Approximation by Bernstein polynomials at the point of discontinuity

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APPROXIMATION BY BERNSTEIN POLYNOMIALS AT THE POINT OF DISCONTINUITY

by

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ABSTRACT

Chlodovsky showed that if x_0 is a point of discontinuity of the first kind of the function f, then the Bernstein polynomials $B_n(f, x_0)$ converge to the average of the one-sided limits on the right and on the left of the function f at the point x_0. In 2009, Telyakovskii in [5] extended the asymptotic formulas for the deviations of the Bernstein polynomials from the differentiable functions at the first-kind discontinuity points of the highest derivatives of even order and demonstrated the same result fails for the odd order case. Then in 2010, Tonkov in [6] found the right formulation and proved the result that was missing in the odd order case. It turned out that the limit in the odd order case is related to the jump of the highest derivative. The proofs in these two cases look similar but have many subtle differences, so it is desirable to find out if there is a unifying principle for treating both cases. In this thesis, we obtain a unified formulation and proof for the asymptotic results of both Telyakovskii and Tonkov and discuss extension of these results in the case where the highest derivative of the function is only assumed to be bounded at the point under study.

DEDICATIONS

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INTRODUCTION

In mathematics, a polynomial is an expression of finite length constructed from variables and constants using only the operations of addition, subtraction, multiplication, and non-negative integer exponentiation. Because of their simple definition, their quick implementation on computer systems, and their ability to approximate and represent a tremendous variety of functions, polynomials are incredibly useful mathematical tools. Polynomials are the simplest kinds of functions. Not only can they be differentiated and integrated easily, but also they can be pieced together to form spline curves that can approximate any function to any preassigned accuracy. These are some of the motivations behind the study of approximations by polynomials of the form:

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n. \tag{1}$$

Here a polynomial is a linear combination of a set of linearly independent monomials $\{1, x, x^2, \ldots, x^n\}$, which is commonly called the power basis.

Since polynomials have many advantages in application, we want to obtain a good approximation for any function at any point in its domain. Most students may easily recall the remarkable theorem in analysis that establish this possibility:

Theorem 1 (Weierstrass Approximation Theorem) *Let I be a closed bounded interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Then for each positive number ϵ, there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that*

$$|f(x) - p(x)| < \epsilon \quad \textit{for all } x \in I. \tag{2}$$

This theorem permits us to approximate any continuous function with polynomials as accurately as we want without any assumption about differentiability. However, it does not give us a practical way to construct desired polynomial. This gives rise to Bernstein polynomials, which offer one way to prove the Weierstrass approximation theorem that every real-valued continuous function on a real interval $[a, b]$ can be uniformly approximated by polynomial functions over $\mathbb{R}$.

Bernstein Polynomials

Let $f(x)$ be a bounded function on $[0, 1]$. For $n = 1, 2, \ldots$, let

$$B_n(f, x) := \sum_{v=0}^{n} f\left(\tfrac{v}{n}\right) p_{n,v}(x), \tag{3}$$

with $p_{n,v}(x) := \binom{n}{v} x^v (1-x)^{n-v}$. We call $B_n(f, x)$ the Bernstein polynomials of f.

It is well known (see, for example, [3], Theorem 1.1.1) that if the function $f(x)$ is continuous at a point x_0, then its Bernstein polynomials at this point converge to $f(x_0)$ as $n \to \infty$. In general, we cannot expect statements of this kind for arbitrary discontinuous functions because $B_n(f, x)$ depends only upon $f(\frac{v}{n})$ whose value does not describe in general the behavior of $f(x)$ at irrational points. However, in certain cases, the behavior of the polynomials $B_n(f, x)$ may be described even for discontinuous functions $f(x)$.

Chlodovsky ([1], Theorem 1.9.1) showed that if L^+ and L^- denote the right and left upper limits of f at x while l^+ and l^- denote the right and left lower limits of f at x, then

$$\frac{l^- + l^+}{2} \leq \liminf_{n\to\infty} B_n(f, x) \leq \limsup_{n\to\infty} B_n(f, x) \leq \frac{L^- + L^+}{2}. \tag{4}$$

It follows that if x_0 is a first-kind discontinuity point of f, then $B_n(f, x_0)$ converge to the arithmetic mean of the left and right limits of the function f at x_0.

If the function $f(x)$ has a derivative of even order $2i, i = 1, 2, ...$, at $x_0 \in [0, 1]$, then, according to the results of E.V. Voronovskaya (for $i = 1$) and of S.N. Bernstein (for $i \geq 2$), the following asymptotic formulas hold for the deviation of $B_n(f, x_0)$ from $f(x_0)$ as $n \to \infty$ (see [3], Section 1.6.1):

$$f(x_0) - B_n(f, x_0) = -\frac{x_0(1-x_0)}{2n} f''(x_0) + o\left(\frac{1}{n}\right), \tag{5}$$

$$f(x_0) - B_n(f, x_0) = -\sum_{j=2}^{2i-1} \frac{T_{n,j}(x_0)}{j!n^j} f^{(j)}(x_0) - \frac{1}{i!}\left(\frac{x_0(1-x_0)}{2}\right)^i \frac{f^{(2i)}(x_0)}{n^i} + o\left(\frac{1}{n^i}\right), \tag{6}$$

where

$$T_{n,j} = \sum_{v=0}^{n} (v - nx)^j p_{n,v}(x).$$

Here the derivatives are in the sense of Peano.

Peano Derivatives

In order to discuss recent results of the asymptotic formulas, we need the following definition.

Definition 1 *Let $(a, b) \subseteq \mathbb{R}$ and $m \in \mathbb{N}$. The function $f : (a, b) \to \mathbb{R}$ is called m times Peano differentiable at $x_0 \in (a, b)$ if there are $\{f_i\}_{i=1}^m \subseteq \mathbb{R}$ such that*

$$P_m(f, x_0 + h, x_0) \equiv f(x_0 + h) - f(x_0) - hf_1 - \cdots - \frac{h^m}{m!} f_m = o(h^m), \tag{7}$$

as $h \to 0$. Then f_i is called the ith Peano derivative of f at x_0.

Since f_i depends on x_0, we denote $f_i := f_i(x_0)$. For more information about differentiability of Peano derivatives, see [2]. If f has an nth Peano derivative $f_n(x_0)$ at x_0, then it has also a kth Peano derivative $f_k(x_0)$, $k = 1, 2, \ldots, n-1$, and $f_1(x_0) = f'(x_0)$, the ordinary first derivative. If f has an ordinary nth derivative, $f^n(x_0)$, at x_0, then Taylor's theorem shows that $f_n(x_0)$ exists and equals the ordinary nth derivative [4].

Now we are ready to discuss some recent results regarding the Asymptotic Formulas.

BACKGROUND

The following sections provide information of asymptotic formulae of even and odd order, respectively.

Asymptotic Formulae for Even Order

In 2009, Telyakovskii [5] extended Bernstein's asymptotic formulas for the differences between the Bernstein polynomials and the functions at the points of first-kind discontinuity of the highest even-order derivative. It is shown that at the point $x_0 \in (0,1)$ where the derivative of the function f of order $2i-1, i = 1, 2, ...$, exists and the derivative $f^{(2i)}$ has a discontinuity of the first kind, the value of the derivative of even order $f^{(2i)}(x_0)$ in formulas (5) and (6) can be replaced by the arithmetic mean of the right and left derivatives of order $2i$ at the point x_0. Telyakovskii also demonstrated that his theorem is not necessarily true for the odd-order case by the example at the end of this subsection.

Theorem 2 (Telyakovskii) *[5] When the function $f(x)$ possesses Peano derivatives up to order $2i-1, i = 1, 2, \ldots,$ at a point $x_0 \in (0,1)$, for all t such that the points $x_0+t \in [0,1]$ and function $v(t) \to 0$ as $t \to 0$, the deviations of the Bernstein polynomials from the function f at the point x_0 satisfy the asymptotic equality as $n \to \infty$*

$$\begin{aligned} f(x_0) - B_n(f, x_0) = &- \sum_{j=2}^{2i-1} \frac{T_{n,j}(x_0)}{j!n^j} f^{(j)}(x_0) \\ &- \frac{1}{i!n^i} \left(\frac{x_0(1-x_0)}{2} \right)^i \frac{f^{(2i)}(x_0+0) + f^{(2i)}(x_0-0)}{2} \\ &+ o\left(\frac{1}{n^i}\right), \end{aligned} \tag{8}$$

whenever

$$f(x_0+t) = \sum_{j=0}^{2i-1} \frac{t^j}{j!} f^{(j)}(x_0) + \frac{t^{2i}}{(2i)!}(f^{(2i)}(x_0+0)+v(t)), \quad t>0, \tag{9}$$

$$f(x_0+t) = \sum_{j=0}^{2i-1} \frac{t^j}{j!} f^{(j)}(x_0) + \frac{t^{2i}}{(2i)!}(f^{(2i)}(x_0-0)+v(t)), \quad t<0. \tag{10}$$

The following example illustrates the result for the even order discontinuous point.

$$h(x) := \begin{cases} x^2 - x + \frac{1}{2} & \text{if } 0 \le x \le \frac{1}{2}, \\ x - x^2 & \text{if } \frac{1}{2} < x \le 1. \end{cases} \tag{11}$$

Since $h''(x)$ is discontinuous at $x = \frac{1}{2}$, we can apply the result and get,

$$h(\tfrac{1}{2}) - B_n(h, \tfrac{1}{2}) = o(\tfrac{1}{n}). \tag{12}$$

which can also be verified by direct computation of $B_n(h, \frac{1}{2})$.

However, the same result fails for the odd order case as illustrated by this example.

$$g(x) := \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2}, \\ 1 - x & \text{if } \frac{1}{2} < x \le 1. \end{cases} \tag{13}$$

In fact,

$$g(\tfrac{1}{2}) - B_n(g, \tfrac{1}{2}) = \tfrac{1}{\sqrt{2\pi n}} + o(\tfrac{1}{\sqrt{n}}). \tag{14}$$

Asymptotic Formulae for Odd Order

Telyakovskii's demonstration that his even-order result is not true for odd-order discontinuity may give some kind of impression that there is no odd-order analog of this result. Yet, surprisingly, in 2010, Tonkov [6] found the right formulation and proved the result that is missing in the odd-order case. He considered the case when an odd-order derivative

$f^{(2i+1)}(x_0)$ has a first-kind discontinuity. By following almost the same steps in [5], the main difference in the result between even and odd cases lies in the leading order term. One is the average of the right and left limits of the highest derivative, the other is half of the jump itself.

Theorem 3 (Tonkov) *[6] When the function $f(x)$ possesses Peano derivatives up to order $2i, i = 0, 1, \ldots,$ at a point $x_0 \in (0,1)$, for all t such that the points $x_0 + t \in [0,1]$ and function $v(t) \to 0$ as $t \to 0$, the deviations of the Bernstein polynomials from the function f at the point x_0 satisfy the asymptotic equality as $n \to \infty$*

$$
\begin{aligned}
f(x_0) - B_n(f, x_0) = &- \sum_{j=2}^{2i} \frac{T_{n,j}(x_0)}{j! n^j} f^{(j)}(x_0) + (f^{(2i+1)}(x_0 - 0) - f^{(2i+1)}(x_0 + 0)) \\
&\times \frac{2^{2i} i!}{(2i+1)! \sqrt{\pi} n^{i+\frac{1}{2}}} \left(\frac{x_0(1-x_0)}{2} \right)^{i+\frac{1}{2}} \\
&+ o\left(\frac{1}{n^{i+\frac{1}{2}}} \right),
\end{aligned} \tag{15}
$$

whenever

$$
f(x_0 + t) = \sum_{j=0}^{2i} \frac{t^j}{j!} f^{(j)}(x_0) + \frac{t^{2i+1}}{(2i+1)!} (f^{(2i+1)}(x_0 + 0) + v(t)), \quad t > 0, \tag{16}
$$

$$
f(x_0 + t) = \sum_{j=0}^{2i} \frac{t^j}{j!} f^{(j)}(x_0) + \frac{t^{2i+1}}{(2i+1)!} (f^{(2i+1)}(x_0 - 0) + v(t)), \quad t < 0. \tag{17}
$$

Motivated by these results, in the present paper we prove a more general formulation, which will include both Theorem 2 and Theorem 3 as special cases.

UNIFYING ASYMPTOTIC FORMULA

We will first give a unified proof of the results in [5] and [6].

Lemma 1 *Suppose that* $x \in (0,1)$, $\delta_n = n^{-\frac{2}{5}}$, *and*

$$\sigma_{n,r}(x) := \sum_{v} \left| \frac{v}{n} - x \right|^r \rho_{n,v}(x), \quad r = 0, 1, 2, \ldots,$$

where the sum is taken over v *satisfying* $0 < \frac{v}{n} - x \leq \delta_n$ *or* $0 < x - \frac{v}{n} \leq \delta_n$. *Then*

$$\lim_{n\to\infty} n^{\frac{r}{2}} \sigma_{n,r}(x) = \frac{\Gamma(\frac{r}{2} + \frac{1}{2})}{2\sqrt{\pi}} [2x(1-x)]^{\frac{r}{2}}. \tag{18}$$

Proof: Consider the case when $0 < \frac{v}{n} - x \leq \delta_n$. The proof for case when $0 < x - \frac{v}{n} \leq \delta_n$ is similar.

By Theorem 1.5.2 from [3], if $\alpha > \frac{1}{3}$, then for each $x \in (0,1)$ satisfying the condition

$$\left| \frac{v}{n} - x \right| \leq \frac{1}{n^\alpha}$$

uniformly with respect to v, the following relation holds:

$$\lim_{n\to\infty} \frac{\rho_{n,v}(x)}{P_{n,v}(x)} = 1,$$

where $P_{n,v}(x) = \left(\frac{1}{2\pi x(1-x)n}\right)^{\frac{1}{2}} \exp\left[-\frac{n}{2x(1-x)}\left(\frac{v}{n} - x\right)^2\right]$.

Therefore, as $n \to \infty$, the following equivalent relation holds uniformly with respect to these v:

$$\left(\frac{v}{n} - x\right)^r \rho_{n,v}(x) \approx \left(\frac{n}{2\pi x(1-x)}\right)^{\frac{1}{2}} \int_{\frac{v}{n}}^{\frac{v+1}{n}} (u-x)^r \exp\left[-\frac{n}{2x(1-x)}(u-x)^2\right] du.$$

This implies that, as $n \to \infty$ and let $v = (u-x)\sqrt{\frac{n}{2x(1-x)}}$,

$$\sigma_{n,r}(x) \approx \left(\frac{n}{2\pi x(1-x)}\right)^{\frac{1}{2}} \int_x^{x+\delta_n} (u-x)^r \exp\left[-\frac{n}{2x(1-x)}(u-x)^2\right] du$$
$$= \frac{1}{\sqrt{\pi}}\left(\frac{2x(1-x)}{n}\right)^{\frac{r}{2}} \int_0^{B_n} v^r e^{-v^2} dv, \tag{19}$$

where $B_n = \delta_n \left(\frac{n}{2x(1-x)}\right)^{\frac{1}{2}}$.

By the choice of δ_n, we have $B_n \to \infty$ as $n \to \infty$. Therefore, it follows from (19) that

$$\lim_{n\to\infty} n^{\frac{r}{2}} \sigma_{n,i}(x) = \frac{1}{\sqrt{\pi}}[2x(1-x)]^{\frac{r}{2}} \int_0^{\infty} v^r e^{-v^2} dv. \tag{20}$$

Since

$$\int_0^{\infty} v^r e^{-v^2} dv = \frac{1}{2}\Gamma\left(\tfrac{r}{2}+\tfrac{1}{2}\right),$$

it follows that relation (20) is equivalent to (18):

$$\lim_{n\to\infty} n^{\frac{r}{2}} \sigma_{n,r}(x) = \frac{\Gamma(\frac{r}{2}+\frac{1}{2})}{2\sqrt{\pi}}[2x(1-x)]^{\frac{r}{2}}.$$

□

Lemma 2 *Let* $x \in (0,1)$, n *and* r *be natural numbers,* $N_+ = \{v \in \mathbb{N} : \; x < \frac{v}{n} \le 1\}$ *and* $N_- = \{v \in \mathbb{N} : \; 0 \le \frac{v}{n} < x\}$. *Then*

$$\lim_{n\to\infty} n^{\frac{r}{2}} \sum_v \left|\frac{v}{n} - x\right|^r \rho_{n,v}(x) = \frac{\Gamma(\frac{r}{2}+\frac{1}{2})}{2\sqrt{\pi}}[2x(1-x)]^{\frac{r}{2}}, \tag{21}$$

where the sum is taken over either $v \in N_+$ *or* $v \in N_-$.

PROOF: Consider the sum

$$n^{\frac{r}{2}} \sum_{v=0}^{n} \left|\frac{v}{n} - x\right|^r \rho_{n,v}(x). \tag{22}$$

Notice that if the number nx is integer, then it belongs to neither N_+ nor N_-. Since the term with $v = nx$ in (22) is zero, we can represent this sum as $\sum_{v\in N_+} + \sum_{v\in N_-}$.

Set $\delta_n = n^{-\frac{2}{5}}$ and split the sum from (21) into four sums. Let $\sum_{(1)}$ be the sum over $v \in N_+$ for which $\frac{v}{n} - x \le \delta_n$, $\sum_{(2)}$ be the sum over $v \in N_+$ for which $\frac{v}{n} - x > \delta_n$, $\sum_{(3)}$ be the sum over $v \in N_-$ such that $x - \frac{v}{n} \le \delta_n$, $\sum_{(4)}$ be the sum over $v \in N_-$ such that $x - \frac{v}{n} > \delta_n$.

By Lemma 1, $\sum_{(1)}$ and $\sum_{(3)}$ are equal and

$$n^{\frac{r}{2}} \sum\nolimits_{(1)} \left|\frac{v}{n} - x\right|^r \rho_{n,v}(x) = \frac{\Gamma(\frac{r}{2} + \frac{1}{2})}{2\sqrt{\pi}} [2x(1-x)]^{\frac{r}{2}}. \tag{23}$$

To estimate the sum $\sum_{(2)}$ and $\sum_{(4)}$, we apply result of [3, Theorem 1.5.3]: If $x \in (0,1)$ and

$$0 \le z \le \tfrac{3}{2}\sqrt{nx(1-x)}, \tag{24}$$

then

$$\sum_v \rho_{n,v}(x) \le 2e^{-z^2}, \tag{25}$$

where the sum is taken over values of v such that

$$|v - nx| \ge 2z\sqrt{nx(1-x)}. \tag{26}$$

Let $z \le \frac{3}{2}[nx(1-x)]^{\frac{1}{4}}$. Then conditions (24) and (26) are of the form

$$[nx(1-x)]^{\frac{1}{4}} \le \sqrt{nx(1-x)} \quad \text{ and } \quad \left|v - nx\right| \ge 3\left(\frac{x^3(1-x)^3}{n}\right)^{\frac{1}{4}}.$$

For the number v appearing in $\sum_{(2)}$, both of these estimates hold if n is sufficiently large. Hence, it follows from (25) that, for such integers n, we have

$$\sum\nolimits_{(2)} \left(\frac{v}{n} - x\right)^r \rho_{n,v}(x) \le \sum\nolimits_{(2)} \rho_{n,v}(x) \le 2\exp\left[-\frac{9}{4}\sqrt{nx(1-x)}\right],$$

which together with (23) yields (21). □

Main Result

The proofs in the even-order and odd-order cases look similar but have many subtle differences, so it is desirable to find out if there is a unifying principle for treating both cases. We now state and prove the asymptotic formula up to any order. Note that the power of the leading order is half of the order of highest derivative, which is parallel to the results in both [5] and [6].

Theorem 4 *Suppose the function $f(x)$ possesses Peano derivatives up to order $r-1$ at a point $x_0 \in (0,1)$, for all t such that the points $x_0+t \in [0,1]$ and function $v(t) \to 0$ as $t \to 0$ with*

$$f(x_0+t) = \sum_{j=0}^{r-1} \frac{t^j}{j!} f^{(j)}(x_0) + \frac{t^r}{r!}(f_r^+ + v(t)), \quad t > 0, \tag{27}$$

$$f(x_0+t) = \sum_{j=0}^{r-1} \frac{t^j}{j!} f^{(j)}(x_0) + \frac{t^r}{r!}(f_r^- + v(t)), \quad t < 0. \tag{28}$$

Then the deviations of the Bernstein polynomials from the function f at the point x_0 satisfy the asymptotic equality as $n \to \infty$

$$\begin{aligned} f(x_0) - B_n(f,x_0) = &- \sum_{j=2}^{r-1} \frac{T_{n,j}(x_0)}{j!n^j} f^{(j)}(x_0) \\ &- \left(f_r^+ + (-1)^r f_r^-\right) \frac{\Gamma(\frac{r}{2}+\frac{1}{2})}{2\sqrt{\pi} r!} \left[\frac{2x_0(1-x_0)}{n}\right]^{\frac{r}{2}} \\ &+ o\left(\frac{1}{n^{\frac{r}{2}}}\right). \end{aligned} \tag{29}$$

PROOF: Suppose that $N_+ = \{v \in \mathbb{N} : x_0 < \frac{v}{n} \leq 1\}$ and $N_- = \{v \in \mathbb{N} : 0 \leq \frac{v}{n} < x_0\}$. First, we compute the difference,

$$f(x_0) - B_n(f,x_0) = \sum_{v=0}^{n} [f(x_0) - f(\tfrac{v}{n})]\rho_{n,v}(x_0). \tag{30}$$

Now we apply the definition of Peano derivative,

$$\begin{aligned}
f(x_0) - B_n(f,x_0) &= -\sum_{v=0}^{n}\left[\sum_{j=1}^{r-1}\frac{f^{(j)}(x_0)}{j!}(\tfrac{v}{n}-x_0)^j\right]\rho_{n,v}(x_0)\\
&\quad - \sum_{v\in N_+}\frac{1}{r!}(\tfrac{v}{n}-x_0)^r\left[f_r^+ + \upsilon(\tfrac{v}{n}-x_0)\right]\rho_{n,v}(x_0)\\
&\quad - \sum_{v\in N_-}\frac{1}{r!}(x_0-\tfrac{v}{n})^r\left[f_r^- + \upsilon(\tfrac{v}{n}-x_0)\right]\rho_{n,v}(x_0)\\
&= -\sum_{j=2}^{r-1}\frac{T_{n,j}(x_0)}{j!n^j}f^{(j)}(x_0)\\
&\quad - \frac{f_r^+}{r!}\sum_{v\in N_+}|\tfrac{v}{n}-x_0|^r\rho_{n,v}(x_0)\\
&\quad - \frac{(-1)^r f_r^-}{r!}\sum_{v\in N_-}|\tfrac{v}{n}-x_0|^r\rho_{n,v}(x_0)\\
&\quad - \sum_{v=0}^{n}\frac{1}{r!}(x_0-\tfrac{v}{n})^r\upsilon(\tfrac{v}{n}-x_0)\rho_{n,v}(x_0).
\end{aligned}$$

By Lemma 2 and since $\upsilon(t)\to 0$ as $t\to 0$ [5, See Lemma 3],

$$\begin{aligned}
f(x_0) - B_n(f,x_0) &= -\sum_{j=2}^{r-1}\frac{T_{n,j}(x_0)}{j!n^j}f^{(j)}(x_0)\\
&\quad - \left(f_r^+ + (-1)^r f_r^-\right)\frac{\Gamma(\frac{r}{2}+\frac{1}{2})}{2\sqrt{\pi}r!}\left[\frac{2x_0(1-x_0)}{n}\right]^{\frac{r}{2}}\\
&\quad + o\left(\frac{1}{n^{i+\frac{1}{2}}}\right).
\end{aligned}$$

□

By the definition of Peano derivative, we have

Corollary 1 *If $f^{(r)}(x)$ exists in a punctured neighborhood of x_0, then $f_r^+ = f^{(r)}(x_0+0)$ and $f_r^- = f^{(r)}(x_0-0)$. Moreover, Theorem 4 will take the forms of theorems of Telyakovskii and Tonkov according to the parity of r.*

GENERALIZATION

In Theorem 4, we assume the existence of the one-sided derivatives (27) — (28). We are interested in the case when the first-kind discontinuity does not exist and obtain the following result.

Definition 2 *We define the following function: for x and $x+t \in [0,1]$,*

$$f^{[r]}(x,t) := \frac{f(x+t) - \sum_{j=0}^{r-1} \frac{t^j}{j!} f^{(j)}(x)}{t^r/r!}.$$

Theorem 5 *Suppose the function $f(x)$ possesses Peano derivatives up to order $r-1$ in $(0,1)$. For the point $x_0 \in (0,1)$, let L^+ and L^- denote the right and left upper limits of $f^{[r]}(x_0,t)$ and let l^+ and l^- denote its right and left lower limits, respectively. Then*

$$\begin{aligned} \frac{l^+ + l^-}{2} &\leq \liminf_{n\to\infty}[B_{n,r}(f,x_0) - f(x_0)]C^{-1}(n,r,x_0) \\ &\leq \limsup_{n\to\infty}[B_{n,r}(f,x_0) - f(x_0)]C^{-1}(n,r,x_0) \leq \frac{L^+ + L^-}{2}, \qquad \textit{if } r \textit{ is even}, \qquad (31) \\ \frac{l^+ - L^-}{2} &\leq \liminf_{n\to\infty}[B_{n,r}(f,x_0) - f(x_0)]C^{-1}(n,r,x_0) \\ &\leq \limsup_{n\to\infty}[B_{n,r}(f,x_0) - f(x_0)]C^{-1}(n,r,x_0) \leq \frac{L^+ - l^-}{2}, \qquad \textit{if } r \textit{ is odd}, \qquad (32) \end{aligned}$$

where

$$B_{n,r}(f,x_0) := B_n(f,x_0) - \sum_{j=2}^{r-1} \frac{T_{n,j}(x_0)}{j!n^j} f^{(j)}(x_0)$$

is an expansion up to order $r-1$ and

$$C(n,r,x_0) := \frac{\Gamma(\frac{r}{2}+\frac{1}{2})}{r!\sqrt{\pi}} \left[\frac{2}{n}x_0(1-x_0)\right]^{\frac{r}{2}}$$

is a factor independent of f.

PROOF: Since $f(x)$ possesses Peano derivatives up to order r in $(0,1)$,

$$\begin{aligned}B_n(f,x_0)-f(x_0) &= \sum_{v=0}^{n}[f(\tfrac{v}{n})-f(x_0)]\rho_{n,v}(x_0)\\ &= \sum_{v=0}^{n}\left[\sum_{j=1}^{r-1}\frac{f^{(j)}(x_0)}{j!}(\tfrac{v}{n}-x_0)^j+\tfrac{f^{[r]}(x_0,\frac{v}{n}-x_0)}{r!}(\tfrac{v}{n}-x_0)^r\right]\rho_{n,v}(x_0).\end{aligned}$$

Now letting $B_{n,r}(f,x_0)=B_n(f,x_0)-\sum_{j=2}^{r-1}\frac{T_{n,j}(x_0)}{j!n^j}f^{(j)}(x_0)$, we have

$$B_{n,r}(f,x_0)-f(x_0)=\sum_{v=0}^{n}\frac{f^{[r]}(x_0,\frac{v}{n}-x_0)}{r!}(\tfrac{v}{n}-x_0)^r\rho_{n,v}(x_0). \tag{33}$$

By assumption, for given $\epsilon>0$, there exists a $\delta>0$ such that

$$\begin{cases} l^+-\epsilon\le f^{[r]}(x_0,\frac{v}{n}-x_0)\le L^++\epsilon & \text{if } 0<\frac{v}{n}-x_0<\delta,\\ l^--\epsilon\le f^{[r]}(x_0,\frac{v}{n}-x_0)\le L^-+\epsilon & \text{if } 0<x_0-\frac{v}{n}<\delta.\end{cases}$$

Let $N_+=\{v\in\mathbb{N}:x_0<\frac{v}{n}\le 1\}$, $N_-=\{v\in\mathbb{N}:0\le\frac{v}{n}<x_0\}$, and

$$C(n,r,x_0)=\frac{\Gamma(\frac{r}{2}+\frac{1}{2})}{r!\sqrt{\pi}}\left[\frac{2}{n}x_0(1-x_0)\right]^{\frac{r}{2}}.$$

We now can express the sum on the right hand side of (33) as $\sum_{v\in N_+}+\sum_{v\in N_-}$. Noting that the dominating parts of the sum $\sum_{v\in N_+}$ and $\sum_{v\in N_-}$ are whenever $|\frac{v}{n}-x_0|<\delta_n$, we modify the proof in Lemma 2 and obtain:

$$\tfrac{l^+-\epsilon}{2}\cdot C(n,r,x_0)\le\sum_{v\in N_+}\frac{f^{[r]}(x_0,\frac{v}{n}-x_0)}{r!}(\tfrac{v}{n}-x_0)^r\rho_{n,v}(x_0)\le\tfrac{L^++\epsilon}{2}\cdot C(n,r,x_0), \tag{34}$$

and if r is even,

$$\tfrac{l^--\epsilon}{2}\cdot C(n,r,x_0)\le\sum_{v\in N_-}\frac{f^{[r]}(x_0,\frac{v}{n}-x_0)}{r!}(\tfrac{v}{n}-x_0)^r\rho_{n,v}(x_0)\le\tfrac{L^-+\epsilon}{2}\cdot C(n,r,x_0), \tag{35}$$

and if r is odd,

$$\tfrac{-L^{-}-\epsilon}{2}\cdot C(n,r,x_0)\le\sum_{v\in N_-}\frac{f^{[r]}(x_0,\frac{v}{n}-x_0)}{r!}(\tfrac{v}{n}-x_0)^r\rho_{n,v}(x_0)\le\tfrac{-l^{-}+\epsilon}{2}\cdot C(n,r,x_0). \qquad (36)$$

We add (35) and (36) to (34), respectively, and obtain

$$\begin{aligned}\frac{l^{+}+l^{-}}{2}&\le\liminf_{n\to\infty}[B_{n,r}(f,x_0)-f(x_0)]C^{-1}(n,r,x_0)\\&\le\limsup_{n\to\infty}[B_{n,r}(f,x_0)-f(x_0)]C^{-1}(n,r,x_0)\le\frac{L^{+}+L^{-}}{2},\\\frac{l^{+}-L^{-}}{2}&\le\liminf_{n\to\infty}[B_{n,r}(f,x_0)-f(x_0)]C^{-1}(n,r,x_0)\\&\le\limsup_{n\to\infty}[B_{n,r}(f,x_0)-f(x_0)]C^{-1}(n,r,x_0)\le\frac{L^{+}-l^{-}}{2}.\end{aligned}$$

□

As a consequence, the asymptotic formulas for the deviation of Bernstein polynomials from functions at the first-kind discontinuity points of the highest derivatives of even order in [5], odd order in [6], and our unifying asymptotic formulas are all special cases of the above Theorem.

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