Propagation Effect In Inhomogeneous Media, Including Media With Light-induced And Fixed Gratings

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PROPAGATION EFFECT IN INHOMOGENEOUS MEDIA, INCLUDING MEDIA WITH LIGHT-INDUCED AND FIXED GRATINGS

by

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Optics and Photonics at the University of Central Florida Orlando, Florida

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Major Professor: Boris Ya. Zeldovich
ABSTRACT

Optical waves propagation in various types of volume gratings, materials with constant impedance and optical fibers are studied. Instability of cross–phase modulation and of Energy transfer via GRON-type (Grating-type Orientational Nonlinearity in Liquid Crystal) Stimulated Scattering is numerically observed. Two diffractive optical elements made of volume gratings are suggested and analyzed. A transmission hologram based on the analogy with Stimulated Raman Adiabatic Passage (STIRAP) in nonlinear optics is proposed. This transmission hologram demonstrates high diffraction efficiency and low sensitivity to polarization and hologram strength. The other is a reflection hologram with two crossed-gratings. It features good angular selectivity in comparison with the poor angular selectivity of conventional Bragg grating mirror. This defense also contains the approximation of Maxwell equations for the description of depolarized light sources and polarization-insensitive detectors. A scalar wave equation, Z-Helmholtz equation, is proposed and discussed in the approximation of constant impedance media. As examples, this equation successfully describes a) Fresnel transmission coefficient, and b) Goose-Hanschen shift in total internal reflection, for depolarized incident light and, at the same time, polarization-insensitive detectors. Evolution of polarization during light propagation in an inhomogeneous locally isotropic medium, and also in a single-mode fiber is described by Rytov’s non-rotation equation. With arbitrary chosen real unit vector, the complete description of polarization change can be described in a single rotation angle obtained from the integral of rotation rate. Based on introduction of this reference frame, a device is suggested as rigid body’s rotation sensor due to polarization change in a twisted fiber.
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There are some good friends I know from different countries during my study in US. Thank them, especially Ion and Mihaela, for a lot of help and enriching my life.
In memory of my grand parents

蔡其章 先生 和 蔡何彩蓮 女士
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<td>GRON</td>
<td>Grating-type Orientational Nonlinearity Definition of Acronym</td>
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<tr>
<td>STIRAP</td>
<td>Stimulated Raman Adiabatic Passage</td>
</tr>
<tr>
<td>RCWA</td>
<td>Rigorous Couple Wave Analysis</td>
</tr>
<tr>
<td>SVEA</td>
<td>Slowly Varying Envelope Approximation</td>
</tr>
<tr>
<td>LC</td>
<td>Liquid Crystal</td>
</tr>
<tr>
<td>GO</td>
<td>Geometric Optics</td>
</tr>
<tr>
<td>ZLS</td>
<td>Zeldovich, Libermann, and Savchenko</td>
</tr>
<tr>
<td>NLC</td>
<td>Nematic Liquid Crystal</td>
</tr>
<tr>
<td>RK4</td>
<td>Runge-Kutta 4th order</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary Differential Equation</td>
</tr>
<tr>
<td>PTR</td>
<td>Photo-Thermo-Refractive</td>
</tr>
<tr>
<td>TIR</td>
<td>Total Internal Reflection</td>
</tr>
<tr>
<td>Z-H</td>
<td>Zeldovich-Helmholtz</td>
</tr>
<tr>
<td>TE</td>
<td>Transverse Electric</td>
</tr>
<tr>
<td>TM</td>
<td>Transverse Magnetic</td>
</tr>
<tr>
<td>a.u.</td>
<td>Arbitrary Unit</td>
</tr>
<tr>
<td>$M$</td>
<td>Hologram strength</td>
</tr>
<tr>
<td>$Q$</td>
<td>Grating Vector</td>
</tr>
<tr>
<td>$e$</td>
<td>Polarization Vector</td>
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<tr>
<td>$\varepsilon$</td>
<td>Dielectric Permittivity</td>
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<td>$\mu$</td>
<td>Magnetic Permeability</td>
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<td>Symbol</td>
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<td>$n$</td>
<td>Refraction Index</td>
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CHAPTER ONE: INTRODUCTION

The light propagation in continuous media has been the main issue of study in optics. Seeking the solutions of Maxwell equations in electrodynamics closely related to the development of optics. A lot of effort in scientific literature [1-3] has built a great foundation as the research resources in optics. One interesting topic is the wave propagation in periodic media. This research leads to many types of diffractive optical elements, including spectroscopy, pulse compressor, beam splitter, multiplexer, scanning devices; the application of gratings is almost everywhere in optics. The early grating theory from the primitive Rayleigh’s scalar expansions, the later development of modal methods [4-9], to integral of Green functions [10-11], always try the attempt to find the solutions of diffracted and transmitted fields. Among all, the couple-wave theory [12] has far reaching influence in the progress of various methods of wave solution in gratings, such as RCWA [13-18] etc.

The technique in deriving the couple-wave theory has the main concerns in this dissertation. We try to design and analyze both two types of transmission and reflection volume hologram based on this approach. These two holograms have merits compared to conventional holographic elements. Before going that, the gratings induced in LC (liquid crystal) by the giant nonlinearity due to rotation of molecules is explored in the first chapter. This work successfully describes how we achieved the beam combining through the Stoke’s waves in such light-induced gratings. Unusually instability was observed for the two-wave’s coupling in the process.
of energy transfer from the pumping beam into the signal beam in the presence of stimulated scattering. This result eventually goes more deep discussion about this phenomenon [19].

The next interesting topic is considering the light propagation in particular inhomogeneous media, in which the impedance is a constant. The reflection behaviors of TE and TM polarized light incident on objects are fully understood in optics. However, the solutions of each wave equation require separately discussion due to the nature of boundary conditions in Maxwell equations. Further observation shows that, in case of constant impedance, the boundary conditions for TE and TM waves will become indistinguishable. This gives us hint that we can approximate the Maxwell vectorial wave equations by a scalar wave equation characterized by constant impedance. We call this new scalar wave equation: Z-Helmholtz equation. We had used this equation to calculate the Fresnel reflections and Goose-Hanschen shift to describe the unpolaried light. The comparison between exact and our approximate results are given in chapter four. The criterion for validity of Z-Helmholtz equation will also be discussed.

As the final part of this dissertation, we study the polarization evolution along the trajectory of a ray in a spatially inhomogeneous, but locally isotropic medium under the approximation of Geometric Optics (GO). Rytov’s work of non-rotation of polarization is reviewed and compared with the ZLS equation derived by Zeldovich, Libermann, and Savchenko [20-23] directly from Maxwell equations in the GO regime. With introduction of a constant vector, a new reference frame is set up and the polarization change after a propagation time can be obtained by calculating the rotation of the reference frame. It simply requires the direct integration of rotate rate of the frame. An application of this scheme is to design a rotation sensor by calculating the
polarization change in a twist fiber attached to a rigid body. The linear relation between the body rotation angle and the polarization has been numerically verified in this chapter.

This dissertation is organized in the following way. Starting from chapter two, it includes five parts of research work. They are: 1) Instability of cross–phase modulation and of energy transfer via GRON-type (Grating-type Orientational Nonlinearity in Liquid Crystal) stimulated scattering. 2) Adiabatic three-wave symmetric volume hologram. 3) Reflection double grating with high angular selectivity. 4) Z-Helmholtz theory: approximate solutions of Maxwell equations in layered media. 5) Evolution of polarization along light propagation in GO approximation: Rytov’s non-rotation law. In the last chapter, the summary and discussion of this dissertation are addressed.
2.1 Coherent beam combining technology to generate high power beam

Many present-day solid-state lasers can generate very large CW power, especially in the regime of Master Oscillator – Power Amplifier (MO-PA) scheme. The task of combining individual beamlets into one high-power beam of diffraction quality is therefore quite important [24-26]. One of the ways of beam combining and clean-up has been suggested [27]. It is based on the previous work of Dr. Zeldovich [28-29] by use of Stimulated Orientational Scattering in a Nematic Liquid Crystal (NLC) [30-31]. Recent experiments demonstrated that high conversion efficiency can be achieved by this scheme [27]. The work here we are going to show in this chapter is devoted to numerical and analytic study of such beam combining.

We first explored the cross-phase modulation of the pumping beam and signal beam by solving the time-dependent coupled equations numerically. The result showed an unexpected instability of power transfer along the longitudinal spatial direction before it reached the steady state. The solutions were verified by comparison with results of analytic consideration of perturbation theory.
2.2 The Stimulated Orientational Scattering in Nematic Liquid Crystal

In nematic liquid crystal, one-dimensional (+ time) model of orientational scattering is described by the coupled-wave equations for amplitudes \( A \) and \( B \) of waves of two polarizations. Here a strong pump wave \( A \) with degraded spatial amplitude and phase distortions, and a coherent high-quality Stokes-shifted weak signal \( B \), are simultaneously incident into the NLC.

The wave equations through nonlinear orientational coupling are:

\[
\frac{\partial A(z,t)}{\partial z} = i\hbar \theta^*(z,t)B(z,t) \tag{1}
\]

\[
\frac{\partial B(z,t)}{\partial z} = i\hbar \theta(z,t)A(z,t) \tag{2}
\]

\[
\frac{\partial \theta(z,t)}{\partial t} + \Gamma \theta = A^*(z,t)B(z,t) \tag{3}
\]

Here \( 1/\Gamma \) is the relaxation time of the grating \( \theta(z, t) \) of molecular orientation in NLC, and the constant \( h \) determines the strength \( \mu \) and \( \nu \) (1/meter) of cross-phase modulation for the pairs of waves \( A_0 \) and \( B_0 \) of the same frequency. That is, if we look at the solution of steady state, then, from eq.(3)

\[
\theta = \frac{A^*(z,t)B(z,t)}{\Gamma} \tag{4}
\]

\[
\frac{\partial A(z,t)}{\partial z} = i\hbar \frac{|B(z,t)|^2}{\Gamma} A(z,t) = i\nu A(z,t), \quad \nu = h \frac{|B(z,t)|^2}{\Gamma} \tag{5}
\]
\[
\frac{\partial B(z,t)}{\partial z} = i\hbar \frac{|A(z,t)|^2}{\Gamma} B(z,t) = i\mu B(z,t), \quad \mu = \hbar \frac{|A(z,t)|^2}{\Gamma}
\] (6)

However, if we tune the frequency of signal wave \(B\) with an optimally Stoke-shift \(\Omega\), the same quantities \(\mu\) and \(\nu\) will determine the gain coefficients (1/meter, with respect to intensity) in the present of the opposite-polarized pumps. And the optimum frequency shift is \(\Omega_{opt} = \Gamma\). The derivation is as the following, with pump and signal beams:

\[
A(z,t) = e^{-i\omega t} A(z), \quad B(z,t) = e^{-i(\omega-\Omega)t} B(z)
\] (7)

Then from equation (3) we have,

\[
\theta(z) = \frac{\Gamma - i\Omega}{\Gamma^2 + \Omega^2} A^*(z)B(z)
\] (8)

\[
\frac{\partial}{\partial z} |A|^2 = \frac{\partial}{\partial z} AA^* = A^* \frac{\partial}{\partial z} A + A \frac{\partial}{\partial z} A^* = -\frac{2\Omega}{\Gamma^2 + \Omega^2} |B|^2 |A|^2 \equiv -g_A |A|^2, \quad g_A (at \Omega = \Gamma) = \nu
\] (9)

\[
\frac{\partial}{\partial z} |B|^2 = \frac{\partial}{\partial z} BB^* = B^* \frac{\partial}{\partial z} B + B \frac{\partial}{\partial z} B^* = \frac{2\Omega}{\Gamma^2 + \Omega^2} |A|^2 |B|^2 \equiv g_B |B|^2, \quad g_B (at \Omega = \Gamma) = \mu
\] (10)

From the above eqs.(8-10), we can see that with a frequency Stoke’s shift of \(B\) respect to \(A\), the intensities of each wave for \(B\) growing up, for \(A\) going down. Thus, we have energy transfer in this case. More carefully observation shows that, from eqs.(9-10), the maximum value of \(g_{A,B}\) occurs at \(\Omega = \Gamma\). It means we can define the optimum frequency shift for the strongest gain coefficient when \(\Omega = \Gamma\). Moreover, for a detailed discussion, the above coupled equations can
easily be generalized to account the transverse \((x, y)\)-structure of the fields and diffraction effects.

### 2.3 The numerical observation of instability during the energy exchange process

For numerical solutions, we solved the system of \(z\)-equations (1-2) with the use of Runge-Kutta 4th order scheme, and the equation (3) for \(\frac{\partial \theta}{\partial t}\) via first order Euler scheme. The reason for the latter is its connection with the relaxation-type character of the equation for \(\frac{\partial \theta}{\partial t}\). As required by Runge-Kutta algorithm, the values of the grating \(\theta\) at the mid-points with respect to \(z\)-mesh were taken as arithmetic averages of \(\theta\)-values at the integer points of the mesh.

The numerical results show an unexpected instability occurred during the power transfer from the pump beam into signal beam along the longitudinal \(z\) direction of the NLC. This instability gradually goes away after passing some time and eventually reaches the steady state. The following are some of our results:
Example 1: Monochromatic waves, no frequency shift, no noise, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t_{\text{max}} = 25$.

Fig. 1: Dependence of the intensity of the “pump” wave $|A|^2$ on the distance $z$ at time $t$, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t = 25$.

Fig. 2: Dependence of the intensity of the “signal” wave $|B|^2$ on the distance $z$ at time $t$, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t = 25$. 
Example 2: B-signal is frequency-shifted at the input, $\Omega = x \cdot \Gamma$, $x = 1$, no noise, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma^{\tilde{\Theta}_{\text{max}}} = 25$.

Fig. 3: Dependence of the intensity of the “pump” wave $|A|^2$ on the distance $z$ at time $t$, $\Omega = x \cdot \Gamma$, $x = 1$, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t = 25$.

Fig. 4: Dependence of the intensity of the “signal” wave $|B|^2$ on the distance $z$ at time $t$, $\Omega = x \cdot \Gamma$, $x = 1$, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t = 25$. 9
Example 3: B-signal is frequency-shifted at the input, $\Omega = x \cdot \Gamma$, $x = 2$, no noise, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t_{\text{max}} = 25$.

Fig. 5: Dependence of the intensity of the “pump” wave $|A|^2$ on the distance $z$ at time $t$, $\Omega = x \cdot \Gamma$, $x = 2$, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t = 25$.

Fig. 6: Dependence of the intensity of the “signal” wave $|B|^2$ on the distance $z$ at time $t$, $\Omega = x \cdot \Gamma$, $x = 2$, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t = 25$. 
Example 4: B-signal is frequency-shifted at the input, $\Omega = x \cdot \Gamma$, $x = 2$, noise=0.1 (a.u.), $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t_{\text{max}} = 25$

Fig. 7: Dependence of the intensity of the “pump” wave $|A|^2$ on the distance $z$ at time $t$, $\Omega = x \cdot \Gamma$, $x = 2$, noise=0.1, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t = 25$.

Fig. 8: Dependence of the intensity of the “signal” wave $|B|^2$ on the distance $z$ at time $t$, $\Omega = x \cdot \Gamma$, $x = 2$, noise=0.1, $|A(z=0)|^2 = 0.99$, $|B(z=0)|^2 = 0.01$, $\Gamma t = 25$. 
One thing should remind the readers here. Instead of dynamic demonstration, the above presentations in (text and graph) dose not show all the vivid and rapid changes of the intensity in the region of energy transfer between $A$- and $B$-waves. Indeed, the dramatic oscillations of energy exchange were observed in the numerical experiment.

2.4 Analysis of instability of energy exchange by perturbation theory

Since the instability nature of solutions of differential equations is closely related to perturbation. We now verified our solutions by comparison with the results of analytic consideration of perturbation theory. As eqs.(4-6) describing two identical frequencies waves with cross-phase modulation, the un-perturbed solutions with small perturbations are suggested in the following form:

\[ A = A_0 e^{i\mu z} \cdot [1 + \alpha(z,t)] \]  \hspace{1cm} (11)  

\[ B = B_0 e^{i\mu z} \cdot [1 + \beta(z,t)] \]  \hspace{1cm} (12)  

\[ \theta(z,t) = (A_0 \cdot B_0 / \Gamma) \cdot e^{i(\mu-\nu)z} \cdot [1 + \psi(z,t)] \]  \hspace{1cm} (13).  

Here $\alpha(z,t)$, $\beta(z,t)$, and $\psi(z,t)$ are the complex amplitudes of small perturbations, and they satisfy: $|\alpha(z,t)|$, $|\beta(z,t)|$, $|\psi(z,t)| << 1$. Substitute the above eqs. (11, 12, 13) into eqs. (1, 2, 3). The resultant equations for $A$, $B$-wave and temporal evolution of grating reduce to
\[
\frac{\partial \alpha}{\partial z} + iv\alpha - iv\psi^* - iv\beta = 0 \quad (14),
\]
\[
\frac{\partial \beta}{\partial z} + i\mu\beta - i\mu\psi^* - i\mu\alpha = 0 \quad (15),
\]
\[
\frac{1}{\Gamma} \frac{\partial \psi}{\partial t} + \psi = \alpha^* + \beta \quad (16).
\]

Since the perturbation is both of amplitude and phase, we therefore can take the complex conjugate of the above linearization equations and treat them as system equations of six independent variables. Next, we seek the solution of these equations of the form:
\[(\alpha, \alpha^*, \beta, \beta^*, \psi, \psi^*) \propto e^{i\tilde{\Omega}t}.\]  Then from eq. (16) we have
\[
\psi = \frac{(\alpha^* + \beta)}{1 + i\tilde{\Omega}/\Gamma} \quad (17).
\]

Denote \(1/(1 + i\tilde{\Omega}/\Gamma) = D\), and substitute eq.(17) into eqs. (14, 15) and their corresponding complex conjugates, we arrive at the following equations,
\[
\frac{\partial \alpha}{\partial z} + iv\alpha - iv\beta - ivD(\alpha + \beta^*) = 0
\]
\[
\frac{\partial \alpha^*}{\partial z} - iv\alpha^* + iv\beta^* + ivD(\alpha^* + \beta) = 0
\]
\[
\frac{\partial \beta}{\partial z} + i\mu\beta - i\mu\alpha - i\muD(\alpha^* + \beta) = 0
\]
\[
\frac{\partial \beta^*}{\partial z} - i\mu\beta^* + i\mu\alpha^* + i\muD(\alpha + \beta^*) = 0 \quad (18).
\]
In a similar fashion, we look up the above $z$-dependence solutions in the form of $(\alpha, \alpha^*, \beta, \beta^*) \propto e^{i\dot{\lambda}z}$. Then the equations (18) reduce to the following set of algebraic equations:

\[
\begin{align*}
[\lambda + \nu(1 - D)]\alpha - \nu\beta - \nu D\beta^* &= 0 \\
[\lambda - \nu(1 - D)]\alpha^* + \nu D\beta + \nu\beta^* &= 0 \\
- \mu\alpha - \mu D\alpha^* + [\lambda + \mu(1 - D)]\beta &= 0 \\
\mu D\alpha + \mu\alpha^* + [\lambda - \mu(1 - D)]\beta^* &= 0
\end{align*}
\]  

Equalizing the determinant of the system (19) to zero, we get the characteristic equation, which turned to be surprisingly simple (with the above notation for $D$):

\[
\dot{\lambda}^2 \left[ \dot{\lambda}^2 - (\mu^2 + \nu^2)(1 - D)^2 - 2\mu\nu(1 - D^2) \right] = 0,
\]  

This gives us four eigenvalues,

\[
\begin{align*}
\Lambda_{1,2} &= 0 \\
\Lambda_{3,4} &= \pm \sqrt{(\mu^2 + \nu^2)(1 - D^2) - 2\mu\nu(1 - D^2)}
\end{align*}
\]  

We note that both $\hat{\Omega}$ and $\dot{\lambda}$ characterize the perturbation or the equivalent noise term in the process of coupling. Therefore, if the imaginary part of $\Lambda_3$ means the growth of noise, i.e. the cause of instability. When only one of the waves has large intensity, e.g. $\mu \gg \nu$, the imaginary part of $\Lambda_3$ describes the process of amplification of a weak signal $\beta$ in the presence of strong pump $|A_0|^2$. 

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In the figure 9, we will show the property of instability of the two monochromatic coupled waves.

\( GR(x) \equiv \text{Re} \{ \Lambda(\hat{\Omega}=\chi \Gamma) \} \) is cross-phase modulation of the perturbation;

\( GI(x) \equiv \text{Im} \{ \Lambda(\hat{\Omega}=\chi \Gamma) \} \) is the spatial growth coefficient of the perturbation

\[ \mu = |A_0|^2/\Gamma = 0.65, \quad \nu = |B_0|^2/\Gamma = 0.35 \]

Fig. 9: The blue line shows the real part of the eigenvalue of perturbation, which converges to 1 as \( x \) increase, showing constant phase modulation of perturbation; the red line shows the imaginary part, which approaches 0 with large \( x \), indicating the exponential decay of the perturbation amplitude.
The blue curve is the difference of actual cross-phase modulation, and its value for $\Omega = 0$ for no Stokes’s phase shift as we are in discussion of two identical frequency waves, as presented in the un-perturbed solution of eqs.(11-13). And we see it approaching to 1 when our $\hat{\Omega}$ reaches to $\Gamma$. That means the phase instability is settled.

On the other hand, the red curve represents the growth of the amplitude of the noise. At the same time, it approaches to 0 as the phase modulation of perturbation becomes constant. That means the instability or the noise will be sufficiently suppressed.

Now we can draw a mini-conclusion: even for the monochromatic pair of waves, there is temporal 1-dimensional instability! This work has been presented in the conference [32-34] and a further research on this topic has also been published [19].
CHAPTER THREE: ADIABATIC THREE-SYMMETRIC VOLUME HOLOGRAM

3.1 Conventional transmission volume hologram

Transmission volume holograms have a variety of applications, such as optical switching, filtering, multiplexing, imaging systems, spectral beam combining, etc. The diffraction efficiency of a single thick transmission grating is \( \eta = [\sin(M)]^2 \) [12, 35-36], which can reach 100%, if the hologram strength \( M = \kappa L = \pi n_1 L / \lambda_{\text{vac}} \) equals \( \pi / 2 \), or \( 3\pi / 2 \), or \( 5\pi / 2 \), etc. Here \( L \) is the effective interaction length, \( L = (e_A \cdot e_B) L_z / [\cos(\theta_{A,\text{med}})\cos(\theta_{B,\text{med}})]^{0.5} \), \((e_A \cdot e_B)\) is the polarization factor, and modulation of refractive index in the volume grating is assumed to be \( \delta n = n_1 \cos(Q \cdot R) \); \( \lambda_{\text{vac}} \) is the wavelength of light in vacuum, \( \theta_{A,B,\text{med}} \) are the angles of \( \mathbf{k} \)-vectors of \( A \)- and \( B \)-waves with z-axis inside the medium.

Disadvantage of this simple volume hologram is obvious, due to rather high sensitivity of \( \eta \) to the hologram strength \( M = \kappa L \). In practice, for the manufacturing concern, it may be rather difficult to achieve such maximum diffraction efficiency. For example, the precise control of recording grating strength and the accurate cut of the thickness of hologram.

We thus seek the possibility of finding a way to reduce this sensitivity of diffraction efficiency \( \eta \) upon the hologram strength \( M \). In this chapter, we propose a new scheme based on the analogy with nonlinear optical phenomenon, STIRAP: Stimulated Raman Adiabatic Passage,
[37], which allows efficient transfer of populations between two Raman sublevels via transition through third resonant level.

In the later sections, we will show the numerical results show of a particular coupling profile that preserves high diffraction efficiency without almost any dependence on the hologram strength, including the suppressed influence of polarization.

3.2 Three coupled waves in volume hologram via two adiabatic gratings

Since STIRAP allows efficient energy transfer for two Raman sublevels via transition through third resonant level. In applying this parallel thinking of STIRAP to holography, a third intermediate wave \( C \) needs to be introduced. And this intermediate wave \( C \) will interact both with the primary wave \( A \) and diffracted wave \( B \) to accomplish the energy transfer from \( A \) to \( B \). Therefore, two gratings are needed to have three coupled waves in our new volume hologram. The two gratings are characterized by the following modulation of index of refraction,

\[
\delta n = n_{CA}(z) \cos(Q_{CA} \cdot R) + n_{CB}(z) \cos(Q_{CB} \cdot R) \\
= 0.5\left[n_{CA}(z) \exp(iQ_{CA} \cdot R) + n_{CB}(z) \exp(iQ_{CB} \cdot R)\right] + \text{compl.conj.}
\]

(22)

Here \( C \) is the intermediate wave, as seen in the Figure 10. A similar discussion by Zhao et al. could be found in [38]*, which described the synthetic index grating consist of two volume index gratings with different wavevectors.
We present our field (for definiteness, of the TE polarization) in the form.

\[
\begin{align*}
E_{\text{real}}(R,t) &= E(R,t) + \left[ E(R,t) \right]^*; \\
E(R,t) &= \frac{\hat{e}_x}{2} \left[ \frac{a(z)}{\cos(\theta_{A,\text{med}})} \exp(ik_A \cdot R) + \frac{b(z)}{\cos(\theta_{B,\text{med}})} \exp(ik_B \cdot R) + \frac{c(z)}{\cos(\theta_{C,\text{med}})} \exp(ik_C \cdot R) \right] \\
&\times \exp[i\delta k_c x - i(\omega + \delta \omega)t]
\end{align*}
\]

(23)

where the parameters are, \( k_{A,B,C} = (\omega n_0/c)[e_x \sin(\theta_{A,B,C,\text{med}}) + e_z \cos(\theta_{A,B,C,\text{med}})] \), and

\[
\begin{align*}
\theta_{B,\text{med}} &= -\theta_{A,\text{med}}, & \theta_{C,\text{med}} &= 0, & Q_{CA} &= k_C - k_A, & Q_{CB} &= k_C - k_B
\end{align*}
\]

(24)

*In their research, a three wave coupled equation with pseudo phase matching also derived, the result analytic solution was found with initial conditions. However, the eigenmodes are less discussed. Therefore, the interesting dark mode in our following result was never mentioned in their case.*

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i.e. for the un-perturbed propagation directions \( \mathbf{k}_A, \mathbf{k}_B, \mathbf{k}_C \) (inside the medium of the hologram)
the Bragg condition is satisfied in assumption that the frequency detuning \( \delta \omega \) and angular detuning \( \delta \theta_{A,\text{air}} \) are zero. It should be noted also this particular configuration is \( A \parallel B \) symmetric, while the general non-symmetric case yields similar results.

The parameter \( \delta k_x \) is influenced both, by the frequency detuning \( \delta \omega \), and by the change of incidence angle \( \delta \theta_{A,\text{air}} \), in the following manner:

\[
\delta k_x = \frac{\delta \omega}{c} \sin(\theta_{A,\text{air}}) + \frac{\omega}{c} \cos(\theta_{A,\text{air}}) \delta \theta_{A,\text{air}}
\]  

(25)

The coupled wave equations are:

\[
\begin{align*}
\frac{da}{dz} &= i[\kappa_{CA}(z)]^*c(z) + i\alpha a(z), \\
\frac{db}{dz} &= i[\kappa_{CB}(z)]^*c(z) + i\beta b(z), \\
\frac{dc}{dz} &= i\kappa_{CA}(z)a(z) + i\kappa_{CB}(z)b(z) + i\gamma c(z),
\end{align*}
\]  

(26)

Here,

\[
\alpha = \left[ \frac{\omega}{c} (\xi - \mu) \frac{\delta \omega}{\omega} - \nu \delta \theta_{A,\text{air}} \right], \quad \beta = \left[ \frac{\omega}{c} (\xi + \mu) \frac{\delta \omega}{\omega} + \nu \delta \theta_{A,\text{air}} \right], \quad \gamma = \frac{\omega}{c} \left[ \frac{\delta \omega}{\omega} \right], \quad \mu = \frac{\sin^2 \theta_{A,\text{air}}}{n \cos \theta_{A,\text{med}}}
\]

\[
\nu = \frac{\sin \theta_{A,\text{air}} \cos \theta_{A,\text{air}}}{n \cos \theta_{A,\text{med}}}, \quad \xi = \frac{n}{\cos \theta_{A,\text{med}}}, \quad \kappa_{CA} = \frac{\omega}{2c \cdot [\cos(\theta_{A,\text{med}})]^{1/2}} n_{CA}, \quad \kappa_{CB} = \frac{\omega}{2c \cdot [\cos(\theta_{A,\text{med}})]^{1/2}} n_{CB}
\]  

(27)
The parameters characterizing the spectral and angular detuning from Bragg regime are $\alpha$, $\beta$, and $\gamma$, respectively, all of dimensions [1/meter], $\theta_{A,\text{air}} = |\theta_A| = |\theta_B|$ is the angle of incidence from air for both $A$- and $B$-waves, which are assumed to propagate symmetrically with respect to $z$-axis.

Analysis of the eigen-solutions of equation (26) of the form $e^{iAz}$ with constant values of $\kappa_{AC}$ and $\kappa_{BC}$ and under the condition of zero detuning ($\delta\omega/\omega = \delta\theta_{A,\text{air}} = 0$, i.e. $\alpha = \beta = \gamma = 0$) yields three eigenvectors. These three eigenmodes are illustrated in the following Figure 11.

![Diagram](image)

Fig. 11: The interpretation of three eigenmodes of a double-recorded volume hologram with constant coupling coefficients. The phase terms are $\sigma_\alpha = e^{i\Lambda L}$, $\sigma_\beta = e^{i\Lambda L}$, $\sigma_\gamma = e^{i\Lambda L}$. 

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That is, if we launch a wave that can be decomposed into three elements: $\alpha$, $\beta$, $\gamma$, after propagation through the hologram, the projection components to these eigenmodes remains constant except with additional phase terms, $e^{i\lambda_x}$, $e^{i\lambda_y}$, $e^{i\lambda_z}$, respectively. Let us substitute the eigen-solutions of the form $e^{i\Lambda z}$ into equation (26). This leads to the following characteristic equation:

\[
\begin{vmatrix}
-i\Lambda & 0 & i\kappa_{AC}^* \\
0 & -i\Lambda & i\kappa_{BC}^* \\
i\kappa_{AC} & i\kappa_{BC} & -i\Lambda
\end{vmatrix} = 0
\] (28)

The eigenvalues and eigenvectors are the following,

\[
\Lambda_1 = 0 \\
\Lambda_2 = +\sqrt{\kappa_{AC}^2 + \kappa_{BC}^2} \\
\Lambda_3 = -\sqrt{\kappa_{AC}^2 + \kappa_{BC}^2}
\]

\[
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} = \begin{bmatrix}
\frac{\kappa_{AC}^*}{\sqrt{\kappa_{AC}^2 + \kappa_{BC}^2}} & \frac{-\kappa_{AC}^*}{\sqrt{\kappa_{AC}^2 + \kappa_{BC}^2}} & \kappa_{BC} \\
\frac{\kappa_{BC}^*}{\sqrt{\kappa_{AC}^2 + \kappa_{BC}^2}} & \frac{-\kappa_{BC}^*}{\sqrt{\kappa_{AC}^2 + \kappa_{BC}^2}} & \kappa_{AC} \\
1 & 1 & 0
\end{bmatrix}
\]
The notion of eigenmodes is, strictly speaking, applicable to the case of the system of equations with constant coefficients only. However, if these coefficients change slowly with $z$, then one can expect adiabatic evolution, i.e. that the amplitude of each mode (not of a wave, but of a mode!) will be preserved in the process of propagation.

Therefore, since any wave can be expressed in terms of these eigenmodes, we thus tried to manipulate the eigenmodes components of $A(z), B(z)$ to obtain the desired output, i.e. high diffraction efficiency. By changing the coupling coefficient $\kappa$ along $z$, the projected components also change accordingly.

As mentioned above, to keep those eigenmodes not mixed when $\kappa$ varies, it needs adiabatic following process, i.e. $d\kappa/dz << |\kappa|^2$ such that eigenmodes are still eigenmodes.

### 3.3 Design of high diffraction efficiency independent on grating strength and polarization

If we take a careful look at the third eigenmode in the previous section, it may give us a clue to help our design. We first obtain the general form of the wave in terms of these eigenmodes, namely,
If there is no $C$ wave in the initial incident waves, then we can exclude the first two eigenmodes by requiring $d_1(z) = d_2(z) = 0$. In the adiabatic following process, say, we want the $A(z)$ convert into $B(z)$ at the end of hologram. It is obviously from the components of the third eigenmode that $\kappa_{BC}$ should be maximum at beginning, minimum at the end; on the contrary, $\kappa_{AC}$ should be minimum at beginning, maximum at the end.

This is essentially the counterintuitive STIRAP-type configuration requiring that the $BC$-grating is “turned on” closer to the input of $A$-wave, $AC$-grating is “turned on” closer to the output of $B$-wave, and two gratings do overlap in the hologram.

Potential advantage of this design is that in the assumption of the validity of adiabatic approximation, the efficiency of $A \rightarrow B$ power transfer is 100% and does not depend on the hologram strength $M$. In this case the values of $M_{CA}$ and $M_{CB}$ are defined as

\[
M_{AC} = \int \kappa_{AC}(z) dz \\
M_{BC} = \int \kappa_{BC}(z) dz
\]  

(30)
We have validated our “adiabatic” reasoning by direct numeric integration of the coupled equations for all the three waves $A$, $B$, and $C$. From our numerical experiments, a preferred profile of $\kappa_{BC}(z)$ and $\kappa_{AC}(z)$ was found,

$$\kappa_{AC}(z) = \frac{2M_{AC}}{L} \cdot \sin^{2}\left(\frac{\pi z}{2L}\right)$$

$$\kappa_{BC}(z) = \frac{2M_{BC}}{L} \cdot \cos^{2}\left(\frac{\pi z}{2L}\right)$$  \hspace{1cm} (31)

Now we would like to show some of our numerical results of such adiabatic-gratings. The dependence of the diffraction efficiency $\eta(A \rightarrow B)$ on the hologram strengths $M_{CA}$ and $M_{CB}$ in the intervals from 0 to 15 is depicted at the Fig. 12 for the case of perfect Bragg matching.

Our data show that diffraction efficiency $\eta$ is above 85%, when $(M_{AC}, M_{BC}) \sim (5.2, 5.2)$, and is above 97.85% when $(M_{AC}, M_{BC}) \sim (7.5, 7.5)$. Efficiency $\eta$ asymptotically approaches 100% for larger hologram strengths.

These results manifest the important feature of our scheme: dramatic reduction of the sensitivity of diffraction efficiency $\eta$ on $M$. Other profiles have also been studied under the same hologram strengths $M_{AC}$ and $M_{BC}$, with similar results; albeit the profile from Equation (31) was the best.
Fig. 12: Diffraction of the output wave as a function of two hologram strengths $M_{AC}$, $M_{BC}$. Here the wavelength of incident wave $A$ is 1.064 μm, holographic glass index refraction $n=1.5$, modulation of index refraction $n_{CA}$, $n_{BC} = 2 \times 10^{-4}$, incident angle $30\degree$ such that $\kappa_{AC} = \kappa_{BC} = 6.082 \times 10^{-4}$ μm$^{-1}$ and the corresponding hologram thicknesses $L$ are from 12.322 to 41.107 mm for $M_{AC}$, $M_{BC} = 7.5$ to 25 respectively.

Consideration of other (TM) polarization of interacting waves in our symmetric geometry results in multiplication of each of the coupling constants (at the given values of $n_{CA}$, $n_{CB}$) by the factor $(e_A \cdot e_C) \hat{U} (e_B \cdot e_C) = \cos(\theta_{A,med})$. This is quite remarkable, since it allows to diffract TM-polarized wave $A$ into wave $B$ propagating even at $90\degree$ to each other due to $(e_A \cdot e_C) \hat{U} (e_B \cdot e_C)$ being 0.707 each.
The following Fig. 13 and Fig. 14 illustrate the idea. They represent the convention transmission hologram and our adiabatic gratings, respectively.

![Illustration of TM polarized beam interacting in the transmission hologram](image)

Fig. 13: Diffraction of TM waves coupling in the transmission hologram. If the wave \( A \) into wave \( B \) propagating at \( 90^\circ \), we can see that the diffraction efficiency \( \eta = 0 \), due to the coupling strength \( \kappa \BAR{\phi}_A \BAR{\phi}_B = 0 \).
Fig. 14: Diffraction of TM waves coupling in the transmission hologram. Even the wave A into wave B propagating at 90°, we can see that the diffraction efficiency $\eta$ still can reach 100% if the hologram strength $M$ is large enough, due to the coupling strength $\kappa \hat{O} (e_A e_C) \hat{U} (e_B e_C) = \cos(\theta_{A,med}) = 0.707$.

This is the task, which could not be achieved by a single-grating hologram. Moreover, the intensity of the diffracted $B$-wave will be the same close to 100% (and even the phase will be the same) for both polarizations, if the modified hologram strength is large enough for the “unfavorable” polarization as well.
3.4 Angular and spectral detuning effect on adiabatic gratings

In reality, the perfect Bragg matching may not always be the case. We studied the effect of detuning on our new type transmission hologram by numerical integration of the coupled wave equations. We kept the same profile (Equation (31)) for the coupling, but took detuning into account. The results of the numerical solution of the equations (26) are shown in Figure 15.

Fig. 15: Diffraction efficiency of the diffracted wave as a function of the angular and spectral detuning values ($d\theta$, in degrees in air; $dw/w$, dimensionless) with relatively small hologram strengths $M_{CA} = M_{CB} = 4$ and thickness $L = 5 \text{ mm}$, $\lambda_{\text{vac}} = 1.06 \text{ mm}$, $\theta_{a,\text{air}} = 50^\circ$. 

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As the figure indicates, our adiabatic hologram possesses both, spectral and angular selectivity, just like an ordinary volume hologram. However, contrary to the ordinary hologram, one cannot substitute change of the angle by the change of wavelength without the loss of diffraction efficiency. This means the compensation of wavelength detuning by the angular detuning is not perfect anymore.

The ratio of width $\delta \theta$ (at optimally tuned $\lambda_{vac}$) to the width $\delta \theta$(at fixed $\lambda_{vac}$) depends on the particular geometry and on the hologram strength. Our numerical experiments have shown this factor to be around 7 to 12. The same may be stated about spectral selectivity. The same ideas of adiabatic $A/B$ interaction via weakly excited intermediate $C$-wave are applicable to the design of a beam splitter $|A|^2 p_A|A|^2 + p_B|B|^2$ with the splitting ratio $p_A/p_B$ being robust with respect to manufacturing errors and polarization.

We may draw a conclusion here. In this chapter, a new type of double-grating volume hologram has been suggested, which transfers energy from the primary wave into the diffracted wave through a weak intermediate wave. We consider the strongly diminished dependence of diffraction efficiency on the hologram strength as one of the advantages of the scheme suggested. Another possible advantage of the suggested scheme is the twice smaller spatial frequency $|Q_{AC}|, |Q_{BC}|$ of each of the gratings, if we have to achieve certain value of deflection angle, $\theta = 2 \arcsin(|k_A - k_B|/2k)$. Yet another possible advantage is to use the same optical element (our double-grating adiabatic hologram) to handle very efficient diffraction of both polarizations. This work has actually been published [39].
CHAPTER FOUR: REFLECTION CROSSED-GRATINGS WITH HIGH ANGULAR SELECTIVITY

4.1 Reflection grating as Bragg mirror

Bragg mirror is one of most widely used optical component in many applications. One of these applications is served as reflection mirror in the laser cavity. Reflected hologram made of PTR glass as Bragg mirror has great advantage of resisting the damage under the working condition of high power laser beam [40-43]. The good spectral selectivity of reflected hologram indeed gives a sharp frequency spectrum in the output laser beam.

However, to improve the beam quality, especially for semi-conductor lasers, the conventional Bragg mirror doesn’t feature good angular selectivity. In this chapter, we propose a new scheme based on double-recorded crossed gratings inside the volume hologram, which shows great improvement of angular selectivity.

A previous scheme considered earlier by M. G. Moharam and coworkers [44], which uses surface hologram with the grooves placed at top of a planar waveguide. It selects beam with normal incident angle and propagates transversely inside the waveguide, and then is normally reflected back with opposite propagation. Our scheme here is functionally analogous to theirs but with high tolerance to strong laser power due to the durability of volume hologram in comparison to surface gratings.
For beams propagation inside the transmission hologram of double-recorded-gratings, some results have been shown in our previous work [39]. In the early work of Alferness and Case [45]**, the angular selectivity of transmission hologram of double grating was fully discussed but not reflection hologram. With respect to reflection hologram, nontrivial behavior of four-waves coupling and the angular selectivity will be presented and analyzed in the following sections.

4.2 Four waves coupling in the reflection hologram of crossed-gratings

As described in the previous section, the angular selectivity for convention Bragg mirror is rather poor. To overcome this weakness, we introduce a set of cross-gratings to increase the angular selectivity via a new way of waves coupling.

Considering the derivation of coupled-wave theory [12, 46], non-resonant terms are dropped off in the phase matching of Bragg wavevector. In the same way, we can consider the phase matching condition of crossed-gratings in Fig. 16 and the beams propagations in Fig. 17.

**Various angular selectivity of thick transmission hologram of double gratings was described in their paper. With equal or unequal coupling strength of two gratings and the relative orientation between these two grating vectors are variable parameters in their calculation. There are two cases in their work; direct coupling deals with separate Bragg conditions and cross coupling deals with common Bragg condition for these two gratings. However, none of their configuration is similar to our horizontal and vertical coupling of four waves in our symmetric crossed gratings.
Fig. 16: Two crossed-gratings were recorded throughout the whole PTR glass.

Fig. 17: Waves propagation in the new design of the two-grating hologram where four waves actually arise in the double-crossed gratings. They are $A_+, A_- B_+, B_-$ respectively, and $A_+$ is the incident wave.
Here four waves are appearing in the hologram, with $A_+$, $A_-$ the incident and reflected waves, $B_+$, $B_-$ the excited intermediate waves. Our crossed-gratings are arranged in such a way that $A_+$ and $A_-$ propagate oppositely in the $z$-direction, while $B_+$ and $B_-$ propagate oppositely in the $x$-direction. This requires that two gratings are mutually perpendicular to each other and orient at $45^\circ$ with respect to incident and reflected beams.

To illustrate how it works, we consider the total field of TE polarized beam in the Maxwell equations. The resultant Helmholtz wave equation is,

$$
\nabla^2 E + \frac{\omega^2}{c^2} \left[ n_0^2 + n_0 (\mu_1 e^{iQ_1 r} + \mu_1^* e^{-iQ_1 r} + \mu_2 e^{iQ_2 r} + \mu_2^* e^{-iQ_2 r}) \right] E = 0
$$

(32)

where $n_0$ is the background index of refraction, $\mu_1$, $\mu_2$ are the grating strength, $Q_1$, $Q_2$ are the grating vectors.

And $E$ is the total field, which is written as

$$
E = A_+ e^{i\omega_{n_0} k_z r} + A_- e^{-i\omega_{n_0} k_z r} + B_+ e^{i\omega_{n_0} k_y r} + B_- e^{-i\omega_{n_0} k_y r}
$$

(33).

For obtaining the coupled-wave equations, we then substitute eq.(33) into eq.(32), apply the Slowly Varying Envelope Approximation (SVEA), and drop off the non-resonant terms by phase matching $k_{\text{out}} = k_{\text{inc}} + Q$. 

34
To pick up the right phase matching condition, we should carefully look at the diagram of our crossed-gratings. According to Fig.13 shown above, it is equivalent to \( \mathbf{k}_{A_+} - \mathbf{k}_{B_+} \hat{\mathbf{Q}}_1, \mathbf{k}_{A_+} - \mathbf{k}_{B_+} \hat{\mathbf{Q}}_2, \mathbf{k}_{A_-} - \mathbf{k}_{B_-} \hat{\mathbf{Q}}_1, \mathbf{k}_{A_-} - \mathbf{k}_{B_-} \hat{\mathbf{Q}}_2 \). Then we obtain,

\[
\exp\left( i \frac{\omega}{c} n_0 k_{g_+} z \right) \left[ 2i \frac{\omega}{c} n_0 k_{g_+} \frac{\partial A_+}{\partial z} + \frac{\omega^2}{c^2} B_+ \cdot n_0 \mu_1 \exp(i \frac{\omega}{c} n_0 (k_{g_+} x - k_{g_+} z) + i \mathbf{Q}_1 \cdot \mathbf{r}) \right] + \\
+ \frac{\omega^2}{c^2} A_+ \cdot n_0 \mu_2 \exp(-i \frac{\omega}{c} n_0 (k_{g_+} x + k_{g_+} z) + i \mathbf{Q}_2 \cdot \mathbf{r})
\]

\[
\exp\left( -i \frac{\omega}{c} n_0 k_{g_+} z \right) \left[ -2i \frac{\omega}{c} n_0 k_{g_+} \frac{\partial A_+}{\partial z} + \frac{\omega^2}{c^2} B_+ \cdot n_0 \mu_2' \exp(i \frac{\omega}{c} n_0 (k_{g_+} x + k_{g_+} z) - i \mathbf{Q}_2 \cdot \mathbf{r}) \right] + \\
+ \frac{\omega^2}{c^2} B_+ \cdot n_0 \mu_1' \exp(i \frac{\omega}{c} n_0 (-k_{g_+} x + k_{g_+} z) - i \mathbf{Q}_1 \cdot \mathbf{r})
\]

\[
\exp\left( i \frac{\omega}{c} n_0 k_{g_-} x \right) \left[ 2i \frac{\omega}{c} n_0 k_{g_-} \frac{\partial B_-}{\partial x} + \frac{\omega^2}{c^2} A_- \cdot n_0 \mu_1 \exp(i \frac{\omega}{c} n_0 (k_{g_-} z - k_{g_-} x) - i \mathbf{Q}_1 \cdot \mathbf{r}) \right] + \\
+ \frac{\omega^2}{c^2} A_- \cdot n_0 \mu_2 \exp(i \frac{\omega}{c} n_0 (-k_{g_-} z - k_{g_-} x) + i \mathbf{Q}_2 \cdot \mathbf{r})
\]

\[
\exp\left( -i \frac{\omega}{c} n_0 k_{g_-} x \right) \left[ -2i \frac{\omega}{c} n_0 k_{g_-} \frac{\partial B_-}{\partial x} + \frac{\omega^2}{c^2} A_- \cdot n_0 \mu_2' \exp(i \frac{\omega}{c} n_0 (k_{g_-} z + k_{g_-} x) - i \mathbf{Q}_1 \cdot \mathbf{r}) \right] + \\
+ \frac{\omega^2}{c^2} A_- \cdot n_0 \mu_1' \exp(i \frac{\omega}{c} n_0 (-k_{g_-} z + k_{g_-} x) + i \mathbf{Q}_1 \cdot \mathbf{r})
\] = 0 \quad (34).

If we add the absorption coefficient \( \varepsilon = n_0 n_1 \omega/c \) (\( n_1 \) is the imaginary part of refraction index) and the first order frequency detuning \( \gamma = n_0 \Delta \omega/c \) in the refraction index and frequency terms of eq.(34), we arrive at the following set of four equations.

\[
\frac{\partial A_+}{\partial z} = i \kappa_1 B_+ + i \kappa_2 B_- + (i \gamma - \varepsilon) A_+ \\
\frac{\partial A_-}{\partial z} = -i \kappa_1^* B_+ - i \kappa_2^* B_- - (i \gamma - \varepsilon) A_- \\
\frac{\partial B_+}{\partial x} = i \kappa_1^* A_+ + i \kappa_2 A_- + (i \gamma - \varepsilon) B_+ \\
\frac{\partial B_-}{\partial x} = -i \kappa_1^* A_- - i \kappa_2^* A_+ - (i \gamma - \varepsilon) B_- \quad (35).
\]
where $\kappa_1 = \mu_1\omega/2c$, $\kappa_2 = \mu_2\omega/2c$.

Next, taking the Fourier components of all the waves with respect to the incidence angle, e.g. $B_\circ \hat{U} e^{iqx}$, where $q = k_x = k \cdot \sin \theta$. Then we can account the effect of angular detuning by analyzing each spatial Fourier component.

Substitute $B_\circ \hat{U} e^{iqx}$ into eq.(35), accordingly, we have

\begin{align*}
B_+ &= \frac{i\kappa_1^*}{iq-i\gamma+i\varepsilon} A_+ + \frac{i\kappa_2}{iq-i\gamma+i\varepsilon} A_- \\
B_- &= -\frac{i\kappa_2^*}{iq+i\gamma-i\varepsilon} A_+ - \frac{i\kappa_1}{iq+i\gamma-i\varepsilon} A_-
\end{align*}

(36)

After substitute eq. (4) back into the first two equations in eq. (3) to obtain the coupled-wave equations for $A$-wave only. For the case $\kappa_1 = \kappa_2 = \kappa = \kappa^*$, we get

\begin{align*}
\frac{\partial A_+}{\partial z} &= PA_+ + QA_- \\
\frac{\partial A_-}{\partial z} &= -QA_+ - PA_-
\end{align*}

(37),

where $Q = i\kappa^2[(1/(q-\gamma-i\varepsilon)-(1/(q+\gamma+i\varepsilon))], \quad P = -\varepsilon+i\gamma+Q$.

It is worthy to make a note here. Although we derived the coupled-wave equation for $A$-wave, there is no direct coupling between $A_+$ and $A_-$. If we carefully take a look at eq.(35), it is the intermediate waves $B_\circ$ directly couples with $A_\circ$ respectively.
This means two Bragg conditions should be satisfied simultaneously for each coupling in order to have good diffraction efficiency. This is the main feature of our new design of hologram, which results the great improvement of angular selectivity.

4.3 Analysis of high angular selectivity by calculation of reflection efficiency

Before going to finding the solution of equation (37), we would like address the property of its coupling coefficients. In reality, for the finite size of incident beams, the angular detuning \( q \) actually plays a critical role in our scheme. As eq.(37) indicates, the coupling strength between \( A_+ \) and \( A_- \) waves is mostly characterized by \( Q \), since \( P \) also \( \bar{O}Q \).

Indeed, if we ignore the absorption and rewrite it as \( Q = ik^2 \left[1/(q-\gamma) - 1/(q+\gamma)\right] = 2i\kappa^2 \gamma/(q^2-\gamma^2) \), then \( Q \bar{O} 1/q^2 \), i.e. the coupling strength is inverse proportional to the square of angular detuning \( q \). As a consequence, this relation illustrates why our new type of reflection hologram has very good angular selectivity.

We rewrite equation (37) in the form,

\[
\begin{bmatrix}
A_+(z) \\
A_-(z)
\end{bmatrix} = \exp\left(\hat{M}z\right)\begin{bmatrix}
A_+(0) \\
A_-(0)
\end{bmatrix}
\] (38).
Here $\hat{M} = \begin{bmatrix} P & Q \\ -Q & -p \end{bmatrix}$. We denote the eigenvalues of the matrix $\hat{M}$ as $l_1, l_2$, they are equal to $\pm \sqrt{p^2 - Q^2}$, respectively. If we define $r = \sqrt{p^2 - Q^2}$ and use the Lagrange interpolation formula for a matrix,

$$\exp(\hat{M} z) = \exp(\lambda_1 z) \frac{\hat{M} - 1 \lambda_2}{\lambda_1 - \lambda_2} + \exp(\lambda_2 z) \frac{\hat{M} - 1 \lambda_1}{\lambda_2 - \lambda_1},$$

then the expression for the exponential of a matrix becomes

$$e^{\hat{M} z} = \frac{1}{2\rho} \begin{bmatrix} P(e^{\rho z} - e^{-\rho z}) + \rho(e^{\rho z} + e^{-\rho z}) & Q(e^{\rho z} - e^{-\rho z}) \\ -Q(e^{\rho z} - e^{-\rho z}) & -P(e^{\rho z} - e^{-\rho z}) + \rho(e^{\rho z} + e^{-\rho z}) \end{bmatrix}$$

The above matrix elements can be written as

$$e^{\hat{M} z} = \begin{bmatrix} \frac{P}{\rho} \sinh \rho z + \cosh \rho z & \frac{Q}{\rho} \sinh \rho z \\ -\frac{Q}{\rho} \sinh \rho z & -\frac{P}{\rho} \sinh \rho z + \cosh \rho z \end{bmatrix} = \hat{H}$$

From equation (38, 39) we can obtain the solution for $A_0(z)$ with given boundary conditions.
Consider the boundary condition for $A_0(z)$ with incident beam $A_+(0) = 1$, and the reflected beam $A_-(L) = 0$, here $L$ being the thickness of hologram. Accordingly, the reflection and transmission efficiency coefficients are respectively defined as:

$$\eta_{ref} = \left| \frac{A_-(0)}{A_+(0)} \right|^2, \quad \eta_{tran} = \left| \frac{A_-(L)}{A_+(0)} \right|^2$$

Now we explicitly write down the solution in terms of $\hat{H}$ matrix, namely

$$\begin{bmatrix} A_+(L) \\ A_-(L) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} A_+(0) \\ A_-(0) \end{bmatrix},$$

or

$$A_+(L) = H_{21}(L)A_+(0) + H_{22}(L)A_-(0)$$
$$A_-(L) = H_{11}(L)A_+(0) + H_{12}(L)A_-(0)$$

(40)

Thus, the reflection and transmission efficiencies $\eta_{ref}$, $\eta_{tran}$, according to eqs. (39, 40) they are,

$$\eta_{ref} = \left| \frac{A_-(0)}{A_+(0)} \right|^2 = \left| \frac{A_-(L)}{A_+(0)} \right|^2 = \left( \frac{Q \sinh \rho L}{P \sinh \rho L + \rho \cosh \rho L} \right)^2$$

$$\eta_{tran} = \left| \frac{A_-(L)}{A_+(0)} \right|^2 = \left| \frac{A_+(L)}{A_+(0)} \right|^2 = \left( \frac{P \sinh \rho L + \cos \rho L}{\rho P \sinh \rho L + \rho^2 \cosh \rho L} \right)^2.$$
The Fig. 18 below shows the dependence of reflection and transmission efficiencies upon the hologram strength $M$, with the angular detuning $q = 0.1$, damping $\varepsilon = 0.01$ and non-detuned free propagation $\gamma = 0$ (i.e. without wavelength detuning).

![Graph showing reflection and transmission coefficients as a function of hologram strength](image)

Fig. 18: The reflection and transmission coefficients as a function of hologram strength with the angular detuning $q = 0.1$, damping $\varepsilon = 0.01$ and no wavelength detuning, $\gamma = 0$.

We see that as long as the hologram $M$ is large enough, the reflection efficiency is good, i.e. is close to 1. Nevertheless, if we choose the hologram strength in the not very large region of the above graph, $M = 0 \sim 0.1$ for example, then the detuning effect appears apparently.
In the next section, we will examine both the spectral and angular detuning of our crossed-hologram. And the comparisons with those results from conventional one-grating Bragg mirror are discussed.

4.4 Results of the spectral and angular selectivity of our crossed-gratings

In this section, we will first review the coupled-wave equation of reflection grating with both the spectral and angular detuning from Bragg regime considered. The apparent improvement of angular selectivity of our new reflection hologram without scarifying the diffraction efficiency will be shown here.

The coupled-wave equations of conventional reflection hologram with grating strength $\mu$, refraction index $n_0$ and its imaginary part $n_I$, are:

$$\frac{\partial A_+}{\partial z} = i(\gamma - \widetilde{q} - \varepsilon)A_+ + i\kappa A_-$$
$$\frac{\partial A_-}{\partial z} = -i\kappa^* A_+ - i(\gamma - \widetilde{q} - \varepsilon)A_-$$

(41),

where $\kappa = \mu \omega/2c$, $\gamma = n_0 d \omega/c$, $\varepsilon = n_0 n_1 \omega/c$. And the angular detuning is $\widetilde{q} = q^2/2k$, with $q = k_x = k \sin q x = n_0 \omega/c$ as can be easily derived from the spatial second derivative in the wave equation.

Now we compare the diffraction efficiency $h_{\text{ef}}$ respectively from equation (37) and (41) in the following graphs.
I. Spectral selectivity of reflection holograms:

a). 

b).

Fig. 19: The reflection efficiency with no angular detuning $q = 0$, absorption $\epsilon = 0.1$, the grating strength $\kappa$ from 0 to 1 ($10^4 \text{ m}^{-1}$), for wavelength $\lambda_{\text{vac}} = 1.06 \text{ mm}$, refraction index $n_0 = 1.5$, $\mu = 0$ to $1.886 \times 10^{-4}$, and the spectral detuning $\gamma = 2 \rho n_0 D_{\lambda}/\lambda^2$ from - 3 to 3, $D_{\lambda} = 3.57 \times 10^{-5} \text{ mm}$. a). 

Convention reflection hologram with thickness $L = 2.5 \text{ mm}$, $D_{\lambda, \text{HWHM}} = 3.6 \times 10^{-5} \text{ mm}$; b). Our crossed hologram with thickness $L = 1 \text{ mm}$, $D_{\lambda, \text{HWHM}} = 2.32 \times 10^{-5} \text{ mm}$.

The above figures 19 a) and b) are depicted with maximum reflection efficiency $h_{\text{ref}} = 0.7996$ for reflection grating and $h_{\text{ref}} = 0.8133$ for our crossed gratings. We can see that both gratings all possess good spectral selectivity.
At first glance, the diffraction efficiency curve of reflection hologram seems sharper. But the reflection hologram should pay the price in increasing its thickness, i.e. larger hologram strength to catch up the same reflection efficiency $h_{ref}$ of crossed hologram. The curve will broaden if the hologram strength decreases and the $h_{ref}$ will also reduce dramatically.

II. Angular selectivity of reflection holograms:

a).

Fig. 20: The reflection efficiency with no spectral detuning $\gamma = 0$, absorption $\epsilon = 0.1$, the grating strength $\kappa$ from 0 to 1 for wavelength $\lambda_{vac} = 1.06$ $\mu m$, refraction index $n_0 = 1.5$, $\mu = 0$ to $1.886 \times 10^{-4}$ and the angular detuning $q = k \sin q$ from -3 to 3, $Dq = 0.08 \times 10^{-2}$ radian. a). Convention reflection hologram $L = 2.5$ $mm$, $Dq_{HWHM} = 5 \times 10^{-3}$ radian; b). Our crossed hologram with thickness $L = 1$ $mm$, $Dq_{HWHM} = 9 \times 10^{-4}$ radian.
The above figures 20 a) and b) are depicted with the same reflection efficiency $h_{\text{ref}} = 0.7996$ for reflection grating and $h_{\text{ref}} = 0.8133$ for our crossed gratings with other parameters kept the same as in Fig. 19. We can see that our crossed grating has a much better angular selectivity. The reflection grating has a pretty flat region of high diffraction efficiency. In compression, our angular selectivity is pretty sharp. Although people may argue that this crossed hologram has a longer interacting length, since the $B_0$ are propagating horizontally. This interacting thickness would be diminished as long as we include the absorption in our calculation.

III. The compensation between spectral and angular detuning:

![Graphs showing compensation between spectral and angular detuning](image)

Fig. 21: The reflection efficiency with compensation between spectral detuning $\gamma$ and angular detuning $q$ both from -3 to 3, with the same previous parameters $\varepsilon = 0.1$, $\kappa = 1$ but thickness $L = 2/m$. a). Convention reflection hologram; b). Our crossed hologram.
From the above Fig. 21 a) and b), it shows the compensation between spectral and angular detuning in our crossed-grating are more in a symmetric manner. As deviated from the central region, this compensation gradually fades away.

On the contrary, this is not the case for reflection hologram. We can see that, along the top bent cure in the two diagonal regions, significant compensation is observed and continuously keeps the effect. Thus it is more unpleasant to obtain the unexpected higher reflection efficiency.
5.1 Review of Maxwell wave equation: Helmholtz equation

There is great recent interest in electromagnetic effects in the media, where effectively both dielectric permittivity \( \varepsilon \) and magnetic permeability \( \mu \) are different from \( \varepsilon_{\text{vac}} \) and \( \mu_{\text{vac}} \): “negative refractive index” media, [47-48], photonic bandgap structures, [49-51], “stealth technology materials”, etc.

From theoretical point of view, it is important to understand, which effects in electrodynamics are governed by the gradients of propagation speed \( v(\mathbf{r}) \) [m/s] and which are controlled by the gradients of impedance \( Z(\mathbf{r}) \) [Ohm]. Here

\[
v(\mathbf{r}) = \left\{1/ [\varepsilon(\mathbf{r})\mu(\mathbf{r})]\right\}^{0.5} \equiv c/n(\mathbf{r}), \quad n(\mathbf{r}) = \left\{[\mu(\mathbf{r})\varepsilon(\mathbf{r})] / [\mu_{\text{vac}}\varepsilon_{\text{vac}}]\right\}^{0.5}, \quad (42)
\]

\[
Z(\mathbf{r}) = \left\{\mu(\mathbf{r})/\varepsilon(\mathbf{r})\right\}^{0.5} \equiv \left\{[\mu(\mathbf{r})/\mu_{\text{vac}}] / [\varepsilon(\mathbf{r})/\varepsilon_{\text{vac}}]\right\}^{0.5} \cdot 377 \text{ Ohm}, \quad (43)
\]

and \( n(\mathbf{r}) \) is the refractive index. This indeed draws us attention to study on this topic. We will first review the scalar Helmholtz equation for homogeneous media and then discuss more general cases.

In electrodynamics, suppose we are dealing with spatially homogeneous media, the Maxwell wave equation indeed equivalent to Helmholtz equation with a scalar function \( u(\mathbf{r}) \):
\[(\nabla \cdot \nabla + k^2) u(\mathbf{r}) = 0 \quad (44)\]

Such Helmholtz equation applies to each Cartesian component of each of four complex vectors \(\mathbf{E}, \mathbf{H}, \mathbf{D},\) and \(\mathbf{B},\) provided the dielectric permittivity and magnetic permeability are both isotropic and constant in all space.

In reality, the property of wave propagation from one medium into the other, such as Fresnel reflection, transmission is of practical interest. For simplicity, we discuss one-dimensional case here, so called layered media.

General case of layered medium with both, \(n(\mathbf{r}) = n(z),\) and \(Z(\mathbf{r}) = Z(z),\) i.e. with the arbitrary dependences of both dielectric permittivity and magnetic permeability on one coordinate only, \(\varepsilon = \varepsilon(z), \mu = \mu(z).\) It allows also separation of two types of polarizations, namely, TE-type and TM-type. However, the wave equations for these two types of waves are generally different.

Let us first consider the TE polarized wave as a solution of Maxwell equations for layered medium with new definition \(\eta(\mathbf{r}) \equiv \varepsilon_{\text{vac}}/\varepsilon(\mathbf{r}),\) \(\nu(\mathbf{r}) \equiv \mu_{\text{vac}}/\mu(\mathbf{r}).\) For TE wave, we can assume the displacement current has the form

\[\mathbf{D} = \nabla \varphi \times \mathbf{z} \quad (45)\]
Then Maxwell equations become,

\[
i\omega \mathbf{B} = \nabla \times \mathbf{E} = \nabla \times (\eta \nabla \varphi \times \mathbf{z}),
\]

\[
\mathbf{B} = (i\omega)^{-1}[\partial_z (\eta \partial_x \varphi) \mathbf{x} + \partial_x (\eta \partial_y \varphi) \mathbf{y} - \eta (\partial_x^2 + \partial_y^2) \varphi \mathbf{z}],
\]

(46),

\[
-i\omega \mathbf{D} = \nabla \times \mathbf{H},
\]

\[
in\omega(\nabla \varphi \times \mathbf{z}) = \nabla \times \{\nu (i\omega)^{-1}[\partial_z (\eta \partial_x \varphi) \mathbf{x} + \partial_x (\eta \partial_y \varphi) \mathbf{y} - \eta (\partial_x^2 + \partial_y^2) \varphi \mathbf{z}]\}
\]

(47).

From eqs. (46-47), equating the three unit vectors \( \mathbf{x}, \mathbf{y}, \mathbf{z} \) for both sides will lead the following:

\[
-\omega^2 \partial_y \varphi = \partial_y [\nu \eta (\partial_x^2 + \partial_y^2) \varphi] + \partial_z [\nu \partial_z (\eta \partial_y \varphi)]
\]

\[
= \partial_y [\nu \eta (\partial_x^2 + \partial_y^2) \varphi] + \partial_z [\nu \partial_x \varphi \partial_z \eta + \varphi \nu \partial^2 \eta + \nu \partial_z \varphi \partial_z \eta + \partial_z \varphi \partial_z (\eta \nu) + \nu \eta \partial^2 \varphi]
\]

\[
= \partial_y [\nu \eta (\partial_x^2 + \partial_y^2) \varphi] + \partial_z [\nu \partial_z (\eta \partial_x \varphi + \varphi \partial_z \eta) + \varphi \partial_z \eta \partial_z \nu + \eta \partial_z \nu \partial_z \varphi]
\]

\[
= \partial_y [\nu \eta (\partial_x^2 + \partial_y^2) \varphi] + \partial_z [\nu \partial_z (\eta \varphi) + \varphi \partial_z \eta \partial_z \nu + \eta \partial_z \nu \partial_z \varphi]
\]

(48).

After some arrangement, the above equation is equivalent to

\[
\nabla \cdot (\nu \nabla(\eta \varphi)) + \omega^2 \varphi = 0
\]

(49).

Similarly, a scalar function to describe TM wave is

\[
\nabla \cdot (\eta \nabla(\nu \psi)) + \omega^2 \psi = 0
\]

(50).
These two equations (49-50), of course, are irreducible to each other. However, in a special layered media where the impedance is constant, i.e. \( Z(z) = \) constant, then both TE and TM waves become indistinguishable.

This generate us the idea, the possibility of using a scalar wave equation to represent the approximate solution of Maxwell equation in inhomogeneous media. As the birth of Z-Helmholtz equation, this will be fully described in the next section.

### 5.2 Z-Helmholtz equation

In acoustics, the equation that describes the longitudinal sound wave propagating in medium is

\[
Z(r)k(r)\nabla \left[ \frac{1}{k(r)Z(r)} \nabla P_1 \right] + k^2(r)P_1 = 0
\]

where \( Z(r) \) is the impedance with the definition \( Z = r_0C \), and \( P_1, k, r_0, C \) are the pressure, wave number, density of medium, speed of sound, respectively. Derivation of this equation may be found in textbooks on Acoustics.

If we consider the above equation (51) with the condition of constant impedance, then it would reduced to
\[ k(r) \nabla \left( \frac{1}{k(r)} \nabla P_1 \right) + k^2(r) P_1 = 0 \] (52).

In a similar fashion for electrodynamics, let’s go back to eqs. (49, 50). If functional dependences of \( \nu(z) \) and \( \eta(z) \) are identical, then the impedance is constant in the whole volume of the medium, while \( k(z) \propto 1/\nu(z) \propto 1/\eta(z) \). If we changing the unknown functions \( \varphi(r) \) and \( \psi(r) \) by:

\[
F(r) = \eta(r) \varphi(r), \quad G(r) = \nu(r) \psi(r)
\] (53)

Then both \( F \) and \( G \) satisfy the following equation:

\[
k(r) \nabla \cdot \left( \frac{1}{k(r)} \nabla u(r) \right) + k^2(r) u(r) = 0 \quad \text{(Z-H)}
\] (54)

This equation we will label it Zeldovich-Helmholtz equation, or simply, Z-Helmholtz equation. This equation stands for wave propagation in constant impedance media and is an approximation for TE and TM light in assumption of not large change of refraction index and small incident angle.

To have a better understanding of Z-Helmholtz equation, we will consider the boundary condition for TE and TM light from rigorous Maxwell equation.
\[ E_i + E_r - E_t = 0 \]
\[ \sqrt{\frac{\varepsilon}{\mu}} (E_i - E_r) \cos \theta_i - \sqrt{\frac{\varepsilon'}{\mu'}} E_r \cos \theta_r = 0 \] (TE) \hspace{1cm} (55)

\[ (E_i - E_r) \cos \theta_i - E_r \cos \theta_r = 0 \]
\[ \sqrt{\frac{\varepsilon}{\mu}} (E_i + E_r) - \sqrt{\frac{\varepsilon'}{\mu'}} E_r = 0 \] (TM) \hspace{1cm} (56)

Here \( E_i, E_r, E_t \) are incident, reflected, and transmitted fields, respectively. We can see that in constant impedance, then \( \sqrt{\frac{\varepsilon}{\mu}} = \sqrt{\frac{\varepsilon'}{\mu'}} \), equations (55) and (56) are equivalent, so both TE and TM are indistinguishable.

Remarkable property of Z-H equation is that for the normal incidence to layered medium, \( k(r) = k(z) \), the following function is an exact solution:

\[ u(z) = \text{const} \cdot \exp \left\{ i \int_{z_i}^z k(z')dz' \right\} \] \hspace{1cm} (57)

In other words, there is no Fresnel reflection at the normal incidence to the impedance-matched medium, even if the refraction index function \( n(z) \) suffers sharp discontinuity.

In next sections, we will see how well of the approximation of Z-H to actual un-polarized light. The Fresnel reflection and Goose-Hanschen effect will be considered separately.
5.3 Fresnel Reflections

We will show below the detailed calculation for the Fresnel reflection for TE, TM polarized wave from Maxwell equations. We will compare them to the reflection coefficient for Fresnel reflection calculated for the scalar wave function from Z-Helmholtz equation, i.e. for the case of the pair of impedance-matched media.

The Fresnel reflection coefficients for both TE and TM cases are well known:

\[
\frac{E_r}{E_i} = \frac{Z_{21} \cos \theta_i - \cos \theta_t}{Z_{21} \cos \theta_i + \cos \theta_t} \quad \text{(TE)} \quad \frac{E_r}{E_i} = \frac{\cos \theta_i - Z_{21} \cos \theta_t}{\cos \theta_i + Z_{21} \cos \theta_t} \quad \text{(TM)}
\]

(58)

where \(Z_{21} = Z_2/Z_1\). Now we proceed to the derivation of the reflection coefficient for the Z-Helmholtz equation. Note that the eq. (54), when applied to this layer medium, is reduced to

\[
k(z) \frac{d}{dz} \left( \frac{1}{k(z)} \frac{du}{dz} \right) + k^2(z)u = 0
\]

(59)

Consider the following configuration of the wave vectors in Fig. 22 for the incident, refracted and reflected waves:
Fig. 22: Depiction of unit amplitude incident wave $k_1$, reflection wave $k_3$ of amplitude $r$, transmitted wave $k_2$ of amplitude $t$.

By requiring the continuity of the tangential component of $k$, of the field $W(r)$ and of the normal derivative of $k(r)^{-1} W(u)/\times$ at the boundary $z = 0$, we get

$$e^{i(k_1 x - k_1 z)} + re^{i(k_1 x + k_1 z)} = te^{i(k_2 x - k_2 z)} \bigg|_{z=0}, k_{1x} = k_{2x}$$

$$\frac{1}{k_1} ik_{1x} (-e^{i(k_1 x - k_1 z)} + re^{i(k_1 x + k_1 z)}) = \frac{1}{k_2} ik_{2x} te^{i(k_2 x - k_2 z)} \bigg|_{z=0}, k_{1x} = k_{2x} \quad (60)$$

With further simplification of equation (60), the reflection formula of $Z$-Helmholtz equation acquires rather simple form:

$$\frac{u_r}{u_1} = \frac{\cos \theta_r - \cos \theta_i}{\cos \theta_r + \cos \theta_i} \quad (61)$$
Based on the above formula, we now compare the intensities of Fresnel reflection for the cases, which we describe as TE, TM, Z-H. We take also the arithmetic average of TE and TM intensities of reflected light, in order to represent the un-polarized incident light and polarization-insensitive detector of the reflected light. Here are the results as shown in Fig. 23,

![Graph showing reflection intensities](image)

Fig. 23: Reflection intensities of TE wave (red line), TM wave (blue dotted line), and their arithmetic average value \([R(TE)+R(TM)]/2\) (green dashed line), compared with Z-H reflection intensity (purple bar-dot line), as functions of incidence angle with refraction indexes \(n_1 = 1, n_2 = 1.34\).
where $R_p$, $R_s$, $R_d$, $R_z$ are the reflection intensities: $R(TE)$, $R(TM)$, $[R(TE)+R(TM)]/2$, $R(Z-H)$, the parameter $t$ is the incidence angle, and both of the two media have the same magnetic permeability.

We assume that the medium #1 has refraction index $n_1 = 1$, medium-2 refraction index $n_2 = 1.34$. As we see from the figure, the approximation of Z-H to the un-polarized light in the layer medium with large incidence angle is very good. As another example, we set $n_2 = 1.5$ as shown in the following figure 24.

![Figure 24](image.png)

Fig. 24: Reflection intensities of TE wave (red line), TM wave (blue dot line), and their arithmetic average value $[R(TE)+R(TM)]/2$ (green dash line), compared with Z-H reflection intensity (purple bar-dot line), as functions of incidence angle with refraction indexes $n_1 = 1$, $n_2 = 1.5$. 

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We see a remarkable property of Z-H equation, i.e. of the approximation of constant impedance. Sure, only Maxwell equations yield exact results, and only Maxwell equations allow to calculate reflection for both polarizations separately. However, the next best thing (if we are not interested in polarizations, i.e. if illuminating light is depolarized and detector is polarization-insensitive) is Z-H equation for one scalar function.

5.4 Goose-Hanschen Effect

In this section, we try to calculate the Goose-Hanschen effect of both electro-magnetic waves and the scalar solution from Z-H equation under the condition where the total internal reflection occurs.

Goose-Hanschen effect is the effect of longitudinal shift of the “center of gravity” of a localized beam under Total Internal Reflection (TIR). As we did it for Fresnel reflection, we start with TE and TM waves first and then calculate the Z - H case. For the total internal reflection, since there is no transmitted wave, we can write eq.(58) in the following form, denoting $k_{1x} = k_{2x} = q$,

$$\frac{E_r}{E_i} |_{z=0} = e^{i\phi(q)} = \frac{Z_{21} \sqrt{1 - \left(\frac{q}{k_1}\right)^2}}{2} \sqrt{1 - \left(\frac{q}{k_2}\right)^2} \left(\frac{q}{k_2} - 1\right) \left(\frac{q}{k_1} + 1\right)$$

(TE) (62)
Now we give the derivation of the displacement of Goose-Hanschen effect. Consider the incident wave $E_i(x,t) = E(x)e^{i(k_x z - \omega t)}$, where

$$
E_i(x) = \int A'(k_x) e^{i k_x x} dk_x , \quad k_x = k_0 x + q
= e^{i k_0 x} \int A(q) e^{iqx} dq
$$

From eqs.(62, 63), we obtain the expression for reflected wave,

$$
E_r(x, z, t) = e^{i(k_0 x - k_0 x + \phi \omega t)} \int A(q) e^{iqx} \cdot e^{i\phi(q)} dq
= e^{i(k_x \omega t)} \int A(q) e^{iqx} \cdot e^{i\phi(q) \frac{d\phi(q)}{dq} dq}
= e^{i[k_x \omega t + \phi(k_0)]} \int A(q) e^{iq[x + \frac{d\phi(q)}{dq}]} dq
= e^{i[k_x \omega t + \phi(k_0)]} E_i(x + \delta x)
$$

Therefore the Goose-Hanschen shift $\delta x = df(q)/dq$ for TE polarized light could be easily calculated by taking the derivative of equation (62). Similarly, for TM case, the relative phase formula is,

$$
\left. \frac{E_r}{E_i} \right|_{\omega=0} = e^{i\phi(q)} \frac{1 - \frac{q}{k_1} + i Z_2}{1 - \frac{q}{k_1} - i Z_2} \left(\frac{q}{k_2} \right)^2 - 1
$$

(TM)
For the Z-H phase change, it is

\[
\frac{u_+}{u_+} |_{z=0} = e^{i\phi(q)} = \frac{i \sqrt{\left(\frac{q}{k_2}\right)^2 - 1} - \sqrt{1 - \left(\frac{q}{k_1}\right)^2}}{i \sqrt{\left(\frac{q}{k_2}\right)^2 - 1} + \sqrt{1 - \left(\frac{q}{k_1}\right)^2}} \quad \text{(Z-H)} \quad (66)
\]

After carrying out the calculation, we found that the Goose-Hanschen shifts for these three cases are,

\[
\begin{align*}
\delta x_{TE} &= -2Z_2 \left(1 - \frac{1}{k_2^2} \right), \\
\delta x_{TM} &= -2Z_2 \left(1 - \frac{1}{k_1^2} \right), \\
\delta x_{Z-H} &= -2 \frac{q}{k_2} - \frac{1}{k_1^2} \sqrt{\frac{q}{k_2} \left(\frac{1}{k_1^2} - 1\right) \left[1 - Z_2^2 (q^2 - k_1^2) + Z_2^2 (q^2 + k_1^2)\right]}
\end{align*}
\]

(67)

To investigate the above relationship of these three shifts, let us express the \(\delta x_{TE}\) and \(\delta x_{TM}\) in terms of \(\delta x_{Z-H}\), that is,

\[
\begin{align*}
\delta x_{TE} &= \delta x_{Z-H} \left(1 + \frac{1}{Z_2^2 k_2^2 \left[1 + k_1^2 q^2 - k_1^2 q^2\right]} \left[\frac{1}{Z_2^2 k_2^2 \left[k_1^2 q^2 - q^2\right] + k_1^2 q^2 \left[q^2 - k_1^2 q^2\right]}\right]\right) \\
\delta x_{TM} &= \delta x_{Z-H} \left(1 + \frac{1}{Z_2^2 k_2^2 \left[k_1^2 - q^2\right] + k_1^2 q^2 \left[q^2 - k_1^2 q^2\right]} \left[\frac{1}{Z_2^2 k_2^2 \left[k_1^2 q^2 - q^2\right] + k_1^2 q^2 \left[q^2 - k_1^2 q^2\right]}\right]\right)
\end{align*}
\]

(68)
The Fig. 25 below shows these Goose-Hanschen shifts as a function of incidence of angle. Here again we explore the cases with the same magnetic permeability for all media and the refraction indexes $n_1 = 1.34$, $Z_{21}=1.34$, $k_2=1$, $n_2 = 1$, where $D_xE$, $D_xM$, $D_xZ$ are the shifts of TE, TM, and Z-H respectively.

![Diagram](image.png)

Fig. 25: The displacement of the “center of gravity” of a localized beam during the total internal reflection, i.e. the Goose-Hanschen shift. Red line is for TE polarized beam, Blue dot line for TM polarized beam, Green dash line for Z-H wave, incidence angle is denoted by $t$, $n_1 = 1.34$, $Z_{21}=1.34$, $k_2=1$, $n_2 = 1$.

As in the previous case, we will compare the arithmetic average of TE and TM shift ($D_xD$) with that of the Z-H, as shown in the Fig. 26.
Fig. 26: Comparison of the Goose-Haschen shift of between the arithmetic average of TE and TM (Red line) waves and that of the Z-H wave (blue dot line) with incidence angle $t$, $n_1 = 1.34$, $Z_{21}=1.34$, $k_2=1$.

As another example we set $n_1 = 1.5$, the results are shown in Fig. 27 and 28.

Fig. 27: The displacement of the “center of gravity” of a localized beam during the total internal reflection, i.e. the Goose-Haschen shift. Red line is for TE polarized beam, Blue dot line for TM polarized beam, Green dash line for Z-H wave, with incidence angle $t$, $n_1 = 1.5$, $Z_{21}=1.5$, $k_2=1$. 

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Fig. 28: Comparison of the Goose-Haschen shift of between the arithmetic average of TE and TM (Red line) waves and that of the Z-H wave (blue dot line) with incidence angle $t$, $n_1 = 1.5$, $Z_{21} = 1.5$, $k_2 = 1$.

The above plots again prove that the behavior of non-polarized waves according to Maxwell equations may be described by Z-Helmholtz scalar wave equation with surprisingly good accuracy provided we are working in the regime of layered media.

We foresee the application of ZH equation for the discussion of the properties of the media with “negative refractive index”. Indeed, while analytic continuation of $n = \sqrt{(\varepsilon' + i\varepsilon'') \cdot (\mu' + i\mu'')}$ to the negative values of $\varepsilon'$ and $\mu'$ at positive $\varepsilon''$ and $\mu''$ yields negative $n$, analytic continuation of impedance $Z = \sqrt{(\mu' + i\mu'')/(\varepsilon' + i\varepsilon''})$ stays with unchanged sign.
To conclude this chapter, we found a wave equation for one scalar function, Z-H eq. (54), which turns to be a surprisingly good substitution for the description of certain effects of electromagnetic propagation, if the illumination is depolarized and the detectors are polarization-insensitive.
6.1 Evolution of polarization in inhomogeneous media

Evolution of polarization is studied in numerous works, [52-57]. In this chapter, we are interested in the light propagation in an inhomogeneous locally isotropic medium, and also in a single-mode fiber, which is polarizationally neutral, but which is smoothly bent and twisted.

Local law of polarization change along the ray’s trajectory may be labeled as “Rytov’s non-rotation”. Consider the trajectory $r = r(\tau)$ of a ray in a spatially inhomogeneous but locally isotropic medium, with $r(\tau)$ being the position of “propagating photon” at the moment $\tau$.

For determination of the polarization evolution, we should describe the trajectory. Therefore, we first introduce the spatial profile of the propagation speed $v(r)$, with $[v] =$ meter/sec. In particular, for the inhomogeneous media,

$$v(r) = \frac{c}{n(r)}, \quad c = \frac{1}{\sqrt{\varepsilon_{\text{vac}}\mu_{\text{vac}}}}, \quad n(r) = \sqrt{\frac{\varepsilon(r)\mu(r)}{\varepsilon_{\text{vac}}\mu_{\text{vac}}}} \quad (69)$$

Here, $\varepsilon(r), \mu(r), \varepsilon_{\text{vac}}, \mu_{\text{vac}}$ are the local values of dielectric permittivity, magnetic permeability, and their vacuum values, respectively, $n(r)$ being the local value of refractive index.
Then the descriptions of trajectory of ray under the approximation of geometrical optics (GO) analog to the equation of motion in mechanics are

\[ \frac{dr(\tau)}{d\tau} = v(\tau) \cdot t(\tau) \]  \hspace{1cm} (70)

\[ \frac{dt(\tau)}{d\tau} = v(\tau) \cdot (\nabla \ln n(\tau) - t(\tau) \cdot \nabla \ln n(\tau)) \]  \hspace{1cm} (71)

Equation (71) defines real unit tangent vector \( t(\tau) \) to the trajectory, and has the property that,

\[ t = t^*, \quad t \cdot t = 1, \quad t = v^{-1} \frac{dr}{d\tau} \]  \hspace{1cm} (72)

In the GO regime, equation (71) is similar to the Second Newton’s law for the photon’s momentum \( p = \hbar \omega t/v \). The derivation of equation (71) may be found in the most textbooks, see e.g. [3].

Given the profile \( n(\tau) \) and initial conditions \( r(\tau = 0) = r_0, \ t(\tau = 0) = t_0 \) (with \( t_0 \cdot t_0 = 1 \)), we can find the trajectory, i.e. functions \( r=r(\tau), \ t=t(\tau) \) via numerical (or any other) integration of the system (70, 71). It is easy to see, that

\[ \frac{d}{d\tau} (t \cdot t) = 0 \]  \hspace{1cm} (73)
as a direct consequence of equations (70, 71), and therefore the condition $t \cdot t = 1$ is preserved all along the trajectory of photon. We can further introduce the length $l(\tau)$ along the trajectory, such that

$$\frac{dl}{d\tau} = v(r(\tau)) \quad (74)$$

and the equations (70, 71) become

$$\frac{dr}{dl} = t(l) \quad (75)$$
$$\frac{dt}{dl} = V(r(l)) - t \{ t \cdot V(r(l)) \} \quad (76)$$

Here we have introduced the vector $V(r)$ \{of dimensions [1/meter]\} of the gradient of logarithm of refractive index.

S.M. Rytov considered evolution of polarization of photon propagating in the medium under GO approximation discussed above in 1938. He used for that purpose the so-called “natural Frenet coordinate system” of unit vectors $t, N, B$, with properties

$$(t \cdot t) = (N \cdot N) = (B \cdot B) = 1$$
$$(t \cdot N) = (N \cdot B) = (B \cdot t) = 0$$
$$t \cdot [N \times B] = 1 \quad (77)$$
This coordinate system $t, N, B$ is defined by the following equations

\[
\frac{dr}{dl} = \frac{1}{v} \frac{dr}{d\tau} = t, \quad \frac{dt}{dl} = \frac{1}{v} \frac{dt}{d\tau} = \frac{1}{R(\tau)} N
\]  

(78)

\[
\frac{dN}{dl} = \frac{1}{v} \frac{dN}{d\tau} = -\frac{1}{R(\tau)} t + \kappa(\tau) B
\]

\[
\frac{dB}{dl} = -\kappa(\tau) N, \quad B = [t \times N]
\]  

(79)

Here $N$ is the unit vector of the normal to the (generally curved) trajectory,

\[
N = \frac{dt}{dl} \left/ \sqrt{\left( \frac{dt}{dl} \cdot \frac{dt}{dl} \right) / \left( \frac{dt}{d\tau} \cdot \frac{dt}{d\tau} \right)} \right.
\]

(80),

and $1/R(\tau)$ of dimensions is the local value of curvature.

Similarly, bi-normal $B$ is unit vector defined as $B = [t \hat{\Omega} N]$, while the value of “torsion” $\kappa(l)$ [1/meter] may be found from

\[
\kappa(l) = t \cdot \left[ \frac{dt}{dl} \times \frac{d^2 t}{dl^2} \right] R^2 \equiv R^2 v^{-6} \frac{dr}{d\tau} \left[ \frac{d^2 r}{dl^2} \times \frac{d^3 r}{d\tau^3} \right]
\]

(81).

Rytov’s law of polarization evolution is stated in the following way. Let us present polarization vector $e$ in the form
\[ \mathbf{e}(l) = \cos \varphi \cdot \mathbf{N} + \sin \varphi \cdot \mathbf{B} + 0 \cdot \mathbf{t} \]  

(82)

Then according to Rytov,

\[ \frac{d\varphi(l)}{dl} = -\kappa(l) \]  

(83)

From the above equations (78-83), we can determine the polarization evolution. In a number of cases, the actual trajectory of a ray may be relatively close to a straight line. That means \( R \to \infty \). Then from Rytov’s law \( d\varphi/dl = -\kappa(l) \) we conclude, that there is a very small actual change of polarization \( \mathbf{e}(l) \) as a result of the compensation of two large contributions.

The first one is the rotation of Frenet frame \((\mathbf{N}, \mathbf{B})\), approximately at a rate of torsion \( \kappa \) (radian/meter), and second one is the counter-rotation of \( \varphi \) with respect to the \((\mathbf{N}, \mathbf{B})\)-frame, with the exact rate \( \kappa \).

Now we had seen that how the polarization evolution related to the light trajectory in the GO regime. In the next section, we will perform the calculation of such an evolution of polarization at a finite part of trajectory with our new reference frame instead of Rytov’s.
6.2 Calculation of polarization change from ZLS equation

Using equations (78, 79), we can verify the following very simple equation that holds for the evolution of polarization vector $e = e(\tau)$:

$$\frac{de}{dl} = -t(\cdot \frac{dt}{dl}), \quad \frac{de}{d\tau} = -t(\cdot \frac{dt}{d\tau}) \quad \text{(ZLS)} \quad (84)$$

Actually, this equation was first derived directly from Maxwell equations in approximation of GO by Zeldovich, Liberman and Savchenko in [20-21]. Therefore we will call it ZLS equation here. It is also known as the “pseudo-parallel transport” of polarization.

The next task is how we use ZLS to find out the polarization change. Here ZLS equation has evident advantage, since it can deal with evolution of polarization in a “laboratory frame” instead of Frenet frame.

Particular calculation of input-to-output transformation in laboratory frame requires introduction of some reference vector. Take an arbitrary real unit vector $a = (a_x, a_y, a_z)$,

$$a \cdot a = 1, \quad a = a^* \quad (85),$$

and consider the trajectory sufficiently characterized by $t(\tau)$ or $t(l)$. 
Then for each time moment $\tau$ (or trajectory point $l$) we will introduce two unit vectors $\alpha$, $\beta$, perpendicular to the tangent vector $t \equiv t(\tau)$:

$$\alpha = \alpha(\tau) = (a - t(a \cdot t))/\sqrt{1 - (a \cdot t)^2}, \quad \beta = \beta(\tau) = [t \times a]$$  \hspace{1cm} (86)

Most general unitary evolution of polarization according to Rytov’s equations (82, 83) or ZLS eq.(84) may then be presented in the form,

$$e(l) = f(l)\alpha(l) + g(l)\beta(l)$$  \hspace{1cm} (87)

Direct substitution $e(l)$ of eq.(87) into ZLS eq.(84), it gives

$$\frac{df}{d\tau} \alpha + f \frac{d\alpha}{d\tau} + \frac{dg}{d\tau} \beta + g \frac{d\beta}{d\tau} = -t \left( e \cdot \frac{dt}{d\tau} \right)$$  \hspace{1cm} (88)

Apply dot product $\alpha$, $\beta$ to both sides of eq.(88) respectively, and use the following property:

$$\frac{d}{d\tau}(\alpha \cdot \alpha) = 2(\frac{d\alpha}{d\tau} \cdot \alpha)$$

$$\frac{d}{d\tau}(\beta \cdot \beta) = 2(\frac{d\beta}{d\tau} \cdot \beta)$$

$$\frac{d}{d\tau}(\alpha \cdot t) = 0 = (\alpha \cdot \frac{dt}{d\tau} + (\frac{d\alpha}{d\tau} \cdot t)$$

$$\frac{d}{d\tau}(\beta \cdot t) = 0 = (\beta \cdot \frac{dt}{d\tau} + (\frac{d\beta}{d\tau} \cdot t)$$  \hspace{1cm} (89)
It generates,

\[
\frac{df}{d\tau} = -g\left(\alpha \cdot \frac{d\beta}{d\tau}\right) \\
\frac{dg}{d\tau} = f\left(\beta \cdot \frac{d\alpha}{d\tau}\right)
\] (90).

And,

\[
\alpha \cdot \frac{d\beta}{d\tau} = \alpha \cdot \frac{d(t \times \alpha)}{d\tau} = \alpha \cdot \left(\frac{d\mathbf{t}}{d\tau} \times \alpha + \mathbf{t} \times \frac{d\alpha}{d\tau}\right) = \mathbf{t} \times \frac{d\alpha}{d\tau} \cdot \alpha
\] (91)

Moreover, it can also show that,

\[
\alpha \cdot \frac{d\beta}{d\tau} \equiv -\left(\beta \cdot \frac{d\alpha}{d\tau}\right)
\] (92)

Substitute expression for \( \alpha \) in eq.(86) into eq.(91), it results

\[
\alpha \cdot \frac{d\beta}{d\tau} = -\alpha \left[ \mathbf{t} \times \frac{d\mathbf{t}}{d\tau} \frac{\mathbf{a} \cdot \mathbf{t}}{1 - (\mathbf{a} \cdot \mathbf{t})^2} \right] = \sigma
\] (93)

Through eqs.(90, 92, 93), it yields very simple equation for the (generally complex values) coefficients \( f(l) \) and \( g(l) \)
\[
\frac{df}{dl} = -\sigma g(l), \quad \frac{dg}{dl} = \sigma g(l),
\]

This is the equations of “rotation” with respect to the axes \(\alpha(l)\) and \(\beta(l)\), including themselves may or may not rotate along the trajectory.

The final expression of polarization projection components of \(f\) and \(g\) are shown in the following formula through the rotation angle \(\gamma\).

\[
f(l) = f(0) \cos(\gamma(l)) - g(0) \sin(\gamma(l))
g(l) = f(0) \sin(\gamma(l)) + g(0) \cos(\gamma(l))
\]

(95)

where the initial conditions are \(f(0) = \cos(q_0)\), \(g(0) = \sin(q_0)\), with \(q_0\) being the initial projection angle for \(e\) in \(\alpha(0), \beta(0)\) coordinates. And the time derivative of \(\gamma\) is actually the rotation rate \(\sigma\),

\[
\sigma = \frac{d\gamma}{d\tau} = -\mathbf{a} \cdot (\mathbf{t}(\tau) \times \frac{d\mathbf{t}}{d\tau}) \frac{\mathbf{a} \cdot \mathbf{t}(\tau)}{1 - (\mathbf{a} \cdot \mathbf{t}(\tau))^2}
\]

(96)

As a result, the problem of polarization reduces to the following integral, if we know the evolution \(t = t(\tau)\) or \(t = t(l)\).

\[
\gamma(\tau_2 \leftarrow \tau_1) = \int_{\tau_1}^{\tau_2} \sigma d\tau = -\int_{\tau_1}^{\tau_2} \mathbf{a} \cdot (\mathbf{t}(\tau) \times \frac{d\mathbf{t}}{d\tau}) \frac{\mathbf{a} \cdot \mathbf{t}(\tau)}{1 - (\mathbf{a} \cdot \mathbf{t}(\tau))^2} d\tau
\]

(97)
Equations (87, 95, 97) constitute the main result of our investigation. There are considerable advantages of the expressions.

First, the basic vectors $\alpha(l)$ and $\beta(l)$ do not depend on the “prehistory” of the trajectory, and are defined by $t(l)$ and a fixed vector $a$ only. Second, the “rotation rate” $\sigma(l)$ depends on the first $l$-derivative of the unit tangent vector $t(l)$. Third, if the changes of trajectory are small, then both, the vectors $\alpha(l)$ and $\beta(l)$, and the “angle” $\gamma(l)$ change very little.

All these advantages would be absent, if so-called “natural Frenet” basic vectors (normal and bi-normal) were used. In particular, if $t(\tau_2) = t(\tau_1)$, i.e. if the $t$-vector evolves over a close trajectory on the sphere ($t \cdot t = 1$) at the interval $\tau_1 < \tau < \tau_2$, then $a(\tau_2) = a(\tau_1)$, $b(\tau_2) = b(\tau_1)$. This would not be the case for Frenet basis.

In such case of $t(\tau_2) = t(\tau_1)$, the angle $\gamma$ has direct meaning of polarization rotation with respect to fixed axes $a$, $b$ in the same plane, the latter being perpendicular to $t(\tau_2) \equiv t(\tau_1)$. In this case $\gamma(\tau_2 \leftarrow \tau_1)$ is equal to the famous Berry’s phase [56], which will be discussed in the next section.

We want to emphasize here three important points.

1). The expressions (87), (95), (97), are not limited to the case $t(\tau_2) = t(\tau_1)$; they yield explicit expression for the polarization at the arbitrary point of trajectory.
2) The expression (87) gives \( e(\tau) \) in terms of basis vectors \( \alpha(\tau) \), \( \beta(\tau) \), the latter being determined by fixed vector \( a \) and local \( t(\tau) \), not by its derivative \( dt/d\tau \).

3) On the contrary, the expressions for the torsion \( \kappa(l) \) and thus for \( N \), \( B \) and \( \phi(l) \) in Frenet frame require calculation of \( dt/dr \propto d^2r/dr^2 \) and \( d^3t/dr^3 \propto d^3r/dr^3 \).

Besides that, calculations for almost straight parts of trajectory, when \( R = \left| dt/dl \right|^{-1} \rightarrow \infty \), may be numerically extremely challenging. The same may be said about numerical evaluation of \( d^3r/dr^3 \).

Our expressions deal with the actual change of polarization only, and contain \( dt/dr \), but not \( d^2t/dr^2 \). Some difficulty may arise in our approach, if \( \left| a \cdot t(\tau) \right| \approx 1 \) at certain intermediate point \( \tau \) of trajectory; then our expressions (96), (97) for \( \gamma \) become rather sensitive to small errors in \( t(\tau) \) at that point. However, generally the trajectory \( t(\tau) \) is a line on a unit sphere. Therefore, if this is not an area-filling fractal-type line, one can always eliminate this difficulty by a proper choice of the fixed reference \( a \)-vector.

However, some special cases of sharp changes of direction do draw our attention. Consider the case where the unit tangent vector \( t(\tau) \) undergoes rather fast change: \( t(\tau_1) = t_1, t(\tau_2) = t_2 \). We can introduce the angle \( \rho \) (radians) of the direction change via

\[
\rho = \arccos(t_1 \cdot t_2)
\] (98)
Assuming that we deal with a “planar piece of trajectory”, one can describe the evolution $t(\tau)$ e.g. by the formula,

$$t(\tau) = \frac{t_1 \sin(\rho(1-x)) + t_2 \sin(\rho x)}{\sin \rho}$$

where $0 < x(\tau) < 1$, $x(\tau) = (\tau - \tau_1)(\tau_2 - \tau_1)$, so that $x(\tau=\tau_1)=0$, $x(\tau=\tau_2)=1$. We can then integrate the expressions eq.(96) or eq.(97) within this interval. This integral looks complicated, but a direct numerical calculation overcomes this difficulty.

Despite that, we also found out the analytical solution of such integral, the result is detailed addressed in the appendix.

**6.3 The design of rotation sensor for a rigid body**

The well-known Gauss-Bonnet theorem from differential geometry, as well as the results by Vladimirskii, Berry, and experimental results by Chiao and Tomita, [57], deal with the change of polarization for the specific case, when the propagation directions $t(l_1)$ and $t(l_2)$ at the beginning ($l = l_1$) and at the end ($l = l_2$) of the trajectory are the same, $t(l_1) = t(l_2)$. Polarization change at this interval may be characterized by rotation angle $\gamma$, or by Berry’s geometrical phase difference between two circularly-polarized components (divided by 2).
Two remarkable facts should be noted for this particular case of $t(l_1) = t(l_2)$. First, the rotation angle does not depend on the choice of the coordinate system in the plane, where the polarization vector $e$ “lives”, i.e. in the plane perpendicular to $t(l_1) = t(l_2)$. Second, this half of Berry’s phase, or half of the rotation angle $g$ (in radians), equals numerically to the solid angle $\Gamma$ (in steradians) subtended by the closed-loop trajectory, which unit vector $t(l)$ draws on the unit sphere.

Based on such nice linear property between the rotation angle $g$, or the change of polarization vector $e$, and the trajectory solid angle $\Gamma$, we try to attach a flexible polarizationally-neutral fiber to a rigid body to make a rotation sensor as shown in the following Fig. 29.

Fig. 29: The configuration of a flexible polarizationally-neutral fiber attached to a rigid body, which was subject to various rotations (depicted by a vector of “angular velocity” $\Omega$) along the z-axis. The part of fiber attached to the object is fixed on the plane subject to rotation along the z-axis. The ends of the fiber were fixed in laboratory frame, so that input and output propagation directions were fixed and opposite to each other.
In our design, the ends of the fiber were fixed in laboratory frame, so that input and output propagation directions were fixed and opposite to each other, while the part of fiber attached to the rigid body was fixed on a plane subject to rotation along the z-axis.

We now try to simulate the above experiment by assuming the trajectory \([x(l), y(l), z(l)]\) or \([x(t), y(t), z(t)]\) of fiber in the following model. There are three sections I, II, III, in the fiber configuration, corresponding to, coming to the object, attached to the object, and leaving the object.

**Section I. 0 ≪ t ≪ T, fiber coming to the object.**

\[
\begin{align*}
    x(t) &= f(t) a \cos \left( \frac{2\pi n}{T} t \right) \\
    y(t) &= f(t) a \sin \left( \frac{2\pi n}{T} t \right), \quad 0 \leq t \leq T \\
    z(t) &= \frac{t}{T}
\end{align*}
\]

(100)

\[
f(t) = \begin{cases} 
    \sin^2 \left( \frac{t}{2\tau} \right) & 0 \leq t \leq \tau \\
    1 & \tau \leq t \leq T - \tau \\
    \sin^2 \left( (t-T) \frac{\pi}{2\tau} \right) & T - \tau \leq t \leq T 
\end{cases}
\]
Section II. $T \ddot{\alpha} t \dddot{\alpha} T_2$. fiber attached to the rotation object

\begin{align*}
x(t) &= u(t - T) a \cos \left( \frac{2\pi n}{T} t \right) \\
y(t) &= f(t) a \sin \left( \frac{2\pi n}{T} t \right), \quad T \leq t \leq T + T_2 \\
z(t) &= 1 + \frac{T_2}{\pi} \sin \left( \frac{\pi}{T_2} (t - T) \right) \\
u(t) &= \frac{t}{T_2} \left( 1 - \frac{t}{T_2} \right) \sin \left( \frac{2\pi}{T_2} \right)
\end{align*}

\begin{equation}
(101)
\end{equation}

Section III. $T + T_2 \dddot{\alpha} t \dddot{\alpha} 2T + T_2$, fiber leaving the object

\begin{align*}
x(t) &= f(t) a \cos \left( \frac{2\pi n}{T} t - 2T - T_2 \right) \\
y(t) &= f(t) a \sin \left( \frac{2\pi n}{T} t - 2T - T_2 \right), \quad T + T_2 \leq t \leq 2T + T_2 \\
z(t) &= \frac{2T + T_2 - t}{T} \\
f(t) &= \\
&\begin{cases} 
\sin^2 \left( \frac{\pi}{2} \left( t - (T + T_2) \right) \right) & T + T_2 \leq t \leq T + T_2 + \tau \\
1 & T + T_2 + \tau \leq t \leq 2T + T_2 - \tau \\
\sin^2 \left( \left( T - T \right) \frac{\pi}{2\tau} \right) & 2T + T_2 - \tau \leq t \leq 2T + T_2 
\end{cases}
\end{align*}

\begin{equation}
(102)
\end{equation}
Here $T$ is the time for light propagation in the first and the third sections of fiber, and $T_2$ is the time in the flat section attached to the object subject to rotation. The parameter $t$ characterizes the asymptotic behavior of the fiber such that the fiber has both fixed ends and is fixed on the rotation plane of the object. And $n$ defines the number of rotation in radian.

The above chosen of configuration for the trajectory for fiber is non-trivial. We should asymptotic functions such that the two ends of fiber are fixed and the portion of the fiber attached to the object should be fixed on the plane of rotation.

Moreover, the junctions at the three portions of the fiber should be continuous not only the trajectory but also the first derivative respect to time. Since we are only working on the case that, there is no sharp change in our trajectory.

Fig. 30 (a), (b), (c) shows our computer generated trajectory for our fiber sensor. They are corresponding to having $n = 10, 10.04, 10.06$ for (a), (b) and (c) respectively, i.e., the values of the rotation number $n$ in equations (100, 101, 102).
a) Rotation number, \( n = 10 \)

![Graph a) Rotation number, \( n = 10 \)]

b) Rotation number \( n = 10.04 \)

![Graph b) Rotation number \( n = 10.04 \)]
c) Rotation number $n = 10.06$

Fig. 30: The rotation of our fiber model according to equations (100-102). In (a) we choose the rotation number $n = 10$, in (b) $n = 10.04$, in (c) $n=10.06$.

As the above figures indicates, our fiber modeling eqs.(100-102) work successfully to have fixed ends and fixed plane subject to z-axis rotation.

Before go to testing the function of our rotation sensor, we will first try such configuration with calculation of polarization evolution both by ZLS or Rytov’s non-rotation equation (84). For verification of our scheme that a unit vector $a$ can be arbitrary chosen as the constitution of “laboratory frame” by eqs (85, 86), we compare the calculation of ZLS equation by two methods. First, we solve it by standard Runge-Kutta 4th order algorithm for ODE (ordinary differential
equation); second, we calculate it by use of equations (87, 95) with the rotation angle $g$ given in equation (97), the integral of rotation rate $\sigma$.

Here we choose the spherical coordinates for the representation of arbitrary unit vector $\mathbf{a}$. Its value is $\mathbf{a} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, where both $q$ and $f$ vary from 20 to 29 degree with input polarization $\mathbf{e} = (1, 5, 0)$, $n=10.2$. The results are shown in Fig 31, 32, 33 and 34.

Fig. 31 The x components of electric field at the output of the fiber. a). Result from Rk4, which is a constant, due to there is no function dependence on coordinate frame; b). Result from rotation integral, we can see that although the unit vector varies in the spherical range, both $q$ and $f$ vary from 20 to 29 degree with polarization input $\mathbf{e}(0) = (1, 5, 0)$, $n=10.2$, the result is almost a constant.
Fig. 32: The y components of electric field at the output of the fiber. a). Result from Rk4, b). result from rotation integral.

Fig. 33: The z components of electric field at the output of the fiber. a). Result from Rk4, b). result from rotation integral.
Fig. 34: The inner product of electric fields and from rotation integral. We can see that it is almost a constant 1 in the spherical range, both $q$ and $f$ vary from 20 to 29 degree. This means that the polarization evolution can totally be characterized by our “laboratory frame” i.e., eqs (85) and (86).

From the above figures, we can conclude that the polarization evolution can be totally described by our rotation integral and arbitrary chosen “laboratory frame”. Next, we will see how our rotation sensor works.

We know that the polarization change is determined by the rotation angle $g$, which is twice of trajectory solid angle $\Gamma$ as mentioned in the beginning of this section. So we therefore first measure the polarization $e$ vector in the output end, then rotate the object, and then measure the new $e$ vector in the output end again.
As a consequence, we have the polarization change $\Delta e$, which is equivalent to the change of rotation angle $\Delta \theta$. This is because both the two ends of the fiber are fixed during object rotation, we can determine $\Delta \theta$ from $\Delta e$ through equations (87) and (95) by fixed $a$, $\alpha$ and $\beta$.

Once we know $\theta$, then $\Delta \Gamma = \Delta \theta / 2$, so the change of objection rotation along z-axis also known. Therefore, we have a rotation sensor.

Let’s verify our idea by numerical experiment based on the model of fiber we suggested above. Suppose we have chosen a unit vector $a$ and the corresponding reference frame $\alpha$ and $\beta$. Now we try to rotate the fiber in the same way as shown in the configuration in Fig. 30 (a), (b) and (c) by changing the rotation number $n$ in equations (100-102).

This is equivalent to change the trajectory solid angle $\Delta \Gamma$. And we calculate the corresponding $\theta$ subject to the various rotation numbers $n$.

Our numerical results are shown in the following Fig. 35 (a), (b).
Fig. 35: The linear relation between the rotation number $n$, i.e. the trajectory solid angle $\Gamma$, and the rotation angle with the condition polarization input $e(0) = (1, 5, 0)$, and the chosen fixed unit vector $(1, 2, 3)$.  a). the plot of $n$-$\gamma$ in unit of $2\pi$ (radians), $n = 10.2$ to $10.6$; b) reset the origin of figure as the plot of $\Delta n$-$\Delta \gamma$ in degree. We can see that $\Delta \Gamma = \Delta \gamma / 2$ and the jump in the figure is actually due to an additional cycle of $2\pi$.

In Fig. 35, we set our unit vector $a = (1, 2, 3)$, and subject to rotation by changing $n$ from $10.2$ to $10.6$. We see that in (a) the $n$-$\gamma$ plot, it keeps as a linear function and so does in (b) for $\Delta n$-$\Delta \gamma$, where we rest the origin from figure (a) and plot it in units of degree instead of radius. The sudden jump in the figure is actually the rotation angle undergoing an additional cycle of $2\pi$. Moreover, the modulus of the slope is exactly equal to $2$ in above figure.
This numerically proves that the Berry phase or the rotation angle $g$ is exactly twice of the trajectory solid angle, i.e., $\Delta \Gamma = \Delta g / 2$.

In short conclusion, we had introduced a new reference frame, which has more advantage compare with Rytov’s Frenet frame. By use the ZLS equation along with this frame, we can calculate the rotation angle and the polarization evolution. One of the achievement is we numerically verified the Rytov-Berry-Chiao theorem. One of the applications is our design of new type of rotation sensor, which determines the object rotation by measuring the polarization evolution of light in a polarizationally-neutral fiber attached to the object.
We had used the light induced gratings in NLC to do the beam combining through stimulated orientational scattering. The energy transfer by the Stoke’s wave causes stability in the diffracted process. We believed that in the scientific literature it is the first time we observed this phenomenon. Through the analysis work by applying the perturbation theory, we conclude that in the Stoke’s regime, the phase noise will be stabilized and the noise amplitude will gradually fade away. This means if we choose the optimized Stoke’s shift, the combined output beam will have a stable output either in wavefront or the intensity. Therefore, our beam combining scheme has the potential of true applicability.

Two types of holographic gratings are also studied in this dissertation. It helps us to have a better understanding the phase matching condition towards the derivation of coupled-wave theory from Maxwell equation. The idea of applying the STIRAP in nonlinear optics to have efficient energy transfer in volume hologram generates an interesting eigenmode, the dark mode. The power flows is counter-intuitive in our adiabatic holographic gratings. The existence of the dark mode actually explains the physical meaning in this counter-intuitive coupling. Our adiabatic holographic gratings proves the merit of diffraction efficiency be hologram strength and polarization insensitive [39]. This will help a lot in transmission hologram manufacturing process and have more capability in optical switching, since it can transfer the energy of the TM polarized waves as well. In the future work, we actually consider the possibility to use the
adiabatic grating in reflection hologram, even in the case of non-symmetric incident and
diffracted waves.

For the perpendicular crossed-grating in the reflection hologram we had suggested, it
demonstrates the good spectral selectivity and excellent angular selectivity. The indirect
coupling behavior for the incident and reflect waves gives us a good explanation of such good
angular selectivity. It rises from the fact that the indirect coupling coefficients are inverse square
angular dependent. Due to four waves coupling inside this volume hologram, the horizontal
waves indeed spread the beam size in order to have large diffraction efficiency. This is the price
we paid for good angular selectivity. On top of that, the diffraction efficiency is very good and
less dependent on the thickness of the hologram. We are looking forward this type of reflection
hologram been manufactured and served as the reflection mirror for laser cavity.

Through our various study of wave propagation in the inhomogeneous media, the Z-
Helmholtz equation describes the undistinguishable TE and TM polarized waves in the constant
impedance media. In reality, in the case of not large index refraction discontinuity and incident
angle for layered media, the Z-H equation is good approximation for unpolarized light and
polarization independent detectors. If we take a good look at the Fresnel reflection formula, we
found that the reflection coefficient is independent of incident angle if the impedance is spatially
constant across the boundary of media. This is remarkable and nontrivial. If we choose a
periodic structure, take the photonic crystal for example, what happens to the band structure if
the whole impedance is constant? And how about the transmission and reflection coefficients?
Are they also independent of incident angle? There are a lot of questions waiting for us to
answer. In fact, there are numerous topics to discuss in the constant impedance media including the negative refractive index material.

As we investigate the waves propagation in inhomogeneous media, although the evolution of polarization can be fully described in Rytov’s formula by use of Frenet natural reference frame. The way we look at this problem is to use the equivalent ZLS equation and seek a better reference frame to avoid the calculus of torsion, which need higher order derivative of tangential vector of the light trajectory. In our new reference frame, the advantage is obvious, as can be seen in our numerical analysis. Nevertheless, during the calculation of polarization change in our new frame, it automatically generates a rotation angle, which is equal to the famous Berry phase. And one of the applications of this rotation angle is help us design a new rotation sensor. As long as the rotation subject to a certain axis, this is the case of 2-D rotation. In the future work, we will see the applicability of this new reference subject to 3-D rotation. In this general case, the relation between the polarization evolution and the trajectory solid angle subject to 2-rotation-axises needs further study and investigation.
APPENDIX: ANALYTICAL SOLUTION OF ROTATION ANGLE IN
SHARP CHANGE OF LIGHT PROPAGATION
The rotation angle integral indeed has the form,

\[
\gamma = \int_{\tau_1}^{\tau_2} \mathbf{a} \cdot \mathbf{t} \times \frac{d\mathbf{t}}{d\tau} \times \frac{\mathbf{a} \cdot \mathbf{t}}{1 - (\mathbf{a} \cdot \mathbf{t})^2} d\tau
\]

(A-1)

In the case of sharp change of wave propagation, the tangential vector of light trajectory turns from \( \mathbf{t}_1 \) into \( \mathbf{t}_2 \), within the intersection angle,

\[\rho = \arccos(\mathbf{t}_1 \cdot \mathbf{t}_2)\]  

(A-2)

We can assume the evolution \( \mathbf{t}(\tau) \) can be expressed in terms of \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \) by the following formula,

\[
\mathbf{t}(\tau) = \frac{\mathbf{t}_1 \sin(\rho(1 - x)) + \mathbf{t}_2 \sin(\rho x)}{\sin\rho}
\]

(A-3)

where \( 0 < x(\tau) < 1 \), \( x(\tau)=(\tau-\tau_1)(\tau_2-\tau_1) \), so that \( x(\tau=\tau_1)=0, x(\tau=\tau_2)=1 \). By change of variable, we can simplify eq.(A-1) into

\[
\gamma = -\mathbf{a} \cdot \left[ \frac{\rho}{\sin\rho} \mathbf{t}_1 \times \mathbf{t}_2 \right] \int_{0}^{1} \frac{\mathbf{a} \cdot \mathbf{t}(x)}{1 - (\mathbf{a} \cdot \mathbf{t})^2} dx
\]

(A-4)

Again, by setting \( \frac{\mathbf{a} \cdot \mathbf{t}_1}{\sin\rho} = p, \frac{\mathbf{a} \cdot \mathbf{t}_2}{\sin\rho} = q \), and \( P = p\sin\rho, Q = q - p\sin\rho, \rho x = \phi \),

We get \( \mathbf{a} \cdot \mathbf{t}(x) \equiv \mathbf{a} \cdot \mathbf{t}(\phi) = P\cos\phi + Q\sin\phi \), such that,
\[
\int_{0}^{1} \frac{\mathbf{a} \cdot \mathbf{t}(x)}{1 - (\mathbf{a} \cdot \mathbf{t})^2} \, dx \quad \rightarrow \quad \frac{1}{\rho} \int_{0}^{\rho} \frac{\mathbf{a} \cdot \mathbf{t}(\phi)}{1 - [\mathbf{a} \cdot \mathbf{t}(\phi)]^2} \, d\phi
\]  

(A-5)

Recognize that \( P \cos \phi + Q \sin \phi = r \cos \theta_2 \), \( r = \sqrt{P^2 + Q^2} \), so the integral becomes

\[
\frac{1}{\rho} \int_{\cos^{-1}(P \cos \rho, Q \sin \rho)}^{\cos^{-1}(P \cos \rho, Q \sin \rho)} \frac{r \cos \theta_2}{1 - r^2 \cos^2 \theta_2} f(\theta_2) \, d\theta_2
\]  

(A-6)

with \( f(\phi) = \frac{r \sin \theta_2}{P \sin \phi - Q \cos \phi} = \pm \frac{r \sin \theta_2}{r |\sin \theta_2|} = \pm 1 \).

Set \( \sin \phi = y \), then

\[
\int \frac{r \cos \theta_2}{1 - r^2 \cos^2 \theta_2} \, d\theta_2 = \int \frac{r}{(1 - r^2) + r^2 y^2} \, dy = \frac{1}{\sqrt{1 - r^2}} \tan^{-1}\left(\frac{r y}{\sqrt{1 - r^2}}\right)
\]  

(A-7)

Finally, from eqs.(A4- A7), we have the analytical solution of the integral \( \gamma \),

\[
\gamma = \pm \frac{\mathbf{a} \cdot \mathbf{t}_1 \times \mathbf{t}_2}{\sin \rho \sqrt{1 - r^2}} \left[ \tan^{-1}\left(\frac{r}{\sqrt{1 - r^2}} \sin^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{t}_2}{r}\right)\right) - \tan^{-1}\left(\frac{r}{\sqrt{1 - r^2}} \sin^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{t}_1}{r}\right)\right) \right]
\]  

(A-8)
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