase begins with creating transcripts of all whole class discussions from the class periods considered for the study. The videos are then viewed by more than one researcher and claims made by either students or the instructor are noted. Toulmin’s model is then used for each claim that was made, identifying the claim, the data, the warrant, and the backing if present. These identifications are done by more than one researcher independently. An argumentation log is created which documents what is viewed to be claims, data, warrants, and backings (see Appendix F for sample). The team of researchers then verifies or refutes each other’s views of the claims, data, and warrants and comes to an agreement.

The second phase of the analysis involves taking the argumentation log and looking across the spectrum of episodes to determine what mathematical ideas become taken-as-shared. There are two criteria Rasmussen and Stephan (in press) identify for determining when ideas become taken-as-shared. First, when backings and/or warrants no longer appear in a students’ explanation, the mathematical idea expressed can be considered taken-as-shared. Second, when any part of the argument switches position within subsequent arguments and are unchallenged, the mathematical ideas expressed are taken-as-shared. For example, when a warrant becomes data or a backing becomes a warrant, that mathematical idea is considered shared knowledge. A mathematical ideas chart (see Appendix G for sample) is then made for each episode which includes ideas that are viewed to be taken-as-shared, what ideas seem to be developing as taken-as-shared that the researchers need to “keep an eye on”, and any additional comments. These mathematical ideas charts are then compared episode by episode to determine where there is
movement from “keep an eye on” to taken-as-shared and to verify or refute claims of conjectures of what ideas are taken-as-shared.

In the third phase of the analysis, the taken-as-shared ideas from the mathematical ideas charts are listed and organized around common mathematical activities that students were engaged in as the ideas became taken-as-shared. This general mathematical activity is defined as classroom mathematical practices (Rasmussen & Stephan, in press). This process for documenting mathematical activity meets Cobb’s (2003) criteria that “the analysis should permit documentation of the developing mathematical reasoning of individual students as they participate in communal classroom processes” (p. 11).

**Limitations**

This study was conducted as a qualitative research study with only 16 students. Due to the qualitative nature of this study, the results are less generalizable to populations beyond the one studied. The cyclical nature of design-based research, however, lends itself to some measure of generalizability. Additionally, the analysis process based on Toulmin (1969) documents only collective, taken-as-shared mathematical ideas. The individual student’s learning is used in examining whole class interactions, but the analysis related to individual student’s learning is outside the scope of this study.

In examining limitations related to the analysis process, it was not always possible to document all the parts of the argumentation scheme. Much of this analysis is subjective and different interpretations are possible. This limitation was addressed by members of the research team independently examining the data and follow up discussions related to the analysis.
Additionally, students’ motivation to explain and justify answers and participate in the classroom discussion may have been motivated by a desire to achieve a specific grade in the course. The course was a four-credit course, making the grade all the more important. This aspect could not be eliminated and may have an effect on the quality of the student interactions and discussions.

**Conclusion**

The study presented here was part of a classroom teaching experiment conducted in a semester-long mathematics education course for preservice elementary school teachers. The students in the course were taught using an instructional sequence designed to support their learning of place value and operations. The whole class interactions were analyzed to determine classroom mathematical practices that developed through the implementation of the instructional sequence. Toulmin’s (1969) argumentation scheme was used for data analysis purposes to determine when mathematical ideas became taken-as-shared in order to then identify classroom mathematical practices. The results of that data analysis are presented in the following chapter.
CHAPTER FOUR: FINDINGS

The research focus of this study was on examining the social aspects of the classroom, namely the a) social norms, b) sociomathematical norms, and c) classroom mathematical practices that facilitated collective student learning regarding place value and operations and the ways in which the instructional sequence and subsequent revisions facilitated that learning. The instructional sequence used in this study framed place value and operations in base eight, where the only symbols that were available to use were 0, 1, 2, 3, 4, 5, 6, and 7. When counting past 7, the next number would be 10, which was called “one-ee-zero”. Similarly, when counting past 77 (seven-ee-seven), the next number would be 100, which was called “one-hundree”. The language used was first developed by Yackel and Karnes (1992, 1993) at Purdue Calumet Univeristy. In cases in which the numbers are base eight, the numeral will be written as 12, meaning 12 in base eight. Since most of the instructional sequence discussed here was in the setting of base eight, when numbers are in base ten, it will be identified as base ten.

The research team met after each class session to discuss the learning goals for the prior session and how they had or had not been realized as well as the plans for the next session based on the future learning goals associated with the hypothetical learning trajectory. The decisions that were made in these research team meetings served to assist in the development of an actual learning trajectory which students created and revisions to the hypothetical learning trajectory and instructional sequence for future implementation towards the goal of developing an instructional theory of how preservice teachers may learn place value and whole number operations. The research team discussed areas in which social and sociomathematical norms were being established as well as conceptual developments that seemed to be present or were
missing. Revisions to the sequence were made on a daily basis and those revisions supported collective student learning of place value and operations and provided experiences which supported the students’ collective mathematical understanding.

**Social and Sociomathematical Norms**

The establishment of social and sociomathematical norms began with the first class session. Although the place value and operations sequence was not begun until the middle of the second class period, the norms began to be established through a problem solving focus that was independent of specific mathematical topics. In the first class session of the semester, students were given general problems and asked to find solutions without using algebra. Many of the problems could have been solved using algebraic concepts; however, a goal was to have the students thinking mathematically and explaining their work. This was more easily accomplished by asking them to use strategies other than algebra. Both the social and sociomathematical norms continued to be established throughout the implementation of the place value and operations sequence. Social norms that were established included the expectation that students would a) explain and justify their answers, b) attempt to make sense of other student’s solutions, and c) ask questions if there was a lack of understanding related to something another student or the instructor said or did. Sociomathematical norms that were established included establishing the criteria for what counted as a different or unique mathematical solution and what was a good explanation (see Table 11).
Table 11: Social and Sociomathematical Norms for this Classroom

<table>
<thead>
<tr>
<th>Social Norms</th>
<th>Sociomathematical Norms</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Explain and justify solutions and solution processes</td>
<td>• What counts as a different or unique solution</td>
</tr>
<tr>
<td>• Attempt to understand other student’s solutions</td>
<td>• What makes a good explanation</td>
</tr>
<tr>
<td>• Ask questions if something is not understood</td>
<td></td>
</tr>
</tbody>
</table>

**Explain and Justify**

A social norm that was established through the course of the classroom teaching experiment was that students were expected to explain and justify their answers and solution processes. The first class session of the semester included problem solving as a method of eliciting student thinking. The goal of these activities was to begin to establish the social and sociomathematical norms for the classroom. The problems given were selected to facilitate discussion of the mathematics, not with specific mathematical goals, but with goals of facilitating the establishment of the social and sociomathematical norms of the classroom. These problems were all in base ten, as the instructional sequence for place value and operations had not yet begun. The problem solving discussion began with the allowance problem which was given as follows:

Mark gets $1.85 a week for an allowance. He always gets 16 coins. The coins are always nickels, dimes, and quarters. How many of each type of coin does he receive?

In solving this problem, students determined various solutions and many went about finding those solutions in different manners. The common thread of explaining one’s work and justifying one’s process was woven throughout, as demonstrated in this episode, the first discussion of a possible solution to the allowance problem.
Kathy: I got 3 quarters, 9 dimes, and 4 nickels.

Instructor: How many of you got that same solution? How did you get that solution, Kathy?

Kathy: I went backwards thinking out the problem. And I knew since we had $1.85 there obviously was one nickel in the solution. So, I subtracted one coin which left me with 15 coins and $1.80 left. And then,

Instructor: Do you all follow that so far?

Student: How did you get 15 coins?

Kathy: One nickel minus 16 coins is 15 coins.

Instructor: So we know we have to have 16 coins. Okay.

Kathy: And then, I did trial and error. I actually started with 4 quarters and found out that was too much so I went down to 3 quarters which then left me with 12 coins and $1.05 left to figure out. Then I saw the $1.05 and subtracted another nickel because obviously only a nickel can make 5 cents. And it left me 11 coins left to play around with and $1.00. And just looking at it I knew that I needed 9 dimes. That equaled 2 coins and 10 cents left, so 2 nickels would be 10 cents.

The student’s solution was explained and her process was justified. The instructor and another student in the class questioned the solution process, prompting Kathy to provide more explanation and justification for her choices. The need to explain was introduced by the instructor as an expectation at the beginning of the course. The expectation was continued to be facilitated by episodes like this one which involved the same allowance problem and occurred later in the first class session. Sarah provided a solution and was asked to explain how she got it by the instructor.

Sarah: Four quarters, five dimes, and seven nickels.

Instructor: How many got that one? How did you get it Sarah?

Sarah: Well, I started with the quarters first. It was just basically all trial and error. Well, I knew that if I had four quarters, I would have to have an odd number of nickels. And so I just played around with the dimes. I went from like five nickels to seven nickels.

Although in this instance, the instructor asked for the explanation, as the classroom teaching experiment progressed, the students began to ask each other for explanations as in this episode
which took place on day 3 of the instructional sequence. The task involved determining how many dots were in 6 groups of 6. The previous task had been 5 groups of 6.

Joe: 37.

Instructor: How did you get it?

Joe: From the other one. We had 6 groups of 6 and I just added the extra one.

Instructor: So you had five groups of six plus one? How many of you did it that way? Anyone have questions? Matt?

Matt: How come you only added one because it is a whole set of six, so you have to add six instead of one to the answer we had last time, so I think the answer should be 36 plus six and in base eight world that is 44.

The request for an explanation came from Matt in this case instead of the instructor. Although the instructor asked if anyone had questions, Matt felt the obligation to ask for an explanation for why Joe believed the answer was 37. As the semester continued, the expectations for explanations and justifications became taken-as-shared by the students and they began asking each other for more explanation or justification. This norm was negotiated by the instructor with the students through whole class discussions that focused on examining student explanations and justifications of solutions. In the following episode at the beginning of the second class session, the students were discussing the solution to the following problem:

A card game for 2 through 6 players has a deck of cards that can always be divided evenly among all the players. What is the smallest possible number of cards that can be in the deck?

One student claimed the solution was 120 and explained that she knew it had to be a multiple of 6 and it also had to be a multiple of 5, so it had to end in a 5 or a 0. She then multiplied 6 and 5 to get 30 and found what she believed to be the smallest multiple of 30 that was divisible by 2, 3, 4, 5, and 6. Kim challenged her by saying “Why didn’t 60 work? That is what I got.” The student responded that 60 was not divisible by 4, providing her justification why she believed 60 did not work. Kim then responded that 4 times 15 was 60, prompting the student to change her answer.
The challenge in this case came directly from another student. Kim asked the student to justify why the answer was not 60 and in the process, the original student changed her solution. The student’s justification was asked for not by the instructor, but by another student. This was a critical step in students accepting explaining and justifying their answers as a legitimate classroom social norm. This norm continued to be established throughout the teaching experiment. On day 3 of the instructional sequence, students were asked to determine how many dots were in 3 groups of 6 using the 10 frames (see figure 2).

![Figure 2: Three groups of six with 10 frames](image)

Katrina: Well, I did the four on top, four, four, four, and then I took two from each, or two from two of the columns and made another four so that is 20 and there is 2 left over so that is 22.

The instructor asked for the answer to the problem and Katrina immediately gave an explanation with her solution. She was not prompted for the explanation of her solution, but freely gave it. Episodes like this one continued to occur, demonstrating that the social norm of explaining one’s solution and justifying the process could be considered taken-as-shared. Again, later in that same class session, Carrie was giving a solution to the problem of five packs of gum in which there was 10 sticks in each pack.

Carrie: I wrote 10 five times because there was five packs and each pack had 10 pieces.
Instructor: Okay.

Carrie: I just added them. I did 10 plus 10 is 20 and 10 plus 10 is 20 and I knew that equaled 40 plus 10 is 50.

Again, Carrie freely gave her explanation of her solution and justified her process. There was not a need on the instructor’s part to ask for more explanation and the students accepted her solution process. This norm of explaining solutions and justifying solution processes had become taken-as-shared.

Making Sense of Other Student’s Solutions

A second social norm that was valued was that students were expected to attempt to make sense of other student’s solutions and solution processes. This often involved asking questions when they did not understand a solution process presented by another student or something the instructor said. This social norm was also supported by the prior norm of explaining and justifying solutions. This began to be established by the instructor of the course saying things like: “Questions so far?” “Do you agree with that?” and “Are there questions for [student]?” In this episode during the first class session, a student provided a solution to the dot problem. The dot problem gave the first three segments of a pattern in which a dot moved from the top left corner of a box in the first cell to the top right corner of the box in the second cell, to the bottom right corner of the box in the third cell (see figure 3).

![Figure 3: The Dot Problem](image-url)
The question asked to find what the 31st cell could look like. A student gave a solution which involved dividing by 4 and taking the remainder to determine which step of the pattern was the 31st. The instructor asked if there were questions and another student spoke up to indicate that he did not understand her solution process.

Instructor: Questions about how [the student] solved this problem?
Tim: I still don’t understand how she solved it.
Instructor: Good, I’m glad you asked.

Tim did not understand the way the student solved the problem and met his obligation to try to make sense of other student’s solutions by asking questions. This was part of the necessity and responsibility of making sense of other solutions by asking questions. The instructor helped establish this norm for the students by asking these types of questions and not allowing students to accept other students’ solutions without adequate understanding. It seemed to the research team, however, that some students were not trying to understand other students’ solutions and were not asking questions when they needed to and were expected to do so. In order to compensate for this, the research team decided in the team meeting after day one of the place value sequence that the instructor would begin to ask students to explain what another student just said. This was a useful strategy in getting the students to ask questions when they did not understand another student’s explanation and provided a way for the instructor to verify if students understood the mathematics involved. This also was useful to see if students were fulfilling their obligation to make sense of each other’s solutions and explanations. This episode took place on day two of the place value and operations instructional sequence. Students were given the task of reorganizing 2 boxes, 4 rolls, and 6 pieces of candy where 1 roll was 10 pieces and 1 box was 10 rolls.
David: One box, 2 rolls, 116 pieces, no 126 pieces.
Instructor: Laura, how did David get that?
Laura: I couldn’t tell you.
Instructor: Then you have some questions to ask David. Ask a question.
Laura: How did you get that?
David: Take the box and turn that into a 100 pieces. Take 2 rolls would be 20 and 6 pieces. So you have 1 box, 2 rolls, and 126 pieces.

This was followed by the instructor asking Laura to explain for herself what David did. She continued to get lost in the reorganization, and her partner helped her out.

Instructor: What did he do, Laura?
Laura: I see what he did.
Instructor: Explain it to me. I got a little lost.
Laura: So there is 10 rolls in the box, and there’s …talking to Kim
Instructor: Talk a little louder so we can all hear you.
Kim: There’s 10 rolls of 10 which equals 100 individual pieces. He separated the bottom box (of the picture on the board) into pieces instead of rolls. 10 rolls, but 100 pieces.

This episode continued until Laura was able to explain for herself what David had done in the original solution. The norm of making sense of other student’s solutions continued to be established throughout the instructional sequence. This norm was considered to be taken-as-shared in the classroom when, at the end of day two of the place value and operations instructional sequence, the instructor was able to say things like:

By your silence, I am assuming you understand how to shift these around and how to reconfigure them. You know it is your responsibility to ask the questions when you need more help.

The students recognized this aspect of their responsibility as students in this classroom. This continued to be established by instances of students asking each other questions to make sense of their solutions as in this episode on day 3 of the instructional sequence. Students were given the problem to find how many candies were in five bags if a machine put 17 candies in each bag.
Katrina: Because it was 17, to make it easier I broke it down into 10 and 7.

Instructor: Hold on, so you broke 17 into 10 plus 7.

Katrina: Yeah, and because we had five bags, I did 10 plus 10 five times and that gives us 50 and then I did the sevens by themselves.

Instructor: So this group represents five groups of 10 and this group represents five groups of seven? Okay.

Katrina: And then I added seven and seven and got 16 and then did that with the next two sevens and then 16 and 16 is 34 plus seven is 43. And then I added the 50 and the 43 to give me 113.

…

Instructor: Any questions on this? Doug?

Doug: I am having a hard time following it.

Instructor: Well, if you are having a hard time following it, maybe you need more explanation. Ask Katrina for some of that? What questions do you have? Lilly looks like she wants to jump in here.

Lilly: Maybe if you wrote after the 10 plus 7 you show how you regrouped the five the 10 and the 7, if you write 5, parentheses 10 plus 7.

Doug: That helps.

Instructor: What helped you understand? What do you understand now?

Doug: Well, I see the process. I kind of see it.

Instructor: What do you understand?

Doug: I understand why she divided the 17 into 10 plus 7. She took five times each of them. Now I see it.

Instructor: What did she do?

Doug: There is 17 pieces in each bag and they split that up into 10 plus 7 and so there’s five bags, so five times 10 came out to 50 and then seven has to be added five times.

Instructor: Why?

Doug: Because there is 10 plus seven in each bag, so 10 times 5 is 50 and there is still 5 times 7 more pieces in there, so 5 times 7 is 43.

Doug did not understand the process that Katrina used to solve this problem, and he was willing to ask for clarification. This helped to establish the social norm of making sense of another student’s thinking, giving more evidence that the norm had become taken-as-shared. The work of establishing social norms began almost immediately with the first class session of the semester.
Work continued throughout the implementation of the instructional sequence on the establishment of sociomathematical norms including what constituted a different solution and a good explanation. Although it is believed that these social norms were established, there were not any cases found in these five class sessions in which a student violated a social norm. This would further substantiate the establishment of the social norms. If a student violated what was believed to be an established social norm, the reaction of the classroom community can provide evidence for that norm having been established. If the students in the classroom did not allow the student to violate the social norm, this provides substantiation for the establishment of the social norm. On the other hand, if the students in the classroom allowed the student to violate the norm, questions may be raised as to whether the social norm was actually established.

**Different Solution**

Sociomathematical norms were established as they related to criteria for what counted as different and unique mathematical solutions to the same problem. This norm began to be established by the instructor often inquiring if other students had a different solution or a different solution strategy. The instructor looked for these different solutions and solution strategies as she walked around the classroom when the students were working in small groups. During the problem solving tasks in the first class session of the semester, she made comments like

What was important to me in solving [the allowance] problem is seeing that there are several different solutions. And what I am trying to communicate with that is it is important to look once you get a solution, that doesn’t necessarily mean it is time to stop. Say, can we solve this in any other ways? Or can we get to any other solutions? So here’s an example of how we got two different solutions and also looked at the different ways of solving them. I’m going to skip you now to the dots problem. As I walked around I looked at you solving this problem … I didn’t see different solutions, but I heard about different ways you solved the problem, so different solution strategies.
By discussing with the class the value of both different solutions and different solution methods, the need for the sociomathematical norm of criteria for different solutions was established. What was more difficult to establish was an understanding of what counted as a different or unique solution or solution method, the sociomathematical norm. This began to be established by the instructor saying things like “Did anybody solve this differently?” and “How many of you got that same solution? Who had something else?” Discussion ensued in several class sessions of whether two solution methods were the same or different.

In an episode on day one of the instructional sequence, students were engaged in a task which involved using a blank number line to find 52 – 23, Darren said:

Darren: I started at 52. Subtract 10. Subtracted another 10. Subtracted another 3 to get to 27.

Kim solved the same problem by saying:

Kim: So I started at the 23 and went up by 10’s until I got to 53. I knew it was one less than 53 because it was 52, so I took that one away from the 10 to get 7. And then 10, 10, and 7 is 27.

A discussion then ensued as to whether these two solution methods were the same or different. The class determined that these solution methods were different because one was taking away the 23 from 52 and one was adding onto 23 until you reached 52, providing initial criteria for what made solutions different. Discussions such as these provided a context for determining different solutions versus the same solution.

Discussion regarding what made solutions different continued throughout the instructional sequence. This episode occurred toward the end of day two of the instructional sequence, during a task in which a given number of rolls and pieces of candy were shown pictorially and the total number of candies was given (see figure 4).
The students were asked to find the number that were missing or covered up (see Appendix A). One student solved the problem by subtracting 51 minus 23 using a traditional regrouping algorithm (see figure 5).

\[
\begin{array}{c}
4 & 11 \\
5 & 4 \\
\hline
2 & 3 \\
\hline
2 & 6 \\
\end{array}
\]

Figure 5: Traditional Regrouping Algorithm for 51 - 23

Sarah counted up from 23 to 51 as in the following episode.

Sarah: The total number was 51, so you have two rolls which is 20, so to get to 50, I first tried to add three more rolls,

Instructor: So to get from 20 to 50 you tried to add 3 more rolls.

Sarah: Yeah, and then I figured out that was going to be too much, you would go over, so I went down to two rolls.

Instructor: What do you mean, from 20 to 50 is 3 rolls?

Sarah: Yeah, it’s three rolls, but you already have three pieces as well and you are trying to get to 51, so you would go over.

Instructor: so three rolls took you over.

Sarah: Right, so I went down to two rolls, which got me to 43. And then I counted on my fingers up to 51.
Instructor: Okay, and how many did you count on your fingers?

Sarah: Six.

Instructor: So what did you decide?

Sarah: Two rolls and six pieces or 26 pieces.

This proceeded with the instructor inquiring as to whether these solutions were the same or different.

Instructor: So are these the same solution strategy, same solution types or different? What do you think?

Kathy: That subtraction one when she first started doing it I was like how is she going to do this? Like I didn’t see it but when she went through it, it just seems like a lot of work as compared to the number line, but maybe it’s just the way I think about it.

Student: When you get to larger numbers, it would be easier.

Kathy: Still, I know my numbers, but it’s easier to count up from 23 than to subtract from 51. I don’t know.

... Instructor: Okay, so we are calling this (51 minus 23 with regrouping algorithm) basic subtraction? What did we do to figure out this one (the counting up)? Did we also subtract?

Carrie: You added.

Instructor: So we added from where to where?

Carrie: From 23 to 51.

Instructor: Okay. And here we didn’t, we subtracted. So we look at this very same problem and what you shared with me were two very different ways to solve the same problem. We want to do a lot of that in here. We want to share different ways of solving the problem, so when you watch and hear someone else and understand the way they solved the problem and you solved it differently, what you would qualify as a different solution – and we will look at some of those and see are they different or aren’t they different, I would like for you to share that so that we can understand your thinking.

The students determined that these were indeed different solutions and part of the criteria of what made them different was the operation involved. In the first case, the operation was subtraction while in the second, the operation was addition. This was the first instance of the instructor delineating that a goal was to establish what made solutions different. The instructor demonstrated her intent to continue to work at establishing these criteria with her statement that
she wanted the students to share what they thought were different solutions so that others can understand the thinking involved, the social norm of making sense of other student’s solutions, and so that it can be determined what makes them different versus the same solution. On day three of the instructional sequence, the discussion of what made solutions different again appeared. The students were given the task of finding how many sticks of gum were in 5 packs of 7. Both Joe and Sarah began the problem with $7 + 7 + 7 + 7 + 7$. Joe solved this by finding a pattern within adding 7 repeatedly, that the 10 digit went up by 1 and the ones digit went down by one each time. Sarah solved this same problem by adding the 7s together in pairs and then partitioning any two digit numbers into 10s and 1s. The discussion then led to what made the solutions different.

Instructor: How many of you solved it like that? Now how neat is that? So they both started with 7 plus 7 plus 7 plus 7 plus 7. Same or different? They each did five groups of seven, they each used repeated addition. Are these the same strategy or different?

David: Different.

Instructor: Why?

David: The one over there found a pattern and he didn’t have to do much work as opposed to a lot of work.

Instructor: So different strategies because a lot of work and not a lot of work? What else might describe these as different strategies?

April: On the first one, he continually counted up so he had to be continually thinking in terms of 17, 20, and so on. The second one she grouped and reorganized things to make it easy to put things together, like the 10s and the 20 with the 10. So while it might be more written work, it certainly seems easier on the brain to think of it.

The solutions were determined to be different by several criteria. One was the amount of work involved in solving the problem. David felt the patterning solution was less work than the partitioning solution. April gave another criteria with the difference between the amount of written work and the amount of mental work involved.
This sociomathematical norm continued to be established throughout the instructional sequence as students developed an understanding of different solutions. The discussion of whether solutions were different tended to drop away and students presented different solutions automatically, as in this episode which occurred on day three of the instructional sequence, with showing 4 groups of 4 with 10 frames.

Instructor: So you saw the four groups of four as $4 + 4 + 4 + 4$. How did you know that was 20?

Laura: 4 plus 4 is 10 and then the other 4 plus 4 is 10 and 10 plus 10 is 20.

Instructor: Okay…. Anyone do it differently? Doug?

Doug: I looked at it and saw the 4 as half of 10, so I saw a half and a half and a half and a half was 2 wholes which was 20.

Doug recognized that his solution was different from Lilly’s and presented his solution method to the class. Doug understood that his solution was different, but the criteria for why his was different was not discussed. Students demonstrated a taken-as-shared understanding of what counted as a different mathematical solution when they were able to successfully determine and explain two different solutions or two different solution methods to the same problem with homework or test questions. Although students were able to present different solutions and seemed to understand criteria for what made them different, that criteria did not always become explicit. In many cases, the instructor asked for different solutions and they were presented and accepted as different without the discussion of what made them different. Some criteria that were established, however, included using a different operation and a difference in the amount of written versus mental work. These criteria were taken-as-shared by the classroom community, but they were not exhaustive criteria. Students were also able to help each other determine what was contained in a good explanation, another sociomathematical norm.
Good Explanations

The research team perceived that establishing what was a good explanation was one of the more difficult sociomathematical norms to develop. Verbal explanations were more easily developed as the students provided verbal explanations to each other during each class period. Verbal and written explanations required different substance, a topic of discussion in the research team meetings, as in this one after day two of the instructional sequence.

RT1: For me there is a difference between a written explanation and the criteria for those, versus a verbal one. What gets constituted as criteria for verbal explanations may change during the course – does change during the course of the semester.

…

RT1: But for a written one. Think about mathematical proofs in journals, and they have certain forms and there are criteria mathematicians just in our discipline have developed for what counts as an acceptable proof.

The criteria for verbal discussions is negotiated between the students in the course and the instructor. As mathematical ideas become taken-as-shared, it is no longer necessary to provide a mathematical explanation or justification for that mathematical idea. For written explanations, however, the student cannot assume that the reader has the same taken-as-shared mathematical ideas. That means that the mathematical ideas that need to be explained and justified in written explanation include ideas that the collective classroom community has determined to be taken-as-shared. Verbal explanations were more easily established through the student discourse in the classroom and students’ participation in the classroom community. For written explanations, however, instructional tasks were discussed by the research team and added to the instructional sequence with the goal of helping students to determine a quality written explanation.

Students’ questioning of each other helped to develop what was a good verbal explanation. Students felt obligated to explaining their solutions and solution processes, so they often asked questions about the depth of explanations. As noted in field notes of researcher-to-
student interactions, some of this seemed to be as a result of their desire to please the instructor and concern for their numerical grades on assignments. They were concerned about what was needed for a good written explanation out of obligation to the instructor, not because they valued the need. For example, in this episode on day three of the instructional sequence, the students were attempting to find out how many sticks of gum were in 5 packs if there were 7 in each pack. Many students began by writing $7 + 7 + 7 + 7 + 7$. Sarah solved it in this way:

Sarah: I had the 7 up there five times, and I found that 7 and 7 was 16 and then the next two 7s were also 16, so I broke down the 16s into a 10 and a 6 each of them.

Instructor: You broke it down now to 10 and 6.

Sarah: The other one as well. And then I still have a seven from the top. And I added those together.

Instructor: What did you add together?

Sarah: Sorry. The two 10s and then ... 

Sarah: I got the six and the six which is 14 and I broke that down into a 10 and a 4 so I have 20 plus 10 plus 4 is 34 and then I added 7.

After she was done, Kathy asked:

Kathy: Would that be enough explanation?

Instructor: This (the writing on the board of what the student did) with what she said with it.

Amy: So then I broke down the 16 into a 10 and a 6 and then I combined the 10.

The students were interested to see what the instructor felt was a good explanation and were beginning to determine for themselves what needed to be included. Their motivation, however, seemed to stem from pleasing the instructor and earning specific grades on their assignments rather than the establishment of the sociomathematical norm of what is needed for a good explanation. Many times, students asked questions about what would be acceptable on the test or on a homework assignment, providing some evidence that grades may have been a motivating factor.
As the students continued through the course of the instructional sequence, their explanations continued to develop as evidenced by their whole class discussions and verbal explanations and by the written explanations on their tests and homework assignments. Written explanations were more difficult for the students to complete and to understand what needed to be included to make it a good explanation. This was partially addressed with an activity at the beginning of day three of the instructional sequence in which students examined middle school students’ explanation of a multiplication word problem in base ten and determined which was the better explanation and why (see Appendix C). This task was added after discussion in the research team meeting regarding the issue of explaining and justifying answers. The research team felt that a task needed to be added in to the instructional sequence to better facilitate students’ development of taken-as-shared understandings of what constituted a good explanation.

Amy’s description of why one explanation was better than the other is presented here.

Amy: Obviously the first one would be really unacceptable because they gave an answer and they gave the operation they used, but it doesn’t tell us anything about how they used the operation to get the answer.

Instructor: Okay.

Amy: The second one, they are getting there because they are kind of showing how they used the operation but not explaining where they got their numbers from. And then the third one, they went into total detail and easily, if you had no idea how to do this problem, by reading that explanation you could figure out how to do it.

Instructor: Other comments? What about that last one made you think that after seeing it you could definitely figure out how to do it.

Amy: They did everything step by step so you could follow it from the beginning to the end and they explained where each number that they used came from and they explained why they used the operation that they did and then at the end, they kind of went back summed it up in the last sentence in one sentence what they did.

The students were then asked to look at an activity which had been given at the end of day two dealing with transactions at the candy shop in base eight and were asked as a group to come up with a written explanation that they believed was a good explanation. Each group’s explanation
was then examined by the whole class and discussion ensued as to what was good about each explanation and what might be missing. One group presented its solution to the problem in which they had a picture of 4 rolls and 4 pieces and were asked how they might unpack some of the rolls to sell 27 candies. They were also asked to determine how many candies they had left.

Kathy: You begin with four rolls which equals 40 pieces plus the four individual pieces. In order to sell 27 pieces, you must unroll three rolls which now gives you 30 individual pieces. To determine how many candies will be left, we subtract 27 from 44 which is the total candies and we did the math.

The students in the class then discussed what may or may not be missing or incomplete and what would need to be included to make it a good explanation.

Instructor: Questions? How did you know that 4 rolls gave you 40 pieces?

Amy: Because you know there is 10 pieces per roll.

Kathy: But we didn’t say that.

Amy: So we should explain how many pieces per roll.

This discussion continued and led to

David: The only thing that I think is missing is explaining how they got 40 pieces. I mean, I know you multiply by 10, but you should explain it.

Kathy: We agree.

Instructor: So, Kathy, why don’t you come up and let the class help you work to a description, an explanation of this.

Kathy: We took for granted our own knowledge.

Through the experiences with this task, the students were beginning to comprehend what was necessary to have a good written explanation, part of which was including what might seem to be knowledge that was taken-as-shared. The need for the explanation was accepted, but many students struggled with what needed to be included in the explanation, especially in the written form. The social norms of explaining, justifying, and questioning were established through this classroom teaching experiment. There is little evidence, however that the sociomathematical
norms of what counted as a different solution and a good explanation was fully established and taken-as-shared. The discussions surrounding the criteria for what counted as different did not take place in ways that would demonstrate all the criteria that was taken-as-shared; however, some criteria was established, namely different operations and different written versus mental work. The same can be said for what counted as a good explanation. Although the criteria were not explicitly discussed, so conclusions cannot be made that these sociomathematical norms were taken-as-shared by the classroom community, the student work demonstrated that they had some understanding of what counted as different solutions and what made a good explanation. These norms, however, did support the development of classroom mathematical practices which were identified using Toulmin’s argumentation scheme (Toulmin, 1969).

**Argumentation**

The first phase of determining classroom mathematical practices which were supported by the social aspects of the classroom in learning place value and operations was to identify *claims*, *data*, and *warrants* based on Toulmin’s argumentation scheme (Toulmin, 1969). The transcripts of each class session were analyzed and instances of claims were identified. When a claim was identified, the transcript was then examined for data supporting that claim. The warrant that gave legitimacy to the claim and the reason that data supported that claim were identified if they were present. Members of the research team were consulted to provide verification that claims, data, warrants, and backings were being identified in consistent and agreeable ways (Rasmussen & Stephan, in press).
On day one of the place value instructional sequence, students were challenged to use the tool of the blank, or empty, number line (Klein et al., 1998) where marks were not initially used to indicate any values to add 37 and 45. Doug presented the way in which he added 37 and 45 as follows:

Doug: I was looking at 37 and made it a 40 and then the 44 and then added. *(Claim)*

Instructor: Say that again.

Doug: Make the 37 a 40. *(Data)*

Instructor: What have I added to it when I made it a 40?

Doug: 1.

Instructor: Okay. And then what?

Doug: Make the 45 a 44 and then add them to make 104. *(Data)*

David: I don’t see it.

Doug: You added one to that and took one away from the other. *(Warrant)*

David: Oh, gotcha.

In this case, the student made a claim that one could make the problem of 37 plus 45 into 40 plus 44. The data to support that claim was that he made the 37 into a 40 and made the 45 into a 44 and then added to get 104. Another student did not connect what he was doing, so he asked for justification. This provided a warrant that the reason he could add in that way was that he added one to the first number and took one away from the other. This warrant gave legitimacy to the fact that this type of compensation was allowed and mathematically correct. The warrant gave the student who questioned the method the information he needed to agree that the student’s claim was legitimate mathematically, so the student did not request a backing. In many instances, both the data and the warrant were only given when another student or the instructor questioned the claim.
In another instance on day two of the instructional sequence, students were identifying how many dots were in a double 10 frame in base eight. The configuration given had six dots in the first frame and three dots in the second frame as shown in figure 6.

![Double 10 Frame](image)

Figure 6: Double 10 for 6 plus 3

The class determined collectively that there were 11 dots in all. The ways in which the students saw 11 were different. One discussion follows here.

Student: I saw the 6 and then I did 7, 10, 11. *(Claim)* After 6 I started counting. *(Data)*

Instructor: So you counted up from 6. Then 7, 10, 11. How did you know when to stop?

Student: There were only three numbers there. *(Warrant)*

In this case, the *claim* was made by the student, that there were 11 dots in the picture. This student provided the *data* that she saw the 6 and then counted up to 11. She was challenged as to how she knew when to stop counting, and she provided the *warrant* that there were only three dots, so she needed to count up three numbers.

As the class continued to progress through the instructional sequence, many warrants seemed to drop away, indicating that mathematical ideas were taken-as-shared. As students’ explanations became more complete, their explanations included the data the other students needed to understand their solution and their justifications and warrants were built into their
explanations. There was less reason to ask for further confirmation, unless the answer was a topic of disagreement to begin with, as shown here.

Instructor: You don’t think that [12 x 10] equals 120?

Kathy: No.

Instructor: Well, Matt does. Ask Matt a question or tell him why.

Kathy: Did you count in base ten on your fingers?

Matt: I didn’t count in base ten on my fingers.

Kathy: But if you have ten twelve times, what do you get? If you add 10, 12 times (Data), what do you get? You get 140. (Claim)

David: She did it twelve times instead of 12 times.

Instructor: David, what question do you have for Kathy?

David: Count 10 12 times instead of

Kathy: 10, 20, 30, 40, 50, 60, 70, 100, 110, 120, 130, (Warrant)

Class: No

Kathy: No, I am not in base ten and yet I am adding in base ten.

Instructor: You are adding in base ten on your fingers.

In this case, Kathy questioned the answer of 120, and discovered her own error by being asked for data and a warrant for her claim. She claimed the answer was 140. Her data was that she did 10, 12 times. She discovered by the request for a warrant that she did 12 in base ten (twelve) instead of base 8. The warrant was only asked for because there was disagreement on the solution. When students seemed to understand another student’s solution strategy and agreed with their answer, there was little request for warrants. This provided some justification for mathematical ideas becoming taken-as-shared. The data tended to be provided automatically based on the norms present in the classroom. The absence of warrants was one indication when mathematical ideas became taken-as-shared. Another indication of mathematical ideas becoming
taken as shared was shifting within an argument, i.e. what was once a claim was now data (Rasmussen & Stephan, in press).

Claims, data, and warrants continued to be identified throughout the transcripts from each class period. An argumentation log which identified claims, data, questions that were asked, warrants, and backings if they appeared was created and a sample log appears in appendix F. The argumentation logs were then used as data themselves to identify when mathematical ideas became taken-as-shared. This was documented through the use of a mathematical ideas chart, a sample of which appears in appendix G. This chart identified which mathematical ideas seemed to be taken-as-shared and which mathematical ideas to keep an eye on. There was also a column for additional comments. This column could also be used to identify when ideas that seemed to be taken-as-shared were later questioned, indicating that there may be some question as to whether the idea was actually taken-as-shared earlier. Once the mathematical ideas chart was completed, the next step in the analysis process was to identify the classroom mathematical practices present in the classroom that were supported by the instructional sequence and hypothetical learning trajectory.

**Classroom Mathematical Practices**

Classroom mathematical practices are defined as those ways of reasoning, explaining, and justifying mathematically that become taken-as-shared in the classroom community as a whole (Stephan & Cobb, 2003). In identifying classroom mathematical practices, the mathematical ideas chart was examined and the mathematical activity in which students were engaged when ideas became taken-as-shared was identified. The classroom mathematical
practices should also relate to the hypothetical learning trajectory used in the instructional sequence. (Cobb et al., 2001). The purpose of the learning trajectory is to anticipate the learning in which students in the class and the classroom community might engage while in the class. “It is feasible to view a conjectured learning trajectory as consisting of an envisioned sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practices” (Cobb et al., 2001, p. 125). With this in mind, the hypothetical learning trajectory was used as a basis for what classroom mathematical practices were expected. The transcripts were analyzed side-by-side with the mathematical ideas chart to determine what classroom mathematical practices were actually formulated. The tasks which supported the mathematical practices were also identified and modifications that were made to the instructional sequence as it was implemented were examined to determine support of collective student learning. As the classroom mathematical practices were identified, the actual learning trajectory of the classroom community could be identified and further modifications and revisions could be made with respect to future implementations of the instructional sequence and hypothetical learning trajectory. The classroom mathematical practices identified and supported by this learning trajectory and instructional sequence were 1) unitizing, 2) flexibly representing numbers, and 3) reasoning about operations.

**Unitizing**

An important mathematical idea that is key to understanding place value and operations is unitizing. Unitizing relates to both concepts of ten and multiplicative reasoning as discussed in chapter 2 (Cobb & Wheatley, 1988; Simon & Blume, 1994; Steffe, 1983, 1994). With respect to
concepts of ten, students who are able to unitize are able to iterate 10 without concrete models. Initial multiplicative reasoning development also relies on unitizing with respect to repeating a given unit to model multiplication. Students demonstrated that they had not yet developed the idea of unitizing in base eight on day one when they tried to make sense of what came after 77. The task in which they were engaged was counting by 2s starting at 2. Some students reached 76 and were not sure what to do from there. The following discussion ensued.

Instructor: Okay, 70. Let’s get there for the people who didn’t get there. 70, 72, 74. Let’s go a little slower, 76, 80, 82

Student: Wait, there is no eight.

... Instructor: Some people went Ooh. What was the “Ooh” all about?

Student: There is no eight and nine.

Instructor: Okay, so, Ooh, we stop at 76 and then what?

Kathy: Why can’t you just go on to a hundred?

Instructor: Let’s go by ones from 70 first and then we will return.

Class: 70, 71, 72, 73, 74, 75, 76, 77

Instructor: What?

Class: Hundred?

Instructor: Why? What symbols would I write here?

Student: 100 (indicating a one, a zero, and a zero)

Instructor: Why?

... Sarah: Because if there is no eight or nine in eight world, you are skipping the eighties and the nineties. And then 100 (one-hundree) would be the next number.

Instructor: What if I am learning to count to begin with? Your explanation wouldn’t help me. I live in base eight, I was born in base eight, and I am starting kindergarten and I want to know how to count to one-hundree before I start, so I get to 70, and then I am going 71, 72, 73, 74, 75, 76, 77, 70-10, 70-11, 70-12, Help me. …

Kathy: Wouldn’t you have to explain the places?

Instructor: What do you mean?
Kathy: Like when you start with 1 that is the ones place and when you get to 70, the 7 is in the tens place.

Instructor: The 10 place.

Kathy: And then when you get to 77 you have to go up to the one hundree place.

Instructor: Why?

Kathy: Because that is the number that comes after it?

David: 77 is the last number in the 10 spot.

Instructor: This (77) is in the 10 spot?

Kathy: If you have 77 apples, and you add one, that is 100.

Sarah: You filled up those places (the 1s and 10s) as far as they can go, so you have to

Instructor: So I am picturing that these are my apple places?

Sarah: So you have seven in each.

Instructor: These are my apples?

Kathy: If you add an apple to the right, you now have to carry,

April: I kind of like the apple thing because it is like if the bucket is full and you try to put another apple on, one is going to roll over to the next bucket, but that one is full, too. So you need another bucket, another place holder.

Instructor: So this one would roll over into this bucket?

April: Right, and then that one is full, so you need one more bucket.

Instructor: So when this one rolls over to this bucket, I had 77 and I added 1. That 1 rolled over to here and that rolled over to here, so then I have 177?

Kathy: no, because your other two buckets of apples are full, so you take those away

David: You put another bucket on top of it. After it is done, you put another bucket on top of it.

Sarah: When it becomes full, you just have to take it away and replace it with an empty one. Wouldn’t that work?

Instructor: So when this became full, when I put this one here, I take it away and I leave an empty one?

This episode demonstrated that unitizing was not a taken-as-shared mathematical idea at this point in the instructional sequence. The buckets of apples seemed to signify equal sizes regardless of place value. Some students modeled it with a bucket of apples in both places as
opposed to seeing it as a bucket of apples in the “ones” place and a bucket of buckets in the “10” place. These students were not recognizing that 10 ones was 1 one-ee-zero. When one bucket of apples was filled, they wanted to just take it away, not move it to the 10s place. Some students appeared to comprehend that they were different, at least on a surface level, as Sarah and Kathy seemed to do here.

Instructor: So I had 7 apples here and 7 apples here. (pointing to two different place holders) Is this right?

Kathy: You have 70 apples in the one?

Instructor: You want me to draw 70 apples?

Sarah: No, it’s the tens place, so each of those apples counts for ten.

Sarah demonstrated that she was beginning to be able to unitize 10 ones into 1 one-ee-zero, but was unable to provide a justification for why that worked mathematically. She recognized that the second bucket was the “tens” place and so each apple counted for “ten”, but failed to recognize that it was really buckets of apples. Students began their development of unitizing 10 in the same manner children do, by counting number sequences as in counting by 1s, 2s, 5s, or 10s. Children do not initially recognize that when counting by 10, they are adding one more set of ten ones. They view the counting sequence of 10, 20, 30, 40, … as simply a sequence of numbers. The concept of unitizing 10 began to be developed with subsequent tasks.

In this same class session, the blank number line was introduced as a tool for students to use for compensation and unitizing through story problems involving addition and subtraction of two- and three-digit numbers in base eight. A goal of the blank number line tasks was to support student development of more efficient counting strategies, or thinking strategies as well as to provide a tool for students to use in reasoning mathematically. There were students who were still counting everything by ones, a strategy that children use when first learning the base ten
number system and manipulating and operating with numbers (Cobb & Wheatley, 1988; Fuson, 1992; Ross, 1986; Steffe, 1983). In the team meeting following the first class session on place value, discussion ensued regarding the placement of the blank number line problems. It was felt by some of the research team that the blank number line was out of place; however, a goal of the blank number line was to facilitate counting strategies, which seemed to fit with the learning goals associated with counting and unitizing in base eight. Some students were reluctant to use the number line as a tool and wanted to use their traditional addition and subtraction algorithms to solve the problems. This reluctance may have been due to the way the research team introduced the blank number line as a tool and the conclusions reached after the class session that the blank number line activity was out of place.

RT2: The reason I wanted to keep them at the number line level is to keep them counting, as oppose to plugging in an algorithm where they should understand place values. But based on our buckets of apples, they weren’t understanding place values. So I didn’t want them to use a place value algorithm that they don’t understand to solve something by just skipping eights and nines.

The problem that ensued with the blank number line was that conceptual development of 10 was skipped. The blank number line asked the students to count efficiently with two and three digit numbers, but their learning of operating with smaller, single digit addends was overlooked.

RT1: The reason I ask about [the number line] as the very next activity after working with number word sequences is because [the number line] kind of assumes conceptually that you can work with those abstract symbols - that you’re taking, when you go up by tens or 10s, that’s a unit for you. You can make those jumps by 10s. And a lot of students, like, I remember I wrote this one down.

RT2: In the back, “I just count up by ones.”

RT1: I just count up by ones. She couldn’t go by leaps of 10.

The students had not been given experiences which supported their conceptualizing units of 10.

RT1: Kind of what we’ve missed is basic facts, because we went from, I just learned how to count to adding two-digit numbers, but I didn’t…

RT3: No, we missed unitizing.
As a result of this discussion, the research team decided to modify the instructional sequence to give students the experiences needed to support unitizing.

The sequence was modified to give students experiences with the tool of the 10 frames. It was felt that giving students the 10 frames first would support their mathematical learning and conceptual development of a unit of 10. The 10 frames gave students experiences with combining single digit numbers and obtaining sums of less than 20. Skipping this task and jumping to the blank number line forced the students to make a conceptual leap that they were often not ready to take (Cobb & Wheatley, 1988; Steffe, 1983). The classroom mathematical practice of unitizing was supported by the taken-as-shared mathematical ideas of 1) creating units of 10, 2) 10 as an iterable unit, and 3) unitizing for multiplication.

**Creating units of 10**

On day two, students were engaged in a task which involved identifying how many dots were in a double 10 frame. An example of this task is given in figure 7. Students were shown the configuration for 2-3 seconds on an overhead projector. After the overhead projector was switched to off, students were then asked how many dots they had seen.

![Figure 7: Double 10 Frame Example](image-url)
Students began to reason that they could move dots around and fill frames to make 10, thus making the counting of the dots easier. Some students began this task by counting all the dots, as Amy did with the following:

![Diagram of two frames, each containing a group of dots.]

Amy: I just saw it and counted really fast. *(data)*

Instructor: So you just saw it and how did you count really fast?

Amy: I counted one at a time. *(warrant)*

Instructor: 1, 2, 3, 4

Amy: 1, 2, 3, 4 and then 5, 6, 7, 10. *(Claim)*

This paralleled children’s initial learning of base ten through counting. Another student added the two 4s together to get 10.

Lilly: 10 *(Claim)*

Instructor: Lilly how did you know?

Lilly: I saw 4 in the first one and four more in the other, but in a different shape, and I just added them. *(Data)*

Then students progressed to adding the dots in each frame in order to combine them together. In the following episode which was two tasks later, there were 3 dots in the first 10 frame and 5 dots in the second 10 frame.
Joe: I saw the three on one side and four and one on the other side. [sic]

Instructor: And then what did you get?

Joe: 10

Instructor: You paused for a moment, what were you doing when you paused?

Joe: I was adding them up in my head.

As this idea of counting became established, students progressed to ideas of making units of 10 by filling 10 frames. Initially, the students who began moving dots around did so to make the frames look like a previous problem that they already knew the answer to; they related one problem to a previous problem by moving the dots from one spot to another, again paralleling developmental levels of children learning base ten. For example, students had already seen a configuration like this and determined that the number of dots was 11, the claim, since you could count up 6, 7, 10, 11, the data. Then, students were shown
Sarah claimed that the solution was 11, as follows:

Sarah: I moved the dot that was on the far left over to the empty space over on the right \(\text{(data)}\), so we had the same shape we had last time. \(\text{(Warrant)}\)

In a subsequent task, students were shown 10 frames with 3 dots on the left side and 5 dots on the right side.

Joe reasoned that the solution was 10, the \textit{claim}, because

Joe: You could have brought the one on the right over to the left. \(\text{(Data)}\)

This gave the same configuration as a prior problem which showed 4 on the left and 4 on the right, so the solution must be the same. The student broke apart a unit of 5 into 4 and 1 and moved the 1 to make a configuration he was familiar with, a mathematical idea which can support unitizing 10. His movement of the dot to make 4 and 4 was not questioned. Joe continued with the strategy that Sarah had presented and was not questioned. The idea of moving dots to make configurations that were recognized from before shifted from being a warrant in Sarah’s case to being data in Joe’s case. This would indicate that the mathematical idea of
moving dots to make the configuration look like a previous configuration was becoming taken-as-shared.

Once students recognized moving the dots around as a legitimate strategy, they began to formulate ideas of unitizing by moving dots to make 10s. In this example, one frame was already filled, and the students recognized that as 10 without counting to verify that fact. This was not questioned by any of the students.

Student: On the first box, they were all full, so we know that is 10. On the second box, one whole side was full which was four plus one and I got 15. (Data)

On a subsequent problem that had 6 in one frame and 7 in the other, Matt said the following:

Matt: I moved two over from the right side to the left side to make 10 and there was one left over the four, so 15. (Data)

He saw that the first frame was missing two to get to 10 and so he moved those over to make 10. Then he looked to see what was left in the other frame and recognized that he already had 10 in the first frame. This was not questioned by other students in the class and he used the taken-as-
shared mathematical idea of moving dots to make new configurations in his reasoning. The students also began to reason by taking away from 10s, as David did to the same problem.

David: The way I saw it was counting the blank spots, and then 20 minus 3. (Data)

Instructor: So, the two 10 frames make 20 and then you took away three from that (Warrant). How did you figure out the answer?

David: I just counted backwards. (Data)

Instructor: Okay, so, 20, 17, 16, 15. (Claim)

David’s claim was that the solution was 15 because you could count the blank spots and then count backwards, his data, and a warrant linking these together, that the two 10 frames make 20 and then taking away from that was given by the instructor. This built on the claim that Doug had made earlier in this same episode when one 10 frame with 7 dots was shown.

Class: Seven. (Claim)

Instructor: How do you know, Doug?

Doug: I saw three rows of two and one more. Well, actually I saw one blank on the bottom. (Data)

Instructor: What did you do with that one blank at the bottom? How did you know that gave you seven?

Doug: Because we are in base eight. (Warrant)

Instructor: Okay. You are in base eight, so we had that 10 with one blank would give you 7. (Warrant)

In Doug’ justification, seeing a full frame missing some dots was given as data with the warrant that 10 with one blank was 7, or counting back from 10. Counting back from 10 was then used
by David as data for saying he counted back from 20 to find out how many were there. The shift of the compensation strategy from the warrant in Doug’ case to the data in David’s case shows that the idea of counting back from a full frame was taken-as-shared.

On day three of the sequence, Lilly went back to creating 10s when dealing with multiplication and the problem of 5 groups of 6, when she said:

Lilly: I moved the dots into the empty spaces and I filled three of them up completely, so I knew I had 30 and I added the six. (Data)

This method was not questioned by the class and the warrant had dropped off, further substantiation that the idea of making 10s was a taken-as-shared mathematical idea. In the same task, for the problem three groups of six, David also created full 10 frames.

David: I counted dots too, but I took the column on the right and I moved the bottom two over to the first column (Data)

Since the first box was only missing two dots, he moved two over to fill that frame, creating a unit of 10. His data was that of filling 10 frames, that which was earlier a warrant. This continued to facilitate the development of unitizing, and the 10 frame became a tool that was
important in students’ reasoning about place value and operations. As students reasoned with the 10 frames and developed the taken-as-shared mathematical idea of creating 10s, iterating 10 began to be established as a taken-as-shared mathematical idea which supported unitizing as well.

**10 as an Iterable Unit**

The idea of unitizing continued to be developed in this same class session and the following session as students engaged in tasks involving the candy factory. The candy factory scenario was introduced with an estimating activity which involved estimating the number of rolls of candy that could be made from a bag of loose candies. This activity was added to the instructional sequence in this iteration based on discussion from the research team meeting following day one of the sequence.

RT1: What you might want to do before candy shop one [which involves composing and decomposing] is go to estimating, because at this point before candy shop one, you’ve just merely introduced the convention that we’re going to put 10 pieces in a roll. And then you’re asking them to work with these pictures as if they represent that for them right away. What you might do is some estimating, like have a bag full of loose cubes, and say, these are candies, a bag of candies. Estimate, if we were to take these out in rolls of 10, about how many rolls would you get out? That helps them start thinking about coordinating singles with the 10s. And then you actually physically count them out. I mean, it’s pain staking. Don’t do all of them. But do at least three or four rolls. Do one, two, three, right in front of them. Students need that physical activity of creating a ten, if it’s going to lead to a conceptual creation of ten singles into ten.

Again, conceptual development of making 10s out of pieces was skipped by overlooking this type of activity. When the students were immediately asked to engage in reconfiguring candy representations, they had not had the opportunity to begin to develop the idea of coordinating units – that 10 pieces is the same as 1 roll, viewing 10 as an iterable or abstract composite unit (Cobb & Wheatley, 1988; Steffe, 1983). The estimating activity might fill that conceptual gap, so it was added to the instructional sequence for this iteration. The estimation activity also
introduced the model of collections of objects as a model of place value concepts using boxes, rolls, and pieces in the context of the candy factory.

In the estimating activity on day two, the bag of cubes contained 12 rolls and 5 pieces of candy. The students were asked to determine how many pieces of candy had been in the bag at the start. One group determined the answer to be 145. Although their claim of 145 was subsequently shown to be incorrect, Katrina reasoned about it by making units of 10.

Kelly: What did we say? 145? \textit{(claim)}

Instructor: Questions?

Matt: How did they get that?

... Katrina: No, I made 12 sticks and counted my 12 sticks and counted by 10s and I didn’t count the eight and nine, so I got 100’s and then I went from there. \textit{(Data)}

Katrina’s data was that she made 12 sticks and counted by 10s. This was not questioned for the legitimacy of counting by 10s. Her claim of 145 was questioned by other students in the class, but not for the purposes of discounting her data.

Katrina: That’s the number of sticks I made \textit{(Data)}. Oh, I have too many?

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{katrina-drawing.png}
\caption{A representation of Katrina's drawing of rolls of candy}
\end{figure}

Instructor: Well count them.

Matt: That is twelve rolls.

Katrina: No, I don’t need these.

Instructor: Where did you get confused?

Student: Remember, we were in eight land the whole time.

Instructor: Oh, so you crossed over. Now count out loud for us.
Katrina: 1, 2, 3, 4, 5, 6, 7, 10
Instructor: Are there 10 there? How many did you want to be there?
Katrina: Eight but there is no eight, so 10.
Instructor: how many rolls are there all together?
Katrina: 12
Amy: So you need two more rolls.
Kelly: 1, 2, 3, 4, 5, 6, 7, 10, 11, 12

Counting by 10s was accepted as a legitimate mathematical activity so long as the counting was done correctly in base eight and students had begun to develop 10 as at least a numerical composite (Steffe, 1983). They were able to count by 10, but there was not evidence at this point that this was beyond a counting sequence. While discussing the same problem, Debbie said:

Debbie: Well, I tried to use the number line but then I got confused because I originally thought there was 145 also.
Instructor: Okay, so could you use the number line now to do it?
Debbie: I am still a little confused, but.
Instructor: Okay
Debbie: yeah I could.
Instructor: Try it for us?
Debbie: (drawing on board) So I would just do 10, 20, (Data1)

... 
Debbie: Group each 10, each roll so that would be 1 roll, 2 rolls,
Instructor: So where would you stop?
... 
Kelly: There should be 12 spaces of 10. (Claim1)
Instructor: Did you all hear Kelly?
Kelly: There should be 12 spaces between each 10, from 10 to 20, there should be 12 of those (pointing at space between 10 and 20). (Claim1)
Debbie: Between here (pointing to between 10 and 20)? Because you have 10, 11, 12, 13, up and so.

...  
Instructor: Let’s go back to Kelly’s 12 between each one….Tell me about that.

Kelly: Well, there are 12 rolls and there is 10 candies in each roll, can you help me. (Data1)

Kathy: You were saying 12 should come between each line, but actually it should be 10 between each lines and 12 lines or spaces total, not individual spaces, but spaces. (Claim2)

Instructor: Do you want to show us what you are talking about?

Kelly: There should be a line here and that would be zero. We don’t have any yet and between here there would be 10 candies and then 10 more candies all the way up here and then you just count the spaces, 1, 2, 3, 4, 5, 6, 7, 10.. well that it’s just there. (Warrant1) That is how you use the number line, it tells you [that] you have 120 candies plus the five. (Claim3)

Instructor: Is she confusing you? Doug, explain to Kelly what you understand she said.

Doug: She is going by sets of 10 since each roll is 10 and she is going by rolls all the way up, one roll at a time with five candies left over. (Data3)

Debbie correlated 10 with 1 roll, identifying that she had established that 10 was 1 roll and 10 pieces simultaneously. She went back and forth between 10 and 1 roll, substantiating that she viewed them as the same, thus creating a composite unit of 10. She then counted by that unit, making it an abstract composite unit (Cobb & Wheatley, 1988; Steffe, 1983). She still used a visual picture of the number line, so there is not evidence that she had transitioned to 10 as an iterable unit. Other students in the class worked with Debbie to understand her solution and identified ways in which the mathematical idea of 10 as an abstract collectible unit was becoming taken-as-shared. Kelly made the claim that counting the number of spaces of 10 and getting 12 spaces would give you 120 candies with 5 left over.

Discussion ensued in the research team meeting following this class session about the students’ development of unitizing and recognizing 10 as an iterable unit.
RT1: But what they were doing is that if they were – I counted 10, 20, 30, 40, 50– 120, 121, 122, 123, and they went naturally from counting in 10s to counting by ones. And that ability to switch like that without really thinking about it is what happens when you’re exploring how many pieces were in there. And I think that’s the first place I really saw that switch so naturally.

The students’ development of 10 as an abstract or iterable unit was facilitated by the experiences using the candy factory and the collection of objects model using boxes, rolls, and pieces supported that development. The collection of objects model provided a visual picture for composing units of 10 and students had a common language with which to work in explaining and justifying their solutions. The candy factory scenario helped them to develop concepts of 10 by giving them contextualized experiences that facilitated their conceptual development in line with the heuristics of Realistic Mathematics Education (RME). At this point, however, the collection of objects model using the boxes, rolls, and pieces was confined to the candy factory scenario.

As the instructional sequence progressed through day two, students engaged in a task in which they were given a configuration of 2 boxes, 4 rolls, and 6 pieces and asked to find different ways in which that amount of candy might have been left on a table. David made the claim of 1 box, 2 rolls, and 126 pieces.

David: One box, 2 rolls, 116 pieces, no 126 pieces. (Claim)

Instructor: Laura, how did David get that?

Laura: I couldn’t tell you.

Instructor: Then you have some questions to ask David. Ask a question.

Laura: How did you get that?

David: Take the box and turn that into 100 pieces. Take 2 rolls would be 20 and 6 pieces. So you have 1 box, 2 rolls, and 126 pieces. (Data)

Instructor: What did he do, Laura?

Laura: I see what he did.

Instructor: Explain it to me…
Laura: So there is 10 rolls in the box, and there’s …talking to Kim

Kim: There’s 10 rolls of 10 which equals 100 individual pieces. He separated the bottom box into pieces instead of rolls. 10 rolls, but 100 pieces. \textit{(Warrant)}

Instructor: Do you see that? How did he get 100 pieces?

Laura: There’s 10 rolls and 10 pieces in each roll, so there’s 100 pieces. \textit{(Warrant)}

Instructor: Okay, then what?

Laura: Took the two rolls which have 10 pieces in each one, so that turns into 20. And then the 6 pieces. \textit{(Warrant)}

Kim: He added the other full box and the two full rolls.

Instructor: Where were they?

Kim: They were all already put together.

Instructor: So one box and two rolls. And where is this?

Kim: The leftover 100 pieces would make up the second box and the 20 pieces would make the 2 rolls and the six pieces. Instead of them being all in the nice box pictures they were in pieces. \textit{(Warrant)}

David’s claim of 1 box, 2 rolls, and 126 pieces was supported by his data of how he broke apart the boxes and rolls into pieces. He recognized that 1 box was simultaneously 100 pieces and that 1 roll was simultaneously 10 pieces. In making sense of David’s solution, Laura and Kim provided a warrant for his claim that he separated one box and one roll into pieces since one box was 100 pieces and one roll was 10 pieces. On another problem in the same task, students were given 3 boxes, 6 rolls, and 4 pieces and asked to determine other configurations.

Matt: I drew three boxes that’s 300, then I drew five rolls that’s 50, and then for the ones I put 14 ones. \textit{(Claim)}

Instructor: So, what did you do with this last roll?

Matt: I took it away and added 10 pieces. \textit{(Data)}

Instructor: So, in a sense, if you had this roll, we are sort of unpacking it, so I can take this roll and, so what did you end up with?

Matt: I ended up with 364. \textit{(Claim)}
Matt used David, Laura, and Kim’s work that 1 box was 100 pieces and 1 roll was 10 pieces in justifying why his 3 boxes, 5 rolls, and 14 pieces gave the same number of pieces, 364. The idea that 1 box was 100 pieces had shifted from being a warrant used by Kim to being part of the data used by Matt. This gives an indication that viewing 1 box as 100 pieces simultaneously was a taken-as-shared mathematical idea. The same could be said for viewing 1 roll as 10 pieces simultaneously, as shown in this episode which immediately followed Matt’s claim of 3 boxes, 5 rolls, and 14 pieces. Carrie claimed another solution was 36 rolls and 4 pieces. She started with the original configuration of 3 boxes, 6 rolls, and 4 pieces.

Carrie: I got 36 rolls and 4 pieces. *(Claim)*

Instructor: How did you get that?

...  
Carrie: I took away the boxes and added 10 rolls for each box. *(Data)*

Instructor: So you unpacked these boxes. We can show that by *(drawing on board).* I’m sharing some notation with you. I unpacked this box and made rolls.

Carrie: I did all three of them. *(Data)*

Instructor: Okay.

Carrie: And then I got 36 rolls and 4 pieces.

Instructor: Amy, can you tell me how she got 36 rolls?

Amy: She knew that there was 10 rolls per box, so then plus the 6 rolls that were already there, so 30 plus 6 is 36 rolls and then the four pieces. *(Data)*

Amy’s use of 10 rolls per box further demonstrated that the idea of 10 as an abstract or iterable unit had become a taken-as-shared idea. Although previous supplied, the warrant for how 10 rolls per box demonstrated that the solution was 36 rolls and 4 pieces was not included and was not asked for, giving indication that the mathematical idea of 10 as an iterable unit was becoming taken-as-shared.
In subsequent tasks on the same day, day two, students were given tasks involving transactions in the candy shop and were asked to determine how many were left after adding or taking away some candies. In this episode, the students were given a picture of 6 rolls and 1 piece and asked how they could sell 33 candies. They were also asked how many candies would be left.

April: What I did was I left the picture of the rolls and the piece up there and I … basically I x’d out one of the rolls and wrote 10 more pieces. (Data)

Instructor: So, for here you wrote 10 pieces

April: And then the question said you were going to sell 33 pieces, so what we did was we basically crossed out 3 rolls and 3 pieces so we drew a line through three of the pieces and counted up what was left which was 2 rolls and 6 pieces, or 26 pieces. (Claim)

In this episode, April was able to view 1 roll as 10 pieces simultaneously as demonstrated by taking one roll and making it 10 pieces. This had previously been a warrant in the case of David, Laura, and Kim and had been used as data by Matt. A warrant was not asked for, further substantiating that 10 as an iterable unit was a taken-as-shared idea. Then, she needed to take away 33 pieces, and she immediately saw that as 3 rolls and 3 pieces. After she had taken those away, she ended up with 2 rolls and 6 pieces, which she immediately said was 26 pieces. She was able to go back and forth with considerable ease, thus demonstrating that she was able to unitize 10. Warrants and backings had dropped off of the argument, verifying this as a taken-as-shared mathematical idea.

**Unitizing in Multiplication**

Unitizing again appeared as a classroom mathematical practice on day three when the task involved multiplication. In this task, students were asked to find how many dots. The 10 frames were again used and given such that there were groups of dots to facilitate the HLT
learning goal of operating in base eight (See Appendix H for an example). Initially, students solved these problems by counting all the dots or by reorganizing them into ways that were easier to count, by 10s or by 4s for example. Laura solved four groups of four by adding as shown here.

Class: 20 (Claim)
Instructor: How did you know? Laura, how did you know that was 20?
Laura: I added them. ...(Data)
Instructor: How did you know this equaled 20?
Laura: I just added the four plus four plus four plus four. (Warrant)

She solved the multiplication problem of 4 x 4 by repeatedly added 4. The instructor recorded these tasks as multiplication problems to further develop the link between the task and multiplicative reasoning.

In a subsequent problem in that same task, students were shown six groups of five dots. Darren broke it down into six groups of four and then six groups of one and combined them. The way he solved this problem demonstrated unitizing as it related to multiplicative reasoning.

Darren: 36. (Claim)
Instructor: How did you know? ...Six groups of five?
Darren: I just added the groups of four so there is six groups of four. (Data)
Instructor: Like 1, 2, 3, 4, 5, 6 groups of four? (pointing to 10 frames)
Darren: Right. So I go 4, 10, 14, 20, 24, 30 and then I add the single dots to get to 36. (Data)
Instructor: How many single dots?
Darren: Six.

Although the task showed 36 dots arranged in 6 groups of 5, Darren unitized by organizing each 5 into 4 and 1. He recognized that 6 groups of 5 was the same as 6 groups of 4 plus 6 groups of 1. He unitized when he found six groups of four by counting by fours. He recognized that four
was a unit for multiplication in finding six groups of four and his warrant included counting by 4s, viewing 4 as a unit for multiplication.

Unitizing with multiplication was also found to be connected with unitizing 10, as demonstrated here. Sarah was explaining her solution to 5 packs of gum with 7 in each pack. She used the unit of 7 for multiplication and found $7 + 7 + 7 + 7 + 7$, and then used units of 10 in solving that addition problem.

Sarah: I had the 7 up there five times, and I found that 7 and 7 was 16 and then the next two 7s were also 16, so I broke down the 16s into a 10 and a 6 each of them. *(Data)*

Instructor: You broke it down now to 10 and 6.

Sarah: The other one as well. And then I still have a seven from the top. And I added those together. *(Data)*

Instructor: What did you add together?

Sarah: Sorry. The two 10s and then...

...Sarah: I got the six and the six which is 14 and I broke that down into a 10 and a 4 so I have 20 plus 10 plus 4 is 34 and then I added 7. *(Data)*

Sarah unitized with multiplication by partitioning 16 into 10 plus 6 and 14 into 10 plus 4, which was her data. Doug solved the same problem by compensating in a different way. This related back to Doug’s warrant earlier of counting backwards from a full 10 frame which was used by David as data. This led to some indication that creating 10s was a taken-as-shared mathematical idea. Doug revisited that mathematical idea by again counting backwards from full 10s, in this case full packs of gum.

Doug: I may have misread it, I knew that there was supposed to be 10 sticks in each pack, so there should be 50 sticks in, that’s how much there should be, and because 7 sticks is one less than a full pack of 10, I just added the sticks and took five, no seven from the five and came up with 43.

Lilly: There was five.

Doug: Well, I just, I said it wrong, I meant five.

Instructor: Where did you get the five from?
Doug: There was, when the machine broke down and put seven sticks in each pack and that is one less in each pack and there are five packs, so that is five less than what should be there which is 50

Instructor: Because there is one less for each pack?

Doug: So that should be 50 sticks but now we are taking five from that and reducing it to 43.

Doug partitioned the 10 into 7 and 1 and found that 5 groups of 7 was 5 groups of 10 minus 5 groups of 1. This, again, is a form of unitizing that contributed to the taken-as-shared mathematical idea of creating 10s and iterating for multiplication. This also contributed to the taken-as-shared model of compensation strategies for solving problems.

The classroom mathematical practice of unitizing was supported by the taken-as-shared mathematical ideas of 1) creating units of 10, 2) 10 as an iterable unit, and 3) unitizing in multiplication. Mathematical justifications moved from being warrants to being a part of data for substantiating claims, giving legitimacy to the view that these were taken-as-shared mathematical ideas. The warrants that initially appeared in explaining solution processes that involved unitizing 10 also dropped off, giving further indication that these ideas were taken-as-shared. Changes made to the instructional sequence supported the learning goals associated with the HLT of counting and understanding base eight and unitizing. As the classroom mathematical practice of unitizing developed, another classroom mathematical practice emerged, that of the flexible representation of numbers and was supported by the student’s concepts of unitizing 10.

**Flexible Representation of Numbers**

As the classroom mathematical practice of unitizing became established, students began to develop proficiency and mathematical understanding related to flexibly representing numbers. This classroom mathematical practice did not occur subsequent to unitizing, but occurred
simultaneously. The introduction of the candy factory scenario and packing and unpacking boxes, rolls, and pieces facilitated development of unitizing concepts at the same time as developing concepts of flexibly representing numbers by composing and decomposing. The overlapping establishment of classroom mathematical practices is consistent with the findings of Stephan and Rasmussen (2002). The classroom mathematical practice of flexible representations of numbers was supported by the mathematical ideas of 1) decomposing numbers and 2) composing numbers.

**Decomposing Numbers**

Day two of the instructional sequence began with students being asked to represent three digit numbers in base eight in different ways. The context of the candy shop was still used. There was discussion at the research team meeting about setting up the scenario for reconfiguring the candy representations in order to give the students a scenario that they could believe, consistent with the Realistic Mathematics Education (RME) instructional theory. It was decided to set up a scenario of the students as workers in the candy factory who had left their post. The owner found the tables with various configurations of boxes, rolls, and pieces. They were asked to find different ways that the owner might have found combinations of boxes, rolls, and pieces. In this first example, students were told that there were 2 boxes, 4 rolls, and 6 pieces left on a table.

David: One box, 2 rolls, 116 pieces, no 126 pieces. *(Claim)*

Instructor: Laura, how did David get that?

Laura: I couldn’t tell you.

Instructor: Then you have some questions to ask David. Ask a question.

Laura: How did you get that?
David: Take the box and turn that into a 100 pieces. Take 2 rolls would be 20 and 6 pieces. So you have 1 box, 2 rolls, and 126 pieces. (Data)

Instructor: What did he do, Laura?

Laura: I see what he did.

Instructor: Explain it to me…

Laura: So there is 10 rolls in the box, and there’s … talking to Kim

…

Kim: There’s 10 rolls of 10 which equals 100 individual pieces. He separated the bottom box into pieces instead of rolls. 10 rolls, but 100 pieces. (Warrant)

Instructor: Do you see that? How did he get 100 pieces?

Laura: There’s 10 rolls and 10 pieces in each roll, so there’s 100 pieces. (Warrant)

Instructor: Okay, then what?

Laura: Took the two rolls which have 10 pieces in each one, so that turns into 20. And then the 6 pieces. (Warrant)

…

Kim: He added the other full box and the two full rolls.

Instructor: Where were they?

Kim: They were all already put together.

Instructor: So one box and two rolls. And where is this?

Kim: The leftover 100 pieces would make up the second box and the 20 pieces would make the 2 rolls and the six pieces. Instead of them being all in the nice box pictures they were in pieces. (Warrant)

David’s data of 2 rolls would be 20 pieces and 1 box would be 100 pieces was supported by Kim and Laura’s warrants that each roll has 10 pieces and each box has 10 rolls which is 100 pieces.

A subsequent question asked for other ways to configure 3 boxes, 6 rolls, and 4 pieces. Matt claimed that a solution would be 3 boxes, 5 rolls, and 14 pieces.

Matt: I drew three boxes, that’s 300, then I drew five rolls that’s 50, and then for the ones I put 14 ones. (Claim)

Instructor: So, what did you do with this last roll?

Matt: I took it away and added 10 pieces. (Data)

Instructor: So, in a sense, if you had this roll, we are sort of unpacking it, so I can take this roll and, so what did you end up with?
Matt: I ended up with 364.

Instructor: If you left this as three boxes, we can say 3 boxes. And here you ended up with

Matt: 5 rolls

Instructor: And here

Matt: 14 pieces (Claim)

Other students got different representations, like Carrie, who said

Carrie: I got 36 rolls and 4 pieces. (Claim)

Instructor: How did you get that?

…

Carrie: I took away the boxes and added 10 rolls for each box. (Data)

Instructor: So you unpacked these boxes. We can show that by \textit{(drawing on board)}. I’m sharing some notation with you. I unpacked this box and made rolls.

Carrie: I did all three of them.

Instructor: Okay.

Carrie: And then I got 36 rolls and 4 pieces. (Claim)

Instructor: Amy, can you tell me how she got 36 rolls?

Amy: She knew that there was 10 rolls per box, so then plus the 6 rolls that were already there, so 30 plus 6 is 36 rolls and then the four pieces. (Data)

After Carrie shared how she got 36 rolls and 4 pieces, Amy was asked to explain her thinking. She was able to explain that she unpacked 3 boxes and made them rolls and got 36 rolls with 4 pieces. The student’s claim was validated by her data that she decomposed 3 of the boxes into 30 rolls and added them to the 6 rolls that were already there. Breaking apart rolls into 10 pieces and boxes into 10 rolls had previously been used as a warrant by Kim and Laura in explaining David’s solution. This shift of function indicates that decomposing numbers was becoming a taken-as-shared mathematical idea. Students continued to develop multiple solutions to these problems by unpacking boxes, rolls, and pieces to create alternative solutions. The need to understand 10 as an iterable unit was critical to beginning to compose and decompose numbers,
so this mathematical practice was dependent upon the prior practice of unitizing being established and was, in fact, established at least in some part alongside unitizing. April presented another solution to the same problem, as follows here.

April: I unpacked one of the boxes and got 10 rolls. I unpacked one of the rolls and got 10 candies out of that. *(Data)*

Instructor: So what did you end up with?

April: 2 boxes, 15 rolls, 14 pieces *(Claim)*

Instructor: Darren, how did she get 15 rolls in this?

…

Darren: Well, she broke down one of the boxes into 10 rolls and then added it to the five existing rolls to equal 15 rolls. *(Data)*

Instructor: But there are 6 rolls.

Darren: She broke down one of the rolls into pieces. *(Data)*

Again, decomposing one box into 10 rolls was used as data after being used as a warrant in David’s explanation. In general, the students only unpacked, or decomposed, numbers in creating these alternative representations. This indicates that decomposing numbers was beginning to become a taken-as-shared mathematical idea. This was further substantiated by this episode, later in the same class session. Students were asked to reconfigure 277 candies given without pictures. Some students drew pictures to support their reasoning while others kept their reasoning more abstract with indicating how many boxes, rolls, and pieces, but not explicitly drawing them.

Instructor: We have 277 candies.

Katrina: I did two boxes, seven rolls, and seven pieces. *(Claim1)*

Instructor: How did you notate that? Did you write out two boxes,

Katrina: I actually drew it. I wrote it underneath. *(Data1)*

Instructor: 2 boxes, 7 rolls, and 7 pieces. I’m going to write it over here. Darren, what did you get?

Darren: One box, 16 rolls, 17 pieces *(Claim2)*

Instructor: Can you help me get there from here?
Darren: Break down one of the boxes into rolls, 10 rolls, add it to the existing 7, so 17 rolls, break one of the rolls down into 10 pieces, and add it to the existing 7 pieces. (Data2)

Instructor: So what did you end up with?

Darren: One box, 16 rolls, 17 pieces.

Again, Darren’s data of breaking down boxes into 10 rolls and breaking down rolls into 10 pieces had previously been used as a warrant by Kim and Laura, providing further confirmation that the mathematical idea of decomposing numbers was taken-as-shared. Additionally, a backing was not requested for why Darren was able to do that mathematically, giving further evidence that decomposing numbers was a taken-as-shared mathematical idea.

As students continued to progress through the instructional sequence, tasks were given in a more abstract situation. The context was still in the candy factory, but the pictures were no longer provided. The students then began reconfiguring candy representations more abstractly with an inventory form which was similar to a place value chart (see figure 9).

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
</table>

Figure 9: Sample Inventory Form

Students demonstrated flexibility in thinking about representations of numbers with the inventory form in ways similar to those they had used pictorially with reconfiguring collections of boxes, rolls, and pieces. Their explanations corresponded to those from initial pictorial representations of packing and unpacking boxes, rolls, and pieces to create new representations of numbers. They continued to use the boxes, rolls, and pieces language as support for their conclusions and as common language with which to explain and justify their work, as in this episode on day two of the instructional sequence in which students were given 2 boxes, 5 rolls, and 6 pieces on an inventory form and asked to find other configurations.
Instructor: Two boxes, five rolls, six pieces. What else did you write down?

Joe: Two boxes, four rolls, 16 pieces. *(Claim1)*

Instructor: How did you do that?

Joe: Well, I left the boxes alone and I took one of the rolls, I took one roll out and

Instructor: What did you do with it?

Joe: I separated it into pieces. *(Data1)*

Instructor: So you have two boxes, four rolls, because you unpacked one of them, and made 16 pieces. *(Data1)* Does anyone have a different solution to this?

Carrie: I did one box, 15 rolls, and 6 pieces. *(Claim2)*

Instructor: Darren, how could she have done that?

Darren: She broke down one of the boxes. *(Data2)*

Instructor: From where – are you looking here or here *(the original or Joe’s solution)*?

Darren: There. The original one.

Instructor: From the original one.

Darren: And added 10 rolls to the existing five to make it 15 and then kept the number of pieces the same. *(Data2)*

The students continued to reason and give data to support their claims in ways that corresponded to the reasoning that was taken-as-shared with the pictorial representations. Their data were accepted by the class and warrants had dropped off. The data given were the same as that given with the pictorial representations which was supported by warrants, namely that 1 box was 10 rolls and 1 roll was 10 pieces, giving further confirmation that the mathematical idea of decomposing numbers was becoming taken-as-shared. As students progressed through development of conceptual understandings of place value, composing numbers became a taken-as-shared mathematical idea as well.
Composing Numbers

As students reconfigured candy representations, they developed the coordination of 10 pieces and 1 roll and continued to unitize. This led to the development of the taken-as-shared mathematical idea of composing numbers. This was established on day two with the task of flashing configurations of rolls and pieces with the overhead projector and asking students to determine how much was there (see Appendix A). The students needed to compose or decompose units of 10, or 1 roll, in order to describe how much was showing. In this episode, students were shown a picture of 4 rolls and 5 pieces and told that there were 71 pieces total.

There are 71 candies in all. How many are missing?

They were asked to find out how much was missing. Kim demonstrated composing 10s in this episode.

Kim: Well, I saw I had five pieces and I needed a one at the end of my number and I couldn’t just make four pieces disappear, so I made 11 pieces total which left ... Kim: I made it so I had 11 pieces, because I already had 5 and I needed a 1 at the end of my total, so you need 6 more pieces to make 11, 5 and 6 is , oh, wait, actually you will need 3 more pieces, and then you have 11 pieces and you need 2 more rolls to make up for the other (Data)

Instructor: Laura, what?

Laura: I thought she needed four more pieces.

Instructor: Wait, you think she needs four more pieces here?

Kim: Oh, yeah, I always skip 10

Instructor: Okay, so you need four more pieces. And here you only need two more rolls? (Data)
Kim: You have 10 pieces to make up that last roll. *(Warrant)*

Instructor: So I can take these and pack them, okay, so these became this *(pointing to 10 pieces becoming 1 roll)* to make how much all together?

Kim: 71 *(Claim)*

Instructor: So what did we end up adding on all together?

Kim: 2 rolls and 4 pieces.

Kim added enough pieces to compose 1 roll out of 10 pieces in order to solve this problem. In this case, a prior claim, that of 10 pieces making 1 roll, was now used as a warrant, changing position and demonstrating that the mathematical idea of composing numbers was becoming taken-as-shared.

Composing numbers was also found to take place in tasks on day three. At the beginning of the day, the students revisited transactions with the candy factory scenario. In this problem, they were given a picture of 2 rolls and 5 pieces and wanted to add 16 more candies. Katrina claimed the solution was 4 rolls and 3 pieces or 43 pieces.

Katrina: Okay, we started with 2 rolls and then we added another roll from the new 16 pieces. *(Data)*

Instructor: Add one roll, 10 pieces, from 16 new pieces *(Data)*

Katrina: And then we put the other 6 pieces, the individual pieces because she made 16 new pieces of candy to the other side. And we all made another roll which contained 10 pieces to make it a little easier to see, giving us 4 rolls and 3 pieces or 43 pieces. *(Data)*

Katrina composed 1 roll from 10 pieces. First, she took 10 pieces from the 16 that were added and made a roll. Then she took the 6 pieces and the 5 original pieces and recognized that would make another roll with 3 pieces left over. This was a presentation of her group’s work, indicating that her group agreed with her representation and claims that were supported by her data. The warrant of why 10 pieces became one roll was no longer required, indicating that unitizing 10 was indeed a taken-as-shared mathematical idea. Additionally, students did not request a warrant
as to why 16 pieces could be partitioned into 10 pieces and 6 pieces, giving further indication that composing and decomposing was a taken-as-shared mathematical idea. When the students were transitioned back into base ten, similar situations were encountered with base ten blocks, and composing and decomposing numbers was further established as a taken-as-shared mathematical idea, as shown in the following episode. This took place on day four of the instructional sequence and involved adding two base ten numbers together using base ten blocks. The problem given was to add 28 and 39 (base ten) using base ten blocks. Most students made 28 with 2 rods (rolls) and 8 pieces and 39 with 3 rods and 9 pieces. Lilly composed 10 by solving this problem in this way:

Lilly: I got five rolls and then I made 10 pieces into one more roll. (Data)

Instructor: How did you get five rolls?

Lilly: The sticks.

Instructor: So you had the … 3 sticks and 2 sticks.

Lilly: Yes.

Instructor: Okay.

Lilly: And then I got the pieces and I made one more roll out of the pieces. (Data)

Instructor: Okay, from what pieces?

Lilly: From the 8 pieces and 2 from the 9. (Warrant)

Instructor: Okay.

Lilly: And then I had seven in pieces. And then I added them all together and got 67. (Claim)

Lilly composed one ten from ten ones in a flexible way, taking pieces from one number and giving them to the other to compose ten. This was similar to the compensation strategies that were developed using the 10 frames as a tool. She moved pieces from one set to match with another set in order to make 10. In another problem in this same episode, students were asked to add 238 and 174 (in base ten).
David: Well, first I did the boxes, and then I did pieces, and then I did rolls altogether. *(Data)*

Instructor: So when you combined the boxes,

David: I had three boxes.

Instructor: And when you combined the pieces, how many pieces did you have?

David: I had 2 left over and made a roll. *(Claim)*

Instructor: But before you exchanged it, how many pieces did you have when you combined them together?

David: What I did with it was I had eight in my hand and then added the four so what I did was I held my fingers like this and those six I could add to make a roll and had two left. *(Data)*

Instructor: So you never did have eight individual pieces and four individual pieces? You never did have 12. So you went straight to recording rolls.

David: So… I had a roll I took to the side and I had two, so I put two down there. Then I had an extra roll, and I added it to all the other things and I had 11 extra, so I put that down and then I left one and changed it to [a box].

David used the pieces from the 238 together with some of the pieces from 174 to make a new roll. He composed one roll out of ten pieces. This was no longer questioned as a legitimate mathematical activity, confirming that the mathematical idea of composing numbers was taken-as-shared. The model of boxes, rolls, and pieces reappeared in the context of base ten, as many students referred to the base ten blocks by the names boxes, rolls, and pieces. The tool of boxes, rolls, and pieces had given the students a common tool to use which carried over from base eight into base ten. The mathematical ideas of composing and decomposing numbers combine to create the classroom mathematical practice of *flexibly representing numbers.*

**Reasoning About Operations**

As the mathematical practices of unitizing and flexibly representing numbers became established, the classroom mathematical practice of *reasoning about operations* began to be established. This practice involved seeing the inverse relationships of addition and subtraction or
multiplication and division; however it also involved seeing and using relationships between addition and multiplication or subtraction and division. This mathematical practice was supported by phases three and four of the HLT, operations within base eight and base ten representations and alternative strategies. The mathematical ideas that supported the classroom mathematical practice included subtraction as the distance between and using one operation to reason about another.

**Subtraction as the Distance Between**

One operation that was examined extensively was that of subtraction. The idea of addition as joining two objects seemed to be taken-as-shared from the very beginning of the teaching experiment. No one questioned that adding two sets of objects together implied joining the sets. Subtraction, however, was not as easily described and was often represented in different ways. The take away model for subtraction was the most commonly used, but other models including distance between became taken-as-shared throughout the course of the instructional sequence. Much of this reasoning began with an activity which used the blank number line as a tool to solve addition and subtraction problems in base eight. An early problem was that of subtracting 52 minus 23 with the blank number line. Darren said

Darren: I started at 52. Subtract 10. Subtracted another 10. Subtracted another 3 to get to 27.

This was clearly a take away model in that he was starting with the total, 52, and taking away 23 to see where he ended up. This model was very common in the explanations that students gave
for subtraction. Less frequently used, but just as important, was the model of the distance between. This began to be established with Kim solving this same problem in the following way.

**Kim: (Drawing on board)** So I started at the 23 and went up by 10’s until I got to 53. I knew it was one less than 53 because it was 52, so I took that one away from the 10 to get 7. And then 10, 10, and 7 is 27.

Kim reasoned that the answer was the distance between 23 and 52, so she counted up on the number line to find out how much space was between the numbers. She also recognized the relationship between addition and subtraction by adding on to 23 to get to 52 and compensated by jumping past her goal of 52 and coming back. Subtraction as the distance between was used more and more, and the justification of it became less and less warranted. When the students were transitioned back into base ten on day four, they were given two scenarios that involved compensation. The first was given as

A student did this to solve $47 - 29$ (base ten)

- $50 - 30 = 20$
- $20 - 4 = 16$

Is this correct? If so, why? If not, what is the error?

In this problem, the student’s error was determined to be improper compensation. The number line helped to the students to describe this in several ways.

**Instructor:** I saw someone (a student in the class) describing it like this and then they went away from it. They had a 29 here and a 47 here (on the number line). And they were focusing on this space (pointing to the space between 29 and 47) (Data). Why were they focusing on that space? They moved on before I could ask this question. What is important about this?

**Class:** That is the answer. *(Claim)*

**Instructor:** Sarah, what?

**Sarah:** That is your answer *(Claim).* That is the difference between them. That is what they are asking. *(Data)*
Instructor: So if this (pointing to the gap between the numbers) is the difference between them...So 29 plus 1 became 30 and 47 plus 3 became 50 and then I saw this. (Data) (drawing on the number line to show the change in the gap between the numbers) Where do we go with this?

April: What is important to notice is if you go from 29 to 30, you are basically shrinking the answer by 1 and

Instructor: This being our answer. (pointing to the bracket indicating the distance between the numbers)

April: Right. That is our answer, so by moving from 29 to 30, you are shrinking the answer by 1. (Claim)

Instructor: Without changing the 47.

April: Right. So you really need to add one back and then you are extending the answer by going up 3 so you really need to subtract those 3 out and if you think of it in those terms you will actually get the right answer. (Data)

The number line became a tool that was used in developing the mathematical idea that subtraction was finding the distance between the numbers. This idea became taken-as-shared as students further explored the Equal Compensation Algorithm (see Appendix B). The second compensation-type problem that students were given involved an algorithm called Equal Compensation (see appendix B). This immediately followed the prior compensation problem. The way this algorithm was presented was as a suggestion of what a child might have done. The students were asked to make sense of this compensation strategy, and the number line became a tool for doing that. Using the number line in this case helped to establish the mathematical idea that subtraction was seen as the distance between the two numbers. The number line had been introduced in the context of base eight and addition and subtraction word problems. With this
activity, students were able to reason with the number line in ways that supported its use for general mathematical activity.

In the Equal Compensation Algorithm, two numbers are being subtracted that require regrouping to use the standard subtraction algorithm. If the ones column requires regrouping, 10 ones is added to the “top number” and one 10 is added to the “bottom number” so that there are enough ones to take away. The same compensation happens in the tens column if regrouping is necessary. As opposed to regrouping within one number, the Equal Compensation Algorithm adds to both numbers, keeping the distance between them the same. In discussing this algorithm, students used the blank number line as a tool for explaining the reason the algorithm is correct. Students were explaining why the following was mathematically correct (see figure 10).

![Figure 10: Equal Compensation Algorithm (base ten)](image)

Kathy: You know that when you add the ten and you add the other ten you are going to shift up 20 spaces so you subtract the ten and the ten to get back to your original bracket. *(Claim)*

Katrina: If you put a piece of paper up there *(instructor then holds up piece of paper next to number line)*, you want to keep it the same borders, so if you are going to go up 10 with one, the bottom one has to go up 10 too to keep it the same. So like that is situated between 257 and 632 and you want to add ten that is going to cause the top to go up 10 too to keep it the same answer. *(Data)*

Instructor: So add ten here?

Katrina: So if you go up ten to the 267, that makes the top part go up too. *(Data)*

Instructor: When we did this, this one went up also *(pointing to 257 going to 267 and 632 going to 642)* *(Data).*
Katrina: To keep the same answer, you have to make sure you are making the same, *(Warrant)*

Instructor: So when we added 10 to our 632,

Katrina: You have to put it on the bottom too. *(Warrant)*

Joe: Or the paper rips. *(talking about the paper from before that was being shifted up and down on the number line)*

Instructor: And then,

Kathy: So you aren’t really adding ten, you are just shifting up ten spaces because you added ten on the bottom. *(Warrant)*

The students came to agreement on the use of the number line for making sense of these problems and the idea that subtraction could be modeled as the distance between the numbers. As students reasoned about the first scenario and made sense of using the number line in that case, the mathematical understanding began to be developed. Then students were able to apply that mathematical understanding to a new scenario in order to make sense of a different type of problem and reasoned about the algorithm using the mathematical idea that subtraction was the distance between the numbers. What was originally a claim was then used as a warrant, making the mathematical idea of *subtraction as the distance between* a taken-as-shared idea. The compensation strategies made sense in light of this mathematical idea. Other operations were also discussed and the relationships between operations were used in reasoning mathematically.

**Using One Operation to Reason About Another**

The relationships between operations are important in mathematical understandings. Multiplication and division concepts rely on addition and subtraction concepts and, in fact, build on student understandings of addition and subtraction. The inverse relationships between addition and subtraction as well as multiplication and division are important for students to
recognize and use in explaining and justifying mathematical ideas. Also of importance are the relationships between addition and multiplication as well as subtraction and division. Understandings related to multiplication and division rely on initial understandings related to addition and subtraction, so recognizing and understanding the relationships between operations is of importance in developing operational fluency.

During day one of the instructional sequence, students were engaged in a task involving adding and subtracting using a blank number line. They were given the problem 52 – 23 to solve on the number line. Two students solved this problem in different ways. Darren’s solution was

Darren: I started at 52. Subtract 10. Subtract another 10. Subtracted another 3 to get to 27.

Kim’s solution was to do the following.

Kim: (Drawing on board) So I started at the 23 and went up by 10’s until I got to 53. I knew it was one less than 53 because it was 52, so I took that one away from the 10 to get 7. And then 10, 10, and 7 is 27.

Darren used a take away model of taking 23 away from 52, one of the more common models used by children. Kim used a type of compensation strategy where she started at 23 and counted up to 52. There was some question initially about the two strategies; however, the class determined that both were legitimate. Students continued to use addition as in the example above to solve what was viewed to be a subtraction problem. This began to lay the foundation for developing relationships between operations and using one operation to reason about another.

Although the relationship between addition and subtraction was important to examine, even more interesting relationships between the operations developed as this mathematical practice continued to be established. The ways in which students reasoned about these relationships provided for interesting explanations and justifications. Students used addition and
subtraction knowledge to reason about multiplication and division and this type of reasoning became taken-as-shared through the course of the instructional sequence.

The research team determined that an initial task for multiplication in base eight should be using the 10 frames to support students grouping objects multiplicatively (see Appendix H). Student thinking was notated as they reasoned about how to determine how many dots. Initial reasoning with the groups of 10 frames included recognizing addition as related to multiplication. In this episode on day 3 of the instructional sequence, students were shown four groups of 4 dots and asked to determine how many dots total. The class determined the solution was 20, and Laura explained

Laura: I added them…. (Data)

Instructor: Sorry. How did you know this equaled 20?

Laura: I just added the four plus four plus four plus four. (Warrant)

Instructor: So you saw the four groups of four as 4 + 4 + 4 + 4. How did you know that was 20?

Laura: 4 plus 4 is 10 and then the other 4 plus 4 is 10 and 10 plus 10 is 20. (Warrant)

Laura solved the multiplication problem of 4 times 4 by adding four plus four plus four plus four. The instructor for the course supported the connection between this task and multiplicative reasoning by notating on the board the groups of objects multiplication problem associated with the dot frames that were shown. For Laura’s example, it was notated that 4 x 4 = 20, making connections between the dot frames and multiplication. This task continued and students were shown different configurations of groups of 10 frames. The sequence was intentionally designed so that subsequent problems could build on prior solutions. For example, the first configuration shown was 4 groups of 4 dots. The next configuration was 5 groups of 4 dots. The students were able to reason that this was just one more group of four, as April did.

April: We had already figured out there were 20 and then just added 4 so 20 + 4. (Data)
She recognized the fact that she could just add another set of 4 to what had already been determined to be 4 times 4 and used addition as a strategy for determining the solution to a multiplication problem. This same type of reasoning continued throughout this task and student’s explanations were accepted as legitimate by the class. In one case, Joe concluded that 6 groups of 6 was 6 groups of 5 plus 1. The other students challenged his conclusion as follows:

Joe: From the other one (5 groups of 6). We had 6 groups of 6 and I just added the extra one. (Claim1)

Instructor: So you had five groups of six plus one (Data1)? How many of you did it that way? Anyone have questions? Matt?

Matt: How come you only added one because it is a whole set of six, so you have to add six instead of one to the answer we had last time, so I think the answer should be 36 plus six and in base eight world that is 44. (Claim2)

Joe: Yeah.

Instructor: Why?

Joe: Because in base eight, like he said it would be 44. And I am still stuck in base ten.

Instructor: I am curious where this one came from (in the 5 groups of 6 plus 1).

…

Kathy: I think he added one being the number of groups rather than a six being the number in each group. (Claim3)

Joe: Yeah, that’s right.

Kathy: So you could use that problem above as five times, five groups times six in each group plus I have an extra group so you could make it, you could make it 6 times 1. (Data3)

Instructor: Is it six groups of one in each group?

Kathy: No, it’s 1 times 6 and then you get 36 plus 6 equals 44. (Data3)

…

Matt: I think he did it in base ten world, 5 times 6 is 30 and six times six would have been 36 and he just added one because it is one more than the last one. Maybe he got stuck in base ten world? (Claim4)

Instructor: It is the one that I am not sure about. Kathy, you want to talk to Joe a little?

Kathy: Would you like me to explain it to you? [The instructor] labeled it above as the number of groups which was the first part and there are five groups and six dots in each group, so that is why when we added that extra group, we should reveal that. She had one whole group which is the number of groups, one group and there is six dots in that group. (Warrant3)
Instructor: So you are talking about this right here (the sixth group of six dots) being one more group with six dots. One group of six dots. (Warrant3)

The fact that the student could add groups of six to get the solution to the multiplication problem was taken-as-shared in this case. The students did not question his strategy of adding groups of six and using addition to solve a multiplication problem. Students used this mathematical idea of adding groups, which Laura previously used as a warrant, as data to convince their fellow student why 6 groups of 6 was 5 groups of 6 plus 1 group of 6 as opposed to 5 groups of 6 plus 1. The classroom mathematical practice of unitizing came into play here as well in that the unit was the group size, six, and that unit had to be iterated. By saying 5 groups of 6 plus 1, the group size was not iterated in the last set. Warrants were not requested to explain why 5 groups of 6 was the same as 6 plus 6 plus 6 plus 6 plus 6, and students used another student’s reasoning to support their own conclusions, thus making the mathematical idea of the relationship between addition and multiplication taken-as-shared, contributing to the classroom mathematical practice of reasoning about operations. The tasks and ways of reasoning students used continued in base ten as it related to the relationship between addition and multiplication, further establishing this as a taken-as-shared mathematical idea. In this episode at the beginning of day four of the instructional sequence, students were reasoning about a division problem, but working through multiplication. There was some question of how much was 10 times 12. Matt claimed that the answer was 120.

Matt: It is the same as if we added a box we knew that would be 100 candies, if there was 10 candies in each roll and 10 rolls in each box that would be 100 candies, so we can just do it the same way we do it now and say if we had 10 rolls of 12 candies that would be 120 and then there would be 5 left over (the answer to the original division problem of 125 divided by 12). Instead of having to do it 10 times, we can just multiply instead of saying 12 plus 12 plus 12, 10 times. (Data)

Instructor: Questions?
Matt: Because multiplication works the same way as addition, especially if you have 10 it makes it easy. *(Backing)*

Matt’s backing here that multiplication works the same way as addition was originally a claim made by students that you could repeatedly add numbers for multiplication. This switch of role from claim to backing provided evidence that the mathematical idea of using addition to reason about multiplication was taken-as-shared. This was further established as a taken-as-shared mathematical idea in base ten when students modeled multiplication problems using base ten blocks and a groups of objects meaning. This episode took place on day five of the instructional sequence when multiplication in base ten was introduced with the problem 4 x 12 (base ten).

Instructor: Lilly, how did you solve this?

Lilly: I got four rolls and eight pieces. *(Claim)*

...  

Instructor: How did you do that?

Lilly: I added the rolls together and all the pieces together. *(Data) Pieces and rolls*

Instructor: So it sounds like you made groups of objects, meaning four groups of 12 objects each. How many of you used that groups of objects meaning? I noticed that this group is not raising your hands, what did you do?

Amy: 12 groups of 4.

Instructor: 12 groups of 4. Is that a groups of objects meaning?

Amy: Yeah.

Instructor: So there is 12 groups of 4 and what did you do with those?

Kathy: Added them together.

Instructor: How?

Amy: 4, 8, 12,

Instructor: So you counted up until you got to 48?

Students used as data that they repeatedly added to find 4 groups of 12 or 12 groups of 4. They recognized the mathematical idea of reasoning about multiplication using addition as taken-as-shared and used it in their explanation of their solution. Additionally, Lilly’s composing and
decomposing within rolls and pieces was not questioned, giving further substantiation that flexibly representing numbers was a classroom mathematical practice.

As students developed multiplication strategies, they also explored division strategies and began to recognize the relationship between subtraction and division. They were given division story problems, some of which represented sharing (partitive) division and some of which represented measurement (quotitive) division. In examining a quotitive division problem, students reasoned by repeatedly subtracting. In this episode on day four of the instructional sequence, students were given the following problem in base eight:

Sarah has 125 candies. She wants to give each of her friends 12 candies. How many friends can she share with? Does she have any candies left for herself? If so, how many?

Students reasoned through the problem in many different ways, one of which was

April: Okay. What I did was I figured we are going to start with 125 and we want to give each one 12 candies and the question was how many friends can she share with and did she have any left for herself and if so, how many. So my feeling was I wanted to basically take away 12 every time because you don’t want to give a friend a partial amount of candies. So I took 125 and subtracted 12 from it and then I basically wrote out here this is one friend. (Backing) And then from here, I just subtracted the ones column here and got 3 and the tens and got 1 and the hundreds just comes down like that, basically like we did yesterday, so that was one friend. And then I took what I had left which was 113 and subtracted another 12 and this is another friend and then keeping them like this, 3 minus 2 is 1, 1 minus 1 is 0 and the hundreds come down. And I basically kept going like that until I didn’t have a whole lot, one whole portion of 12 left. (Data)

\[
\begin{array}{c}
125 \\
-12 \\
\hline
113 \\
-12 \\
\hline
101
\end{array}
\]

(1 friend) (1 friend)

Instructor: So how many times did you have to do this?

April: Basically I got a total of 10 friends ...

April: Well, I got a total of 10 friends with 5 left over. (Claim)

Lilly questioned April’s answer.

Lilly: I didn’t get that answer.

Instructor: So then do you have a question for April?
Lilly: Yeah, how did you end up with only 10 friends?

April: 10 friends? When I got this one I got 55 and then 55 minus 12 and I got 43 and 43 minus 12 is 31… and since there was only five left I couldn’t subtract 12 again, so that is 1, 2, 3, 4, 5, 6, 7, 10. *(Warrant)*

<table>
<thead>
<tr>
<th>125</th>
<th>-12</th>
<th>(1 friend)</th>
<th>67</th>
<th>-12</th>
<th>(1 friend)</th>
<th>31</th>
<th>-12</th>
<th>(1 friend)</th>
</tr>
</thead>
<tbody>
<tr>
<td>113</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>113</td>
<td>-12</td>
<td>(1 friend)</td>
<td>55</td>
<td>-12</td>
<td>(1 friend)</td>
<td>17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>101</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>101</td>
<td>-12</td>
<td>(1 friend)</td>
<td>43</td>
<td>-12</td>
<td>(1 friend)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>67</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>31</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The student was asked to provide a warrant for her solution process of subtracting 12 each time, questioning the relationship between division and subtraction. The warrant included the continuation of the process and demonstrated what she counted to get the answer, which was the number of times she took away 12, substantiating her claim that she could subtract groups of 12 to solve the division problem of 125 divided by 12. Matt responded to April’s method with a more efficient solution strategy that still involved subtracting.

Matt: I did 10 times 12 and that was 120. So that went in exactly 12 times *(Data)*, so she could give 10 friends, and then there was 5 candies that didn’t go in evenly *(Claim)*, so she had five candies left over.

In this case, the mathematical reasoning of subtracting a given number of groups in one chunk was not questioned. The warrant from April’s argument as to why she could subtract groups of 12 had dropped off in Matt’s argument, so the relationship between subtraction and division began to be taken-as-shared. In another scenario, a child wanted to give 14 stickers to each friend and had 236 stickers to give away. Kathy reasoned this way.

Kathy: You know he wants to give them 14 stickers each and I just picked a random number, well it wasn’t really random. We had talked about it at our table. So I subtracted 60 from 236 and I also kept a tally since I was subtracting 60, that is four times. And I saw that I had to take away,
regroup and I just kept going until I got down to 16 and I knew that I couldn’t subtract 60 so I
subtracted 14 and had 2 left. (Data)

Kathy’s reasoning about division as subtraction was not questioned by the students in the class.

A warrant was not requested for this claim and data, despite the warrant being requested in
April’s case. The warrant had dropped away, providing further evidence that this mathematical
idea was becoming taken-as-shared. These reasoning strategies continued as the students
transitioned back into base ten. When given the task to divide 758 by 63 (base ten), students
reasoned through using subtraction as it related to division.

Instructor: Lilly, I want you to share that first way you started doing it that you got frustrated
because it took so much time.

Lilly: Oh, I was subtracting 63 from 758. (Data)

Instructor: So let’s take a look at this. So what I saw on your page was that you had 758 and you
had minus 63. and then what?

Lilly: 695.

Instructor: Why would you subtract 63 if we are trying to divide 758 by 63?

Lilly: To see how many times 63 will go into 758. (Warrant)

Instructor: Any questions for Lilly?

Matt: Wouldn’t that take a really long time because,

Lilly: That is why I stopped.

Instructor: And that is exactly what Lilly said, she did this maybe four or five times and I said
what happened and she said, well, it would work, but it would take a really long time, Matt’s
words exactly,

Lilly: And a lot of paper.

Instructor: And then you moved on to looking at other things. I noticed that, April, you did
something similar to this, but I don’t think it took so much time. Can you come up and share it?

April: We had the same idea that Lilly had, is that you want to see how many times 63 goes into
758. (Data) So my first idea was I am going to subtract 63 and so we were talking as a group and
Matt was the one who said, well, if we take 63 times 10, that would be 630, so if we take 758
minus 630, which is basically 10 63s we get 128 and then we can kind of guess in our head that’s
2 63’s so we take 126 which is 2 63’s and we get 2 left, so we had 10 63s here and 2 63s here and
we added them and got 12 63s remainder of 2. (Claim)
Lilly was able to provide a warrant for why you could subtract groups of 63 to solve the division problem of 758 divided by 63. April then used Lilly’s explanation of subtracting groups of 63 to justify how she solved the same problem. Finding the solution to a division problem by subtracting shifted from a warrant to data, further substantiating that the relationship between division and subtraction was taken-as-shared. This mathematical idea, together with using addition to reason about multiplication, contributed to the establishment of the mathematical practice of reasoning about operations.

**Conclusion**

The classroom teaching experiment methodology and emergent perspective provided the basis for analysis of the data in this study. The data were analyzed to determine social and sociomathematical norms and classroom mathematical practices that developed over the course of the classroom teaching experiment. The classroom mathematical practices were established through the whole class discussions which took place throughout the instructional sequence and facilitated student’s understandings of place value and operations. The tools that were used were often initially introduced by the instructor of the course, but were later used by students independent of the instructor’s influence.

The social norms which were established in this classroom environment were that (a) students were expected to explain and justify their solutions and solution processes and (b) students were expected to make sense of each other’s solutions and solution strategies by asking questions. These social norms supported the sociomathematical norms of what counted as a different solution and what made a good explanation. Data did not support the sociomathematical
norms being taken-as-shared ways of participating in the classroom, although there was some indication that students understood the criteria for what counted as a different solution and what made a good explanation. The fact that students expected each other to explain their solutions provided the need for the sociomathematical norm of what counted as a good explanation, although there was some indication that students felt obligated to provide a good explanation for the purposes of achieving a specific grade in the class.

The classroom mathematical practices that emerged included 1) unitizing, 2) flexibly representing numbers, and 3) reasoning about operations. The ways in which the instructional sequence supported these classroom mathematical practices as well as modifications to both the HLT and the instructional sequence for further iterations will be discussed in the final chapter.
CHAPTER FIVE: CONCLUSION

Using design-based research methodology, this study used qualitative research methods to document a classroom teaching experiment. The research focus was two-fold, including the ways in which the social context of the classroom and the instructional sequence and subsequent revisions facilitated preservice teachers’ development of place value and operation concepts. The social and sociomathematical norms were negotiated cooperatively by both the instructor for the course and the students. The social norms that were established in this classroom were that of 1) explaining and justifying one’s solution and solution processes and 2) making sense of other students’ solutions by asking questions of classmates or the instructor. The sociomathematical norms that were sought to be established through the teaching experiment were determining the criteria of 1) what constituted a different mathematical solution and 2) what made a good explanation. The sociomathematical norm of what counted as a different solution included the criteria of using different operations or differing amounts of written or mental work involved. These criteria were taken-as-shared by the collective classroom and supported the findings of Yackel (2001) related to classroom social and sociomathematical norms. The sociomathematical norm of what made a good explanation was not fully established in this classroom, but student written explanations were detailed enough that it appeared that students understood the criteria that made a good explanation. It was not clear, however, if students created good explanations because they agreed with the social and sociomathematical norms of explaining and justifying and criteria for a good explanation or because they wanted to please the instructor and were concerned about their grades in the course. The norms of the classroom contributed to the ways in which the classroom mathematical practices were established. The classroom mathematical
practices that were established in this classroom were (1) unitizing, (2) flexibly representing numbers, and (3) reasoning about operations. The development of these classroom mathematical practices was determined and analyzed using Toulmin’s (1969) argumentation scheme as a basis for determining when mathematical ideas became taken-as-shared. The mathematical activity present in the classroom when ideas became taken-as-shared was then examined to determine the classroom mathematical practices that had been established.

The ways in which the hypothetical learning trajectory (HLT) and instructional sequence were implemented and modified was an important aspect of examining the classroom structure and in determining the next iteration of the HLT and instructional sequence, keeping with the tenets of design-based research. A second research focus of this study was examining the ways in which the HLT and instructional sequence facilitated the collective student learning with regards to place value and operations. The revisions to the HLT and instructional sequence which took place in this iteration were examined as they related to design-based research methodology and the collective student learning in the classroom. Changes that were made to the instructional sequence during the classroom teaching experiment facilitated the establishment of classroom mathematical practices. The actual learning trajectory and instructional sequence which was realized with the collective classroom community and how it related to or was different from the hypothetical learning trajectory upon which the research team based decisions related to the instructional sequence is an important conclusion that will be discussed here.
The hypothetical learning trajectory and instructional sequence underwent revisions as a result of implementation in this classroom teaching experiment. The initial hypothetical learning trajectory for this study which the instructional sequence was based was discussed in chapter 3, and is shown here in table 12.

**Table 12: Initial Hypothetical Learning Trajectory**

<table>
<thead>
<tr>
<th>HLT Phase</th>
<th>Learning Goal</th>
<th>Supporting Tasks for Instructional Sequence</th>
<th>Supporting Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase One</td>
<td>Counting objects within base eight and unitizing</td>
<td>Counting objects; Skip counting; Dot Frames; Empty Number Line</td>
<td>Snap Cubes; 10 frames; Number Line</td>
</tr>
<tr>
<td>Phase Two</td>
<td>Flexible representations of numbers in base eight</td>
<td>Candy Factory Scenario involving: Estimating; Packing and Unpacking Candy; Inventory Forms</td>
<td>Pictorial representations of boxes, rolls, and pieces; inventory forms</td>
</tr>
<tr>
<td>Phase Three</td>
<td>Operations within base eight</td>
<td>Candy Factory Transactions; Inventory Forms; Dot Frames; Dot Arrays</td>
<td>Pictorial representations of boxes, rolls, and pieces; inventory forms; 10 frames; Snap Cubes</td>
</tr>
<tr>
<td>Phase Four</td>
<td>Base ten representations and alternative strategies</td>
<td>Base-ten Blocks; Arrays; Number Lines</td>
<td>Base-ten blocks; Number Line; Dot Arrays</td>
</tr>
</tbody>
</table>

The hypothetical learning trajectory was developed with children’s developmental progressions in mind and was built off a hypothetical learning trajectory developed for children (Bowers, 1996; Bowers et al., 1999; McClain et al., 1998; Yackel & Karnes, 1992, 1993). Through prior research (McClain, 2003) and the pilot for this study, it was determined that preservice teachers could progress through similar developmental progressions as children when supported by instructional tasks that were based on children’s development, but were placed in
base eight instead of base ten. The developmental progressions of children, then, were important to examine. With these progressions in mind, the classroom mathematical practices were examined in light of the developmental progressions of children.

The first classroom mathematical practice that was identified was that of unitizing. This was related to the hypothetical learning trajectory goal of counting objects in base eight and flexibly representing numbers in base eight. In order to flexibly represent numbers, one must recognize that one 10 is 10 ones simultaneously, which is viewing 10 as an abstract or iterable unit (Cobb & Wheatley, 1988; Steffe, 1983). The conceptual development of unitizing 10 was important for the preservice teachers to develop. This was supported by tasks involving developing counting sequences, 10 frames, the blank number line, and the initial candy shop tasks. It was determined that the 10 frames should be moved in the sequence to immediately follow the counting sequences in order to better assist students in the development of 10 as an abstract or iterable unit. This supported the developmental progressions in that they were given tasks involving small, single digit addends with sums less than 20 before they were asked to develop counting strategies with the blank number line. The instructional sequence as it was implemented here attempted to jump the students to counting efficiently with multidigit numbers before allowing them the experiences needed to assist in the development of units of 10 with smaller numbers. This change was supported by the developmental progressions seen in children as they learn concepts of ten (Cobb & Wheatley, 1988; Ross, 1986; Steffe, 1983) and supported the preservice teachers’ conceptual development of 10. The HLT learning goal was then modified to be that of unitizing and counting efficiently in base eight and the instructional sequence which supported that hypothetical learning trajectory was altered as a result of this
iteration to place tasks in a more developmentally appropriate order, namely including the 10 frames tasks before the blank number line.

The second mathematical practice that was established was that of flexibly representing numbers. This mathematical practice was related to both phase two, flexible representations of numbers in base eight, and phase three, operations within base eight, of the initial HLT. The mathematical practice of flexibly representing numbers was established alongside the first mathematical practice of unitizing. This substantiates findings of Stephan and Rasmussen (2002) that classroom mathematical practices can develop simultaneously as opposed to consecutively as had been determined in previous classroom teaching experiments (Bowers, 1996; Bowers et al., 1999; Cobb et al., 2001). The tasks which supported the mathematical practice of flexibly representing numbers included tasks involving the candy factory scenario with reorganizing pictorial and abstract configurations of boxes, rolls, and pieces and beginning transactions with problems set in the candy shop scenario. The transactions were given in such a way that the students needed to reorganize the candy in order to either take away candy that was sold or add candy that was made. The tasks to supporting this learning goal were selected with children’s development of place value and initial whole number operations in mind. Research has shown that children need to develop place value understandings related to composing and decomposing numbers, which leads to flexibly representing numbers (Ross, 1986). Children who were unable to represent numbers using fewer unit blocks than necessary had not developed flexibility in representing numbers. This was a critical developmental step for future development of whole number operations. Addition and subtraction with two-digit numbers was introduced while composing and decomposing numbers was still developing, as supported by Fuson’s (1990) research. Situations were initially given in the scenario of the candy shop, which supported
students need for modeling situations in developing concepts of whole number operations (Fuson, 1990) and were in line with the Realistic Mathematics Education (RME) instructional theory. The preservice teachers tended to follow the developmental progressions in which children progress by initially counting all the objects by ones, then counting on from one addend, and eventually creating groups of 10 (Cobb & Wheatley, 1988; Steffe, 1983). The tasks supporting the hypothetical learning trajectory goal and classroom mathematical practice of flexibly representing numbers in base eight supported these conceptual developments and the preservice teachers were found to progress through similar developmental levels with respect to adding and subtracting whole numbers, a concept that depends greatly on flexibility in representing numbers. There were few changes that were made to this section of the hypothetical learning trajectory and supporting instructional sequence. The preservice teachers’ conceptual development was supported by the tasks as they were given and the progression of tasks supported their learning in the ways that were expected.

The third mathematical practice that was established in this classroom teaching experiment was reasoning about operations. This mathematical practice was supported by phases three and four of the hypothetical learning trajectory, operations within base eight and base ten representations and alternative strategies. The initial instructional sequence and hypothetical learning trajectory used in this study included returning to base ten to develop alternative strategies for whole number operations. This was determined to be unnecessary for the preservice teachers’ conceptual development of operations. The development of these two classroom mathematical practices could have been supported by remaining in base eight and discussing alternative strategies for whole number operations in base eight. In fact, it was hypothesized that students’ reasoning about whole number operations may have been facilitated
even more by staying in base eight for the entire sequence. The initial reasons for returning to base ten were to give the students opportunities to examine children’s thinking and alternative strategies for whole number operations in an attempt to develop some initial aspects of pedagogical content knowledge. The research team discussed these goals and determined that examining children’s thinking and alternative strategies could still be done while remaining in base eight. The classroom mathematical practice of reasoning about operations was supported by tasks in both phase three and phase four of the hypothetical learning trajectory. Students reasoned about operations in both base eight and base ten by 1) viewing subtraction as the distance between, 2) relating addition with subtraction, 3) relating addition with multiplication, and 4) relating subtraction with division as was discussed in chapter four. These concepts began to be developed in base eight and continued to be applied in base ten. Subtraction as the distance between was supported primarily by tasks in base ten, namely examining children’s compensation strategies and examining alternative subtraction algorithms including the Equal Compensation Algorithm discussed in chapter four. This could have been accomplished in base eight as well as base ten, so returning to base ten was eliminated from the hypothetical learning trajectory for future iterations of this instructional sequence.

At the conclusion of this study, these changes and results were taken into account in modifying the hypothetical learning trajectory. The new hypothetical learning trajectory is presented in table 13.

Table 13: Revised Hypothetical Learning Trajectory for Place Value and Operations

<table>
<thead>
<tr>
<th>HLT Phase</th>
<th>Learning Goal</th>
<th>Supporting Tasks for Instructional Sequence</th>
<th>Supporting Tools</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase One</td>
<td>Unitizing and Counting objects efficiently</td>
<td>Counting strategies and representations; 10 dot frames</td>
<td>Snap Cubes; 10 dot frames; Blank Number Line</td>
</tr>
</tbody>
</table>
The instructional sequence was also revised to accommodate the changes to the hypothetical learning trajectory and the changes that were made throughout the implementation of the instructional sequence including the reorganization of tasks supporting phase one of the HLT and examining children’s thinking and alternative strategies in base eight instead of base ten.

**Implications and Future Research**

This study sought to add to the research base related to preservice teachers’ development of profound understanding of place value and operations (Ma, 1999; McClain, 2003). Research into children’s learning of place value and operations was important for the development of the hypothetical learning trajectory and instructional sequence used in this study, as demonstrated in chapter two. Although there is research indicating that preservice teachers are lacking in their conceptual understandings of place value and whole number operations, little has been suggested for ways in which to improve that understanding. This study sought to present one way in which preservice teachers’ collective development of a profound understanding of place value and operations was supported through the use of a hypothetical learning trajectory and instructional
sequence implemented using a design-based research methodology in a classroom teaching experiment. This study has implications for determining effective ways of teaching preservice elementary education students mathematics for understanding. The classroom environment and the ways in which it supported collective classroom learning were presented here.

Although the ways in which the social aspects of the classroom supported the development of classroom mathematical practices were described here, in order to gain a full picture of the preservice teachers’ learning and understanding of place value and operations, the individual student learning must also be examined. Through the lens of the emergent perspective, this study sought to describe the social side of the interpretive framework. The psychological side, which relies on the social influence, has yet to be examined in detail. That analysis must be done before a full picture of preservice teachers’ profound understandings can be realized. This study is just the first step in that process. Further analysis and research must be completed in order to determine the individual student learning that is supported by not only the instructional sequence and hypothetical learning trajectory, but also the social and sociomathematical norms and classroom mathematical practices identified here. This meets the final criteria that Cobb (2003) cited that “the results from the analyses should feed back to improve the instructional designs” (p. 11).

Additionally, the preservice teachers in this course are also required to complete a mathematics methods course which focuses on effective methods of teaching mathematics at the elementary school level. The structure of the methods course differs from the structure of the content course studied here. There is potential for redeveloping the methods course to better align with the content course and the ways in which students have come to expect learning to take place. The course studied here focused on developing the mathematics content in specific
ways, but not on teaching elementary school students using similar methods. Additionally, internship experiences may need to be addressed to better connect the ways in which the preservice teachers have been taught mathematics for understanding and the ways in which they are expected to teach in their internship experiences. There may be a disconnect between students desiring to teach in different ways and the support they may need to do so. This disconnect may need to be addressed.

**Conclusion**

The hypothetical learning trajectory was realized in many ways. The students were able to unitize and create efficient counting strategies in base eight. This was supported by tasks involving the use of 10 frames and the blank number line. As students developed concepts of 10, they began to compose and decompose numbers, creating flexible representations for numbers. The students realized this goal as well and demonstrated fluency with representing numbers in base eight and base ten in different ways. This led to operational fluency in base eight and the development of alternative strategies for operations in base ten. It was concluded that the entire sequence may be better implemented to include the alternative strategies in the base eight scenario. The tasks of the instructional sequence supported the development of classroom mathematical practices, namely unitizing, flexibly representing numbers, and reasoning about operations. Student learning was supported by the tasks provided through the instructional sequence and the classroom community supported the development of classroom mathematical practices related to place value and operations. The results of this study have implications for
how preservice teachers may be taught elementary school mathematics content to better prepare them to be educators.
APPENDIX A: SAMPLE ACTIVITIES FROM INSTRUCTIONAL SEQUENCE
Candy Shop 1

You own a candy shop in Base 8 World. Candy comes packaged in boxes, rolls, and individual pieces.

<table>
<thead>
<tr>
<th>Box</th>
<th>Roll</th>
<th>Piece</th>
</tr>
</thead>
</table>

There are 10 candy pieces in a roll and 10 rolls in a box.

Use this information to complete the following:

1. Show two different ways to represent the following:

2. Give two different ways to represent 426 candies.
There are 51 total candies.
Broken Machine

Ms. McLaughlin operates a machine that puts 10 sticks of gum in a pack.

1. How many sticks of gum are in 5 packs?

The machine breaks down and now only puts 7 sticks in a pack.

2. How many sticks of gum are in 5 packs?
3. How many sticks of gum are in 7 packs?

After Ms. McLaughlin tries to fix the machine, she makes a mistake and now the machine only puts 6 sticks in a pack.

4. How many sticks of gum will be in 5 packs?
5. How many sticks of gum will be in 6 packs?

1. Ricky rabbit jumps 5 inches every time he takes one hop. For each of the following problems below, show two different ways a child might solve the problem. Explain each way thoroughly. Remember that these are in the 8 world.

   a. How many inches has Ricky traveled if he takes 5 hops?
   b. How many inches has Ricky traveled if he takes 6 hops?
   c. How many inches has Ricky traveled if he takes 3 hops?
   d. Ricky has traveled 24 inches already. How many more hops must he take to travel a total of 43 inches?
   e. Ricky has jumped some already. He takes 3 more hops. If he travels 55 inches altogether, how many hops did he take to start with?
Division Word Problems

1. Katrina brings 52 marbles to school to give to her friends. She plans to give each of 10 friends the same number of marbles. How many marbles will each friend get? Will Katrina have any marbles left? If so, how many?

2. Jason has 43 pencils to share with some of his class. There are 5 students in his class that he would like to give his pencils to. How many pencils does each friend get?

3. Sarah has 125 candies. She wants to give each of her friends 12 candies. How many friends can she share with? Does she have any candies left for herself? If so, how many?

4. Micah has some friends he wants to share his stickers with. He has 236 stickers. How many friends can he share them with if he wants each friend to get 14 stickers? How many stickers, if any, does Micah have left?
APPENDIX B: ALTERNATIVE ALGORITHMS
The algorithm can be done from right to left or from left to right.

Left to right:

\[
\begin{array}{cccc}
2 & 8 & 6 \\
+ & 8 & 5 & 7 \\
\hline
1 & 0 & 0 & 0 \\
1 & 3 & 0 & \text{(80 + 50)} \\
1 & 3 & \text{(6 + 7)} \\
\hline
1 & 1 & 4 & 3
\end{array}
\]

Right to left:

\[
\begin{array}{cccc}
2 & 8 & 6 \\
+ & 8 & 5 & 7 \\
\hline
1 & 3 & \text{(6 + 7)} \\
1 & 3 & 0 & \text{(80 + 50)} \\
1 & 0 & 0 & 0 \\
\hline
1 & 1 & 4 & 3
\end{array}
\]

Figure 11: Partial Sums Algorithm

\[
\begin{array}{ccc}
2 & 5 & 7 \\
+ & 8 & 5 \\
\hline
2 & 13 & 12 & \text{Sum of each column} \\
2 & 14 & 2 & \text{Regroup the 1 in 12 with the 13 in column 2} \\
3 & 4 & 2 & \text{Regroup the 1 in 14 with the 2 in column 2}
\end{array}
\]

Figure 12: Column Addition
Figure 13: Partial Differences Algorithm

\[
\begin{array}{cccc}
6 & 4 & 1 & 7 \\
-4 & 3 & 6 & 9 \\
\hline
\end{array}
\]

\begin{array}{ll}
-2 & 7 - 9 \\
-5 & 10 - 60 \\
1 & 400 - 300 \\
2 & 6000 - 4000 \\
\hline
2 & 0 & 4 & 8 \\
2000 + 100 - 50 - 2
\end{array}

Figure 14: Equal Compensation Algorithm

\[
\begin{array}{cc}
12 \\
6 & 3 & \underline{2} \\
- & \underline{2} & \underline{5} & 7 \\
\hline
& 5 & 12 - 7
\end{array}
\]

\textbf{Step 1}

\[
\begin{array}{cc}
13 & 12 \\
6 & \underline{3} & \underline{2} \\
3 & 6 \\
- & \underline{2} & \underline{5} & 7 \\
\hline
7 & 5 & 13 - 6
\end{array}
\]

\textbf{Step 2}

\[
\begin{array}{cc}
1 & 1 \\
6 & 3 & 2 \\
3 & 6 \\
- & \underline{2} & \underline{5} & 7 \\
\hline
3 & 7 & 5 & 6 - 3
\end{array}
\]

\textbf{Step 3}
The algorithm can be done from right to left or from left to right.

Right to left:

```
2 4
X 4 2
8 2 x 4
 4 0 2 x 20
 1 6 0 40 x 4
 8 0 0 40 x 20
```

```
1 0 0 8
```

Left to right:

```
2 4
X 4 2
8 0 0 40 x 20
 1 6 0 40 x 4
 4 0 2 x 20
 8 2 x 4
```

```
1 0 0 8
```

Figure 15: Partial Products Algorithm

Problem: $2845 \div 45$

$2845 - 45 = 2800$ (one group of 45)
$2800 - 450 = 2350$ (ten groups of 45)
$2350 - 45 = 2305$ (one group of 45)
$2305 - 45 = 2260$ (one group of 45)
$2260 - 225 = 2035$ (five groups of 45)
$2035 - 45 = 1990$ (one group of 45)
$1990 - 90 = 1900$ (two groups of 45)
$1900 - 900 = 1000$ (twenty groups of 45)
$1000 - 900 = 100$ (twenty group of 45)
$100 - 90 = 10$ (two groups of 45)

**Answer:** $1 + 10 + 1 + 5 + 1 + 2 + 20 + 20 + 2 = 63$ remainder 10

Figure 16: Partial Differences Algorithm
APPENDIX C: EXPLANATION EXAMPLE
Problem: There are 5 members of the Green family. Each person eats an apple at least 5 times a week. How many apples does the Green family eat in a year?

Explanation 1: 1300. Because I worked it out with multiplication.

Explanation 2: 5x5 is 25. 25 x 52 is 1300.

Explanation 3: I did 5x5 because there are 5 people and each ate 5 apples, so the family ate 25 apples a week. Then, I did 25x52 because there are 52 weeks in a year and I wanted to find how many apples they ate in one year. 25 apples a week times 52 weeks is 1300 apples a year.
APPENDIX D: IRB APPROVAL FORM
THE UNIVERSITY OF CENTRAL FLORIDA
INSTITUTIONAL REVIEW BOARD (IRB)

IRB Committee Approval Form

PRINCIPAL INVESTIGATOR(S): Janet Andreasen
Debra Wheeldon
IRB #: 05-2548

PROJECT TITLE: Preservice Elementary School Teachers' Understanding of Number and Operation through the Use of an Emerging Instructional Sequence

[X] New project submission  [ ] Resubmission of lapsed project #
[ ] Continuing review of lapsed project #  [ ] Continuing review of #
[ ] Study expired ______  [ ] Initial submission was approved by expedited review
[ ] Initial submission was approved by full board review but continuing review can be expedited
[ ] Suspension of enrollment email sent to PI, entered on spreadsheet, administration notified

Chair
[X] Expedited Approval
Dated: 6 APRIL 2005
Cite how qualifies for expedited review: minimal risk and ______

Signed: ____________________________
Dr. Sophia Dziegielewski

[ ] Exempt
Dated: ____________________________
Cite how qualifies for exempt status: minimal risk and ______

Signed: ____________________________
Dr. Jacqueline Byers

[ ] Expiration
Date: 5 APRIL 2006
[ ] Waiver of documentation of consent approved
[ ] Waiver of consent approved

NOTES FROM IRB CHAIR (IF APPLICABLE): Researcher will verify via e-mail all participants are 18 and older.

Received verification from PI and supervisor 4/8/05 6w
APPENDIX E: INFORMED CONSENT LETTER
May 1, 2005

Dear Student:

We are graduate students at the University of Central Florida. As part of our coursework, we are conducting a study, the purpose of which is to investigate the ways in which elementary preservice teachers understand the number and operation strand in elementary school mathematics. We are asking you to participate in this study because you have been identified as a student in one of the elementary mathematics content courses at UCF. Researchers will observe and videotape the Elementary School Mathematics course (MAE 2801). Selected groups may also be audiotaped during class discussions.

Selected students will be asked to participate in several interviews lasting no longer than 30 minutes each. You will not have to answer any question you do not wish to answer. The interviews will be conducted at your convenience on campus after we have received a copy of this signed consent from you. With your permission, we will also audiotape these interviews.

Only we will have access to the audio and video tapes, which may be professionally transcribed, removing any identifiers during transcription. The tapes will be kept in a locked file cabinet until the completion of this study. They will then be erased. After we have received a copy of this signed consent from you, you will be asked to complete a questionnaire about your mathematics content knowledge at the beginning and the end of the course. Your name will not appear on the questionnaires, but a unique code will be used for identification purposes. Only the researchers will have access to the identification codes. Copies of your course assignments may be used as data for this study. Additionally, video tape segments may be used in presentations and/or publications related to this study. Your identity will be kept confidential and will not be revealed in the final manuscript(s) or any related presentations.

There are no anticipated risks, compensation or other direct benefits to you as a participant in this study. You are free to withdraw your consent to participate and may discontinue your participation in the study at any time without consequence.

If you have any questions about this research project, please contact Janet Andreasen at (407) 823-5430. Our faculty supervisor is Dr. Juli K. Dixon. Questions or concerns about research participants’ rights may be directed to the UCFIRB office, University of Central Florida Office of Research, Orlando Tech Center, 12443 Research Parkway, Suite 207, Orlando, FL 32826. The phone number is (407) 823-2901.

Please sign and return one copy of this letter. A second copy is provided for your records. By signing this letter, you give us permission to videotape you and report your responses anonymously in the final manuscript(s) to be submitted to our faculty supervisor as part of our coursework. You also give us permission to use videotape segments as a part of related publications and/or presentations.

Sincerely,

Janet Andreasen         Debra Wheeldon

I have read the procedure described above for this research study.

I voluntarily agree to participate in the study.

I voluntarily agree to be videotaped in MAE 2801

I voluntarily consent to have videotape segments be used in publications and/or presentations.

/  

Participant         Date

186
APPENDIX F: SAMPLE ARGUMENTATION LOG
<table>
<thead>
<tr>
<th>Claim</th>
<th>Data</th>
<th>Question</th>
<th>Warrant</th>
<th>Backing</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Base Eight 10 Frames</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Class: Five</td>
<td>I saw three and then two.</td>
<td>And when you saw three and you saw two, what did you do with those?</td>
<td>I added them</td>
<td></td>
</tr>
<tr>
<td>Class: Seven.</td>
<td>I saw three rows of two and one more. Well, actually I saw one blank on the bottom.</td>
<td>What did you do with that one blank at the bottom? How did you know that gave you seven?</td>
<td>Because we are in base eight.</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>I saw 4 in the first one and four more in the other, but in a different shape, and I just added them</td>
<td>So you just saw it and how did you count really fast?</td>
<td>I counted one at a time.</td>
<td>1, 2, 3, 4 and then 5, 6, 7, 10</td>
</tr>
<tr>
<td>What do you see? How did you see 11?</td>
<td>I saw the four in the first one. And then there was another four with an extra one.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Claim</td>
<td>Data</td>
<td>Question</td>
<td>Warrant</td>
<td>Backing</td>
</tr>
<tr>
<td>----------------</td>
<td>-------------------------------------------</td>
<td>-----------------------------------</td>
<td>-------------------------------------------------------------------------</td>
<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>I saw it in a picture.</td>
<td>What do you mean by that?</td>
<td>They way they were set up, sort of how she did it, but I didn’t count. I just saw the four in the square, the other four made a square, and there is one left over. So how she did it, but I looked at it as a picture, and I counted it after I figured out what kind of a shape it made.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Class: 10</td>
<td>I saw the three on one side and four and one on the other side.</td>
<td>You paused for a moment, what were you doing when you paused?</td>
<td>I was adding them up in my head.</td>
<td></td>
</tr>
<tr>
<td>Class: 11</td>
<td>I just remembered the dots and counted them after you turned it off.</td>
<td>Okay, how did you count them after I turned it off? Count out loud for me like you did.</td>
<td>Well, I counted normally and then I You counted normally and then I went back because it was not in base ten. So I had to go back.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>how might you have counted this without going back to base ten?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Claim</td>
<td>Data</td>
<td>Question</td>
<td>Warrant</td>
<td>Backing</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td>----------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>Just know that that was six and three on the other side and that was 11.</td>
<td>How did that give you 11? Do you just know that $6 + 3$ is 11 or was there counting involved?</td>
<td>There is no eight or nine, so you don’t count those, then you just go up. Count out loud. How would you have done this?</td>
<td>1, 2, 3, 4, 5, 6, 7, 10, 11</td>
<td></td>
</tr>
<tr>
<td>I did three sets of three. And I know that $3 \times 3$ is 9, but not nine, because it’s base eight, so 11.</td>
<td>What could you have done without translating? Using the thinking you used?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>You know you have six on the right side. And you just say there is half of that on the left side, so you just say 3 and 6 is 11 in base eight.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I saw the 6 and then I did 7, 10, 11. After 6 I started counting.</td>
<td>So you counted up from 6. Then 7, 10, 11. How did you know when to stop?</td>
<td>There were only three numbers there.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Claim</td>
<td>Data</td>
<td>Question</td>
<td>Warrant</td>
<td>Backing</td>
</tr>
<tr>
<td>--------------</td>
<td>----------------------------------------------------------------------</td>
<td>--------------------------------------------------------------------------</td>
<td>-------------------------------------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>Class: 11</td>
<td>I moved the dot that was on the far left over to the empty space over on the right, so we had the same shape we had last time</td>
<td>Where did you use your fingers?</td>
<td>Well, I took the five and then I added four to it. So I was just, 6, 7, 10, 11.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>I saw the first one as four and the second one as 5 and then I used my fingers a lot</td>
<td>The four in what was described earlier as the shape of sevens?</td>
<td>No, I saw the two patterns. I saw the left side pattern and the right side pattern with just one left over.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>I saw two patterns of four and then I added that last one to make 11.</td>
<td>And then how did you figure out that was 10?</td>
<td>Well in our world it would be eight, but in their world it is 10.</td>
<td></td>
</tr>
<tr>
<td>Was it 10?</td>
<td>I thought I saw 3 on this side and 5 on that side.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>You could have brought the one on the right over to the left.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>I counted three then four then one more.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Claim</td>
<td>Data</td>
<td>Question</td>
<td>Warrant</td>
<td>Backing</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td>----------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>Class: 15</td>
<td>On the first box, they were all full, so we know that is 10. On the second box, one whole side was full which was four plus one and I got 15</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>I saw one filled, so that is 1 and the other was 5 so 15.</td>
<td>Why wouldn’t that be six?</td>
<td>Well, I just thought one, because, I am thinking e in Spanish is and, so I see one e five</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>I did the two…The first three rows and then the other side was one.</td>
<td>Okay, so then when you saw the three rows, what did that mean?</td>
<td>I had to count saw. I started with 6 and then. 6, 7, 10, 11, 12, 13, 14 plus 1.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>I counted across the two top rows so it represented 10. Then I counted the four and plus one.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>I moved two over from the right side to the left side to make 10 and there was one left over the four, so 15.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Claim</td>
<td>Data</td>
<td>Question</td>
<td>Warrant</td>
<td>Backing</td>
</tr>
<tr>
<td>-------</td>
<td>---------------</td>
<td>----------</td>
<td>------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>The way I saw it was counting the blank spots, and then 20 minus 3.</td>
<td>So, the two 10 frames make 20 and then you took away three from that. How did you figure out the answer?</td>
<td>I just counted backwards.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Okay, so, 20, 17, 16, 15.</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX G: MATHEMATICAL IDEAS CHART
5/19/05

**10 frames**

<table>
<thead>
<tr>
<th>Taken-as-shared</th>
<th>Keep an eye on</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adding to find total number</td>
<td></td>
<td>10 Frame introduced as tool</td>
</tr>
<tr>
<td>of dots</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adding from larger addend</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Filling 10 Frames to make 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Taking away from full frames – subtracting</td>
</tr>
<tr>
<td></td>
<td></td>
<td>from more than you have</td>
</tr>
</tbody>
</table>

**Candy Shop Estimating**

<table>
<thead>
<tr>
<th>Taken-as-shared</th>
<th>Keep an eye on</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting by 10s as strategy,</td>
<td></td>
<td>Snap Cube Sticks introduced as tool</td>
</tr>
<tr>
<td>adding 10 without counting</td>
<td></td>
<td>Number Line as tool</td>
</tr>
<tr>
<td>up by ones (10 + 10 is 20)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>100 pieces in a box</td>
<td>Boxes, Rolls, and Pieces introduced as tool</td>
</tr>
</tbody>
</table>

**Candy Shop Configurations**

<table>
<thead>
<tr>
<th>Taken-as-shared</th>
<th>Keep an eye on</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unpacking and</td>
<td>Unpacking and</td>
<td>Number Line as Tool to add</td>
</tr>
<tr>
<td>repacking</td>
<td>repacking</td>
<td></td>
</tr>
<tr>
<td>boxes and rolls</td>
<td>boxes and rolls</td>
<td></td>
</tr>
<tr>
<td>to reconfigure</td>
<td>with rolls and</td>
<td></td>
</tr>
<tr>
<td>arrangements</td>
<td>pieces with</td>
<td></td>
</tr>
<tr>
<td></td>
<td>pieces</td>
<td></td>
</tr>
<tr>
<td>Counting to</td>
<td>Adding by place</td>
<td>Boxes Rolls and Pieces</td>
</tr>
<tr>
<td>verify addition</td>
<td>values – adding</td>
<td></td>
</tr>
<tr>
<td>facts</td>
<td>rolls with rolls</td>
<td></td>
</tr>
<tr>
<td></td>
<td>and pieces with pieces</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Unpacking rolls to take away</td>
<td></td>
</tr>
</tbody>
</table>

**Inventory Forms**

<table>
<thead>
<tr>
<th>Taken-as-shared</th>
<th>Keep an eye on</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 box can be</td>
<td></td>
<td>Boxes, Rolls, Pieces</td>
</tr>
<tr>
<td>100 pieces or</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 rolls</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.
APPENDIX H: SAMPLE 10 FRAMES FOR MULTIPLICATION
LIST OF REFERENCES


