Behavior of the solutions to a functional equation which equates a function's inverse to its reciprocal

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Behavior of the Solutions to a Functional Equation Which Equates a Function’s Inverse to its Reciprocal

BY

ROBERT RUDOLPH ANSCHUETZ II
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THESIS
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ABSTRACT

This thesis explores the behavior of solutions of the functional equation

\[ f^{-1}(x) = \frac{1}{f(x)} \text{ for } x \in \text{Dom}(f), \]

where \( f \) is a real-valued function of a real variable. It is quite common to mistake the notation \( f^{-1} \), which means the inverse of \( f \) with respect to composition, with the inverse of \( f \) with respect to multiplication, usually denoted by \( \frac{1}{f} \). This thesis shows that although \( f^{-1} \) and \( \frac{1}{f} \) are usually different functions, they do indeed sometimes represent the same function. This thesis will also provide methods of generating solutions of the functional equation \( f^{-1}(x) = \frac{1}{f(x)} \text{ for } x \in \text{Dom}(f). \)
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INTRODUCTION

Functional equations are equations, both sides of which are terms constructed from a finite number of unknown functions of a finite number of variables and from a finite number of independent variables.

Some examples of functional equations are:

a) \[ f(x + y) = f(x) + f(y) \],

b) \[ f(x)^2 = f(x + y)f(x - y) \],

c) \[ f(x + y) + f(x - y) = 2f(x)f(y) \].

Many functional equations have known solutions. For example, the functional equation \( f(x + y) = f(x) + f(y) \) has many solutions, including \( f(x) = ax \). The purpose of this thesis is to study the behavior of solutions to the functional equation:

\[
f^{-1}(x) = \frac{1}{f(x)} \quad \text{for all } x \in \text{Dom}(f),
\] (1)

where \( f \) is a real-valued function of a real variable.
FUNDAMENTAL PROPERTIES

Let us begin by defining elementary properties for the functional equation described by (1).

Proposition 1

A function $f$ satisfies (1) if and only if for every $a \in \text{Dom}(f)$ there exists $b \in \mathbb{R}$ such that $f$ satisfies the following set of equations:

$$\begin{align*}
    f(a) &= b, \\
    f(b) &= \frac{1}{a}, \\
    f\left(\frac{1}{a}\right) &= \frac{1}{b}, \\
    f\left(\frac{1}{b}\right) &= a
\end{align*}$$

(2)

Proof:

1) Assume $f$ satisfies (1) and let $a \in \text{Dom}(f)$.

Then there exists $b \in \mathbb{R}$ such that $f(a) = b$. Moreover, $f(a) = b$ implies that $a = f^{-1}(b) = \frac{1}{f(b)}$, which implies that

\[ f(b) = \frac{1}{a}. \]

Also, $f(b) = \frac{1}{a}$ implies that

\[ b = f^{-1}\left(\frac{1}{a}\right) = \frac{1}{f\left(\frac{1}{a}\right)}, \]

which implies that
Finally, $f\left(\frac{1}{a}\right) = \frac{1}{b}$. Therefore, for every $a \in Dom(f)$ there exists $b \in \mathbb{R}$ such that $f$ satisfies (2).

2) Assume that for every $a \in Dom(f)$ there exists $b \in \mathbb{R}$ such that $f$ satisfies (2). First let us examine the function $f$ to see if it is one-to-one and, thus, has an inverse. Suppose $x_1, x_2 \in Dom(f)$ such that $f(x_1) = f(x_2) = b$. We need to show that $x_1 = x_2$ to establish that $f$ is one-to-one. Since the function $f$ satisfies (2), we have

$$f(b) = \frac{1}{x_1} \quad \text{and} \quad f(b) = \frac{1}{x_2}.$$ 

Therefore, $\frac{1}{x_1} = \frac{1}{x_2}$, so $x_1 = x_2$. Therefore, $f$ is one-to-one and has an inverse. Let $x \in Dom(f)$. From the assumption, there exists $y = f(x) \in \mathbb{R}$ such that

$$\begin{align*}
f(x) &= y, \\
f(y) &= \frac{1}{x},
\end{align*}$$

such that $f\left(\frac{1}{x}\right) = \frac{1}{y}$. From this set of equations, we have $\frac{1}{f(x)} = \frac{1}{y} = f^{-1}(x)$, so $f$ satisfies (1).

**Proposition 2**

If $f$ satisfies (1), then $Dom(f) = Ran(f)$.

**Proof:**
1) Suppose $x \in \text{Dom}(f)$ and we define $f(x) = y$. Then by Proposition 1,

$$f\left(\frac{1}{y}\right) = x,$$
so $x \in \text{Ran}(f)$.

2) Suppose $y \in \text{Ran}(f)$, then there exists $x \in \text{Dom}(f)$ such that $f(x) = y$. By Proposition 1, $f(y) = \frac{1}{x}$, so $y \in \text{Dom}(f)$.

**Proposition 3**

Suppose $f$ satisfies (1) and $f(a) = b$. Then

a) $a = \pm 1$ if and only if $b = \pm 1$,

b) If $a = b$ then $a = \pm 1$,

c) If $a = \frac{1}{b}$ then $a = \pm 1$,

d) $0 \notin \text{Dom}(f)$ and $0 \notin \text{Ran}(f)$, i.e., $a \neq 0$ and $b \neq 0$.

Proof of a): Notice that $a = \pm 1$ if and only if $a = \frac{1}{a}$ if and only if $b = \pm 1$.

Proof of b): Suppose $a = b$. Then

$$a = f\left(\frac{1}{b}\right) = f\left(\frac{1}{a}\right) = \frac{1}{b} = \frac{1}{a},$$

which implies that $a = \pm 1$.

Proof of c): Suppose $a = \frac{1}{b}$. Then

$$a = f\left(\frac{1}{b}\right) = f(a) = b = \frac{1}{a},$$

which implies that
\[ a = \pm 1. \]

Proof of d): Since \( f \) satisfies (1), \( f(x)f^{-1}(x) = 1 \) for all \( x \in \text{Dom}(f) \). So,

\( f(x) \neq 0 \) and \( f^{-1}(x) \neq 0 \) for all \( x \in \text{Dom}(f) \). Therefore, \( 0 \notin \text{Ran}(f) \) and \( 0 \notin \text{Dom}(f) \), i.e., \( a \neq 0 \) and \( b \neq 0 \).

**Proposition 4**

If \( f \) satisfies the functional equation (1), then so does \( f^{-1} \), the reflection of \( f \) about the line \( y = x \).

Proof: Let \( g = f^{-1} \) and let \( x \in \text{Dom}(g) \). Then \( g(x) = f^{-1}(x) = \frac{1}{f(x)} \), so

\[ f(x) = \frac{1}{g(x)}. \]

Since \( g = f^{-1} \), we know that \( g^{-1} = f \), so \( g^{-1}(x) = f(x) = \frac{1}{g(x)} \).

Therefore, \( g^{-1}(x) = \frac{1}{g(x)} \) for all \( x \in \text{Dom}(g) \).

**Proposition 5**

If \( f \) satisfies the functional equation (1) then so does \( -(-f)^{-1} \), the reflection of \( f \) about the line \( y = -x \).

Proof: Let \( g = -(-f)^{-1} \). We must show that \( g^{-1}(x) = \frac{1}{g(x)} \) for all \( x \in \text{Dom}(g) \).

First, we need to examine whether both \( g^{-1}(x) \) and \( \frac{1}{g(x)} \) exist for all \( x \in \text{Dom}(g) \). We know that the function \( f \) has an inverse, so \( f \) must be one-to-one. It follows then that
\(-f\) must also be one-to-one, and, in turn, that \((-f)^{-1}\) is also one-to-one. Finally, 

g = -(-f)^{-1} must be one-to-one. Therefore, we know that \(g^{-1}(x)\) exists for each 
\(x \in \text{Dom}(g)\). Since \(f\) satisfies the functional equation (1), we note that \(0 \notin \text{Dom}(f)\), 
so \(0 \notin \text{Dom}(-f)\), which implies that \(0 \notin \text{Ran}((-f)^{-1})\), and finally 
\(0 \notin \text{Ran}((-f)^{-1}) = \text{Ran}(g)\). Therefore, for any \(x \in \text{Dom}(g)\), \(g(x) \neq 0\), so \(\frac{1}{g(x)}\) exists 
for each \(x \in \text{Dom}(g)\).

Suppose \(a \in \text{Dom}(g)\) and let \(g(a) = b\). Then \((a, b) \in g\). Since \(g\) and \(f\) are 
reflections of each other about the line \(y = -x\), we have \((-b, -a) \in f\), i.e., \(f(-b) = -a\).

In other words, \(-b = f^{-1}(-a)\). But \(f\) satisfies (1), so \(-b = f^{-1}(-a) = \frac{1}{f(-a)}\). Thus, 
\(f(-a) = -\frac{1}{b}\). Since \((-a, -\frac{1}{b}) \in f\) and since \(g\) and \(f\) are reflections of one another 
about the line \(y = -x\), we conclude that \(\left(\frac{1}{b}, a\right) \in g\). But then \(g\left(\frac{1}{b}\right) = a\), so \(g^{-1}(a) = \frac{1}{b}\).

Finally, we conclude that \(g(a) = b = \frac{1}{g^{-1}(a)}\). This proves that \(g\) satisfies (1).
Let us now examine some examples of one, two, and four point functions that satisfy the functional equation (1). Using Proposition 1, it is easy to verify that each of the following functions satisfies (1):

a) \( f = \{ (1,1) \} , \)

b) \( f = \{ (-1,-1) \} , \)

c) \( f = \{ (1,-1), (-1,1) \} , \)

d) \( f = \{ (\pi, e), \left( e, \frac{1}{\pi} \right), \left( \frac{1}{\pi}, e \right), \left( \frac{1}{e}, \pi \right) \} . \)
DEFINITIONS AND RELATIONSHIPS

Let us now define some terms and state some relationships which will help us better understand the functional equation (1).

Definitions

It is helpful to introduce the following definitions in order to describe further properties of solutions to (1).

Iterates

The iterates \( f^n(x) \) of an invertible function \( f \) with \( \text{Dom}(f) = \text{Ran}(f) \) are defined by

\[
\begin{align*}
  f^0(x) &= x, \\
  f^1(x) &= f(x), \\
  f^{n+1}(x) &= f(f^n(x)), & x &\in \text{Dom}(f), \quad n = 0, 1, 2, \ldots, \\
  f^{n-1}(x) &= f^{-1}(f^n(x)), & x &\in \text{Dom}(f), \quad n = 0, -1, -2, \ldots
\end{align*}
\]

Equivalence

An equivalence relation can be defined on \( \text{Dom}(f) \) as follows: two points \( x, y \in \text{Dom}(f) \) are said to be equivalent under \( f \) if and only if there exist non-negative integers \( m \) and \( n \) such that \( f^m(x) = f^n(y) \). According to Kuczma [1968], this equivalence relation was first introduced by Kazimierz Kuratowski. The equivalence
relation is reflexive, symmetric, and transitive, so the set $\text{Dom}(f)$ can be split into disjoint sets of equivalent elements.

**Orbit**

The set of all points which are equivalent under $f$ to a given $x \in \text{Dom}(f)$ will be called the *orbit of $x$ under $f$*, denoted $\Omega_f(x)$, or simply $\Omega(x)$. Since $f$ is invertible, $x$ and $y$ are equivalent if and only if there exists an integer $k$ (positive, negative, or zero) such that $f^k(x) = y$. Thus, the orbit $\Omega(x)$ consists of points of the form $y = f^k(x)$ for all integer values of $k$. The set of all orbits under $f$ is denoted $\Omega$.

**Complement**

The *complement* of an orbit $\Omega(x) \subseteq \mathbb{R}$, denoted $\Omega(x)'$, consists of all of the real numbers not included in the set $\Omega(x)$.

**Functional Relationships**

From Proposition 1, we can see that for $f$ satisfying (1) the following functional relationships are true:

a) $f\left(\frac{1}{x}\right) = \frac{1}{f(x)}$,

b) $f(f(x)) = f^2(x) = \frac{1}{x}$,
c) \( f(f(f(x))) = f^3(x) = f(f^2(x)) = f\left(\frac{1}{x}\right) = \frac{1}{f(x)} \),

d) \( f(f(f(f(x)))) = f^4(x) = f(f^3(x)) = f\left(f\left(\frac{1}{x}\right)\right) = f^2\left(\frac{1}{x}\right) = x \),

e) \( f(f(f(f(f(x))))) = f^5(x) = f(f^4(x)) = f(x) \).

In general, we have:

\[
\begin{cases}
  f^{4n}(x) = f^0(x) = x, \\
  f^{4n+1}(x) = f^1(x) = f(x), \\
  f^{4n+2}(x) = f^2(x) = \frac{1}{x}, & \text{for } n = 0,1,2,\ldots. \\
  f^{4n+3}(x) = f^3(x) = \frac{1}{f(x)}
\end{cases}
\]

From these relationships we see that the orbit of \( x \) under a solution \( f \) of (1) is given by

\[
\Omega(x) = \left\{ x, \frac{1}{x}, f(x), \frac{1}{f(x)} \right\}.
\]  

(3)
INFINITE CARDINALITY DOMAIN

For a function \( f \) satisfying (1), the following statements can be made about the collection, \( \mathcal{Q} \), of all orbits \( \Omega(x) \) such that \( x \in \text{Dom}(f) \):

a) If \( y \in \Omega(x) \in \mathcal{Q} \), then \( \frac{1}{y} \in \Omega(x) \),

b) If \( \Omega(x), \Omega(y) \in \mathcal{Q} \), then \( \Omega(x) \cap \Omega(y) = \emptyset \) or \( \Omega(x) = \Omega(y) \),

c) If \( 1 \in \Omega(x) \in \mathcal{Q} \), then \( \Omega(x) = \{1\} \) or \( \Omega(x) = \{1,-1\} \),

d) If \( -1 \in \Omega(x) \in \mathcal{Q} \), then \( \Omega(x) = \{-1\} \) or \( \Omega(x) = \{1,-1\} \),

e) If \( x \in \Omega(x) \in \mathcal{Q} \) and \( x \neq \pm 1 \), then \( \Omega(x) \) has exactly four elements.

Proof of a): Suppose \( y \in \Omega(x) \in \mathcal{Q} \). Then \( y = f^k(x) \) for some \( k \in \mathbb{Z} \). But then

\[
\frac{1}{y} = f^2(y) = f^2(f^k(x)) = f^{k+2}(x). \text{ Hence, } \frac{1}{y} \in \Omega(x).
\]

Proof of b): Suppose \( \Omega(x), \Omega(y) \in \mathcal{Q} \). Now suppose \( \Omega(x) \cap \Omega(y) \neq \emptyset \). Then there exists \( z \in \Omega(x) \cap \Omega(y) \). Thus, there exists integers \( m \) and \( n \) such that

\[
z = f^n(x) = f^m(y). \text{ Therefore, } y = f^{n-m}(x) \text{ and } x = f^{m-n}(y). \text{ Let } w \in \Omega(x), \text{ then there exists an integer } k \text{ such that } w = f^k(x) = f^{k+m-n}(y). \text{ Therefore, } w \in \Omega(y), \text{ and,}
\]
thus, $\Omega(x) \subseteq \Omega(y)$. Let $u \in \Omega(y)$, then there exists an integer $j$ such that

$u = f^j(y) = f^{j+n-m}(x)$. Therefore, $u \in \Omega(x)$, and, thus, $\Omega(y) \subseteq \Omega(x)$. Thus,

$\Omega(x) = \Omega(y)$. It is now clear that either $\Omega(x) \cap \Omega(y) = \emptyset$ or $\Omega(x) = \Omega(y)$.

Proof of c): Suppose $1 \in \Omega(x) \in \mathcal{Q}$. By Proposition 3, we know that if $f$ satisfies

(1) and $1 \in \Omega(x) \in \mathcal{Q}$, then either $f(1) = 1$ or $f(1) = -1$. For $f(1) = 1$ and from Proposition 1, we can see that $f^k(1) = 1$, for all $k \in \mathbb{Z}$. Using b) we obtain

$\Omega(x) = \Omega(1) = \{1\}$. For $f(1) = -1$ and from Proposition 1, we can see that

$f^k(1) = \begin{cases} 1, \text{ for all even } k \in \mathbb{Z}, \\ -1, \text{ for all odd } k \in \mathbb{Z} \end{cases}$. Using b) we obtain $\Omega(x) = \Omega(1) = \{1,-1\}$.

Proof of d): Suppose $-1 \in \Omega(x) \in \mathcal{Q}$. By Proposition 3, we know that if $f$ satisfies (1) and $-1 \in \Omega(x) \in \mathcal{Q}$, then either $f(-1) = -1$ or $f(-1) = 1$. For $f(-1) = -1$

and from Proposition 1, we can see that $f^k(-1) = -1$, for all $k \in \mathbb{Z}$. Using b) we obtain

$\Omega(x) = \Omega(-1) = \{-1\}$. For $f(-1) = 1$ and from Proposition 1, we can see that

$f^k(-1) = \begin{cases} -1, \text{ for all even } k \in \mathbb{Z}, \\ 1, \text{ for all odd } k \in \mathbb{Z} \end{cases}$. Using b) we obtain $\Omega(x) = \Omega(-1) = \{1,-1\}$.

Proof of e): Suppose $x \in \Omega(x) \in \mathcal{Q}$ and $x \neq \pm 1$. Then $\Omega(x) = \left\{x, \frac{1}{x}, y, \frac{1}{y}\right\}$,

where $y = f(x)$. Observe that $x = \frac{1}{x}$ only when $x = \pm 1$ and $y = \frac{1}{y}$ only when $y = \pm 1$. 
But since $x \neq \pm 1$ and $y \neq \pm 1$, we know that $x \neq \frac{1}{x}$ and $y \neq \frac{1}{y}$. So, it only needs to be shown that $x \neq y$ and $x \neq \frac{1}{y}$ to show that $\Omega(x) = \left\{ x, \frac{1}{x}, y, \frac{1}{y} \right\}$ has four distinct elements.

In Proposition 3, we have shown that if either $x = y$ or $x = \frac{1}{y}$ then $x = \pm 1$. But $x \neq \pm 1$, so $x \neq y$ and $x \neq \frac{1}{y}$. So, $\Omega(x) = \left\{ x, \frac{1}{x}, y, \frac{1}{y} \right\}$ has four distinct elements.
CONSTRUCTION OF A FUNCTION WHICH SATISFIES (1)

Suppose \(\mathcal{G}\) is a family of subsets of \(\mathbb{R}\) satisfying:

a) If \(y \in O \in \mathcal{G}\), then \(\frac{1}{y} \in O\),

b) If \(O_1, O_2 \in \mathcal{G}\), then \(O_1 \cap O_2 = \emptyset\) or \(O_1 = O_2\),

c) If \(1 \in O \in \mathcal{G}\), then \(O = \{1\}\) or \(O = \{1, -1\}\),

d) If \(-1 \in O \in \mathcal{G}\), then \(O = \{-1\}\) or \(O = \{1, -1\}\),

e) If \(x \in O \in \mathcal{G}\) and \(x \neq \pm 1\), then \(O\) has exactly four elements.

Then there is a function \(f\) satisfying (1) with \(\text{Dom}(f) = \bigcup \mathcal{G}\), i.e.,

\[\bigcup \mathcal{G} = \{x : x \in O \text{ for some } O \in \mathcal{G}\}\].

Proof: For each \(O \in \mathcal{G}\) we define a function \(f_O\) satisfying (1). Then we show that \(f = \bigcup_{O \in \mathcal{G}} f_O\) is a function satisfying (1) with domain \(\bigcup \mathcal{G}\) and, moreover,

\[\{\Omega(x) : x \in \bigcup \mathcal{G}\} = \mathcal{G}\].

To this end, let \(O \in \mathcal{G}\). By c), d), and e), it is clear that i) \(O = \{1\}\), or ii) \(O = \{-1\}\), or iii) \(O = \{1, -1\}\), or iv) \(O\) has exactly four elements.

Suppose \(O = \{1\}\). We define \(f_O\) as follows: \(f_O(1) = 1\). Clearly

\[f_O^{-1}(1) = 1 = \frac{1}{f_O(1)}\]. Hence \(f_O\) satisfies (1).
Suppose $O = \{-1\}$. We define $f_0$ as follows: $f_0(-1) = -1$. Clearly

$$f_0^{-1}(-1) = -1 = \frac{1}{-1} = \frac{1}{f_0(-1)}.$$ Hence $f_0$ satisfies (1).

Suppose $O = \{1, -1\}$. We define $f_0(x) = -x$ for $x \in O$. Then

$$f_0(1) = -1 = \frac{1}{-1} = \frac{1}{f_0^{-1}(1)} \quad \text{and} \quad f_0(-1) = 1 = \frac{1}{1} = \frac{1}{f_0^{-1}(-1)}.$$ Hence $f_0$ satisfies (1).

Suppose $O$ has exactly four elements. Let $m = \min(O)$ and

$$M = \min\left(O \cap \left\{m, \frac{1}{m}\right\}\right)'.$$ From a), we define $f_o$ as follows: $f_0(x) = \begin{cases} M, & \text{if } x = m, \\ \frac{1}{m}, & \text{if } x = M, \\ m, & \text{if } x = \frac{1}{M}. \end{cases}$

Then $f_0^{-1}(M) = m = \frac{1}{1/M} = \frac{1}{f_0(M)}$, $f_0^{-1}(\frac{1}{m}) = M = \frac{1}{1/M} = \frac{1}{f_0(\frac{1}{m})}$,

$$f_0^{-1}\left(\frac{1}{M}\right) = \frac{1}{m} = \frac{1}{f_0\left(\frac{1}{M}\right)}.$$ and $f_0^{-1}(m) = \frac{1}{M} = \frac{1}{f_0(m)}$. Thus $f_0$ satisfies (1).

Let $f = \bigcup_{0 \in \mathcal{O}} f_0$. Observe, from b), that this is a disjoint union, so $f$ is a function.

Clearly $\text{Dom}(f) = \bigcup_{0 \in \mathcal{O}} f$. Thus, for each $x \in \text{Dom}(f)$, there is exactly one $O \in \mathcal{O}$ such that $x \in O$ and $f^{-1}(x) = f_0^{-1}(x) = \frac{1}{f_0(x)} = \frac{1}{f(x)}$. So, $f$ satisfies (1).
FINITE CARDINALITY DOMAIN

**Proposition 6**

A function \( f \) satisfying (1) whose domain is finite can have any natural number as the cardinality of its domain except those of the form \( 4n + 3 \), where \( n \) is a non-negative integer.

Proof: The cardinality of the domain of \( f_0 = \{(1,1)\} \) is 1. The cardinality of the domain of \( f_1 = \{(1,-1),(-1,1)\} \) is 2. Now, \( f_2 = \left\{ \left(2, -\frac{1}{2}\right), \left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, -2\right), (-2, 2) \right\} \) shows that the cardinality of \( f \) can be 4. In fact, if \( n \) is any integer greater than 1, then \( f_n = \left\{ \left(n, -\frac{1}{n}\right), \left(-\frac{1}{n}, n\right), \left(\frac{1}{n}, -n\right), (-n, n) \right\} \) is a solution of (1) whose domain has cardinality 4. Notice that if \( i \neq j \) and \( \{i, j\} \neq \{0, 1\} \), then \( \text{Dom}(f_i) \cap \text{Dom}(f_j) = \emptyset \).

Thus, \( f = f_2 \cup f_3 \cup \ldots \cup f_n \cup f_{n+1} \) is a function satisfying (1) whose domain has cardinality \( 4n \). \( f = f_0 \cup f_2 \cup f_3 \cup \ldots \cup f_n \cup f_{n+1} \) is a function satisfying (1) whose domain has cardinality \( 4n + 1 \). Also, \( f = f_1 \cup f_2 \cup f_3 \cup \ldots \cup f_n \cup f_{n+1} \) is a function satisfying (1) whose domain has cardinality \( 4n + 2 \).

It remains to be shown that every function \( f \) satisfying (1) whose domain is finite must have a domain whose cardinality is different from a non-negative integer of
the form $4n + 3$. This can be shown using a proof by descent.

First let us consider the simple case where $n = 0$, i.e., show that no function $f$ satisfying (1) has a domain whose cardinality is equal to 3. Assume the cardinality of $\text{Dom}(f)$ is equal to 3. Since the cardinality of $\text{Dom}(f) > 2$, there exists $x \in \text{Dom}(f)$ such that $x \neq \pm 1$. Therefore, $\Omega(x)$ has 4 elements, which contradicts the assumption. Therefore no function $f$ satisfying (1) has a domain whose cardinality is equal to 3.

Now, consider the cases where $n \geq 1$. Define the set $W$ as the set of all non-negative integers. Let $S = \{n \in W: \text{there exists a function } f \text{ satisfying (1) where the cardinality of } \text{Dom}(f) \text{ is } 4n + 3\}$. If $S \neq \emptyset$, then $S$ has a least element. Call this least element $k$. If it can be shown that a number $m \in S$ exists which is smaller than $k$, then we have a contradiction to $S \neq \emptyset$. Finding such a number $m \in W$ would imply that no function $f$ satisfying (1) whose domain is finite can have a domain whose cardinality is $4n + 3$ for any whole number $n$.

Assume there exists a function $f$ satisfying (1) where the cardinality of $\text{Dom}(f)$ is $4k + 3$. Let $m = k - 1$, then $m$ is smaller than $k$. We will show that there exists a function $g$ where the cardinality of $\text{Dom}(g)$ is $4m + 3 = 4k - 1$. Since $4m + 3 > 2$, we know that there exists an $a \in \text{Dom}(f)$ such that $a \neq -1, 1$. Let $\Omega(a) = \{a, \frac{1}{a}, b, \frac{1}{b}\}$ and let $F = \Omega(a) \cap \text{Dom}(f)$. Now, $F$ has $4k - 1$ elements. Construct a function $g$ such
that \( \text{Dom}(g) = F \) as follows: \( g(x) = f(x), \ x \in F. \) Thus, \( \text{Dom}(g) \) has cardinality \( 4m + 3 = 4k - 1. \)

Let \( c \in \text{Dom}(g) \). Let \( g(c) = d \). Now, \( c \notin \Omega(a) \), so \( \Omega(c) \cap \Omega(a) = \emptyset \). Since \( f \) satisfies (1) and \( \text{Dom}(g) = \Omega(a)' \cap \text{Dom}(f) \), by Proposition 1 we have the following:

\[
\begin{align*}
g(c) &= f(c) = d, \\
g(d) &= f(d) = \frac{1}{c}, \\
g\left(\frac{1}{c}\right) &= f\left(\frac{1}{c}\right) = \frac{1}{d}.
\end{align*}
\]

Now, \( d \notin \Omega(a) \), \( \frac{1}{c} \notin \Omega(a) \), and \( \frac{1}{d} \notin \Omega(a) \). So,

\[
\begin{align*}
g\left(\frac{1}{b}\right) &= f\left(\frac{1}{d}\right) = c \\
d \in \text{Dom}(g), \ \frac{1}{c} \in \text{Dom}(g), \ \text{and} \ \frac{1}{d} \in \text{Dom}(g). \ \text{Therefore, } g \text{ satisfies (1).}
\end{align*}
\]

We have shown that \( m \in S \) and \( m < k \), so this contradiction implies that \( S = \emptyset \).

Therefore, by proof by descent we have shown that there is no function \( f \) satisfying (1) whose domain has cardinality equal to \( 4n + 3 \) for \( n \) a non-negative integer.
PROPERTIES OF SOLUTIONS CONTINUOUS ON \((0, \infty)\)

The propositions and examples given so far all cover cases with finite or infinite number of points in the domain, but which are not necessarily continuous. We now seek solutions of (1) which are continuous on the interval \((0, \infty)\).

**Proposition 7**

Suppose \(f\) is a solution of (1) that is continuous on \((0, \infty)\). Then \(f(x) < 0\) for all \(x > 0\).

**Proof:** Suppose that for some \(a > 0\), we have \(f(a) > 0\). Since \(f(x) \neq 0\) for all \(x\), it follows from the Intermediate Value Theorem that \(f(x) > 0\) for all \(x > 0\). In particular, \(b = f(2) > 0\). From Proposition 3, we know that \(b \neq 1\). Thus either \(b > 1\) or \(b < 1\).

Suppose \(b > 1\). Then, since \(f(2) = b\), we have \(f(b) = \frac{1}{2}\). Now \(\frac{1}{2} < 1 < b\), so by the Intermediate Value Theorem there is some \(c\) between 2 and \(b\) such that \(f(c) = 1\). We know that \(f(1) = 1\), so \(c = 1\), which is a contradiction since \(1 < 2\) and \(1 < b\).

Suppose \(b < 1\). Then, since \(f(2) = b\), we have \(f(b) = \frac{1}{2}\), \(f\left(\frac{1}{2}\right) = \frac{1}{b}\), and
\[ f\left(\frac{1}{b}\right) = 2. \] Now \( b < 1 < 2 \), so by the Intermediate Value Theorem there is some \( d \) between \( \frac{1}{b} \) and 2 such that \( f(d) = 1 \). We know that \( f(1) = 1 \), so \( d = 1 \), which is a contradiction since \( 1 < \frac{1}{b} \) and \( 1 < 2 \).

Since we have shown that \( f(a) \) cannot be either 0 or positive, \( f(a) \) must be negative.

**Proposition 8**

If \( f \) is a continuous function on \((0, \infty)\) and \( f \) satisfies the functional equation (1), then \( f \) is either increasing or decreasing from \((0, \infty)\) into \((-\infty, 0)\) with \( f(1) = -1 \).

Proof: Suppose \( f \) is a continuous function on \((0, \infty)\) and \( f \) satisfies the functional equation (1). By Proposition 7, we have \( f(x) < 0 \) for all \( x \in (0, \infty) \), so \( f \) is a function from \((0, \infty)\) into \((-\infty, 0)\). Since \( f \) satisfies (1), \( f \) has an inverse, and is, therefore, one-to-one. By the Intermediate Value Theorem, we conclude that \( f \) is either increasing or decreasing. We have also shown earlier that if \( f \) satisfies (1), then either \( f(1) = 1 \) or \( f(1) = -1 \). But since \( f(x) < 0 \) for all \( x \in (0, \infty) \), we must have \( f(1) = -1 \).
EXAMPLES OF SOLUTIONS CONTINUOUS ON \((0, \infty)\)

The following examples illustrate functions which satisfy the functional equation (1) and are continuous on the interval \((0, \infty)\).

a) Let \( f(x) = \begin{cases} -x, & \text{if } x > 0, \\ \frac{1}{x}, & \text{if } x < 0. \end{cases} \) Then \( f \) is a solution to the functional equation (1) which is continuous over the interval \((0, \infty)\).

Proof of a): Suppose \( x > 0 \). Then \( -x < 0 \), \( \frac{1}{x} > 0 \), and \( \frac{1}{-x} < 0 \). By the definition of \( f \), we have \( f(x) = -x \), \( f(-x) = \frac{1}{-x} \), \( f\left(\frac{1}{x}\right) = \frac{1}{x} \), and \( f\left(\frac{1}{-x}\right) = x \). Suppose \( x < 0 \). Then \( -x > 0 \), \( \frac{1}{x} < 0 \), and \( -x > 0 \). By the definition of \( f \), we have

\[ f(x) = -x, \quad f\left(\frac{1}{x}\right) = \frac{1}{x}, \quad f\left(\frac{1}{-x}\right) = -x, \quad \text{and } f(-x) = x. \]

By Proposition 1, \( f \) is a solution to the functional equation (1). Also, \( f \) is continuous over the interval \((0, \infty)\). A graph of this function is shown below:
Figure 1. Graph of the Function \( f(x) = \begin{cases} -x, & \text{if } x > 0, \\ \frac{1}{x}, & \text{if } x < 0. \end{cases} \)
b) Let \( f(x) = \begin{cases} -x, & \text{if } x < 0, \\ -\frac{1}{x}, & \text{if } x > 0. \end{cases} \) Then \( f \) is a solution to the functional equation (1)

which is continuous over the interval \((0, \infty)\).

Proof of b): Since \( f(x) = \begin{cases} -x, & \text{if } x < 0, \\ -\frac{1}{x}, & \text{if } x > 0 \end{cases} \) is the inverse of \( g(x) = \begin{cases} -x, & \text{if } x > 0, \\ -\frac{1}{x}, & \text{if } x < 0 \end{cases} \) and \( g \) satisfies (1), then by Proposition 4, \( f \) is a solution to the functional equation (1).

Also, \( f \) is continuous over the interval \((0, \infty)\). A graph of this function is shown below:
Figure 2. Graph of the Function $f(x) = \begin{cases} -x, & \text{if } x < 0, \\ \frac{1}{x}, & \text{if } x > 0 \end{cases}$
c) For a fixed $s \neq 0$, $f(x) = \begin{cases} -x^s, & \text{if } x > 0, \\ \left(\frac{1}{x}\right)^{1/s}, & \text{if } x < 0 \end{cases}$ is a solution to the functional equation (1) which is continuous over the interval $(0, \infty)$.

Proof of c): Suppose $x > 0$. Then we have $-x^s < 0$, $\frac{1}{x} > 0$, and $\frac{1}{-x^s} < 0$. By the definition of $f$, we have

\[
\begin{align*}
 f(x) &= -x^s, \\
 f(-x^s) &= \left(-\frac{1}{-x^s}\right)^{1/s} = \frac{1}{x},
\end{align*}
\]

Suppose $x < 0$. Then we have

\[
\begin{align*}
 f\left(\frac{1}{x}\right) &= -\left(\frac{1}{x}\right)^s = \frac{1}{-x^s}, \\
 f\left(\frac{1}{-x^s}\right) &= \left(-(-x^s)\right)^{1/s} = x
\end{align*}
\]

$\frac{1}{(-x)^{1/s}} > 0$, $\frac{1}{x} < 0$, and $(-x)^{1/s} > 0$. By the definition of $f$, we have

\[
\begin{align*}
 f(x) &= \left(-\frac{1}{x}\right)^{1/s} = \frac{1}{(-x)^{1/s}}, \\
 f\left(\frac{1}{(-x)^{1/s}}\right) &= -\left(\frac{1}{(-x)^{1/s}}\right)^s = \frac{1}{x}, \\
 f\left(\frac{1}{x}\right) &= (-x)^{1/s}, \\
 f\left((-x)^{1/s}\right) &= -\left((-x)^{1/s}\right)^s = x
\end{align*}
\]

Also, $f$ is continuous over the interval $(0, \infty)$. A graph of this function for $s = 1,2,3$ is shown below:
Figure 3. Graph of the Function $f(x) = \begin{cases} 
-x^s, & \text{if } x > 0, \\
\left(-\frac{1}{x}\right)^{1/s}, & \text{if } x < 0 \end{cases}$, for $s = 1, 2, 3$. 
d) For a fixed $s \neq 0$, $f(x) = \begin{cases} (-x)^{1/s}, & \text{if } x < 0, \\ \left(-\frac{1}{x}\right)^{s}, & \text{if } x > 0 \end{cases}$ is a solution to the functional equation (1) which is continuous over the interval $(0, \infty)$.

Proof of d): Since $s \neq 0$, $f(x) = \begin{cases} (-x)^{1/s}, & \text{if } x < 0, \\ \left(-\frac{1}{x}\right)^{s}, & \text{if } x > 0 \end{cases}$ is the inverse of

$s \neq 0$, $g(x) = \begin{cases} -x^s, & \text{if } x > 0, \\ \left(-\frac{1}{x}\right)^{1/s}, & \text{if } x < 0 \end{cases}$, then by Proposition 4, $f$ is a solution to the functional equation (1). Also, $f$ is continuous over the interval $(0, \infty)$. A graph of this function for $s = 1, 2, 3$ is shown below:
Figure 4. Graph of the Function \( f(x) = \begin{cases} (-x)^{1/s}, & \text{if } x < 0, \\ \left(-\frac{1}{x}\right)^s, & \text{if } x > 0 \end{cases} \), for \( s = 1, 2, 3 \).
CHARACTERIZATION OF SOLUTIONS CONTINUOUS ON \((0, \infty)\)

We now intend to characterize all solutions of (1) that are continuous on the interval \((0, \infty)\). From Propositions 7 and 8, we see that any solution \(f\) of (1) that is continuous on \((0, \infty)\) attains only negative values, satisfies \(f(1) = -1\), and, in addition, \(f\) is either:

1) decreasing from \((0,1]\) into \([-1,0)\)

or

2) increasing from \((0,1]\) into \((-\infty, -1]\).

The following two propositions show that, in fact, we may prescribe \(f\) arbitrarily on \((0,1]\) within the constraints that \(f\) is continuous on \((0,1]\), satisfies \(f(1) = -1\), and satisfies either 1) or 2). Under these conditions, \(f\) will satisfy (1) and be continuous on \((0, \infty)\). Thus, we will have characterized all solutions of (1) which are continuous on \((0, \infty)\).

**Proposition 9**

Suppose \(g\) is a function which is continuous and decreasing from \((0,1]\) into
$[-1,0)$ with $g(1) = -1$. Then $f(x) = \begin{cases} 
 g(x), & \text{if } 0 < x \leq 1, \\
 \frac{1}{g\left(\frac{1}{x}\right)}, & \text{if } 1 \leq x < \infty,
 \frac{1}{g^{-1}(x)}, & \text{if } g(1) \leq x < g(0^+), \\
g^{-1}\left(\frac{1}{x}\right), & \text{if } \frac{1}{g(0^+)} < x \leq \frac{1}{g(1)}
\end{cases}$

satisfies (1) and is continuous on $(0, \infty)$.

Proof: Suppose $a \in (0,1]$ and let $b = f(a)$. Then, by definition of $f$,

$b \in [g(1), g(0^+))$ and $f(a) = g(a) = b$. Thus, $f(b) = \frac{1}{g^{-1}(b)} = \frac{1}{g^{-1}(g(a))} = \frac{1}{a} \in [1, \infty)$.

This implies that $f\left(\frac{1}{a}\right) = \frac{1}{g(a)} = \frac{1}{b} \in \left(\frac{1}{g(0^+)}, \frac{1}{g(1)}\right)$. Therefore,

$f\left(\frac{1}{b}\right) = g^{-1}(b) = g^{-1}(g(a)) = a$.

Suppose $a \in [1, \infty)$ and let $b = f(a)$. Then, by definition of $f$,

$b \in \left(\frac{1}{g(0^+)}, \frac{1}{g(1)}\right)$ and $f(a) = \frac{1}{g\left(\frac{1}{a}\right)} = b$. Thus,

$f(b) = g^{-1}\left(\frac{1}{b}\right) = g^{-1}\left(g\left(\frac{1}{a}\right)\right) = \frac{1}{a} \in (0,1]$. This implies that

$f\left(\frac{1}{a}\right) = g\left(\frac{1}{a}\right) = \frac{1}{b} \in [g(1), g(0^+))$. Therefore, $f\left(\frac{1}{b}\right) = \frac{1}{g^{-1}\left(\frac{1}{b}\right)} = \frac{1}{g^{-1}\left(g\left(\frac{1}{a}\right)\right)} = a$.

Suppose $a \in [g(1), g(0^+))$ and let $b = f(a)$. Then, by definition of $f$, $b \in [1, \infty)$
and \( f(a) = \frac{1}{g^{-1}(a)} = b \). Thus, \( f(b) = \frac{1}{g\left(\frac{1}{b}\right)} = \frac{1}{g(g^{-1}(a))} = \frac{1}{a} \in \left(\frac{1}{g(0^+)} , \frac{1}{g(1)}\right) \). This implies that \( f\left(\frac{1}{a}\right) = g^{-1}(a) = \frac{1}{b} \in (0,1] \). Therefore, \( f\left(\frac{1}{b}\right) = g\left(\frac{1}{b}\right) = g(g^{-1}(a)) = a \).

Finally, suppose \( a \in \left(\frac{1}{g(0^+)} , \frac{1}{g(1)}\right) \) and let \( b = f(a) \). Then, by definition of \( f \),

\[
\begin{align*}
b & \in (0,1] \text{ and } f(a) = g^{-1}\left(\frac{1}{a}\right) = b. \text{ Thus, } f(b) = g(b) = g\left(\frac{1}{g^{-1}(\frac{1}{a})}\right) = \frac{1}{a} \in [g(1),g(0^+)).
\end{align*}
\]

This implies that \( f\left(\frac{1}{a}\right) = \frac{1}{g^{-1}\left(\frac{1}{a}\right)} = \frac{1}{b} \in [1,\infty) \). Therefore,

\[
\begin{align*}
f\left(\frac{1}{b}\right) & = \frac{1}{g(b)} = \frac{1}{g\left(\frac{1}{g^{-1}\left(\frac{1}{a}\right)}\right)} = a.
\end{align*}
\]

We have shown that for any \( a \in \text{Dom}(f) \), if \( b = f(a) \), then \( f(a) = b \), \( f(b) = \frac{1}{a} \), \( f\left(\frac{1}{a}\right) = \frac{1}{b} \), and \( f\left(\frac{1}{b}\right) = a \). Thus, by Proposition 1, \( f \) satisfies (1). Since \( g(x) \) is continuous for \( x \in (0,1] \), \( f(x) = g(x) \) is continuous for \( x \in (0,1] \). And, since \( g\left(\frac{1}{x}\right) \) is continuous for \( x \in [1,\infty) \) and \( g\left(\frac{1}{x}\right) \neq 0 \) for \( x \in [1,\infty) \), \( f(x) = \frac{1}{g\left(\frac{1}{x}\right)} \) is continuous for \( x \in [1,\infty) \). Hence, \( f \) is continuous on \((0,\infty)\).
In a similar fashion, it can be shown that the following conditions for \( g \) will also generate solutions to (1):

a) \( g \) is continuous and decreasing from \([1, \infty)\) into \((-\infty, -1]\) with \( g(1) = -1 \),

b) \( g \) is continuous and increasing from \([-1, 0)\) into \([1, \infty)\) with \( g(-1) = 1 \),

c) \( g \) is continuous and increasing from \((-\infty, -1]\) into \((0,1]\) with \( g(-1) = 1 \).

**Proposition 10**

Suppose \( g \) is a function which is continuous and increasing from \((0,1]\) into \((-\infty,-1]\) with \( g(1) = -1 \). Then \( f(x) = \begin{cases} 
g(x), & \text{if } 0 < x \leq 1, \\
\frac{1}{g(x)}, & \text{if } 1 \leq x < \infty,
\end{cases} 
\)

is a function \((-\infty,-1]\) with \( g(1) = -1 \). This implies that \( f \) is a function which satisfies (1) and is continuous on \((0,\infty)\).

**Proof:** Suppose \( a \in (0,1] \) and let \( b = f(a) \). Then, by definition of \( f \),

\[ b \in (g(0^+),g(1)] \text{ and } f(a) = g(a) = b. \text{ Thus, } f(b) = \frac{1}{g^{-1}(b)} = \frac{1}{g^{-1}(g(a))} = \frac{1}{a} \in [1,\infty). \]

This implies that \( f\left(\frac{1}{a}\right) = \frac{1}{g(a)} = \frac{1}{b} \in \left[\frac{1}{g(1)}, \frac{1}{g(0^+)}\right] \). Therefore,

\[ f\left(\frac{1}{b}\right) = g^{-1}(b) = g^{-1}(g(a)) = a. \]
Suppose \( a \in [1, \infty) \) and let \( b = f(a) \). Then, by definition of \( f \),

\[
b \in \left[ \frac{1}{g(l)}, \frac{1}{g(0^+)} \right] \text{ and } f(a) = \frac{1}{g\left(\frac{1}{a}\right)} = b. \text{ Thus,}
\]

\[
f(b) = g^{-1}\left(\frac{1}{b}\right) = g^{-1}\left(\frac{1}{a}\right) = \frac{1}{a} \in (0,1]. \text{ This implies that}
\]

\[
f\left(\frac{1}{a}\right) = g\left(\frac{1}{a}\right) = \frac{1}{b} \in (g(0^+), g(1)]. \text{ Therefore, } f\left(\frac{1}{b}\right) = \frac{1}{g^{-1}\left(\frac{1}{b}\right)} = \frac{1}{g^{-1}\left(\frac{1}{a}\right)} = a.
\]

Suppose \( a \in \left[ \frac{1}{g(l)}, \frac{1}{g(0^+)} \right] \) and let \( b = f(a) \). Then, by definition of \( f \),

\[
b \in (0,1] \text{ and } f(a) = g^{-1}\left(\frac{1}{a}\right) = b. \text{ Thus, } f(b) = g(b) = g\left(g^{-1}\left(\frac{1}{a}\right)\right) = \frac{1}{a} \in (g(0^+), g(1)].
\]

This implies that \( f\left(\frac{1}{a}\right) = \frac{1}{g^{-1}\left(\frac{1}{a}\right)} = \frac{1}{b} \in [1, \infty). \text{ Therefore,}
\]

\[
f\left(\frac{1}{b}\right) = \frac{1}{g(b)} = \frac{1}{g\left(g^{-1}\left(\frac{1}{a}\right)\right)} = a.
\]

Finally, suppose \( a \in (g(0^+), g(1)] \) and let \( b = f(a) \). Then, by definition of \( f \),

\[
b \in [1, \infty) \text{ and } f(a) = \frac{1}{g^{-1}(a)} = b. \text{ Thus,}
\]

\[
f(b) = \frac{1}{g\left(\frac{1}{b}\right)} = \frac{1}{g\left(g^{-1}(a)\right)} = \frac{1}{a} \in \left[ \frac{1}{g(1)}, \frac{1}{g(0^+)} \right]. \text{ This implies that}
\]
\[ f \left( \frac{1}{a} \right) = g^{-1}(a) = \frac{1}{b} \in (0,1]. \] Therefore, \[ f \left( \frac{1}{b} \right) = g \left( \frac{1}{b} \right) = g(g^{-1}(a)) = a. \]

We have shown that for any \( a \in \text{Dom}(f) \), if \( b = f(a) \), then \( f(a) = b \), \( f(b) = \frac{1}{a} \), \( f \left( \frac{1}{a} \right) = \frac{1}{b} \), and \( f \left( \frac{1}{b} \right) = a \). Thus, by Proposition 1, \( f \) satisfies (1). Since \( g(x) \) is continuous for \( x \in (0,1] \), \( f(x) = g(x) \) is continuous for \( x \in (0,1] \). And, since \( g \left( \frac{1}{x} \right) \) is continuous for \( x \in [1,\infty) \) and \( g \left( \frac{1}{x} \right) \neq 0 \) for \( x \in [1,\infty) \), \( f(x) = \frac{1}{g \left( \frac{1}{x} \right)} \) is continuous for \( x \in [1,\infty) \). Hence, \( f \) is continuous on \((0,\infty)\).

In a similar fashion, it can be shown that the following conditions for \( g \) will also generate solutions to (1):

a) \( g \) is continuous and increasing from \([1,\infty)\) into \([-1,0)\) with \( g(1) = -1 \),

b) \( g \) is continuous and decreasing from \([-1,0)\) into \((0,1]\) with \( g(-1) = 1 \),

c) \( g \) is continuous and decreasing from \((-\infty,-1]\) into \([1,\infty)\) with \( g(-1) = 1 \).

**Example of a Solution Which is Continuous and Decreasing on \((0,\infty)\)**

The function \( f(x) = \begin{cases} -(x^2 + x)/2, & \text{if } 0 < x \leq 1, \\ 2/(-(1/x)^2 - 1/x) & \text{if } 1 \leq x < \infty, \\ 2/(-1 + \sqrt{1-8x}) & \text{if } -1 \leq x < 0, \\ (-1 + \sqrt{1-8/x}))/2, & \text{if } -\infty < x \leq -1 \end{cases} \) satisfies (1).
Proof: Let \( g(x) = -(x^2 + x) / 2 \), \( 0 < x \leq 1 \). Since \( g \) is a function which is continuous and decreasing from \((0, 1] \) into \([-1, 0) \) with \( g(1) = -1 \), then by Proposition 9,

\[
\begin{align*}
    f(x) = & \begin{cases} 
    g(x), & \text{if } 0 < x \leq 1, \\
    \frac{1}{g\left(\frac{1}{x}\right)}, & \text{if } 1 \leq x < \infty,
    \end{cases} \\
    & \text{satisfies (1). We now only have to determine the function } g^{-1}. \text{ Let } y = g(x) = -(x^2 + x) / 2. \text{ Interchanging } x \text{ and } y \text{ and solving for } x \\
    & \text{using the quadratic formula gives } x = g^{-1}(y) = \left(-1 \pm \sqrt{1 - 8y}\right) / 2. \text{ To determine which value of } g^{-1} \text{ to use, we must use the fact that } g(1) = -1, \text{ which implies that } g^{-1}(-1) = 1. \text{ Therefore, } g^{-1}(x) = \left(-1 + \sqrt{1 - 8x}\right) / 2. \text{ Using Proposition 9 implies that the function}
\end{align*}
\]

\[
\begin{align*}
    f(x) = & \begin{cases} 
    -(x^2 + x) / 2, & \text{if } 0 < x \leq 1, \\
    2 \left(\left(1 / x\right)^2 - 1 / x\right) & \text{if } 1 \leq x < \infty,
    \end{cases} \\
    & \text{satisfies (1). A graph of this function is shown below:}
\end{align*}
\]
Figure 5. Graph of the Function $f(x) =$ \[
\begin{cases} 
-(x^2 + x)/2, & \text{if } 0 < x \leq 1, \\
2/(-(1/x)^2 - 1/x), & \text{if } 1 \leq x < \infty, \\
2/(-1 + \sqrt{1-8x}), & \text{if } -1 \leq x < 0, \\
(-1 + \sqrt{1-8x})/2, & \text{if } -\infty < x \leq -1.
\end{cases}
\]
Example of a Solution Which is Continuous and Increasing on \((0, \infty)\)

The function \(f(x) = \begin{cases} \ln(x) - 1, & \text{if } 0 < x \leq 1, \\ \frac{1}{-\ln(x)} - 1, & \text{if } 1 \leq x < \infty, \\ e^{(x+1)/x}, & \text{if } -1 \leq x < 0, \\ \frac{1}{e^{x+1}}, & \text{if } -\infty < x \leq -1 \end{cases}\) satisfies (1).

Proof: Let \(g(x) = \ln(x) - 1\), \(0 < x \leq 1\). Since \(g\) is a function which is continuous and increasing function from \((0,1]\) into \((-\infty,-1]\) with \(g(1) = -1\), then by Proposition 10,

\[
f(x) = \begin{cases} g(x), & \text{if } 0 < x \leq 1, \\ \frac{1}{g(x)}, & \text{if } 1 \leq x < \infty, \\ g^{-1}\left(\frac{1}{x}\right), & \text{if } \frac{1}{g(1)} \leq x < \frac{1}{g(0^+)} , \\ \frac{1}{g^{-1}(x)}, & \text{if } g(0^+) < x \leq g(1) \end{cases}\]

satisfies (1). We now only have to determine the function \(g^{-1}\). Let \(y = g(x) = \ln(x) - 1\). Solving for \(x\) gives \(x = g^{-1}(y) = e^{y+1}\). So,

\(g^{-1}(x) = e^{x+1}\). Using Proposition 10 implies that the function

\[
f(x) = \begin{cases} \ln(x) - 1, & \text{if } 0 < x \leq 1, \\ \frac{1}{-\ln(x)} - 1, & \text{if } 1 \leq x < \infty, \\ e^{(x+1)/x}, & \text{if } -1 \leq x < 0, \\ \frac{1}{e^{x+1}}, & \text{if } -\infty < x \leq -1 \end{cases}\]

satisfies (1). A graph of this function is shown below:
Figure 6. Graph of the Function $f(x) = \begin{cases} 
\ln(x) - 1, & \text{if } 0 < x \leq 1, \\
\frac{1}{-\ln(x) - 1}, & \text{if } 1 \leq x < \infty, \\
e^{(x+1)/x}, & \text{if } -1 \leq x < 0, \\
\frac{1}{e^{x+1}}, & \text{if } -\infty < x \leq -1 
\end{cases}$
POSITIVE-VALUED SOLUTIONS CONTINUOUS ON $[a, b]$, WHERE

$1 < a < b < \infty$

If a solution $f$ of (1) is required to be continuous on all of $(0, \infty)$, then we know that $f(x) < 0$ for each $x \in (0, \infty)$. For $[a, b] \subseteq (1, \infty)$, it is possible to find solutions of (1) which are continuous and positive on $[a, b]$ under very reasonable conditions on $f$ given in the following two propositions.

**Proposition 11**

Suppose $g$ is a positive-valued function which is continuous and decreasing from $[a, b]$ into $[g(b), g(a)]$, where $1 < a < b < \infty$. Then

$$f(x) = \begin{cases} 
  g(x), & \text{if } a \leq x \leq b, \\
  \frac{1}{g\left(\frac{1}{x}\right)}, & \text{if } \frac{1}{b} \leq x \leq \frac{1}{a}, \\
  \frac{1}{g^{-1}(x)}, & \text{if } g(b) \leq x \leq g(a), \\
  g^{-1}\left(\frac{1}{x}\right), & \text{if } \frac{1}{g(a)} \leq x \leq \frac{1}{g(b)}
\end{cases}$$

is a function which satisfies (1) and is continuous on the intervals $[a, b]$, $[\frac{1}{b}, \frac{1}{a}]$, $[g(b), g(a)]$, and $\left[\frac{1}{g(a)}, \frac{1}{g(b)}\right]$, provided these intervals are pairwise disjoint.
Proof: Suppose $x \in [a, b]$ and let $y = f(x)$. Then, by definition of $f$,

$$y \in [g(b), g(a)] \text{ and } f(x) = g(x) = y.$$ Thus, $f(y) = \frac{1}{g^{-1}(y)} = \frac{1}{g^{-1}(g(x))} = \frac{1}{x} \in \left[ \frac{1}{b}, \frac{1}{a} \right]$.

This implies that $f\left(\frac{1}{x}\right) = \frac{1}{g(x)} = \frac{1}{y} \in \left[ \frac{1}{g(a)}, \frac{1}{g(b)} \right]$. Therefore,

$$f\left(\frac{1}{y}\right) = g^{-1}(y) = g^{-1}(g(x)) = x.$$ 

Suppose $x \in \left[ \frac{1}{b}, \frac{1}{a} \right]$ and let $y = f(x)$. Then, by definition of $f$,

$$y \in \left[ \frac{1}{g(a)}, \frac{1}{g(b)} \right] \text{ and } f(x) = \frac{1}{g\left(\frac{1}{x}\right)} = y.$$ Thus,

$$f(y) = g^{-1}\left(\frac{1}{y}\right) = g^{-1}\left(g\left(\frac{1}{x}\right)\right) = \frac{1}{x} \in [a, b].$$ This implies that

$$f\left(\frac{1}{x}\right) = g\left(\frac{1}{y}\right) = \frac{1}{y} \in [g(b), g(a)].$$ Therefore, $f\left(\frac{1}{y}\right) = \frac{1}{g^{-1}\left(\frac{1}{y}\right)} = \frac{1}{g^{-1}\left(g\left(\frac{1}{x}\right)\right)} = x.$

Suppose $x \in [g(b), g(a)]$ and let $y = f(x)$. Then, by definition of $f$, $y \in \left[ \frac{1}{b}, \frac{1}{a} \right]$ and $f(x) = \frac{1}{g^{-1}(x)} = y$. Thus, $f(y) = \frac{1}{g\left(\frac{1}{y}\right)} = \frac{1}{g(g^{-1}(x))} = \frac{1}{x} \in \left[ \frac{1}{g(a)}, \frac{1}{g(b)} \right]$. This implies that $f\left(\frac{1}{x}\right) = g^{-1}(x) = \frac{1}{y} \in [a, b]$. Therefore, $f\left(\frac{1}{y}\right) = g\left(\frac{1}{y}\right) = g(g^{-1}(x)) = x.$
Finally, suppose \( x \in \left[ \frac{1}{g(a)}, \frac{1}{g(b)} \right] \) and let \( y = f(x) \). Then, by definition of \( f \),

\[ y \in [a, b] \text{ and } f(x) = g^{-1}\left( \frac{1}{x} \right) = y. \]

Thus, \( f(y) = g(y) = g\left( g^{-1}\left( \frac{1}{x} \right) \right) = \frac{1}{x} \in [g(b), g(a)] \).

This implies that \( f\left( \frac{1}{x} \right) = \frac{1}{g\left( \frac{1}{x} \right)} = \frac{1}{y} \in \left[ \frac{1}{b}, \frac{1}{a} \right] \). Therefore,

\[
\frac{1}{y} = \frac{1}{g(y)} = \frac{1}{g\left( g^{-1}\left( \frac{1}{x} \right) \right)} = x.
\]

We have shown that for any \( x \in \text{Dom}(f) \), if \( y = f(x) \), then \( f(x) = y \),

\[
f(y) = \frac{1}{x}, \quad f\left( \frac{1}{x} \right) = \frac{1}{y}, \text{ and } f\left( \frac{1}{y} \right) = x. \]

Thus, by Proposition 1, \( f \) satisfies (1).

In a similar fashion, it can be shown that the following conditions for \( g \) will also generate solutions to (1) provided the appropriate intervals are pairwise disjoint:

a) \( g \) is continuous and decreasing from \( \left[ \frac{1}{b}, \frac{1}{a} \right] \) into \( \left[ \frac{1}{g(a)}, \frac{1}{g(b)} \right] \), where \( 1 < a < b < \infty \),

b) \( g \) is continuous and increasing from \( [g(b), g(a)] \) into \( \left[ \frac{1}{b}, \frac{1}{a} \right] \), where \( 1 < a < b < \infty \),

c) \( g \) is continuous and increasing from \( \left[ \frac{1}{g(a)}, \frac{1}{g(b)} \right] \) into \([a, b]\), where
1 < a < b < \infty.

Proposition 12

Suppose \( g \) is a positive-valued function which is continuous and increasing from \([a, b]\) into \([g(a), g(b)]\), where \(1 < a < b < \infty\). Then

\[
f(x) = \begin{cases} 
    g(x), & \text{if } a \leq x \leq b, \\
    \frac{1}{g(x)}, & \text{if } \frac{1}{b} \leq x \leq \frac{1}{a}, \\
    g^{-1}(\frac{1}{x}), & \text{if } \frac{1}{g(b)} \leq x \leq \frac{1}{g(a)}, \\
    \frac{1}{g^{-1}(x)}, & \text{if } g(a) \leq x \leq g(b)
\end{cases}
\]

is a function which satisfies (1) and which is continuous on the intervals \([a, b]\), \([\frac{1}{b}, \frac{1}{a}]\), \([\frac{1}{g(b)}, \frac{1}{g(a)}]\), and \([g(a), g(b)]\), provided these intervals are pairwise disjoint.

Proof: Suppose \( x \in [a, b] \) and let \( y = f(x) \). Then, by definition of \( f \),

\( y \in [g(a), g(b)] \) and \( f(x) = g(x) = y \). Thus,

\[
f(y) = \frac{1}{g^{-1}(y)} = \frac{1}{g^{-1}(g(x))} = \frac{1}{x} \in \left[ \frac{1}{g(b)}, \frac{1}{g(a)} \right].
\]

This implies that \( f\left(\frac{1}{x}\right) = \frac{1}{g(x)} = \frac{1}{y} \in \left[ \frac{1}{g(b)}, \frac{1}{g(a)} \right] \). Therefore,

\[
f\left(\frac{1}{y}\right) = g^{-1}(y) = g^{-1}(g(x)) = x.
\]

Suppose \( x \in \left[ \frac{1}{b}, \frac{1}{a} \right] \) and let \( y = f(x) \). Then, by definition of \( f \),
\[ y \in \left[ \frac{1}{g(b)}, \frac{1}{g(a)} \right] \text{ and } f(x) = \frac{1}{g\left(\frac{1}{x}\right)} = y. \text{ Thus,} \]

\[ f(y) = g^{-1}\left(\frac{1}{y}\right) = g^{-1}\left(g\left(\frac{1}{x}\right)\right) = \frac{1}{x} \in [a, b]. \text{ This implies that} \]

\[ f\left(\frac{1}{x}\right) = g\left(\frac{1}{x}\right) = \frac{1}{y} \in [g(a), g(b)]. \text{ Therefore, } f\left(\frac{1}{y}\right) = \frac{1}{g^{-1}\left(\frac{1}{y}\right)} = \frac{1}{g^{-1}\left(g\left(\frac{1}{x}\right)\right)} = x. \]

Suppose \( x \in \left[ \frac{1}{g(b)}, \frac{1}{g(a)} \right] \) and let \( y = f(x) \). Then, by definition of \( f \),

\[ y \in [a, b] \text{ and } f(x) = g^{-1}\left(\frac{1}{x}\right) = y. \text{ Thus, } f(y) = g(y) = g\left(g^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{x} \in [g(a), g(b)]. \]

This implies that \( f\left(\frac{1}{x}\right) = \frac{1}{g^{-1}\left(\frac{1}{x}\right)} = \frac{1}{y} \in \left[ \frac{1}{b}, \frac{1}{a} \right] \). Therefore,

\[ f\left(\frac{1}{y}\right) = \frac{1}{g(y)} = \frac{1}{g\left(g^{-1}\left(\frac{1}{x}\right)\right)} = x. \]

Finally, suppose \( x \in [g(a), g(b)] \) and let \( y = f(x) \). Then, by definition of \( f \),

\[ y \in \left[ \frac{1}{b}, \frac{1}{a} \right] \text{ and } f(x) = \frac{1}{g^{-1}(x)} = y. \text{ Thus,} \]

\[ f(y) = \frac{1}{g\left(\frac{1}{y}\right)} = \frac{1}{g(g^{-1}(x))} = \frac{1}{x} \in \left[ \frac{1}{g(b)}, \frac{1}{g(a)} \right]. \text{ This implies that} \]
\[ f \left( \frac{1}{x} \right) = g^{-1}(x) = \frac{1}{y} \in [a, b] \text{. Therefore, } f \left( \frac{1}{y} \right) = g \left( \frac{1}{y} \right) = g(g^{-1}(x)) = x \, . \]

We have shown that for any \( x \in \text{Dom}(f) \), if \( y = f(x) \), then \( f(x) = y \, . \)

\[ f(y) = \frac{1}{x} \, , \quad f \left( \frac{1}{x} \right) = \frac{1}{y} \, , \quad \text{and } f \left( \frac{1}{y} \right) = x \, . \text{ Thus, by Proposition 1, } f \text{ satisfies (1).} \]

In a similar fashion, it can be shown that the following conditions for \( g \) will also generate solutions to (1) provided the appropriate intervals are pairwise disjoint:

a) \( g \) is continuous and increasing from \( \left[ \frac{1}{b}, \frac{1}{a} \right] \) into \( \left[ \frac{1}{g(b)}, \frac{1}{g(a)} \right] \), where \( 1 < a < b < \infty \, . \)

b) \( g \) is continuous and decreasing from \( \left[ g(a), g(b) \right] \) into \( \left[ \frac{1}{b}, \frac{1}{a} \right] \), where \( 1 < a < b < \infty \, . \)

c) \( g \) is continuous and decreasing from \( \left[ \frac{1}{g(b)}, \frac{1}{g(a)} \right] \) into \( [a, b] \), where \( 1 < a < b < \infty \, . \)

Example of a Positive-Valued Solution Which is Continuous and Decreasing on \([a, b] \). Where \( 1 < a < b < \infty \)
The function \( f(x) = \begin{cases} 
\frac{1}{\ln(x)}, & \text{if } 3 \leq x \leq 4, \\
-\ln(x), & \text{if } \frac{1}{4} \leq x \leq \frac{1}{3}, \\
\frac{1}{e^{\frac{1}{x}}}, & \text{if } \frac{1}{\ln(4)} \leq x \leq \frac{1}{\ln(3)}, \\
e^x, & \text{if } \ln(3) \leq x \leq \ln(4)
\end{cases} \) satisfies (1). 

Proof: Let \( g(x) = \frac{1}{\ln(x)}, 3 \leq x \leq 4 \). Since \( g \) is a function which is continuous and decreasing on \([3,4]\), then by Proposition 11, \( f(x) = \begin{cases} 
g(x), & \text{if } a \leq x \leq b, \\
\frac{1}{g\left(\frac{1}{x}\right)}, & \text{if } \frac{1}{b} \leq x \leq \frac{1}{a}, \\
g^{-1}\left(\frac{1}{x}\right), & \text{if } \frac{1}{g(b)} \leq x \leq \frac{1}{g(a)},
\end{cases} \) satisfies (1). Since, \( g^{-1}(x) = e^{1/x} \), by Proposition 11 we have

\[ f(x) = \begin{cases} 
\frac{1}{\ln(x)}, & \text{if } 3 \leq x \leq 4, \\
-\ln(x), & \text{if } \frac{1}{4} \leq x \leq \frac{1}{3}, \\
\frac{1}{e^{\frac{1}{x}}}, & \text{if } \frac{1}{\ln(4)} \leq x \leq \frac{1}{\ln(3)}, \\
e^x, & \text{if } \ln(3) \leq x \leq \ln(4)
\end{cases} \]. Since it is easily verified that the intervals are pairwise disjoint, we conclude that \( f \) is a function which satisfies (1). A graph of this function is shown below:
Figure 7. Graph of the Function $f(x) = \begin{cases} \frac{1}{\ln(x)}, & \text{if } 3 \leq x \leq 4, \\ -\ln(x), & \text{if } \frac{1}{4} \leq x \leq \frac{1}{3}, \\ \frac{1}{e^{ux}}, & \text{if } \frac{1}{\ln(4)} \leq x \leq \frac{1}{\ln(3)}, \\ e^x, & \text{if } \ln(3) \leq x \leq \ln(4) \end{cases}$
Example of a Positive-Valued Solution Which is Continuous and Increasing on 
\([a, b]\),

Where \(1 < a < b < \infty\)

The function \(f(x) = \begin{cases} \sqrt{x}, & \text{if } 2 \leq x \leq 3 \text{ or } \frac{1}{3} \leq x \leq \frac{1}{2}, \\ \frac{1}{x^2}, & \text{if } \frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{2}} \text{ or } \sqrt{2} \leq x \leq \sqrt{3} \end{cases} \)

satisfies (1).

Proof: Let \(g(x) = \sqrt{x}, \ 2 \leq x \leq 3\). Since \(g\) is a function which is continuous and increasing on \([2,3]\), then by Proposition 12, \(f(x) = \begin{cases} g(x), & \text{if } a \leq x \leq b, \\ \frac{1}{g\left(\frac{1}{x}\right)}, & \text{if } \frac{1}{b} \leq x \leq \frac{1}{a}, \\ g^{-1}\left(\frac{1}{x}\right), & \text{if } \frac{1}{g(b)} \leq x \leq \frac{1}{g(a)}, \\ \frac{1}{g^{-1}(x)}, & \text{if } g(a) \leq x \leq g(b) \end{cases} \)

satisfies (1). Since \(g^{-1}(x) = x^2\), by Proposition 12 we have

\(f(x) = \begin{cases} \sqrt{x}, & \text{if } 2 \leq x \leq 3 \text{ or } \frac{1}{3} \leq x \leq \frac{1}{2}, \\ \frac{1}{x^2}, & \text{if } \frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{2}} \text{ or } \sqrt{2} \leq x \leq \sqrt{3} \end{cases} \). Since it is easily verified that the intervals are pairwise disjoint, we conclude that \(f\) is a function which satisfies (1). A graph of this function is shown below:
Figure 8. Graph of the Function $f(x) = \begin{cases} \sqrt{x}, & \text{if } 2 \leq x \leq 3 \text{ or } \frac{1}{3} \leq x \leq \frac{1}{2}, \\ \frac{1}{x^2}, & \text{if } \frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{2}} \text{ or } \sqrt{2} \leq x \leq \sqrt{3}. \end{cases}$
**EXTENSIONS**

We can extend our statement (1) to more general cases as follows:

a) \( f^{-n}(x) = \frac{1}{f^n(x)} \) for all \( x \in \text{Dom}(f) \) for any integer \( n \),

b) \( f(x) = \frac{\beta}{f^{-1}\left(\frac{x}{\alpha}\right)} \) for all \( x \in \text{Dom}(f) \) for any integers \( \alpha \) and \( \beta \).

The functional equation a) is a natural extension of (1), with additional levels of composition. The functional equation b) is derived as follows:

The functional equation (1) has many solutions, but there is some sense in which the study of this functional equation reduces to the study of the single, well-defined map

\[
\Phi: \left( \mathbb{R} \setminus \{0\} \right)^2 \to \left( \mathbb{R} \setminus \{0\} \right)^2 \text{ given by } \Phi(x, y) = \left( y, \frac{1}{x} \right), \quad x \neq 0.
\]

One can replace \( \Phi \) by other maps. For example, \( \Phi(x, y) = \left( \alpha y, \frac{\beta}{x} \right), \quad x \neq 0 \) for any integers \( \alpha \) and \( \beta \), which leads to the functional equation \( f(x) = \frac{\beta}{f^{-1}\left(\frac{x}{\alpha}\right)} \). This functional equation can be represented in the complex plane by the map \( \Psi: \mathbb{C}^2 \to \mathbb{C}^2 \) defined by

\[
\Psi(x, y) = \left( e^{\alpha} y, e^{\beta} \frac{x}{x} \right), \quad x \neq 0 \text{ for any integers } \alpha \text{ and } \beta.
\]

This leads to the functional
equation \[ f(x) = \frac{e^{ib}}{f^{-1}(e^{-ka} x)}. \]

One might guess that the solutions of the functional equations a) and b) have properties similar to those of the solutions to (1). However, we will not examine such solutions and properties here.
REFERENCES

