Modification to Einstein's field equations imposed by string theory and consequences for the classical tests of general relativity

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MODIFICATION TO EINSTEIN'S FIELD EQUATIONS IMPOSED BY STRING THEORY AND CONSEQUENCES FOR THE CLASSICAL TESTS OF GENERAL RELATIVITY

by

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ABSTRACT

String theory imposes slight modifications to Einstein's equations of general relativity (GR). In [4] the authors claim that the gravitational field equations in empty space, which in GR are just $R_{\mu\nu} = 0$, should hold one extra term which is first order in the string constant α' and proportional to the Riemann curvature tensor squared. They do admit, however, that this simple modification is just schematic. In [1] the authors use modified equations which are coupled to the dilaton field. We show that the equations given in [4] do not admit an isotropic solution; justification of these equations would require sacrificing isotropy. We thus investigate the consequences of the coupled equations from [1] and the black-hole solution they give there. We calculate the additional perihelion precession of Mercury, the added deflection of photons by the sun, and the extra gravitational redshift which should be present if these equations hold. We determine that additional effects due to string theory in each of these cases are quite minuscule.

DEDICATIONS

For my family.

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INTRODUCTION

In Steven Weinberg's classic text [5], he makes a strong effort to develop general relativity logically from a minimal set of assumptions. To begin, he quickly develops special relativity by (i) assuming Newtonian mechanics governs the behavior of a particle in its rest frame and (ii) imposing Lorentz covariance to determine the laws obeyed by moving particles. To make the transition to general relativity, he imposes the Principle of Equivalence, in which he states that "at every spacetime point in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system such that, within a sufficiently small region [around] the point in question, the laws of nature take the same form [as in] special relativity." Thus the dynamics of a particle in a given gravitational field lies on strong logical foundations.

However, upon "derivation" of the differential equations that determine the gravitational field created by matter, Weinberg notes that the logical footing is not quite as strong. His method of choice for the derivation involves first choosing a locally inertial frame around the matter, in which the gravitational field is weak near the matter and hence described by linear partial differential equations. Then we can use a coordinate transformation to find, with respect to an arbitrary frame, the field in the vicinity of matter. This field obeys nonlinear partial differential equations in general since the gravitational field itself carries energy and momentum, thus acting as its own source. Since our empirical information about the weak field equations is limited due to the extreme weakness of gravitational waves, some guess work is inevitable in carrying out the first step in the derivation.

Indeed, for a metric $g_{\mu\nu}$, Newton's law of gravitation for nonrelativistic mass amounts to

$$\nabla^2 g_{00} = -8\pi G\, T_{00}\,,$$

from which we guess that for a general distribution $T_{\alpha\beta}$ of energy and momentum, the weak-field equations take the form

$$(G_0)_{\mu\nu} = -8\pi G \, (T_0)_{\mu\nu} \, ,$$

where $(G_0)_{\mu\nu}$ is a linear combination of $g_{\mu\nu}$ and its first and second derivatives. Using the Principle of Equivalence to move to the second step of the derivation, we find

$$G_{\mu\nu} = -8\pi G \, T_{\mu\nu}$$

in an arbitrary frame, where $G_{\mu\nu} \to (G_0)_{\mu\nu}$ for weak fields. A few further considerations detailed in [5], including the scale invariance of gravity (which is by no means well established) and the symmetry of $T_{\mu\nu}$, lead to the implication that

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} \, R \, ,$$

so that in regions devoid of matter, we have Einstein's field equations reducing simply to

$$R_{\mu\nu} = 0 \, . \tag{1}$$

Now, the guesswork in the above derivation has not been ignored by relativists, and many alternative theories of gravity have been proposed throughout the decades. Perhaps the easiest one to state is the so-called $f(R)$ gravity. It turns out that (1) can be derived from an action principle, in which $g_{\mu\nu}$ must extremize

$$S = -\frac{1}{16\pi G} \int \sqrt{-g} \, R \, d^4x \, .$$

Instead, one might choose to take an action with a more general scalar acting as the Lagrangian density:

$$S = -\frac{1}{16\pi G} \int \sqrt{-g}\, f(R)\, d^4x\,.$$

This leads to the field equations

$$\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R - \partial_\mu \partial_\nu\right) f'(R) + \frac{1}{2}\, g_{\mu\nu}\, f(R) = 0$$

in the absence of matter [2]. Such equations can remove gravity's scale invariance. For instance, if we choose $f(R) = R + c\,R^2$ for some constant c, dimensional analysis shows that this will only cause changes to classical gravity on small length scales, since $[R] = L^{-2}$.

Alternatively, Brans and Dicke postulated the existence of a scalar field ϕ which mediates long-range forces; after all, we have a vector field A_μ associated with electrodynamics and a second-rank tensor field $g_{\mu\nu}$ corresponding to gravity. In fact, their theory is inspired by Mach's principle, which states that an object's inertia may depend on its motion with respect to the mass distribution of the entire universe. The Brans-Dicke theory is outlined in [5, 2], with field equations given by

$$\Box^2 \phi = \frac{8\pi}{3 + 2\omega}\, T^\mu_\mu\,,$$

$$\begin{aligned} R_{\mu\nu} - \frac{R}{2}\, g_{\mu\nu} = \frac{8\pi}{\phi}\, T_{\mu\nu} &+ \frac{\omega}{\phi^2}\, (\partial_\mu \phi\, \partial_\nu \phi - \frac{1}{2}\, g_{\mu\nu}\, \partial_\alpha \phi\, \partial^\alpha \phi) \\ &+ \frac{1}{\phi}\, (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu}\, \Box^2 \phi)\,. \end{aligned}$$

The first term on the right of the second equation above makes it clear that ϕ plays the role of $1/G$, so that the strength of gravitational coupling becomes a dynamical variable.

Thus it is no surprise that string theory imposes its own modifications to Einstein's theory of gravitation. String theory postulates that our universe is actually a manifold with several extra dimensions in addition to the three spatial and one temporal directions we experience with our senses. The modifications to Einstein's equations come about when we seek effective four-dimensional equations that govern phenomena in spacetime when the extra dimensions are considered to have negligible effect. Green, Schwarz, and Witten suggested one modification to Einstein's equations in [4]; we spend a portion of this thesis investigating consequences of that modification. We then move on to determine the effects of the generally accepted equations written in [1] on well-studied phenomena in our solar system.

BACKGROUND RELATIVITY

Einstein's theory of general relativity, along with any modification thereof, is written in the language of differential geometry. Here we include a short summary of Riemannian geometry (as it relates to gravity) in which we review all formulae which will become necessary for calculations presented later in this thesis.

We remind the reader that spacetime is postulated to be a smooth manifold, i.e. a topological space endowed with a cover (called an atlas) consisting of open sets (called coordinate charts) homeomorphic to open balls in $\mathbb{R}^4$. On the intersection of two charts, the two corresponding homeomorphisms may be combined to become a map from one open set of $\mathbb{R}^4$ to another; such transition maps are required to be smooth.

A vector is associated with a point on the manifold and lives in a vector space called the tangent plane at that point. Thus we do not picture vectors as arrows on the manifold but instead as arrows tangent to the manifold. Such vectors can be defined as equivalence classes of directional derivatives along curves, as agrees with our intuition. In addition, there exists a dual vector space at each point which consists of all linear maps of vectors at that point to the field of real numbers; such is called the cotangent plane. Instead of stopping here, we can consider multi-linear maps from any number of vectors and covectors to the reals; such objects are called tensors.

Choosing a coordinate chart at any given point defines a coordinate basis for the vectors at that point. This in turn specifies a dual basis for the covectors, and we can continue the trend to find a coordinate basis for any tensor at that point. Tensors then have components written $T^{\mu\nu\cdots}{}_{\rho\lambda\cdots}$ with respect to the coordinate basis, where we label the coordinates x^μ or x^ν, etc., with Greek indices taking values $0, 1, 2, 3$.

We notice an immediate problem when we go to differentiate a vector field on a manifold: the definition of partial differentiation requires us to subtract vectors at two different points — from two different vector spaces. This has no meaning. We thus require a structure on the manifold which allows us to transport a vector from one tangent space to another; then we can perform the subtraction. The connection coefficients $\Gamma^{\mu}{}_{\nu\lambda}$ allow us to do this, and we end up with a way to perform covariant differentiation:

$$\nabla_{\mu} A^{\nu} = \partial_{\mu} A^{\nu} + \Gamma^{\nu}{}_{\lambda\mu} A^{\lambda} .$$

In this equation, we have employed the Einstein summation convention in which Greek indices that are repeated (once upstairs, once downstairs) in a term are implicitly summed from 0 to 3. A connection allows us to define a geodesic, i.e. a curve whose tangent vector is parallel-transported into itself along the curve. In symbols:

$$\frac{d^2 x^{\mu}}{dp^2} + \Gamma^{\mu}{}_{\nu\lambda} \frac{dx^{\nu}}{dp} \frac{dx^{\lambda}}{dp} = 0 ,$$

where p parametrizes the path. In fact, this is the equation that governs the motion of a particle through curved spacetime, i.e. in the presence of gravitational fields. The second derivative present allows us to make analogy with Newtonian mechanics and associate the $\Gamma^{\mu}{}_{\nu\lambda}$'s with a sort of force.

With the concept of parallel transport, there also follows a notion of curvature. If a vector is parallel-transported around a parallelogram on the manifold, it may not return to its initial state when it finds its way back to the starting point. Such a discrepancy between the initial and final vectors in this process is determined by the Riemann-curvature tensor:

$$\Delta A_{\mu} = \frac{1}{2} R^{\sigma}{}_{\mu\nu\rho} A_{\sigma} \oint x^{\rho} \, dx^{\nu}$$

in which the integration is along the closed path and

$$R^{\lambda}{}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}{}_{\mu\kappa}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}{}_{\mu\nu}}{\partial x^{\kappa}} + \Gamma^{\eta}{}_{\mu\kappa}\, \Gamma^{\lambda}{}_{\nu\eta} - \Gamma^{\eta}{}_{\mu\nu}\, \Gamma^{\lambda}{}_{\kappa\eta}\,.$$

Repeatedly contracting indices of the Riemann-curvature tensor produces first the Ricci tensor, then the scalar curvature:

$$R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa}\ , \qquad R^{\mu}{}_{\mu} \equiv R\,.$$

In general relativity, it is conventional to put additional structure on the spacetime manifold: a smooth tensor field $g_{\mu\nu}$ which is traditionally associated with defining infinitesimal lengths on the manifold through

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}\,.$$

For covariant differentiation to be compatible with this metric on the manifold, our connection coefficients must be given by

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{g^{\mu\rho}}{2} \left(\frac{\partial g_{\rho\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\rho\lambda}}{\partial x^{\nu}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\rho}} \right)\,.$$

The derivatives in this equation show that, since we make the association $\Gamma^{\mu}{}_{\nu\lambda} \sim$ gravitational force, we also interpret $g_{\mu\nu} \sim$ gravitational potential. Since specifying $\Gamma^{\mu}{}_{\nu\lambda}$ in turn defines $R^{\mu}{}_{\nu\lambda\rho}$, the metric $g_{\mu\nu}$ pins down all geometric objects we have discussed here.

This brings us to the results stated in the Introduction: Einstein's field equations. The geometry of spacetime in the presence of matter is given by a metric which satisfies

$$R_{\mu\nu} - \frac{R}{2}\, g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}$$

where $T_{\mu\nu}$ is the energy-momentum tensor for the matter in question. In empty space we simply have the vanishing of the Ricci tensor.

A SCHEMATIC MODIFICATION TO GRAVITY

According to [4], in a region of spacetime devoid of mass Einstein's field equations become

$$R_{\mu\nu} + \lambda \, R_{\mu\kappa\rho\tau} \, R_{\nu}{}^{\kappa\rho\tau} = O(\lambda)^2 \, , \tag{2}$$

where $\lambda = \alpha'/2$ is proportional to the string parameter; there they dealt with bosonic string theory. Though the authors have since admitted that this equation is merely schematic of the modifications imposed on classical gravity, it is nonetheless interesting to study.

No Static Isotropic Solution

We first prove there is no static, isotropic solution to (2). The most general such metric can be written

$$ds^2 = -B(r)\, dt^2 + A(r)\, dr^2 + r^2\, d\theta^2 + r^2\, \sin^2\theta\, d\phi^2 \, . \tag{3}$$

Putting this into the left side of (2) we find that only the diagonal elements do not vanish identically; they are

$$R_{tt} \; + \; \lambda \left[\frac{-B'(r)^2}{r^2\, A(r)^2\, B(r)} - \frac{\left\{ A(r)\, B'(r)^2 \; + \; B(r) \left[A'(r)\, B'(r) - 2\, A(r)\, B''(r) \right] \right\}^2}{8\, A(r)^4\, B(r)^3} \right] \, , \tag{4}$$

$$R_{rr} \; + \; \lambda \left[\frac{A'(r)^2}{r^2\, A(r)^3} + \frac{\left\{ A(r)\, B'(r)^2 \; + \; B(r) \left[A'(r)\, B'(r) - 2\, A(r)\, B''(r) \right] \right\}^2}{8\, A(r)^3\, B(r)^4} \right] \, , \tag{5}$$

$$R_{\theta\theta} \; + \; \frac{\lambda}{2\, A(r)^4} \left[\frac{4 \left[A(r) - 1 \right]^2 A(r)^2}{r^2} + A'(r)^2 + \frac{A(r)^2\, B'(r)^2}{B(r)^2} \right] \, , \tag{6}$$

$$R_{\phi\phi} \; + \; \frac{\lambda \sin^2\theta}{2\, A(r)^4} \left[\frac{4 \left[A(r) - 1 \right]^2 A(r)^2}{r^2} + A'(r)^2 + \frac{A(r)^2\, B'(r)^2}{B(r)^2} \right] \, ; \tag{7}$$

we will call these expressions (tt), (rr), $(\theta\theta)$, and $(\phi\phi)$ respectively. Here $R_{\mu\nu}$ has nonvanishing components which are given in [5] :

$$\begin{aligned}
R_{tt} &= \frac{B''(r)}{2\,A(r)} - \frac{A'(r)\,B'(r)}{4\,A(r)^2} - \frac{B'(r)^2}{4\,A(r)\,B(r)} + \frac{B'(r)}{r\,A(r)}\,, \\
R_{rr} &= -\frac{B''(r)}{2\,B(r)} + \frac{A'(r)\,B'(r)}{4\,A(r)\,B(r)} + \frac{B'(r)^2}{4\,B(r)^2} + \frac{A'(r)}{r\,A(r)}\,, \\
R_{\theta\theta} &= 1 - \frac{1}{A(r)} + \frac{r\,A'(r)}{2\,A(r)^2} - \frac{r\,B'(r)}{2\,A(r)\,B(r)}\,, \\
R_{\phi\phi} &= R_{\theta\theta}\,\sin^2\theta\,.
\end{aligned}$$

Right away, we see that we can disregard the $(\phi\phi)$ component of (2) as redundant, leaving three equations for us to solve. Through (2), (4), and (5) we also see that

$$O(\lambda)^2 = \frac{(rr)}{A(r)} + \frac{(tt)}{B(r)} = \frac{A'(r)\,B(r) + B'(r)A(r)}{r\,A(r)^2\,B(r)} + \lambda\,\frac{A'(r)^2\,B(r)^2 - B'(r)^2\,A(r)^2}{r^2\,A(r)^4\,B(r)^2}\,. \tag{8}$$

At this point, we move from the general metric (3) to a more specific form. The Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2\,M\,G}{r}\right)\,dt^2 + \left(1 - \frac{2\,M\,G}{r}\right)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 \tag{9}$$

solves (2) to zeroth order in λ ; we seek a static, isotropic solution to (2) correct to first order in λ . This must have the form

$$ds^2 = -\left[1 - \frac{2\,M\,G}{r} + \lambda\,b(r)\right]\,dt^2 + \left[1 - \frac{2\,M\,G}{r} + \lambda\,a(r)\right]^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 \tag{10}$$

which amounts to setting

$$B(r) = 1 - \frac{2\,M\,G}{r} + \lambda\, b(r) \qquad \text{and} \qquad A(r) = \left[1 - \frac{2\,M\,G}{r} + \lambda\, a(r)\right]^{-1} \tag{11}$$

in (3) . Plugging these into (8) we find that

$$\lambda\, \frac{A'(r)^2\, B(r)^2 - B'(r)^2\, A(r)^2}{r^2\, A(r)^4\, B(r)^2} \;=\; O(\lambda^2)\,.$$

Thus to enforce (8) it suffices to set

$$0 = A'(r)\, B(r) + A(r)\, B'(r) \qquad \text{so that} \qquad A(r)\, B(r) = 1 + \lambda\, k = \text{const}\,. \tag{12}$$

The constant of integration must be be unity to first order in λ ; this comes by direct calculation using (11) or by requiring spacetime to be asymptotically Minkowskian (at least approximately). Some might argue that spacetime should become exactly Minkowskian at infinity — this is done in [1] . Being interested in the most general case now, we can simply set $k = 0$ later if desired.

Now, (11) and (12) imply that

$$b(r) = a(r) + k\, \left(1 - \frac{2\,G\,M}{r}\right)\,. \tag{13}$$

With this relationship between $b(r)$ and $a(r)$ enforced, (4) is simply a consequence of (5), so we can henceforth disregard the former, leaving us with two equations.

We are thus left with the (rr) and $(\theta\theta)$ components of (2). Using (11) and (13) in (6) yields

$$r\, a'(r) + a(r) = \frac{12\, G^2\, M^2}{r^4}$$

with solution

$$a(r) = \frac{c}{r} - \frac{4\,G^2\,M^2}{r^4}\,, \tag{14}$$

keeping the constant of integration c for now arbitrary. This determines $b(r)$ through (13):

$$b(r) = \frac{c}{r} - \frac{4\,G^2\,M^2}{r^4} + k\left(1 - \frac{2\,G\,M}{r}\right)\,. \tag{15}$$

This should conclude the calculation; however, we must check to see whether this solution is consistent with the equation $(rr) = 0$. Plugging (11) into (5) using (14) and (15) gives

$$(rr) = \frac{36\,\lambda}{r^4}\left(\frac{G\,M}{r}\right)^2\left(1 - \frac{2\,G\,M}{r}\right)^{-1} \neq 0\,.$$

It thus appears we have an inconsistent system of differential equations.

This result may surprise the reader, but keep in mind that we attempted to solve a set of three differential equations with two unknown functions. Let us remember the analogous calculation in classical gravity, where we can plug the general metric (3) into $R_{\mu\nu} = 0$ and find the Schwarzschild solution (9). In that case, after determining that $A(r) = 1/B(r)$ (i.e. our equation (12) with λ set to zero), one can determine that

$$R_{rr} - \frac{1}{2\,r\,B(r)}\,\frac{dR_{\theta\theta}}{dr} = 0\,,$$

reducing the number of independent equations to two. In our problem, if we plug

$$A(r) = \frac{1 + \lambda\,k}{B(r)}$$

from (12) above into (5) and (6) we find that

$$(rr) - \frac{1}{2\,r\,B(r)}\,\frac{d(\theta\theta)}{dr} \;=\; \frac{\lambda}{2\,r^4\,B(r)}\,\Big\{4 + 4\,B(r)^2 + 2\,r^2\,B'(r)^2 - 4\,B(r)\,\big[r\,B'(r) + 2\big] - k\,r^4\,B''(r) + r^4\,B''(r)^2 - 2\,r\,B'(r)\,\big[r^2\,B''(r) + k\,r^2 - 2\big]\Big\}\,,$$

which does not vanish but is of order λ. It thus seems that we have three independent equations to solve. In contrast to the classical problem of determining the static, isotropic metric which obeys Einstein's equations, our system in the perturbed problem is overdetermined.

Analogue to Birkhoff's Theorem

We can take this result one step further by proving that, in fact, no isotropic solution of (2) exists, dropping the static condition. We do this by showing that any isotropic solution of (2) must necessarily be static. The analogous result in classical general relativity is known as Birkhoff's theorem; we prove it here for this theory of modified gravity.

The most general isotropic metric can be written

$$ds^2 = -B(r,t)\,dt^2 + A(r,t)\,dr^2 + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\phi^2\,.$$

Since we know that Birkhoff's theorem holds true to zeroth order in λ, any isotropic solution to (2) can be written as

$$ds^2 = -\left[1 - \frac{2\,M\,G}{r} + \lambda\,b(r,t)\right]\,dt^2 + \left[1 - \frac{2\,M\,G}{r} + \lambda\,a(r,t)\right]^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\phi^2\,;$$

this defines more $A(r,t)$ and $B(r,t)$ more specifically. Plugging this metric into the left side of (2) gives some nonvanishing off-diagonal components, e.g.

$$O(\lambda)^2 = (tr) = \lambda\,\frac{\dot{a}(r,t)}{2\,G\,M - r}\,;$$

we let dots denote differentiation with respect to time, primes with respect to the radial coordinate. This equation is very convenient, as it implies $a = a(r)$ and $A = A(r)$, thus simplifying remaining equations. Now expanding

$$O(\lambda)^2 = \frac{(tt)}{B(r,t)} + \frac{(rr)}{A(r)}\,,$$

we find again something proportional to

$$A'(r)\,B(r,t) + A(r)\,B'(r,t) = O(\lambda)^2\,;$$

this time we get $A(r)\,B(r,t) = f(t)$, for some function of t. But no matter the functional form of $f(t)$, through a change of coordinates in which

$$dt' = \frac{dt}{f(t)}\,,$$

this function may be absorbed into $B(r,t)$. Let us assume we began the calculation in such a coordinate system (so we can avoid an overflow of primes); we are then left with $A(r)\,B(r,t) = \text{const.}$ This can only hold when $B = B(r)$, leaving us with a static field.

We move now to the generally accepted gravitational equations in the next section and abandon (2) for the remainder of this thesis.

MODIFICATION WITH DILATON COUPLING

According to [1], string theory imposes modifications to Einstein's field equations of general relativity, which to leading order in the string parameter α' take the form

$$R_{\mu\nu} + 2\,\nabla_\mu\,\nabla_\nu\,\Phi + \lambda\,R_{\mu\alpha\beta\gamma}\,R_\nu{}^{\alpha\beta\gamma} \;=\; O(\lambda)^2 \tag{16}$$

$$\Box\Phi - (\nabla\Phi)^2 + \frac{1}{4}\,R + \frac{1}{8}\,\lambda\,R_{\alpha\beta\gamma\delta}\,R^{\alpha\beta\gamma\delta} \;=\; O(\lambda)^2\,. \tag{17}$$

Here, the gravitational field is coupled to the scalar dilaton field, and variations in the two cannot be separated [3]. Again, we are using $\lambda = \alpha'/2$ for bosonic string theory, but in [1] it is noted that $\lambda = \alpha'/4$ in heterotic string theory and $\lambda = 0$ for supersymmetric strings. The corresponding static isotropic spacetime metric outside a mass m is determined in [1] as well. They begin by assuming

$$ds^2 = -f(r)^2\,dt^2 + g(r)^2\,dr^2 + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\phi^2 \tag{18}$$

with

$$f(r) \;=\; f_0(r)\left[1 + \lambda\,\mu(r)\right], \tag{19}$$

$$g(r) \;=\; g_0(r)\left[1 + \lambda\,\epsilon(r)\right], \tag{20}$$

$$\Phi(r) \;=\; \Phi_0 + \lambda\,\varphi(r)\,, \tag{21}$$

where

$$f_0(r)^2 = \frac{1}{g_0(r)^2} = 1 - \frac{2Gm}{r} \qquad \text{and} \qquad \Phi_0 = \text{ const.}$$

To solve (16)-(17) it is beneficial to obtain an equation containing only $\varphi(r)$, its derivatives, and known functions of r. To do this we contract (16) to obtain an expression for the scalar curvature R, then plug that into (17) to get

$$\Box^2\Phi - 2\,(\nabla\Phi)^2 - \frac{\lambda}{4}\,R_{\alpha\beta\gamma\delta}\,R^{\alpha\beta\gamma\delta} = O(\lambda)^2\,. \tag{22}$$

Plugging in (21) we get the ordinary differential equation

$$\frac{r-2Gm}{r}\,\varphi''(r) + \frac{2r-2Gm}{r^2}\,\varphi'(r) = \frac{12G^2m^2}{r^6}$$

with solution

$$\varphi(r) = -\frac{2Gm}{3r^3} - \frac{1}{2r^2} - \frac{1}{2Gmr}\,.$$

Constant of integration are eliminated throughout by requiring spacetime to be asymptotically Minkowskian and keeping the Schwarzschild radius at $r = 2Gm$. The second requirement can always be imposed through a rescaling of the radial coordinate; the first condition seems to be the most reasonable behavior at infinity

Having solved for $\varphi(r)$, we are left with the equations (16), of which only the diagonal components do not vanish identically. We follow the exact same routine to solve these equations as we tried in the previous section, only here there is no inconsistency. We get the result:

$$f(r) = \sqrt{1-\frac{2\,G\,m}{r}}\left[1-\frac{\lambda}{r^2}\left(\frac{23\,r}{24\,G\,m}+\frac{11}{12}+\frac{G\,m}{r}\right)\right] \tag{23}$$

$$g(r) = \frac{1}{\sqrt{1-\frac{2\,G\,m}{r}}}\left[1-\frac{\lambda}{r^2}\left(\frac{r}{24\,G\,m}+\frac{7}{12}+\frac{5\,G\,m}{3\,r}\right)\right]. \tag{24}$$

In passing, we can ask what would have happened if we had searched for an isotropic solution which is not necessarily static. Would we discover another analogue to Birkhoff's theorem? Plugging in a general static metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\,\Big(1 + \alpha'\,\mu(r,t)\Big)\,dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\,\Big(1 + \alpha'\,\epsilon(r,t)\Big)\,dr^2 + r^2\,d\Omega^2\,.$$

into (16), we get nonvanishing off-diagonal again. But this time

$$O(\lambda)^2 = (tr) = \lambda\,\left[\frac{2GM\,\dot{\varphi}}{(2GM - r)\,r} + \frac{2\,\dot{\epsilon}}{r} + 2\,\frac{\partial^2\varphi}{\partial x\,\partial t}\right]\,;$$

we cannot isolate the time derivative of $\epsilon(r,t)$ since $\varphi(r,t)$ may depend on time. Indeed, (22) becomes

$$r^7\,\ddot{\varphi} - r^5\,(r - 2Gm)^2\,\varphi'' - 2r^4\,(2G^2m^2 - 2Gmr + r^2)\,\varphi' + 12G^2m^2r - 24G^3m^3 = 0\,.$$

Thus we conjecture that no analogue to Birkhoff's theorem exists for the gravitational field equations (16)-(17) .

Now in classical general relativity, Birkhoff's theorem implies that a pulsating star, for instance, causes no gravitational radiation. With the new possibility for "isotropic gravitational radiation", perhaps it will one day be possible to search in the radiation from the Big Bang, when string theory would have had large effect on physics, for a component matching the prediction given by string theory.

CLASSICAL TESTS OF GENERAL RELATIVITY

We are interested in using this result to calculate the deviations from Einstein's theory that should occur in phenomena know as the classical tests of general relativity. All these calculations deal with bodies moving in the geometry given by (18) with (23) and (24). Thus we will recall here from [5] how to determine the trajectory of a particle in a static, isotropic geometry.

Consider, once again, the metric given by

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\,d\Omega^2 \tag{25}$$

where we leave $A(r)$ and $B(r)$ arbitrary for now and write $d\Omega^2$ instead of $d\theta^2 + \sin^2\theta\,d\phi^2$ for brevity. To determine a particle's motion in this geometry, we need to solve the geodesic equations

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu{}_{\nu\lambda}\,\frac{dx^\nu}{dp}\,\frac{dx^\lambda}{dp} = 0 \tag{26}$$

with $\Gamma^\mu{}_{\nu\lambda}$ computed in terms of $A(r)$ and $B(r)$ from the metric. By the spherical symmetry of the problem, we can choose $\theta = \pi/2$ for all times, simplifying the equations. This leaves three components of (26):

$$\begin{aligned}
0 &= \frac{d^2t}{dp^2} + \frac{B'(r)}{B(r)}\,\frac{dt}{dp}\,\frac{dr}{dp}\,, \\
0 &= \frac{d^2r}{dp^2} + \frac{A'(r)}{2\,A(r)}\left(\frac{dr}{dp}\right)^2 - r\,\frac{\sin^2\theta}{A(r)}\left(\frac{d\phi}{dp}\right)^2 + \frac{B'(r)}{2\,A(r)}\left(\frac{dt}{dp}\right)^2\,, \\
0 &= \frac{d^2\phi}{dp^2} + \frac{2}{r}\,\frac{d\phi}{dp}\,\frac{dr}{dp}\,.
\end{aligned}$$

The t- and ϕ-equations can each be written as the vanishing of a total derivative. One of the constants of motion resulting from these equations is absorbed into the parametrization of the path, while the other becomes

$$J = r^2 \, \frac{d\phi}{dp} \,. \tag{27}$$

Manipulation of the r-equation then produces

$$A(r) \left(\frac{dr}{dp} \right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} \quad = \quad - \, E = \ \text{const.} \tag{28}$$

where, upon computation of the interval ds^2, we discover that E is strictly positive for massive particles and zero for massless particles. Since we are primarily concerned with the shapes of trajectories, we trade dp for $d\phi$ in (28) using (27) to find

$$\frac{A(r)}{r^4} \left(\frac{dr}{d\phi} \right)^2 + \frac{1}{r^2} - \frac{1}{J^2 \, B(r)} = - \frac{E}{J^2} \,. \tag{29}$$

Rearrangement and integration gives the equation we will use when considering the first two classical tests:

$$\phi - \phi_0 = \pm \int \frac{dr}{r^2} \sqrt{\frac{A(r)}{\frac{1}{J^2 \, B(r)} - \frac{E}{J^2} - \frac{1}{r^2}}} \,. \tag{30}$$

We are henceforth concerned with situations in which the gravitational potential Gm/r is quite small. Indeed, just outside our sun, we have $Gm/r \simeq 2 \times 10^{-6}$ in units where $c = 1$. Thus, keeping only the leading-order corrections to the Schwarzschild metric, we will use

$$ds^2 = - \left(1 - \frac{2 \, G \, m}{r} - \frac{23 \, \lambda}{12 \, G \, m \, r} \right) dt^2 + \left(\frac{1}{1 - \frac{2 \, G \, m}{r}} - \frac{\lambda}{12 \, G \, m \, r} \right) dr^2 + r^2 \, d\Omega^2 \tag{31}$$

instead of (23) and (24). This defines the functions $A(r)$ and $B(r)$ to substitute in (30).

Deflection of Light

Let us begin by calculating the additional deflection of light by a star that should occur due to string theory. This is a scattering problem in which we are interested in the total change in ϕ as a photon approaches the sun from infinity, then returns to infinity. We use (30) to calculate this. Since we deal with light, we must use $E = 0$. We can also find J from (29) since $dr/d\phi = 0$ when $r = r_0$, the minimum distance between the photon and the star's center. Since the geodesic equation (26) is invariant under "time reversal" $p \mapsto -p$, the trajectory of the photon is also symmetric around its point of closest approach to the sun. Hence, we only need to calculate the change in ϕ over the half the path, then double it. If the sun did not deflect light at all, we would expect a change in ϕ of precisely π radians. Putting all these considerations together, the total deflection of a photon as it passes the sun is

$$\Delta\phi = 2\left| \int_{r_0}^{\infty} \frac{dr}{r}\sqrt{\frac{A(r)}{\left(\frac{r}{r_0}\right)^2\left(\frac{B(r_0)}{B(r)}\right) - 1}} \right| - \pi\,,$$

using $A(r)$ and $B(r)$ from (31). Consider expanding this integrand in powers of λ. The λ^0 term would integrate to give the classical result of

$$\Delta\phi = \frac{4\,m\,G}{r_0}\,;$$

this is correct to zeroth order in λ and first order in the potential. For light passing near the surface of our sun, this gives $1.75''$. Now integrating the λ^1 term gives the small deviation $\delta\phi$ from the classical result. To leading order in the potential,

$$\delta\phi \;=\; \frac{\lambda}{12\,G\,m\,r_0}\int_{r_0}^{\infty}\frac{\left(23\,r^2 - r\,r_0 - r_0^2\right)dr}{(r+r_0)\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} \;=\; \frac{11\,\lambda}{6\,G\,m\,r_0}\,. \tag{32}$$

Since $\delta\phi$ is positive, the total deflection is slightly greater than the classical deflection. For light just grazing our sun, $\delta\phi$ is of order 10^{-82} radians. Therefore, stringy effects should shift the reception point of the photon on Earth by an additional 10^{-71} meters. We will comment on the size of these corrections in the Conclusions, after considering all phenomena.

Precession of Perihelia

We proceed to calculate in a similar way the additional precession of planetary orbits due to string theory. We determine E and J from (29) since $dr/d\phi = 0$ when $r = r_\pm$, the radii corresponding to the aphelion and perihelion of planet's orbit. As above, we integrate over only half the path length and double the result. In this case we subtract 2π from the integral, because deviation from 2π implies a precession in the orbit-ellipse. Using these ideas, we determine from (30) that the total precession of an orbit perihelion is given by

$$\Delta\phi \;=\; 2\left|\;\int_{r_-}^{r_+}\frac{\sqrt{A(r)}\,dr}{r^2}\Bigg/\sqrt{\frac{r_-^2\left[1/B(r)\;-\;1/B(r_-)\right]-r_+^2\left[1/B(r)-1/B(r_+)\right]}{r_+^2\,r_-^2\left[1/B(r_+)-1/B(r_-)\right]}-\frac{1}{r^2}}\;\right|-2\pi$$

with substitution for $A(r)$ and $B(r)$ from (31). Again, consider expanding this integrand in powers of λ. The λ^0 term would integrate to give the classical result of

$$\Delta\phi = 3\,\pi\,m\,G\left(\frac{1}{r_+}+\frac{1}{r_-}\right)\;\frac{\text{radians}}{\text{revolution}}\,;$$

this is correct to zeroth order in λ and first order in the potential. For Mercury orbiting our sun, this gives $43.03''$ per century. Now integrating the λ^1 term gives the small deviation $\delta\phi$ from the classical result. To leading order in the potential,

$$\delta\phi \;=\; \frac{\lambda}{12\,G\,m\,\sqrt{r_-\,r_+}}\int_{r_-}^{r_+}\frac{(23\,r\,r_-+23\,r\,r_+-r_-\,r_+)\,dr}{r^2\,\sqrt{(r_+-r)\,(r-r_-)}}\;=\;\frac{15\,(r_-+r_+)\,\pi\,\lambda}{8\,G\,m\,r_+\,r_-}\,. \qquad (33)$$

Since $\delta\phi$ is positive, string theory implies a slightly faster precession than classical general relativity. For Mercury, $\delta\phi$ is of order 10^{-83} radians per revolution. It would take roughly 10^{26} times the age of the universe for Mercury to advance an extra Planck-length due to stringy effects.

Gravitational Redshift

We now move on to calculate the gravitational redshift of light as it travels away from a massive body, which is a consequence of gravitational time dilation. Proper time

$$\Delta t = \sqrt{-g_{\mu\nu}\, dx^\mu\, dx^\nu}$$

governs the ticking of clocks, beating of hearts, and frequency of light waves. For a clock at rest, only time-components of dx^μ do not vanish, leaving

$$dt = \frac{\Delta t}{\sqrt{-g_{00}}}\,.$$

Now consider two atoms in a static gravitational field, one at x_1 and the other at x_2, separated by some spatial distance. If we sit at x_1 we measure the frequency of light coming from atomic transitions at x_1 and x_2 to be

$$\frac{1}{\nu_1} = \frac{\Delta t}{\sqrt{-g_{00}(x_1)}} \qquad \text{and} \qquad \frac{1}{\nu_2} = \frac{\Delta t}{\sqrt{-g_{00}(x_2)}}$$

respectively, where here we take Δt as the proper period of light from the transitions. To be sure, $1/\nu_2$ (equivalently, dt_2) must be given by the second expression above; the static field ensures the travel time for wavefronts between x_1 and x_2 is constant, so the difference between arrival times at x_1 is just the difference between departure times at x_2. Thus the ratio of frequencies received at x_1 is

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}}\,.$$

To zeroth order in λ, we find the classical result

$$\frac{\Delta\nu}{\nu} = \phi(x_2) - \phi(x_1)$$

where $\phi(x)$ here is the gravitational potential at x; this result is correct to first order in the potential. For light received on Earth from atomic transitions on the sun, this becomes roughly -2×10^{-6}; here, the negative result signifies the decrease in frequency as light travels away from the sun. To lowest order in the potential, the λ^1 term in this expression gives the small correction

$$\frac{\delta\nu}{\nu} = -\frac{23\lambda}{24GM}\left(\frac{1}{r} - \frac{1}{R}\right) \tag{34}$$

due to string theory; here r is the radius of the sun, while R is the distance between Earth's surface and the sun's center. The negativity of $\delta\phi$ shows that, once again, the extra shift due to string theory is in the same direction as the classical effect. From Sun to Earth, this result becomes of order 10^{-82}.

CONCLUSIONS

We have investigated consequences of two modifications to classical general relativity. For the schematic change in Einstein's field equations

$$R_{\mu\nu} + \lambda \, R_{\mu\alpha\beta\gamma} \, R_{\nu}{}^{\alpha\beta\gamma} = O(\alpha')^2$$

suggested in [4], we showed not only that no static isotropic solution exists, but also that a parallel to Birkhoff's theorem holds for these equations. Together, these facts imply that the above equations do not even have an isotropic solution.

While discomforting that no isotropic solutions exist for the above equations, this alone is no proof for their invalidity. With the tiny anisotropies found in the cosmic microwave background, one could always claim that we have no need for perfectly isotropic geometry, since it does not exist in our universe.

We then calculated the small changes in the results of the classical tests of general relativity that we should expect if the equations coming out of string theory are indeed true. The results (32), (33), (34) we found in each case were unimaginably small, and unfortunately, there are not many options to rescue them and make observation possible. Even if, instead of near the sun, we let these phenomena occur in the vicinity of a black hole at the center of the observable universe, this will not change our chances at measuring any of these effects. Indeed, the observable universe is about 10^{26} meters across and $MG/r = 1/2$ at the event horizon of a black hole. In the best cases, these will increase the effects we see on Earth "only" by a factor of about 10^{21}. This does not get us into the measurable region above the Planck length. To increase the effects we must consider very small black holes to decrease the radii present in (32), (33), and (34). Still, the radius required is many orders of magnitude smaller than the proton's radius. It is a safe claim, then, that these effects can never be observed in our present universe.

In fact, while the calculations given in the previous section are fine mathematically, there is a physical inconsistency hiding there. The four-dimensional gravitational field equations we have used are effective equations, assuming the extra dimensions have negligible effects on our systems. What we have found is that these calculations lie outside the region of validity of such an approximation. But even though the precise values of the shifts we calculated above may lack physical meaning, they show that modifications imposed on the classical tests of general relativity are not observable above the Planck scale. These phenomena thus are not the right place to look for experimental tests of string theory.

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