# Categorical Properties Of Lattice-valued Convergence Spaces

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# CATEGORICAL PROPERTIES OF LATTICE-VALUED CONVERGENCE SPACES

by

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## ABSTRACT

This work can be roughly divided into two parts. Initially, it may be considered a continuation of the very interesting research on the topic of Lattice-Valued Convergence Spaces given by Jäger [2001, 2005]. The alternate axioms presented here seem to lead to theorems having proofs more closely related to standard arguments used in Convergence Space theory when the Lattice is $L = \{0, 1\}$.Various Subcategories are investigated. One such subconstruct is shown to be isomorphic to the category of Lattice Valued Fuzzy Convergence Spaces defined and studied by Jäger [2001]. Our principal category is shown to be a topological universe and contains a subconstruct isomorphic to the category of probabilistic convergence spaces discussed in Kent and Richardson [1996] when $L = [0, 1]$. Fundamental work in lattice-valued convergence from the more general perspective of monads can be found in Gähler [1995]. Secondly, diagonal axioms are defined in the category whose objects consist of all the lattice valued convergence spaces. When the latter lattice is linearly ordered, a diagonal condition is given which characterizes those objects in the category that are determined by probabilistic convergence spaces which are topological.

Certain background information regarding filters, convergence spaces, and diagonal axioms with its dual are given in Chapter 1. Chapter 2 describes Probabilistic Convergence and associated Diagonal axioms. Chapter 3 defines Jäger convergence and proves that Jäger's construct is isomorphic to a bireflective subconstruct of SL-CS. Furthermore, connections between the diagonal axioms discussed and those given by Gähler are explored. In Chapter 4, further categorical properties of SL-CS are discussed and in particular, it is shown that SL-CS is topological, cartesian closed, and extensional. Chapter 5 explores connections between diagonal axioms for objects in the sub construct $\delta(PCS)$ and SL-CS.

Finally, recommendations for further research are provided.

This dissertation is dedicated to the memory of my father-in-law, Victor Manuel Puente. He was a man who believed so much in the American dream that he sacrificed his career as a mathematics teacher in his native Cuba to bring his family to the United States. While the responsibilities for financially supporting his family in his newly adopted country made his dreams of teaching mathematics impossible to realize, he never lost sight of the importance of education. He inculcated this mindset in his daughter, Sandra–my lovely wife. I remember with fond recollections my many conversations with Manolo about mathematics, education, and the amazing reality that is America.

Additionally, this dissertation is dedicated to my first granddaughter–Elena Benson, who was born while this dissertation was completed. I love you already, Elena. May God bless you forever.

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## LIST OF ABBREVIATIONS

| | |
|---|---|
| A,X | sets |
| $2^X$ | power set of X: the collection of all subsets of X |
| $\Lambda$ | indexing set |
| $I$ | unit interval $[0,1]$ |
| $\mathfrak{F}(X)$ | set of all filters on X |
| $\mathfrak{U}(X)$ | set of all ultrafilters on X |
| $\dot{x}$ | fixed ultrafilters on X |
| $\mathfrak{F} \longleftrightarrow (x_n)$ | filter $\mathfrak{F}$ is the elementary filter of a sequence $(x_n)$ |
| $\mathfrak{F} \leq \mathfrak{G} (\mathfrak{G} \geq \mathfrak{F})$ | filter $F$ is coarser than filter $G$ (filter $G$ is finer than filter $F$); i.e., $F \subseteq G (G \supseteq F)$ |
| $F \vee G$ | supremum of filters $F$ and $G$ whenever it exists |
| $F \wedge G$ | infimum of filters $F$ and $G$ |
| $F \xrightarrow{q} x$ | filter $q$-converges to $x$ |
| $K\sigma^{\rightarrow}F$ | "compression filter", $K\sigma^{\rightarrow}F = \bigcup_{F \in \mathfrak{F}} \bigcap_{y \in F} \sigma(y)$ where $F$ is a filter on a set $J$ and $\sigma$: $J \rightarrow F(X)$ is a mapping |

# CHAPTER 1.
# FILTERS, CONVERGENCE SPACES AND DIAGONAL AXIOMS WITH ITS DUAL

This chapter contains some background information which led to the convergence space study in this dissertation. Section 1.1 provides some preliminaries on filters. In Section 1.2, the concept of fuzzy sets and a brief background on the historical development of fuzzy mathematics is provided. Sections 1.3 and 1.4 provide backgrounds on fuzzy filters and some preliminary information on diagonal conditions. A brief introduction to probabilistic convergence is provided in Section 1.5.

## 1.1 Filters

The notion of a filter of subsets introduced by Cartan [1937] has been used as a valuable tool in the development of topology and its applications. Filters can be viewed as a generalization of sequences. Concepts such as points of closure and compactness that are extremely important in general topology theory cannot be described using sequences, but can be described using general filter theory. Convergence structures are described in terms of filter convergence.

**Definition 1** *Let $X$ be a nonempty set and let $2^X$ represent the power set of $X$. A subcollection $\mathfrak{F}$ of $2^X$ is said to be a* ***filter*** *provided:*

(1) $\mathfrak{F} \neq \varnothing$ and $\varnothing \notin \mathfrak{F}$

(2) $A$, $B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$

(3) $A \in \mathfrak{F}$ and $B \supseteq A$ imply $B \in \mathfrak{F}$.

The collection of all filters on a set $X$ is denoted by $\mathfrak{F}(X)$.

## 1.2 Fuzzy Sets

When mathematical analysis is utilized to describe real life physical situations where uncertainty may exist, non-deterministic models or approaches are appropriate. Such is the case with classical probability theory where random variables and their associated probability distribution functions are used to describe inherent uncertainty present in some deterministic reference frame. Typically, as will be indicated later through careful exposition, the reference frame utilized in probability theory is modeled as a metric space, with some known deterministic metric. Many mathematicians have suggested more satisfactory results could have been obtained if uncertainties were built into the geometric reference frame rather than keeping the reference frame so rigid or non-probabilistic. Probabilistic metric spaces suggested by Menger (1942) and probabilistic convergence spaces investigated by Richardson and Kent (1996) provided more generalizable and relevant examples.

In 1965, L.A. Zadeh developed an approach where uncertainty could be built into the underlying reference frame geometry. The rigid reference frame geometry is replaced by mathematical structures that incorporate ’fuzzy sets’. Fuzzy sets permit the addressing of situations where impreciseness might not be probabilistic, meaning it is not due to some error in measurement. Fuzzy set theory identifies uncertainty as a function of classification, and not error in measurement. Linguistics has utilized many elements of fuzzy theoretic concepts. In particular, our lexicon makes distinctions among words and concepts like large, medium, small, petite, etc. difficult to quantify. These ideas are subjective. Fuzzy set theory allows for mathematical structures which directly model this type of uncertainty.

### 1.3 Fuzzy Filters and Convergence

Lowen [1979] defined the concept of a prefilter as a subset of $[0,1]^X$ in order to study the theory of fuzzy topological spaces. Later, Lowen et al. [1991] used prefilters to define the notion of an $L$-fuzzy convergence space, when $L = [0,1]$, and showed that the category of all such objects has several desirable properties, such as being Cartesian closed, not possessed by the category of all fuzzy topological spaces. Höhle [1997] introduced the idea of a (stratified) $L$-filter as a descriptive map from $L^X$ into $L$ rather than a subset of $L^X$ in the investigation of MV-algebras. Stratified $L$-filters are shown by Höhle and Sostak [1999] to be a fruitful tool employed in the development of general lattice-valued topological spaces. Some basic concepts are listed below.

Unless mentioned otherwise, it is assumed throughout this work that $L = (L, \wedge, \vee)$ is a fixed underlying complete lattice with least (largest) element $0(1)$ which obeys the distributive law $a \wedge (\vee b_{b\in B}) = \vee_{b\in B}(a \wedge b)$ for each $a \in L$ and $B \subset L$, respectively. The above conditions are sometimes referred to in the literature as a **complete Heyting algebra**. Moreover, for each nonempty subset $X$, $L^X$ denotes the complete lattice of all maps from $X$ into $L$ equipped with the product order. Each element of $L^X$ is called a fuzzy subset of $X$. In particular, given $\alpha \in L$ and $A \subset X$, define the fuzzy subset

$$\alpha 1_A(x) := \begin{cases} \alpha, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}, \quad x \in X.$$

**Definition 2** *Given a nonempty set $X$, a map $\mathfrak{F} : L^X \to L$ is called a* ***stratified $L$-filter*** *provided for each $\alpha \in L$ and $a, b \in L^X$:*

(a) $\mathfrak{F}(1_\phi) = 0, \mathfrak{F}(\alpha 1_X) \geq \alpha$

(b) $\mathfrak{F}(a) \leq \mathfrak{F}(b)$ whenever $a \leq b$

(c) $\mathfrak{F}(a) \wedge \mathfrak{F}(b) \leq \mathfrak{F}(a \wedge b)$.

Note that (b) implies equality in $(c)$. Let $\mathfrak{F}_{\mathrm{SL}}(X)$ denote the set of all stratified $L$-filters and $\mathfrak{F}(X)$ the set of all Cartan-filters defined on $X$. When $L = \{0,1\}$, $\Delta : \mathfrak{F}_{\mathrm{SL}}(X) \to \mathfrak{F}(X)$ defined by $\Delta(\mathfrak{F}) := \{A \subseteq X : \mathfrak{F}(1_A) = 1\}$ is a bijection. For a general $L$, define $\mathfrak{F} \leq \mathfrak{G}$ by $\mathfrak{F}(a) \leq \mathfrak{G}(a)$, $(\wedge_{j\in J}\mathfrak{F}_{\mathrm{j}})(a) := \wedge_{j\in J}\mathfrak{F}_{\mathrm{j}}(a)$, $[x](a) := a(x)$ and $\mathfrak{F}_{\mathrm{o}}(a) := \wedge\{a(y) : y \in X\}$, for each $x \in X$ and $a \in L^X$. Then $(\mathfrak{F}_{\mathrm{SL}}(X), \leq)$ is a poset having least element $\mathfrak{F}_0$. Note that when $L = \{0,1\}$, $\Delta([x]) = \dot{x}$ and $\Delta(\mathfrak{F}_0) = \dot{X}$, where $\dot{A}$ denotes the filter of all oversets of $A$.

Let $f : X \to Y$ be a map, $a \in L^X$ and $b \in L^Y$. The **image of $a$ under** $f$ is defined by $f^{\to}(a)(y) := \vee\{a(x) : f(x) = y\}$ provided $y$ belongs to the range of $f$; otherwise, $f^{\to}(a)(y) = 0$. Conversely, $f^{\leftarrow}(b) := b \circ f$ is called the **inverse image of $b$ under** $f$. Moreover, given $\mathfrak{F} \in \mathfrak{F}_{\mathrm{SL}}(X)$ and $\mathfrak{G} \in \mathfrak{F}_{\mathrm{SL}}(Y)$, the **image of $\mathfrak{F}$ under** $f$ is defined as $f^{\to}\mathfrak{F}(b) := \mathfrak{F}(f^{\leftarrow}(b))$and the **inverse image of $\mathfrak{G}$ under** $f$ is given by $f^{\leftarrow}\mathfrak{G}(a) := \vee\{\mathfrak{G}(b) : f^{\leftarrow}(b) \leq a\}$ whenever the latter is a stratified $L$-filter, for each $a \in L^X$ and $b \in L^Y$. Let $\psi \in \mathfrak{F}(X)$ and denote

$$\phi_{\mathfrak{F}} := \{A \subseteq X : \mathfrak{F}(1_A) = 1\} \tag{1.1}$$

$$\mathfrak{F}_\psi(a) := \begin{cases} 1, & \text{if } \iota\, a \in \psi \\ 0, & \text{otherwise} \end{cases}, \quad \text{where}$$

$\iota\, a := \{x \in X : a(x) \neq 0\},\ a \in L^X$.

**Lemma 3** *Given the notations defined in (1.1), assume that* $f : X \to Y$ *is a map,* $\mathfrak{F} \in \mathfrak{F}_{SL}(X)$, $\mathfrak{G} \in \mathfrak{G}_{SL}(Y)$ *and* $\psi \in \mathfrak{F}(X)$. *Then*

(a) $f^{\to}\mathfrak{F} \in \mathfrak{F}_{SL}(Y)$

(b) $f^{\leftarrow}\mathfrak{G} \in \mathfrak{F}_{SL}(X)$ iff $\mathfrak{G}(b) = 0$ whenever $f^{\leftarrow}(b) = 1_{\phi}$

(c) $\phi_{\mathfrak{F}} \in \mathfrak{F}(X)$

(d) $\phi_{[x]} = \dot{x}, x \in X$

(e) $\phi_{\mathfrak{F}_o} = \dot{X}$

(f) $\mathfrak{F}_{\psi} \in \mathfrak{F}_{SL}(X)$ provided $L$ is linearly ordered

(g) $\phi_{\mathfrak{F}_{\psi}} = \psi$ whenever $L$ is linearly ordered

(h) $f^{\to}(\phi_{\mathfrak{F}}) = \phi_{f^{\to}\mathfrak{F}}$.

Verification of Lemma 3(b) is given in Proposition 3.5 of Jäger [2001] and the other parts follow easily from the definitions.

## 1.4 Convergence Spaces and Diagonal Conditions

The aim of the theory of convergence spaces is to generalize traditional concepts in topology while including convergence structures without restrictions imposed by local coherence conditions contained in topologies. This analytic perspective is preferable to a geometric one dependent on intuitive notions like open sets and accumulation points. It should be noted that these concepts are consistent with metric spaces but meaningless in more general spaces. Formal treatments of filters and convergence date back to Frechet whose arguments were based on countable sequences, thereby limiting its usefulness. Subsequent work replaced sequences with nets and research by Bourbaki showed that filters provide a better substitution for nets. Contemporary convergence space theory evolved from work by Choquet [1948], Kowalsky [1954], and Fischer [1959].

Metric space theory has been useful in many applications, but the category $MET$ of metric spaces is not sufficiently inclusive enough to permit important properties. As an example, pointwise convergence in a function space induced by a topology is not metrizable. In fact, in $TOP$–the category of topological spaces–useful convergences like convergence almost everywhere and continuous convergence on a function space are not well defined. It is true, however, convergence structures do provide useful mechanisms to define both concepts. $MET$ is not a topological category since it fails in general to have initial structures (or uncountably infinite products). Furthermore, $MET$ and $TOP$ are neither extensional nor cartesian closed. Additionally, quotient maps are not productive in these categories.

Convergence space theory has proved useful in a variety of mathematical fields including functional analysis and algebraic topology. As an example, some areas of functional analysis depend on relationships between a complete regular topological space $X$ and the R-algebra $C_{co}(X)$ of all continuous real-valued functions defined on $X$ equipped with the topology of uniform convergence on compact subsets. A vast majority of these investigations are hampered by the fact that the evaluation map $e : C_{co}(X)$ x $X \rightarrow \mathbb{R}$ defined as follows $e(f, x) = f(x)$ is not, in general, continuous and furthermore, $C_{co}(X)$ may not be complete. There is a coarsest convergence structure on $C(X)$ that overcomes these problems. Papers by Cook and Fischer [1967] and Binz [1975] have addressed issues in these areas. Work by Frolicher and Bucher [1966] and Frolicher and Kriegl [1985] have researched areas where convergence spaces play significant roles.

**Definition 4** *A convergence structure on a set is a rule which assigns each filter to a set containing the points to which the filter converges. More precisely, let $X$ be a set, $\mathfrak{F}(X)$ denote the set of all filters on $X$ with a function $q : \mathfrak{F}(X) \to 2^X$ subject to the following axioms:*

(1) $x \in q(\dot{x})$ for each $x \in X$, where $\dot{x}$ denotes the ultrafilter containing $\{x\}$;

(2) $\mathfrak{F} \leq \mathfrak{G}$ (that is, $\mathfrak{F} \subseteq \mathfrak{G}$) implies $q(\mathfrak{F}) \subseteq q(\mathfrak{G})$;

(3) $x \in q(\mathfrak{F})$ implies $x \in q(\mathfrak{F} \cap \dot{x})$;

(4) $q(\mathfrak{F}) \cap q(\mathfrak{F}) \subseteq q(\mathfrak{F} \cap \mathfrak{G})$ for all $\mathfrak{F}$, $\mathfrak{G} \in \mathfrak{F}(X)$;

(5) For each $\mathfrak{F} \in \mathfrak{F}(X)$, $x \in q(\mathfrak{F})$ if $x \in q(\mathfrak{G})$, for every ultrafilter $\mathfrak{G} \geq \mathfrak{F}$;

(6) $x \in q(\mathfrak{V}_q(x))$, for all $x \in X$;

where $\mathfrak{V}_q(x) := \cap\{\mathfrak{F} : x \in q(\mathfrak{F})\} = \cap\{\mathfrak{F} : \mathfrak{F} \text{ is an ultrafilter and } x \in q(\mathfrak{F})\}$;

Then $q$ is said to be a

**convergence structure** if it satisfies (1) and (2);

**Kent-convergence structure** if it satisfies (1), (2), and (3);

**limit structure** if it satisfies (1), (2), and (4);

**pseudo-topology** if it satisfies (1), (2), and (5);

**pretopology** if it satisfies (1), (2), and (6).

Note that (6) implies (5) implies (4) implies (3), hence pretopology $\Longrightarrow$pseudo-topology $\Longrightarrow$limit structure $\Longrightarrow$ Kent-convergence structure $\Longrightarrow$convergence structure. The pair $(X, q)$ is called a **convergence** (respectively, **Kent-convergence**, **limit**, **pseudo-topology**, **pretopological**) **space,** respectively. A filter $\mathfrak{F}$ is said to $q - converge\ to\ x$ when $x \in q(\mathfrak{F})$, and is denoted by $\mathfrak{F} \xrightarrow{q} x$.The pair $(X, q)$ is called a **convergence space**. A map $f : (X, q) \to (Y, p)$ between two convergence spaces is called **continuous** whenever $\mathfrak{F} \xrightarrow{q} x$ implies that $f^{\to}\mathfrak{F} \xrightarrow{q} f(x)$, where $f^{\to}\mathfrak{F}$ denotes the filter whose base is $\{f(F) : F \in \mathfrak{F}\}$. Let **CON** denote the category whose objects consist of all the convergence spaces and whose morphisms are all the continuous maps between objects.

### 1.5 Probabilistic Convergence

Probabilistic convergence spaces were introduced by Florescu [1989] as an extension of the notion of a probabilistic metric space which arose from the work of Menger [1942]. Replacing the axioms involving nets with the more compatible filter theory gives the following definition:

**Definition 5** *Let* $L = [0, 1]$, $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}(X)$ *and* $\alpha, \beta \in L$. *The pair* $(X, \overline{Q})$, *where* $\overline{Q} = (Q_\alpha)_{\alpha \in L}$, *is called a* ***probabilistic convergence space*** *provided:*

(a) $\dot{x}_\alpha \xrightarrow{Q_\alpha} x$ and $\dot{X} \xrightarrow{Q_0} x$ for each $x \in X$

(b) $\mathfrak{G} \supseteq \mathfrak{F} \xrightarrow{Q_\alpha} x$ implies $\mathfrak{G} \xrightarrow{Q_\alpha} x$

(c) $\mathfrak{F} \xrightarrow{Q_\alpha} x$ implies $\mathfrak{F} \xrightarrow{Q_\beta} x$ whenever $\beta \leq \alpha$.

The probability of $\mathfrak{F}$ converging to $x$ being at least $\alpha$ is the interpretation given by $\mathfrak{F} \xrightarrow{Q_\alpha} x$. A map $f : (X, \overline{Q}) \to (Y, \overline{P})$ is said to be **continuous** whenever $\mathfrak{F} \xrightarrow{Q_\alpha} x$ implies that

$f^{\rightarrow}\mathfrak{F} \stackrel{P_\alpha}{\rightarrow} f(x)$ for each $\mathfrak{F} \in \mathfrak{F}(X)$, $x \in X$ and $\alpha \in L$. Let **PCS** denote the construct whose objects consist of all the probabilistic convergence spaces and whose morphisms are all the continuous maps between objects. Replacing $\mathfrak{F}(X)$ in Definition 5 with $\mathfrak{F}_{SL}(X)$ gives the following, where $L$ is a general lattice. Properties of the category PCS whenever $L = [0,1]$ can be found in Brock and Kent [1997(a) , 1997(b)] and Kent and Richardson [1996].

# CHAPTER 2.
# STRATIFIED L-CONVERGENCE AND ASSOCIATED DIAGONAL AXIOMS

This chapter provides descriptions of stratified convergence structures and related diagonal axioms. In Section 2.1, stratified lattice convergence is defined and an important theorem is proved showing that probability convergence spaces (PCS) are fully embedded in the category of stratified lattice convergence spaces (SL-CS). A Choquet modification to SL-CS is made, and this author shows that this subcategory is bicoreflective in SL-CS consistent with the definition provided in Preuss [2002]. Section 2.2 provides analysis of the category SL-CS, and in particular the Kowalsky compression operator is defined. This is done in an effort to provide elucidation on SL-CS as the author proves several original theorems (Theorems 7, 8, 10, 12, and 13), as well as Lemma 11. Among original results are a proof that a subconstruct of SL-CS is topological and bireflective. Additionally, this author shows that the subcategory of SL-CS that satisfies version F2 of Kowalsky's compression operator is also pretopological. Results are extended and referred to in subsequent chapters in this work.

## 2.1 Stratified L-Convergence

**Definition 6** *Assume that* $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}_{SL}(X)$ *and* $\alpha, \beta \in L$. *The pair* $(X, \overline{q})$, *where* $\overline{q} = (q_\alpha)_{\alpha \in L}$, *is called a* ***stratified*** $L$***-convergence space*** *whenever the following conditions are satisfied:*

(a) $[x] \stackrel{q_\alpha}{\to} x$, $\mathfrak{F}_\circ \stackrel{q_0}{\to} x$ for each $x \in X$

(b) $\mathfrak{G} \geq \mathfrak{F} \stackrel{q_\alpha}{\to} x$ implies $\mathfrak{G} \stackrel{q_\alpha}{\to} x$

(c) $\mathfrak{F}\xrightarrow{q_\alpha}x$ implies $\mathfrak{F}\xrightarrow{q_\beta}x$ whenever $\beta \leq \alpha$.

A map $f : (X, \overline{q}) \to (Y, \overline{p})$ is called **continuous** provided $\mathfrak{F}\xrightarrow{q_\alpha}x$ implies that $f^{\to}\mathfrak{F}\xrightarrow{p_\alpha}f(x)$, for each $\mathfrak{F} \in \mathfrak{F}_{\mathrm{SL}}(X)$, $x \in X$ and $\alpha \in L$. Denote by **SL-CS** the construct whose objects consist of all the stratified $L$-convergence spaces and whose morphisms are all the continuous maps between objects.

Suppose that $L = [0, 1]$ and $(X, \overline{Q}) \in |\mathrm{PCS}|$. Define $\overline{q} = (q_\alpha)_{\alpha \in L}$ as follows:

$$\mathfrak{F}\xrightarrow{q_\alpha}x \quad \text{iff} \quad \phi_{\mathfrak{F}}\xrightarrow{Q_\alpha}x, \quad \alpha \in L. \tag{2.1}$$

Employing Lemma 3(d),(e), it easily follows that $(X, \overline{q}) \in |\text{SL-CS}|$.

Define

$$\begin{array}{c} \delta : \mathrm{PCS} \to \text{SL-CS} \quad \text{as follows:} \\ \delta(X, \overline{Q}) = (X, \overline{q}) \quad \text{and} \quad \delta(f) = f. \end{array} \tag{2.2}$$

**Theorem 7** *Given the notations used in (2.1)-(2.2), let* $L = [0, 1]$. *Then* $\delta : PCS \to SL - CS$ *is a full-embedding functor.*

**Proof.** First, it is shown that $\delta$ is a functor, where $\delta(X, \overline{Q}) = (X, \overline{q})$. Assume that $f : (X, \overline{Q}) \to (Y, \overline{P})$ is continuous. It must be shown that $f : (X, \overline{q}) \to (Y, \overline{p})$ is also continuous. If $\mathfrak{F}\xrightarrow{q_\alpha}x$, then $\phi_{\mathfrak{F}} \overset{Q_\alpha}{\to} x$ and consequently $f^{\to}(\phi_{\mathfrak{F}}) \overset{P_\alpha}{\to} f(x)$. According to Lemma 3 (h), $\phi_{f^{\to}\mathfrak{F}} = f^{\to}(\phi_{\mathfrak{F}}) \overset{P_\alpha}{\to} f(x)$ and thus $f^{\to}\mathfrak{F}\xrightarrow{p_\alpha}f(x)$. Hence $\delta$ is a functor. Conversely, suppose that $f : (X, \overline{q}) \to (Y, \overline{p})$ is continuous; it remains to show that $f : (X, \overline{Q}) \to (Y, \overline{P})$ is also continuous. If $\psi \overset{Q_\alpha}{\to} x$, then by Lemma 3 (g), $\phi_{\mathfrak{F}_\psi} = \psi \overset{Q_\alpha}{\to} x$ and thus $\mathfrak{F}_\psi\xrightarrow{q_\alpha}x$. Hence $f^{\to}\mathfrak{F}_\psi\xrightarrow{p_\alpha}f(x)$ and $\phi_{f^{\to}\mathfrak{F}_\psi}\xrightarrow{P_\alpha}f(x)$. It follows that $\phi_{f^{\to}\mathfrak{F}_\psi} = f^{\to}(\phi_{\mathfrak{F}_\psi}) = f^{\to}(\psi)\xrightarrow{P_\alpha}f(x)$ and thus $f : (X, \overline{Q}) \to (Y, \overline{P})$ is continuous. Finally, $\delta$ is injective. Indeed, assume that

$(X,\overline{Q}) \neq (X,\overline{P})$ and $\psi \overset{Q_\alpha}{\to} x$ but $\psi \overset{P_\alpha}{\to} x$ fails for some $\alpha \in L$. Since $\phi_{\mathfrak{F}_\psi} = \psi$, $\mathfrak{F}_\psi \overset{q_\alpha}{\to} x$ but $\mathfrak{F}_\psi \overset{p_\alpha}{\to} x$ fails and thus $j$ is injective. ■

Given

$$(X,\overline{q}) \in |\text{SL-CS}|, \text{ define } (X,\overline{R}) \text{ as follows:} \tag{2.3}$$

$$\psi \overset{R_\alpha}{\to} x \text{ iff there exists } \mathfrak{F} \in \mathfrak{F}_{\text{SL}}(X)$$

$$\text{such that } \mathfrak{F} \overset{q_\alpha}{\to} x \text{ and } \psi \supseteq \phi_{\mathfrak{F}}.$$

Verification that $(X,\overline{R}) \in |\text{PCS}|$ is straightforward.

**Theorem 8** *Suppose that $L = [0,1]$. Then $\delta(PCS)$ is bireflective in SL-CS.*

***Proof.*** *Given $(X,\overline{q}) \in |SL\text{-}CS|$, denote $(X,\overline{R}) \in |PCS|$ as defined in (2.3). Let $(X,\overline{r}) = \delta(X,\overline{R})$; then $\mathfrak{F} \overset{q_\alpha}{\to} x$ iff $\phi_{\mathfrak{F}} \overset{R_\alpha}{\to} x, \alpha \in L$. Note that id: $(X,\overline{q}) \to (X,\overline{r})$ is continuous. Indeed, it follows from (2.3) that if $\mathfrak{F} \overset{q_\alpha}{\to} x$, then $\phi_{\mathfrak{F}} \overset{R_\alpha}{\to} x$ and thus $\mathfrak{F} \overset{r_\alpha}{\to} x, \alpha \in L$. Hence id: $(X,\overline{q}) \to (X,\overline{r})$ is continuous. Next, assume that $(Y,\overline{p}) = \delta(Y,\overline{P})$ and $f : (X,\overline{q}) \to (Y,\overline{p})$ is continuous. It must be shown that $f : (X,\overline{r}) \to (Y,\overline{p})$ is continuous. Suppose that $\mathfrak{F} \overset{r_\alpha}{\to} x$; then $\phi_{\mathfrak{F}} \overset{R_\alpha}{\to} x$ and thus there exists $\mathfrak{G} \overset{q_\alpha}{\to} x$ such that $\phi_{\mathfrak{F}} \supseteq \phi_{\mathfrak{G}}$. Since $f : (X,\overline{q}) \to (Y,\overline{p})$ is continuous, $f^{\to}\mathfrak{G} \overset{p_\alpha}{\to} f(x)$ and hence $f^{\to}(\phi_{\mathfrak{G}}) = \phi_{f^{\to}\mathfrak{G}} \overset{P_\alpha}{\to} f(x)$. Since $\phi_{f^{\to}\mathfrak{F}} = f^{\to}(\phi_{\mathfrak{F}}) \supseteq f^{\to}(\phi_{\mathfrak{G}}) \overset{P_\alpha}{\to} f(x)$, it follows that $f^{\to}\mathfrak{F} \overset{p_\alpha}{\to} f(x)$ and thus $f : (X,\overline{r}) \to (Y,\overline{p})$ is continuous. Therefore $\delta(PCS)$ is bireflective in SL-CS.* ■

This author is unable to show that $\delta$(PCS) is bicoreflective in SL-CS but the corresponding result for their “Choquet modifications" is valid and is given in Theorem 10. Object $(X,\overline{q}) \in |\text{SL-CS}|$ is said to be a **stratified $L$-Choquet convergence space** provided $\mathfrak{F} \overset{q_\alpha}{\to} x$ whenever $\mathfrak{G} \overset{q_\alpha}{\to} x$ for each stratified $L$-ultrafilter $\mathfrak{G} \geq \mathfrak{F}$. The full-subconstruct of

SL-CS consisting of all the stratified $L$-Choquet convergence spaces as objects is denoted by **SL-C-CS**.

Define

$$\rho : \text{SL-CS} \to \text{SL-C-CS as follows:} \tag{2.4}$$

$$\rho(f) = f$$

$$\rho(X, \overline{q}) = (X, C\overline{q}), \text{ where}$$

$$\mathfrak{F} \xrightarrow{Cq_\alpha} x \text{ iff } \mathfrak{G} \xrightarrow{q_\alpha} x \text{ for each}$$

$$\text{stratified } L\text{-ultrafilter } \mathfrak{G} \geq \mathfrak{F}.$$

It easily follows that $(X, C\overline{q}) \in |\text{SL-C-CS}|$. Observe that $\rho$ is a functor. Indeed, assume that $f : (X, \overline{q}) \to (Y, \overline{p})$ is continuous in SL-CS, $\mathfrak{F} \xrightarrow{Cq_\alpha} x$ and $\mathfrak{G} \geq f^{\to}\mathfrak{F}$ is a stratified $L$-ultrafilter on $Y$. According to Theorem 9(a) below there exists a stratified $L$-ultrafilter $\mathfrak{H} \geq \mathfrak{F}$ for which $f^{\to}\mathfrak{H} = \mathfrak{G}$. Then $\mathfrak{H} \xrightarrow{q_\alpha} x$, $f^{\to}\mathfrak{H} = \mathfrak{G} \xrightarrow{p_\alpha} f(x)$ and thus $f^{\to}\mathfrak{F} \xrightarrow{Cq_\alpha} f(x)$. Hence $f : (X, C\overline{q}) \to (Y, C\overline{p})$ is also continuous.

Given

$$(X, \overline{q}) \in |\text{SL-CS}|, \text{ define} (X, \overline{S}) \text{ as follows:} \tag{2.5}$$

$$\psi \xrightarrow{S_\alpha} x \text{ iff } \mathfrak{F} \xrightarrow{q_\alpha} x \text{ for each stratified}$$

$$L\text{-ultrafilter } \mathfrak{F} \text{ obeying } \phi_{\mathfrak{F}} \supseteq \psi.$$

A straightforward argument shows that $(X, \overline{S}) \in |\text{PCS}|$; denote $(X, \overline{s}) = \delta(X, \overline{S})$. For sake of convenience, the following results given in Lemma 3.7 and 4.1 in Jäger [2002] are summarized in the next theorem.

**Theorem 9** *(Jäger [2002]). Let $f : X \to Y$ be a map, $\mathfrak{F} \in \mathfrak{F}_{SL}(X)$ and $\mathfrak{G}$ a stratified $L$-ultrafilter on $Y$. Then,*

(a) if $\mathfrak{G} \geq f^{\rightarrow}\mathfrak{F}$, there exists a stratified $L$-ultrafilter $\mathfrak{H} \geq \mathfrak{F}$ such that $f^{\rightarrow}\mathfrak{H} = \mathfrak{G}$

(b) whenever $L = [0,1]$, $\psi = \phi_{\mathfrak{G}}$ is an ultrafilter on $X$ and $\mathfrak{F}_{\psi} = \mathfrak{G}$ is the only stratified $L$-ultrafilter on $X$ satisfying $\phi_{\mathfrak{F}_{\psi}} = \psi$.

**Theorem 10** . *Suppose that $L = [0,1]$ and $\rho$ is the functor defined in (2.4). Then $(\rho \circ \delta)$(PCS) is bicoreflective in SL-C-CS.*

**Proof.** Given $(X, \overline{q}) \in |\text{SL-C-CS}|$, let $(X, \overline{S})$ be as defined in (2.5) and $(X, \overline{s}) = \delta(X, \overline{S})$. Note that id: $(X, \overline{s}) \to (X, C\overline{q})$is continuous. Indeed, assume that $\mathfrak{F} \overset{s_\alpha}{\to} x$ and $\mathfrak{G} \geq \mathfrak{F}$ is a stratified $L$-ultrafilter on $X$. Then by definition of $s_\alpha$, $\phi_{\mathfrak{F}} \overset{S_\alpha}{\to} x$. Since $\phi_{\mathfrak{G}} \supseteq \phi_{\mathfrak{F}}$, it follows from (2.5) that $\mathfrak{G} \overset{q_\alpha}{\to} x$ and thus $\mathfrak{F} \overset{Cs_\alpha}{\to} x$. Hence id: $(X, \overline{s}) \to (X, C\overline{q})$ is continuous and since $\rho$ is a functor, id: $(X, C\overline{s}) \to (X, C\overline{q})$ is also continuous. Next, suppose that $(X, \overline{p}) \in |\delta(\text{PCS})|$ and $f : (X, \overline{p}) \to (X, \overline{q})$ is continuous. It is shown that $f : (X, \overline{p}) \to (X, C\overline{s})$ is continuous. Assume that $\mathfrak{F} \overset{p_\alpha}{\to} y$ and it must be shown that $f^{\rightarrow}\mathfrak{F} \overset{Cs_\alpha}{\to} f(y) = x$. It suffices to show that $\mathfrak{G} \overset{s_\alpha}{\to} x$ whenever $\mathfrak{G} \geq f^{\rightarrow}\mathfrak{F}$ is a stratified $L$-ultrafilter on $X$; equivalently, $\phi_{\mathfrak{G}} \overset{S_\alpha}{\to} x$. Employing the definition of $S_\alpha$ in (2.5), it must be shown that $\mathfrak{H} \overset{q_\alpha}{\to} x$ whenever $\mathfrak{H}$ is a stratified $L$-ultrafilter such that $\phi_{\mathfrak{H}} = \phi_{\mathfrak{G}}$. According to Theorem 9, $\mathfrak{H} = \mathfrak{G}$ and there exists a stratified $L$-ultrafilter $\mathfrak{K} \geq \mathfrak{F}$ obeying $f^{\rightarrow}\mathfrak{K} = \mathfrak{G}$. Since $\mathfrak{K} \overset{p_\alpha}{\to} y$, $\mathfrak{G} = f^{\rightarrow}\mathfrak{K} \overset{q_\alpha}{\to} x$ and thus $\phi_{\mathfrak{G}} \overset{S_\alpha}{\to} x$. Therefore $\mathfrak{G} \overset{s_\alpha}{\to} x$ and $f^{\rightarrow}\mathfrak{F} \overset{Cs_\alpha}{\to} x$; hence $f : (Y, \overline{p}) \to (X, C\overline{s})$ and $f : (Y, C\overline{p}) \to (X, C\overline{s})$ are continuous. Since $C\overline{q} = \overline{q}$, it follows that $(\rho \circ \delta)$(PCS) is bicoreflective in SL-C-CS. ■

## 2.2 Diagonal Axioms with Its Dual

Kowalsky [1954] and Cook and Fischer [1967] investigated diagonal axiom $F$ and its dual $R$ in the category of convergence spaces. They showed that a convergence space is topological iff it obeys $F$ and regular iff it satisfies $R$. The author here extends these axioms to the category $SL - CS$ whose objects consist of all the stratified $L$-convergence spaces. Properties of these axioms are investigated and comparisons are made to the diagonal axioms studied by Gähler [1992,1999]. Categorical terminology used here follows that given by Preuss [2002]. A key component for the diagonal axioms in the category **CON** is the notion of Kowalsky's "compression operator" $K : \mathfrak{F}(\mathfrak{F}(X)) \to \mathfrak{F}(X)$, defined by $K(H) := \bigcup_{A \in H} \bigcap \{\mathfrak{F} \in A\}$, where $H \in \mathfrak{F}(\mathfrak{F}(X))$. For sake of convenience, these axioms are listed below for $(X, q) \in |\text{CON}|$.

(F): Let $J$ be any set, $\psi : J \to X$, $\Delta : J \to \mathfrak{F}(X)$ such that $\Delta(y) \xrightarrow{q} \psi(y)$ for each $y \in J$. If $\mathfrak{F} \in \mathfrak{F}(J)$ such that $\psi^{\to}\mathfrak{F} \xrightarrow{q} x$, then $K\Delta^{\to}\mathfrak{F} \xrightarrow{q} x$.

(R): Let $J$ be any set, $\psi : J \to X$, $\Delta : J \to \mathfrak{F}(X)$ such that $\Delta(y) \xrightarrow{q} \psi(y)$ for each $y \in J$. If $\mathfrak{F} \in \mathfrak{F}(J)$ such that $K\Delta^{\to}\mathfrak{F} \xrightarrow{q} x$, then $\psi^{\to}\mathfrak{F} \xrightarrow{q} x$.

As mentioned earlier, $(X, q)$ is topological (regular) iff axiom $F$ (R) is satisfied, respectively. These axioms are readily extended to $(X, \overline{q}) \in |PCS|$, where $\overline{q} = (q_\alpha)_{\alpha \in L}$, as follows:

(F1): Same as $F$ with $q$ replaced by $Q_\alpha$, for each $\alpha \in L$.

(R1): Same as $R$ with $q$ replaced by $Q_\alpha$, for each $\alpha \in L$.

Axiom F1 (R1) also characterizes whenever $(X, \overline{q})$ is topological (regular) as is shown in Kent and Richardson [1996] and Brock and Kent [1997], respectively. These axioms are extended to the category SL-CS.

The difficulty in extending axioms F1 and R1 to the category SL-CS lies in defining the compression operator. Let $J$ be any set, $\sigma : J \to \mathfrak{F}_{SL}(X)$, $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ and $\phi_{\mathfrak{F}}$ given in (2.5A). Then **Kowalsky's compression operator** on $\sigma^{\to}\mathfrak{F} \in \mathfrak{F}_{SL}(\mathfrak{F}_{SL}((X))$ is defined as

$$K\sigma^{\to}\mathfrak{F} := \bigvee_{A \in \phi_{\mathfrak{F}}} \bigwedge_{y \in A} \sigma(y). \tag{2.6}$$

Using the compression operator defined in (2.6), an extension of the diagonal axioms F1 and R1 to $(X, \overline{q}) \in |SL - CS|$ is given as follows:

(F2) Let $J$ be any set, $\psi : J \to X$, $\sigma : J \to \mathfrak{F}_{SL}(X)$ such that $\sigma(y) \xrightarrow{q_\alpha} \psi(y)$ for each $y \in J$. If $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ obeys $\psi^{\to}\mathfrak{F} \xrightarrow{q_\alpha} x$, then $K\sigma^{\to}\mathfrak{F} \xrightarrow{q_\alpha} x$, $\alpha \in L$.

(R2) Let $J$ be any set, $\psi : J \to X$, $\sigma : J \to \mathfrak{F}_{SL}(X)$ such that $\sigma(y) \xrightarrow{q_\alpha} \psi(y)$ for each $y \in J$. If $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ and $K\sigma^{\to}\mathfrak{F} \xrightarrow{q_\alpha} x$, then $\psi^{\to}\mathfrak{F} \xrightarrow{q_\alpha} x$, $\alpha \in L$.

**Lemma 11** *Assume that $f : X \to Y$, $\sigma : J \to \mathfrak{F}_{SL}(X)$, $\sigma_1 = f^{\to} \circ \sigma$ are each maps, $a \in L^X$ and $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$. Then*

(a) $K\sigma^{\to}\mathfrak{F} \in \mathfrak{F}_{SL}(X)$

(b) $K\sigma^{\to}\mathfrak{F}(a) = \vee\{\wedge_{y \in A}\sigma(y)(a) : A \in \phi_{\mathfrak{F}}\}$

(c) $f^{\to}(K\sigma^{\to}\mathfrak{F}) = K\sigma_1^{\to}\mathfrak{F}$.

**Proof.** (a): Jäger's [2001, Lemma 3.3] characterization as to when the supremum of a collection of SL-filters exists in $\mathfrak{F}_{SL}(X)$ is used here. Indeed, suppose that $A_i \in \phi_{\mathfrak{F}}$ and $a_i \in L^X$ satisfy $\overset{n}{\underset{i=1}{\wedge}} a_i = 1_\phi$. Denote $B = \overset{n}{\underset{i=1}{\cap}} A_i \in \phi_{\mathfrak{F}}$; then $\overset{n}{\underset{i=1}{\wedge}}[\underset{y \in A_i}{\wedge}\sigma(y)(a_i)] \leq \overset{n}{\underset{i=1}{\wedge}}[\underset{y \in B}{\wedge}\sigma(y)(a_i)] = \underset{y \in B}{\wedge}\sigma(y)(\overset{n}{\underset{i=1}{\wedge}} a_i) = 0$ since $\overset{n}{\underset{i=1}{\wedge}} a_i = 1_\phi$ and $\underset{y \in B}{\wedge}\sigma(y) \in \mathfrak{F}_{SL}(X)$. Hence $K\sigma^{\to}\mathfrak{F} \in \mathfrak{F}_{SL}(X)$.

(b): Lemma 3.3 in Jäger [2001] is used again with $a \in L^X$. Then $K\sigma^{\rightarrow}\mathfrak{F}(a) = \vee\{\overset{n}{\underset{i=1}{\wedge}}(\underset{y\in A_i}{\wedge}\sigma(y)(a_i)) : A_i \in \phi_{\mathfrak{F}}, \overset{n}{\underset{i=1}{\wedge}} a_i \leq a, n \geq 1\} = \vee\{\overset{n}{\underset{i=1}{\wedge}}(\underset{y\in A_i}{\wedge}\sigma(y)(a_i)) : B \in \phi_{\mathfrak{F}}, \overset{n}{\underset{i=1}{\wedge}} a_i \leq a, n \geq 1\} = \vee\{\underset{y\in B}{\wedge}\sigma(y)(a) : B \in \phi_{\mathfrak{F}}\}$.

(c) Assume that $b \in L^Y$. Then $f^{\rightarrow}(K\sigma^{\rightarrow}\mathfrak{F})(b) = K\sigma^{\rightarrow}\mathfrak{F}(f^{\leftarrow}(b)) = \vee\{\underset{y\in B}{\wedge}\sigma(y)(f^{\leftarrow}(b)) : B \in \phi_{\mathfrak{F}}\} = \vee\{\underset{y\in B}{\wedge}f^{\rightarrow}(\sigma(y))(b) : B \in \phi_{\mathfrak{F}}\} = \vee\{\underset{y\in B}{\wedge}\sigma_1(y)(b) : B \in \phi_{\mathfrak{F}}\} = K\sigma_1^{\rightarrow}\mathfrak{F}(b)$ according to part (b) above. ■

An object $(X, \overline{q}) \in |SL - CS|$ is said to be **pretopological** provided that $\wedge_{j\in J}\mathfrak{F}_j \overset{q_\alpha}{\rightarrow} x$ whenever $\mathfrak{F}_j \overset{q_\alpha}{\rightarrow} x$, $j \in J$ and $\alpha \in L$. The full subcategory of SL-CS whose objects consist of all the pretopological spaces is denoted by **SL-P-CS**. Moreover, denote the full subcategory of SL-CS consisting of all objects obeying F2 (R2) by **SL-F2-CS** (**SL-R2-CS**), respectively.

**Theorem 12** *Suppose that $(X, \overline{q}) \in |SL - F2 - CS|$; then $(X, \overline{q}) \in |SL - P - CS|$.*

**Proof.** Assume that $\mathfrak{G}_j \overset{q_\alpha}{\rightarrow} x$, $j \in J$, are all the SL-filters on $X$ which $q_\alpha$-converge to $x$. Define $\psi : J \rightarrow X$ and $\sigma : J \rightarrow \mathfrak{F}_{SL}(X)$ by $\psi(j) = x$ and $\sigma(j) = \mathfrak{G}_j$, $j \in J$. Recall that $\mathfrak{F}_o$ denotes the coarsest member of $\mathfrak{F}_{SL}(J)$ and $\mathfrak{F}_o(a) = \wedge_{j\in J}a(j)$, $a \in L^J$. Note that $\phi_{\mathfrak{F}_o} = \{J\}$ since $\mathfrak{F}_o(1_A) = 1$ iff $A = J$. If $b \in L^X$, $\psi^{\leftarrow}(b)(j) = (b \circ \psi)(j) = b(x)$ and thus $\psi^{\leftarrow}(b) = b(x)1_J$. It follows that $\psi^{\rightarrow}\mathfrak{F}_o(b) = \mathfrak{F}_o(\psi^{\leftarrow}(b)) = \mathfrak{F}_o(b(x)1_J) = b(x) = [x](b)$ and thus $\psi^{\rightarrow}\mathfrak{F}_o = [x] \overset{q_\alpha}{\rightarrow} x$. Since $(X, \overline{q}) \in |SL - F2 - CS|, K\sigma^{\rightarrow}\mathfrak{F}_o = \vee_{A\in\phi_F} \wedge_{j\in A}\sigma(j) = \wedge_{j\in J}\mathfrak{G}_j \overset{q_\alpha}{\rightarrow} x$, $\alpha \in L$. Hence $(X, \overline{q}) \in |SL - P - CS|$. ■

It is shown later in Theorem 26 that SL-CS is a topological category that is cartesian closed and extensional. It is shown below that SL-F2-CS and SL-R2-CS are topological and bireflective in SL-CS.

**Theorem 13** *The subconstruct SL-F2-CS (SL-R2-CS) is topological and bireflective in SL-CS, respectively.*

**Proof.** It is shown that SL-F2-CS has initial structures. Assume that $f_j : X \to (Y_j, \overline{p}_j)$, where $j \in J$, $J$ an index class and $(Y_j, \overline{p}_j) \in |SL - F2 - CS|$. Denote $\overline{p}_j = (p_{j\alpha})_{\alpha \in L}$ and let $\overline{q} = (q_\alpha)_{\alpha \in L}$ be the initial structure of the above in SL-CS; that is, $\mathfrak{F} \overset{q_\alpha}{\to} x$ iff $f_j^{\to} \mathfrak{F} \overset{P_{j\alpha}}{\to} f_j(x)$ for each $j \in J$. Suppose that $\psi : J \to X$, $\sigma : J \to \mathfrak{F}_{SL}(X)$ such that $\sigma(y) \overset{q_\alpha}{\to} \psi(y)$ for each $y \in J$, $\alpha \in L$ fixed. Let $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ such that $\psi^{\to} \mathfrak{F} \overset{q_\alpha}{\to} x$. It remains to show that $K\sigma^{\to} \mathfrak{F} \overset{q_\alpha}{\to} x$; equivalently, $f_j^{\to}(K\sigma^{\to} \mathfrak{F}) \overset{P_{j\alpha}}{\to} f_j(x)$ for each $j \in J$. Denote $\psi_j = f_j \circ \psi : J \to Y_j$ and $\sigma_j = f_j^{\to} \circ \sigma : J \to \mathfrak{F}_{SL}(Y_j)$ for each $j \in J$. Then $\sigma_j(y) = f_j^{\to}(\sigma(y)) \overset{P_{j\alpha}}{\to} f_j(\psi(y)) = \psi_j(y)$ and $\psi_j^{\to} \mathfrak{F} = f_j^{\to}(\psi^{\to} \mathfrak{F}) \overset{P_{j\alpha}}{\to} f_j(x)$ since $f_j$ is continuous, for each $y \in J$ and $j \in J$. It follows from Lemma 11 (c) that $f_j^{\to}(K\sigma^{\to} \mathfrak{F}) = K\sigma_j^{\to} \mathfrak{F}$ and $K\sigma_j^{\to} \mathfrak{F} \overset{P_{j\alpha}}{\to} f_j(x)$ since $(Y_j, \overline{p}_j) \in |SL - F2 - CS|$, for each $j \in J$. Hence $K\sigma^{\to} \mathfrak{F} \overset{q_\alpha}{\to} x$ and thus $(X, \overline{q}) \in |SL - F2 - CS|$. Therefore SL-F2-CS is topological since SL-CS is topological. Next, it is shown that SL-F2-CS is bireflective in SL-CS. Let $(X, \overline{q}) \in |SL - CS|$. According to the proof given above, the supremum in SL-CS of all $\overline{s} \leq \overline{q}$ with $(X, \overline{s}) \in |SL - F2 - CS|$ exists and is denoted by $(X, \overline{r})$. Moreover, $(X, \overline{r}) \in |SL - F2 - CS|$. In particular, $\overline{r}$ is the finest structure coarser than $\overline{q}$ satisfying $(X, \overline{r}) \in |SL - F2 - CS|$. Hence id: $(X, \overline{q}) \to (X, \overline{r})$ is continuous. Assume that $f : (X, \overline{q}) \to (Y, \overline{p})$ is continuous, where $(Y, \overline{p}) \in |SL - F2 - CS|$. Let $\overline{s}$ denote the initial structure for $f : X \to (Y, \overline{p})$. Then $(X, \overline{s}) \in |SL - F2 - CS|$ and $\overline{s}$ is the coarsest structure such that $f : (X, \overline{s}) \to (Y, \overline{p})$ is continuous. It follows that $\overline{s} \leq \overline{q}$ and also $\overline{s} \leq \overline{r}$. Consequently, $f : (X, \overline{r}) \to (Y, \overline{p})$ is continuous and thus SL-F2-CS is bireflective in SL-CS. Minor changes in the argument given above shows that SL-R2-CS is also a topological construct that is bireflective in SL-CS. ■

**Remark:** According to Theorem 2.2.12 (Preuss[2002]), a full subconstruct that is bireflective in a topological construct is also topological. Hence it was only necessary to prove that $SL-F2-CS$ $(SL-R2-CS)$ is bireflective in $SL-CS$. However, the proof that $SL-F2-CS$ is topological was used in the bireflective proof.

# CHAPTER 3.
# JÄGER'S CONVERGENCE AND FURTHER CONNECTIONS TO DIAGONAL AXIOMS

This chapter provides a description of the stratified lattice fuzzy convergence spaces studied extensively by Jäger [2001,2005]. In Section 3.1, it is shown that a subsconstruct of SL-CS and Jäger's category (SL-FCS) are isomorphic. This is a significant result that extends knowledge about subcategories of SL-CS that are proved to be both topological and bireflective. Additionally, in Section 3.2, further regularity conditions are provided that further define subcategories of SL-CS as they relate to Gahler's compression operator. This author proves three original theorems (Theorems 16, 19, and 21), as well as Lemmas 18 and 20.

Jäger [2001,2005] defined and investigated the notion of a fuzzy convergence structure when $L$ is a complete Heyting algebra. It is proved in this section that Jager's construct is isomorphic to a bireflective subconstruct of SL-CS.

## 3.1 Stratified L-Fuzzy Convergence Spaces

**Definition 14** . *Let* $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}_{SL}(X)$. *Then* $(X, \lim)$ *is said to be a* ***stratified*** $L$***-fuzzy convergence space*** *provided* $\lim : \mathfrak{F}_{SL}(X) \to L^X$ *obeys the following:*

(a) $(\lim[x])(x) = 1$ for each $x \in X$

(b) $\mathfrak{F} \leq \mathfrak{G}$ implies $\lim \mathfrak{F} \leq \lim \mathfrak{G}$.

A map $f : (X, \lim_X) \to (Y, \lim_Y)$ is called **continuous** whenever $(\lim_X \mathfrak{F})(x) \leq (\lim_Y f^{\to}\mathfrak{F})(f(x))$ for each $\mathfrak{F} \in \mathfrak{F}_{SL}(X)$ and $x \in X$. Since the composition of two continuous functions is continuous, **SL-FCS** is the category whose objects consist of all the stratified $L$-

fuzzy convergence spaces and whose morphisms are all the continuous maps between objects. The grade of convergence of $\mathfrak{F}$ to $x$ is the interpretation given $\lim \mathfrak{F}(x)$.

**Definition 15** . *Object $(X, \overline{q}) \in |SL\text{-}CS|$ is called **left-continuous** provided $\mathfrak{F} \xrightarrow{q_\alpha} x$ iff there exists $A \subseteq L$ such that $\vee A = \alpha$ and $\mathfrak{F} \xrightarrow{q_\beta} x$ for each $\beta \in A$.*

The full-subcategory of SL-CS consisting of all the left-continuous objects is denoted by **SL-LC-CS**. Given $(X, \overline{q}), (Y, \overline{p}) \in |\text{SL-CS}|$, where $\overline{q} = (q_\alpha)_{\alpha \in L}$ and $\overline{p} = (p_\alpha)_{\alpha \in L}$, define $\overline{p} \leq \overline{q}$ provided $p_\alpha \leq q_\alpha$ for each $\alpha \in L$; that is, $\mathfrak{F} \xrightarrow{q_\alpha} x$ implies that $\mathfrak{F} \xrightarrow{p_\alpha} x$.

Define

$$\begin{aligned} &\theta : \text{SL-FCS} \to \text{SL-LC-CS by } \theta(f) = f \text{ and} \\ &\theta(X, \lim) = (X, \overline{q}), \quad \text{where } \overline{q} = (q_\alpha)_{\alpha \in L} \\ &\text{and } \mathfrak{F} \xrightarrow{q_\alpha} x \text{ iff } (\lim \mathfrak{F})(x) \geq \alpha. \end{aligned} \tag{3.1}$$

It easily follows from (3.1) that $(X, \overline{q}) \in |\text{SL-LC-CS}|$.

Conversely, define

$$\begin{aligned} &\psi : \text{SL-LC-CS} \to \text{SL-FCS} \\ &\text{by } \psi(f) = f \quad \text{and} \\ &\psi(X, \overline{q}) = (X, \lim_{\overline{q}}), \quad \text{where} \\ &(\lim_{\overline{q}} \mathfrak{F})(x) = \vee\{\beta \in L : \mathfrak{F} \xrightarrow{q_\beta} x\}. \end{aligned} \tag{3.2}$$

Likewise, it follows from (3.2) that $(X, \lim_{\overline{q}}) \in |\text{SL-FCS}|$.

**Theorem 16** . *Given the notations defined in (3.1) and (3.2),*

(a) $\theta : \text{SL-FCS} \to \text{SL-LC-CS}$ is an isomorphism

(b) SL-LC-CS is bireflective in SL-CS.

**Proof.** (a) First, observe that $\theta$ is a functor. Indeed, assume that $f : (X, \lim_X) \to (Y, \lim_Y)$ is continuous, $\theta(X, \lim_X) = (X, \overline{q})$ and $\theta(Y, \lim_Y) = (Y, \overline{p})$. Suppose that $\mathfrak{F} \xrightarrow{q_\alpha} x$; then by (3.1), $\alpha \le \lim_X \mathfrak{F}(x) \le (\lim_Y f^\to \mathfrak{F})(f(x))$ and thus $f^\to \mathfrak{F} \xrightarrow{p_\alpha} f(x)$. Hence $f : (X, \overline{q}) \to (Y, \overline{p})$ is continuous and $\theta$ is a functor. Conversely, assume that $f : (X, \overline{q}) \to (Y, \overline{p})$ is continuous; it must be shown that $f : (X, \lim_{\overline{q}}) \to (Y, \lim_{\overline{p}})$ is also continuous. Let $\mathfrak{F} \in \mathfrak{F}_{SL}(X)$; then $(\lim_{\overline{q}} \mathfrak{F})(x) = \vee \{\beta \in L : \mathfrak{F} \xrightarrow{q_\beta} x\} \le \vee \{\alpha \in L : f^\to \mathfrak{F} \xrightarrow{p_\alpha} f(x)\} = (\lim_{\overline{p}} f^\to \mathfrak{F})(f(x))$ and thus $f : (X, \lim_{\overline{q}}) \to (Y, \lim_{\overline{p}})$ is continuous. Hence $\psi$ is a functor. It remains to show that $\psi \circ \theta = id_{\text{SL-FCS}}$ and $\theta \circ \psi = id_{\text{SL-LC-CS}}$. Let $(X, \lim) \in |\text{SL-FCS}|$, $\theta(X, \lim) = (X, \overline{q})$ and $\psi(X, \overline{q}) = (X, \lim_{\overline{q}})$ as given in (3.1) and (3.2). Then $\mathfrak{F} \xrightarrow{q_\alpha} x$ iff $(\lim \mathfrak{F})(x) \ge \beta$ and hence $(\lim_{\overline{q}} \mathfrak{F})(x) = \vee \{\beta \in L : \mathfrak{F} \xrightarrow{q_\beta} x\} = \vee \{\beta \in L : (\lim \mathfrak{F})(x) \ge \beta\} = (\lim \mathfrak{F})(x)$, for each $x \in X$. Hence $\lim_{\overline{q}} = \lim$ and therefore $\psi \circ \theta = id_{\text{SL-FCS}}$. Next, it is shown that $\theta \circ \psi = id_{\text{SL-LC-CS}}$. Suppose that $(X, \overline{q}) \in |\text{SL-LC-CS}|$, $\psi(X, \overline{q}) = (X, \lim_{\overline{q}})$ and $\theta(X, \lim_{\overline{q}}) = (X, \overline{p})$. It must be shown that $\overline{q} = \overline{p}$. If $\mathfrak{F} \xrightarrow{q_\alpha} x$, then $\lim_{\overline{q}} \mathfrak{F}(x) \ge \alpha$ and thus $\mathfrak{F} \xrightarrow{p_\alpha} x$. Hence $q_\alpha \ge p_\alpha$ for each $\alpha \in L$ and thus $\overline{q} \ge \overline{p}$. Conversely, suppose that $\mathfrak{F} \xrightarrow{p_\alpha} x$; then $\alpha \le (\lim_{\overline{q}} \mathfrak{F})(x) = \vee \{\beta \in L : \mathfrak{F} \xrightarrow{q_\beta} x\}$. Denote $B = \{\beta \in L : \mathfrak{F} \xrightarrow{q_\beta} x\}$; then $\vee_{\beta \in B} (\alpha \wedge \beta) = \alpha \wedge (\vee_{\beta \in B} \beta) = \alpha$ and since $q_\beta \ge q_{\alpha \wedge \beta}$, $\mathfrak{F} \xrightarrow{q_{\alpha \wedge \beta}} x$ for each $\beta \in B$. Since $(X, \overline{q})$ is left-continuous, it follows that $\mathfrak{F} \xrightarrow{q_\alpha} x$. Hence $p_\alpha \ge q_\alpha$ for each $\alpha \in L$ and thus $\overline{p} = \overline{q}$ and $\theta \circ \psi = id_{\text{SL-LC-CS}}$. Therefore $\theta : \text{SL-FCS} \to \text{SL-LC-CS}$ is an isomorphism.

(b) Given $(X, \overline{q}) \in |\text{SL-CS}|$, define $(X, LC\overline{q})$ as follows: $\mathfrak{F} \xrightarrow{LCq_\alpha} x$ iff there exists $A \subseteq L$ such that $\vee A = \alpha$ and $\mathfrak{F} \xrightarrow{q_\beta} x$ for each $\beta \in A$. It must be shown that $(X, LC\overline{q}) \in |\text{SL-LC-CS}|$.

Since $LCq_\alpha \leq q_\alpha$ for each $\alpha \in L, [x] \overset{LCq_\alpha}{\longrightarrow} x$ and $\mathfrak{F}_\circ \overset{LCq_0}{\longrightarrow} x$. Clearly $\mathfrak{G} \geq \mathfrak{F} \overset{LCq_\alpha}{\longrightarrow} x$ implies that $\mathfrak{G} \overset{LCq_\alpha}{\longrightarrow} x$. Next, suppose that $\beta \leq \alpha$ and $\mathfrak{F} \overset{LCq_\alpha}{\longrightarrow} x$; then there exists $A \subseteq L$ such that $\vee A = \alpha$ and $\mathfrak{F} \overset{q_\alpha}{\rightarrow} x$ for each $\delta \in A$. Define $B = \{\beta \wedge \delta : \delta \in A\}$ and note that $\vee B = \beta$. Since $q_\delta \geq q_{\delta\wedge\beta}$, $\mathfrak{F} \overset{q_{\delta\wedge\beta}}{\longrightarrow} x$ for each $\delta \in A$ and thus by definition of $LCq_\beta$, $\mathfrak{F} \overset{LCq_\beta}{\longrightarrow} x$. Hence $LCq_\alpha \geq LCq_\beta$ whenever $\beta \leq \alpha$. Finally, it is shown that $(X, LC\overline{q})$ is left-continuous. Assume that $A \subseteq L$, $\vee A = \alpha$ and $\mathfrak{F} \overset{LCq_\beta}{\longrightarrow} x$ for each $\beta \in A$. Fix $\beta \in A$. Since $\mathfrak{F} \overset{LCq_\beta}{\longrightarrow} x$ there exists $A_\beta \subseteq L$ such that $\vee A_\beta = \beta$ and $\mathfrak{F} \overset{q_\delta}{\rightarrow} x$ for each $\delta \in A_\beta$. Denote $D = \cup\{A_\beta : \beta \in A\}$; then $\vee D = \alpha$ and since $\mathfrak{F} \overset{q_\delta}{\rightarrow} x$ for each $\delta \in D$, it follows that $\mathfrak{F} \overset{q_\delta}{\rightarrow} x$. Hence $(X, LC\overline{q}) \in |\text{SL-LC-CS}|$. Since $\overline{q} \geq LC\overline{q}$, $id : (X, \overline{q}) \to (X, LC\overline{q})$ is continuous. Suppose that $(Y, \overline{p}) \in |\text{SL-LC-CS}|$ and $f : (X, \overline{q}) \to (Y, \overline{p})$ is continuous. It is shown that $f : (X, LC\overline{q}) \to (Y, \overline{p})$ is also continuous. Assume that $\mathfrak{F} \overset{LCq_\alpha}{\longrightarrow} x$; then there exists $A \subseteq L$ such that $\vee A = \alpha$ and $\mathfrak{F} \overset{q_\beta}{\rightarrow} x$ for each $\beta \in A$. It follows that $f^{\rightarrow}\mathfrak{F} \overset{p_\beta}{\rightarrow} f(x)$ for each $\beta \in A$ and since $(Y, \overline{p})$ is left-continuous, $f^{\rightarrow}\mathfrak{F} \overset{p_\alpha}{\rightarrow} f(x)$. Therefore $f : (X, LC\overline{q}) \to (Y, \overline{p})$ is continuous and SL-LC-CS is bireflective in SL-CS. ■

## 3.2 Connections between Diagonal Axioms and Gähler Axioms

The purpose of this section is to give connections between the diagonal axioms discussed in chapters 1 and 2 and those given by Gähler [1992,1999]. Gähler defined diagonal axioms for pretopological, lattice-valued convergence spaces. However, in view of Theorem 19 (b) below each $(X, \overline{q}) \in |SL - CS|$ which obeys Gähler's axiom F3 is pretopological. Let $J$ be any set $\sigma : J \to \mathfrak{F}_{SL}(X)$ and $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$. Then **Gähler's compression operator** is defined as:

$$G(\sigma^{\rightarrow}\mathfrak{F})(\mathfrak{a}) := \mathfrak{F}(e_a \circ \sigma), \tag{3.3}$$

where $e_a : \mathfrak{F}_{SL}(X) \to L$ is given by $e_a(G) := G(a)$, for each $a \in L^X$. It is easily verified that $G(\sigma^{\to}\mathfrak{F}) \in \mathfrak{F}_{SL}(X)$ whenever $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$. The diagonal axioms using Gähler's compression operator are listed below for $(X, \overline{q}) \in |SL - CS|$:

(F3) Let $J$ be any set, $\psi : J \to X$, $\sigma : J \to \mathfrak{F}_{SL}(X)$ such that $\sigma(y) \overset{q_\alpha}{\to} \psi(y)$ for each $y \in J$. If $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ obeys $\psi^{\to}\mathfrak{F} \overset{q_\alpha}{\to} x$, then $G(\sigma^{\to}\mathfrak{F}) \overset{q_\alpha}{\to} x$, $\alpha \in L$.

(R3) Let $J$ be any set, $\psi : J \to X$, $\sigma : J \to \mathfrak{F}_{SL}(X)$ such that $\sigma(y) \overset{q_\alpha}{\to} \psi(y)$ for each $y \in J$. If $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ obeys $G(\sigma^{\to}\mathfrak{F}) \overset{q_\alpha}{\to} x$, then $\psi^{\to}\mathfrak{F} \overset{q_\alpha}{\to} x$.

Axiom F2* (R2*, F3*, R3*) differs from F2 (R2, F3, R3) in that $\sigma(y) \in \mathfrak{U}_{SL}(X)$ for each $y \in J$, respectively.

Some connections between the compression operators $K(\sigma^{\to}\mathfrak{F})$ and $G(\sigma^{\to}\mathfrak{F})$ are given below. For sake of convenience, the following result by Jäger [2002, Lemma 4.1] is listed. Recall the definition of $\phi_{\mathfrak{F}}$ and $\mathfrak{F}_\psi$ given in (1.1).

**Lemma 17** *Jäger[2002] Assume that $L$ is linearly ordered, $\mathfrak{F} \in \mathfrak{U}_{SL}(X)$, and $\psi = \phi_{\mathfrak{F}}$. Then $\psi \in \mathfrak{U}(X)$ and $\mathfrak{F} = \mathfrak{F}_\psi$.*

**Lemma 18** *Let $\sigma : J \to \mathfrak{F}_{SL}(X)$ and $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$. Then*

(a) $G(\sigma^{\to}\mathfrak{F}) \geq K(\sigma^{\to}\mathfrak{F})$

(b) $G(\sigma^{\to}\mathfrak{F}) = K(\sigma^{\to}\mathfrak{F})$ whenever $\sigma(y) \in \mathfrak{U}_{SL}(X)$ for each $y \in J$, $\mathfrak{F} \in \mathfrak{U}_{SL}(J)$ and $L$ is linearly ordered

(c) $G(\sigma^{\to}\mathfrak{F}) = K(\sigma^{\to}\mathfrak{F})$ provided $L = \{0, 1\}$.

**Proof.** (a): Given any $A \in \phi_{\mathfrak{F}}$ and $a \in L^J$, denote $\alpha = \wedge_{y \in A}\sigma(y)(a)$. Then $(e_a \circ \sigma)(y) = \sigma(y)(a) \geq \alpha$ for each $y \in A$ and thus $e_a \circ \sigma \geq \alpha 1_A$. Hence $G(\sigma^{\to}\mathfrak{F})(a) = \mathfrak{F}(e_a \circ \sigma) \geq$

$\mathfrak{F}(\alpha 1_A) \geq \alpha$ since $A \in \phi_{\mathfrak{F}}$. Therefore $G(\sigma^{\rightarrow}\mathfrak{F})(a) \geq \vee\{\wedge_{y\in A}\sigma(y)(a) : \mathfrak{A} \in \phi_F\} = K(\sigma^{\rightarrow}\mathfrak{F})(a)$ by Lemma 11 (b) and thus $G(\sigma^{\rightarrow}\mathfrak{F}) \geq K(\sigma^{\rightarrow}\mathfrak{F})$.

(b): Given $a \in L^X$, denote $B = \{y \in J : \sigma(y)(a) = 1\}$. Since $\sigma(y) \in \mathfrak{U}_{SL}(X)$ for each $y \in J$, it follows from Lemma 17 that $\sigma(y)(a) = 0$ whenever $y \in B^c$ and thus $e_a \circ \sigma = 1_B$. Likewise, $\mathfrak{F} \in \mathfrak{U}_{SL}(J)$ implies that $G(\sigma^{\rightarrow}\mathfrak{F})(a) = \mathfrak{F}(e_a \circ \sigma) = \mathfrak{F}(1_B) = \begin{cases} 1, & B \in \phi_F \\ 0, & B \notin \phi_F \end{cases}$. First, assume that $B \in \phi_{\mathfrak{F}}$; then $K(\sigma^{\rightarrow}\mathfrak{F})(a) \geq \wedge_{y\in B}\sigma(y)(a) = 1$ and thus $K(\sigma^{\rightarrow}\mathfrak{F})(a) = G(\sigma^{\rightarrow}\mathfrak{F}) = 1$. Next, suppose that $B \notin \phi_{\mathfrak{F}}$; then $\mathfrak{F}(1_B) = 0$ and thus $G(\sigma^{\rightarrow}\mathfrak{F})(a) = 0$. It follows from part (a) that $K(\sigma^{\rightarrow}\mathfrak{F})(a) = 0$ and hence $G(\sigma^{\rightarrow}\mathfrak{F}) = K(\sigma^{\rightarrow}\mathfrak{F})$.

(c): According to (a) and the assumption that $L = \{0, 1\}$, it suffices to show that if $a \in L^X$ and $G(\sigma^{\rightarrow}\mathfrak{F})(a) = 1$, then $K(\sigma^{\rightarrow}\mathfrak{F})(a) = 1$. Denote $B = \{y \in J : \sigma(y)(a) = 1\}$. Since $= \{0, 1\}$, $e_a \circ \sigma = 1_B$ and the argument given in (b) shows that $K(\sigma^{\rightarrow}\mathfrak{F})(a) = 1$. Hence $G(\sigma^{\rightarrow}\mathfrak{F}) = K(\sigma^{\rightarrow}\mathfrak{F})$. ■

Additional properties of objects obeying one of the diagonal axioms are listed below.

**Theorem 19 .** *(a) Let $(X, \overline{q}) \in |SL - F2 - CS|$ ($|SL - R3 - CS|$). Then $(X, \overline{q}) \in |SL - F3 - CS|$ ($|SL - R2 - CS|$), respectively.*

(b) $(X, \overline{q}) \in |SL - F3 - CS|$ implies that $(X, \overline{q}) \in |SL - P - CS|$

(c) $SL - F3 - CS(SL - R3 - CS)$ is a topological construct that is also bireflective in SL-CS, respectively.

**Proof.** (a): Assume that $(X, \overline{q}) \in |SL - R3 - CS|$, $\psi : J \to X$, $\sigma : J \to \mathfrak{F}_{SL}(X)$ such that $\sigma(y) \overset{q_\alpha}{\to} \psi(y)$ for each $y \in J$ and $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ for which $K(\sigma^{\rightarrow}\mathfrak{F}) \overset{q_\alpha}{\to} x$. Employing

Lemma 18 (a), $G(\sigma^{\rightarrow}\mathfrak{F})\overset{q_\alpha}{\rightarrow}x$ and since $(X,\overline{q}) \in |SL-R3-CS|$, $\psi^{\rightarrow}\mathfrak{F}\overset{q_\alpha}{\rightarrow}x$. Hence $(X,\overline{q}) \in |SL-R2-CS|$. A similar argument is valid for the other part.

(b): Assume that $\mathfrak{G}_j\overset{q_\alpha}{\rightarrow}x$, $j \in J$, are all the $q_\alpha$-convergent SL-filters. Define $\psi : J \rightarrow X$ and $\sigma : J \rightarrow \mathfrak{F}_{SL}(X)$ by $\psi(j) = x$ and $\sigma(j) = \mathfrak{G}_j$, $j \in J$. As shown in Theorem 12, $\psi^{\rightarrow}\mathfrak{F}_\circ = [x]\overset{q_\alpha}{\rightarrow}x$. Given $a \in L^X$, $G(\sigma^{\rightarrow}\mathfrak{F}_\circ)(a) = \mathfrak{F}_\circ(e_a \circ \sigma) = \wedge_{j\in J}(e_a \circ \sigma)(j) = \wedge_{j\in J}e_a(G_j) = \wedge_{j\in J}G_j(a)\overset{q_\alpha}{\rightarrow}x$ since $(X,\overline{q}) \in |SL-F3-CS|$. Hence $(X,\overline{q}) \in |SL-P-CS|$.

(c): Verification is deleted here since the argument involves minor changes in the proof of Theorem 13. ■

Recall the definitions of F2, R2, F3 and R3 defined earlier. Given $(X,\overline{q}) \in |SL-CS|$ and $\alpha \in L$, denote

$$T_\alpha = \{(G,z) : G \in \mathfrak{F}_{SL}(X) \quad \text{and} \quad G\overset{q_\alpha}{\rightarrow}z\},$$

and let $\pi_i$ be the $i^{th}$ projection map defined on $T_\alpha$ $i = 1,2$.

Gähler [1992] defines **regularity** of $(X,\overline{q})$ as follows:

$$\text{if } H \in \mathfrak{F}_{SL}(T_\alpha) \text{ such that } G(\pi_1^{\rightarrow}H)\overset{q_\alpha}{\rightarrow}x, \text{ then } \pi_2^{\rightarrow}H\overset{q_\alpha}{\rightarrow}x, \text{ for each } \alpha \in L.$$

Theorem 21 below shows that the above diagonal axioms are satisfied whenever they are valid for $J = T_\alpha$, $\psi = \pi_2$ and $\sigma = \pi_1$, for each $\alpha \in L$. The following lemma is needed.

**Lemma 20** . *Assume that* $(X,\overline{q}) \in |SL-CS|$, $\psi : J \rightarrow X$, $\sigma : J \rightarrow \mathfrak{F}_{SL}(X)$ *such that* $\sigma(y) \overset{q_\alpha}{\rightarrow} \psi(y)$ *for each* $y \in J$, $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ *and define* $\rho : J \rightarrow T_\alpha$ *by* $\rho(y) = (\sigma(y),\psi(y)), y \in J$. *Then*

(a) $K\sigma^{\rightarrow}\mathfrak{F} = K[\pi_1^{\rightarrow}(\rho^{\rightarrow}\mathfrak{F})]$

(b) $G\sigma^{\rightarrow}\mathfrak{F} = G[\pi_1^{\rightarrow}(\rho^{\rightarrow}\mathfrak{F})]$.

**Proof.** (a): Note that $\pi_1 \circ \rho = \sigma$, $\pi_2 \circ \rho = \psi$ and recall that for any map $f : X \to Y$, $f^{\rightarrow}(\phi_{\mathfrak{F}}) = \phi_{f^{\rightarrow}\mathfrak{F}}$. Then $K\sigma^{\rightarrow}\mathfrak{F} = \vee_{A\in\phi_F} \wedge_{y\in A} \sigma(y) = \vee_{A\in\phi_F} \wedge_{y\in A} \pi_1(\rho(y)) = \vee_{A\in\phi_F} \wedge_{\rho(y)\in\rho(A)} \pi_1(\rho(y)) = \vee_{B\in\phi_{\rho^{\rightarrow}\mathfrak{F}}} \wedge_{z\in B} \pi_1(z) = K\pi_1^{\rightarrow}(\rho^{\rightarrow}\mathfrak{F})$.

(b) The result follows from $\pi_1 \circ \rho = \sigma$. ■

**Theorem 21** . *Let* $(X, \overline{q}) \in |SL - CS|$. *Then*

(a) $(X, \overline{q})$ obeys R2 (R3,F2,F3) iff R2 (R3,F2,F3) is satisfied whenever $J = T_\alpha$, $\psi = \pi_2$ and $\sigma = \pi_1$ for each $\alpha \in L$, respectively.

(b) $(X, \overline{q})$ obeys F2 iff $(X, \overline{q})$ is pretopological and satisfies F2 for the special case whenever $J = X$ and $\psi$ is the identity map.

**Proof.** (a): Assume that $(X, \overline{q})$ satisfies R2 whenever $J = T_\alpha$, $\psi = \pi_2$ and $\sigma = \pi_1$. It must be shown that R2 is valid for any $J$, $\psi : J \to X$ and $\sigma : J \to \mathfrak{F}_{SL}(X)$ such that $\sigma(y) \xrightarrow{q_\alpha} \psi(y)$ for each $y \in J$. Suppose that $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ such that $K\sigma^{\rightarrow}\mathfrak{F}\xrightarrow{q_\alpha}x$ and note that if $(G, z) \in T_\alpha$, then $\pi_1(G, z) = G\xrightarrow{q_\alpha}z = \pi_2(G, z)$. Define $\rho : J \to T_\alpha$ by $\rho(y) = (\sigma(y), \psi(y))$ for each $y \in J$. Then $\rho^{\rightarrow}\mathfrak{F} \in \mathfrak{F}_{SL}(T_\alpha)$ and according to Lemma 20 (a), $K\pi_1^{\rightarrow}(\rho^{\rightarrow}\mathfrak{F}) = K\sigma^{\rightarrow}\mathfrak{F}\xrightarrow{q_\alpha}x$. The hypothesis implies that $\pi_2^{\rightarrow}(\rho^{\rightarrow}\mathfrak{F})\xrightarrow{q_\alpha}x$. Since $\psi = \pi_2 \circ \rho$, $\psi^{\rightarrow}\mathfrak{F}\xrightarrow{q_\alpha}x$ and thus $(X, \overline{q})$ obeys R2. Verification of R3, F2 and F3 is proved in a similar manner by employing Lemma 20.

(b): Suppose that $(X, q_\alpha)$ is pretopological and obeys F2 whenever $J = X$ and $\psi$ is the identity map. Assume that $\psi : J \to X$, $\sigma : J \to \mathfrak{F}_{SL}(X)$ such that $\sigma(y) \xrightarrow{q_\alpha} \psi(y)$ for each $y \in J$ and let $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ for which $\psi^{\rightarrow}\mathfrak{F}\xrightarrow{q_\alpha}x$. It must be shown that $K\sigma^{\rightarrow}\mathfrak{F}\xrightarrow{q_\alpha}x$. Define $\Sigma : X \to \mathfrak{F}_{SL}(X)$ by $\Sigma(y) = \mathfrak{U}_{q_\alpha}(y)$ for each $y \in X$, where $\mathfrak{U}_{q_\alpha}(y)$ is the SL-neighborhood filter at $y$. Note that $\mathfrak{U}_{q_\alpha}(y)\xrightarrow{q_\alpha}y$ since $(X, q_\alpha)$ is pretopological. It is shown that $K\sigma^{\rightarrow}\mathfrak{F} \geq K\Sigma^{\rightarrow}\mathfrak{U}_{q_\alpha}(x)$. Because $\sigma(y) \overset{q_\alpha}{\rightarrow} \psi(y)$, $\sigma(y) \geq \Sigma(\psi(y))$ for each $y \in J$ and also $\psi^{\rightarrow}(\phi_{\mathfrak{F}}) = \phi_{\psi^{\rightarrow}\mathfrak{F}} \supseteq \phi_{\mathfrak{U}_{q_\alpha(x)}}$ since $\psi^{\rightarrow}\mathfrak{F}\overset{q_\alpha}{\rightarrow}x$. Hence $K\sigma^{\rightarrow}\mathfrak{F} = \bigvee_{A\in\phi_{\mathfrak{F}}} \bigwedge_{y\in A} \sigma(y) \geq \bigvee_{A\in\phi_{\mathfrak{F}}} \bigwedge_{y\in A} \Sigma(\psi(y)) = \bigvee_{A\in\phi_{\mathfrak{F}}} \bigwedge_{z\in\psi(A)} \Sigma(z) = \bigvee_{B\in\phi_{\psi^{\rightarrow}\mathfrak{F}}} \bigwedge_{z\in B} \Sigma(z) \geq \bigvee_{B\in\phi_{\mathfrak{U}_{q\alpha(x)}}} \bigwedge_{z\in B} \Sigma(z) = K\Sigma^{\rightarrow}\mathfrak{U}_{q\alpha}(x) \overset{q_\alpha}{\rightarrow} x$ according to the hypothesis and the fact that $(X, q_\alpha)$ is pretopological. Therefore, $K\sigma^{\rightarrow}\mathfrak{F} \overset{q_\alpha}{\rightarrow} x$ and thus $(X, \overline{q})$ satisfies F2. ■

Given a set $X$, $\tau \subseteq L^X$ is called a **stratified L-topology** if it satisfies the following:

(a) $\alpha 1_X \in \tau$ for each $\alpha \in L$

(b) $a, b \in \tau$ implies that $a \wedge b \in \tau$

(c) $a_j \in \tau, j \in J$, implies that $\vee_{j\in J} a_j \in \tau$.

The pair $(X, \tau)$ is said to be a **stratified L-topological Space**. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is **continuous** provided $f^{\leftarrow}(b) \in \tau$ whenever $b \in \sigma$. Let **SL-TOP** denote the category whose objects consist of all the stratified $L$-topological space and whose morphisms are all the continuous functions between objects. Let $(X, \tau) \in |SL-TOP|$, define the SL-neighborhood filter at $x$ by $\mathfrak{V}_\tau(x)(a) := \vee\{b(x) : b \in \tau \text{ and } b \leq a\}$, where $a \in L^X$. Note that $\mathfrak{V}_\tau(x) \in \mathfrak{F}_{SL}(X)$ for each $x \in X$. Given $(X, \overline{q}) \in |SL-CS|$, where $\overline{q} = (q_\alpha), \alpha \in L$; then $(X, \overline{q})$ is **topological** provided there exists $(X, \tau_\alpha) \in |SL-TOP|$ such that $\mathfrak{F}\overset{q_\alpha}{\rightarrow}x$ iff $\mathfrak{F} \geq \mathfrak{V}_{\tau_\alpha}(x)$ for each $x \in X$ and $\alpha \in L$. The next result follows from Gähler [1999, Proposition 30] and Theorem 19 (b).

**Theorem 22** *[Gähler, 1999]. Let $(X, \overline{q}) \in |SL-CS|$, where $\overline{q} = (q_\alpha)_{\alpha\in L}$. Then $(X, \overline{q})$ is topological iff $(X, q_\alpha)$ obeys axiom F3 for each $\alpha \in L$.*■

# CHAPTER 4.
# CATEGORICAL PROPERTIES OF SL-CS

This chapter develops and extends key categorical properties of SL-CS. Section 4.1 provides key definitions of these properties and an original proof is given showing that SL-CS is topological, cartesian closed, and extensional. Section 4.2 outlines important subconstructs of SL-CS based on Choquet, L-Kent, and pretopological modifications. Original references are provided indicating the historical significance of each definition. All of this is done as a precursor to the next chapter when two extremely significant original results regarding SL-CS and its subcategories are provided. This author states and proves two original theorems (Theorems 26 and 28), as well as Lemma 24. Results are extended and referred to in subsequent chapters in this work.

## 4.1 Topological, Cartesian Closed, and Extensional Properties of SL-CS

Jäger, et al. [2002] proved that the category SL-FCS discussed in Chapter 3 is a cartesian closed topological construct and studied several important subconstructs. It is shown in this section that SL-CS is topological, cartesian closed and extensional. Moreover, properties of some interesting subconstructs are listed.

Given objects $(X, \overline{q})$ and $(Y, \overline{p})$ in |SL-CS|, let $C(X, Y)$ be the set of all continuous functions from $(X, \overline{q})$ into $(Y, \overline{p})$ and denote the evaluation map $e : C(X, Y) \times X \to Y$ by $e(f, x) = f(x)$, for each $f \in C(X, Y)$ and $x \in X$. Define

$$\overline{c} = (c_\alpha) \tag{4.1}$$

as follows:

$$\Phi \overset{c_\alpha}{\to} f \text{ iff whenever } \mathfrak{F}\overset{q_\beta}{\longrightarrow}x, \beta \leq \alpha,$$

$$e^{\to}(\Phi \times \mathfrak{F})\overset{p_\beta}{\longrightarrow}f(x), \text{ where}$$

$$\Phi \in \mathfrak{F}_{\mathrm{SL}}(C(X,Y)) \text{ and } f \in C(X,Y).$$

Observe that $(C(X,Y), \overline{c}) \in$ |SL-CS|. Indeed, if $\mathfrak{F}\overset{q_\beta}{\longrightarrow}x$ for some $\beta \leq \alpha$, then it follows from Lemma 8.2 in Jäger[2001] that $e^{\to}([f] \times \mathfrak{F}) \geq f^{\to}\mathfrak{F}$ and thus $e^{\to}([f]) \times \mathfrak{F})\overset{p_\beta}{\longrightarrow}f(x)$. Hence $[f] \overset{c_\alpha}{\to} f$ for each $\alpha \in L$. The remaining axioms are easily verified.

Consider the category SL-FCS with $(X, \lim_X)$ and $(Y, \lim_Y) \in$ |SL-FCS|. Jäger [2001] defined

$$(C(X,Y), \lim) \in |\text{SL-FCS}| \tag{4.2}$$

as follows:

$$(\lim \Phi)(f) = \wedge_{\mathfrak{F},x} \vee \{\lambda \in L : (\lim_X \mathfrak{F})(x) \wedge \lambda \leq (\lim_Y e^{\to}(\Phi \times \mathfrak{F}))(f(x)),$$

$$\text{where } \Phi \in \mathfrak{F}_{SL}(C(X,Y)), \ \mathfrak{F} \in \mathfrak{F}_{SL}(X) \text{ and } x \in X.$$

The following remark shows that conditions (4.1) and (4.2) are compatible; verification is a straightforward application of the definitions.

**Remark 23** . *Suppose that* $(X, \lim_X)$, $(Y, \lim_Y) \in$ *|SL-FCS| and* $\theta$ *: SL-FCS* $\to$ *SL-CS is the embedding functor defined in (3.1). Then* $\theta(C(X,Y), \lim) = (C(X,Y), \overline{c})$.

Assume that $Z$ is a subset of $X, (X, \overline{q}) \in$ |SL-CS| and $i_Z : Z \to X$ is the injection $i_Z(y) = y$, for each $y \in Z$. Given $\mathfrak{F} \in \mathfrak{F}_{SL}(Z)$ and $\mathfrak{G} \in \mathfrak{F}_{SL}(X)$, denote $[\mathfrak{F}] : = i_Z^{\to}(\mathfrak{F})$ and $G_Z := i_Z^{\leftarrow}(G)$, provided the latter is a stratified $L$-filter.

**Lemma 24** . *Given the notations defined above with $\mathfrak{F} \in \mathfrak{F}_{SL}(Z)$, $\mathfrak{G} \in \mathfrak{F}_{SL}(X)$ and $a \in L^Z$. Then*

(a) $[\mathfrak{F}]_Z = \mathfrak{F}$

(b) $G_Z(a) = G(a^*)$, where $a^*(x) = \begin{cases} a(x), & x \in Z \\ 1, & x \in X - Z \end{cases}$, provided $G(b) = 0$ whenever $i_Z^{\leftarrow}(b) = 1_\phi$, $b \in L^X$.

Standard arguments can be employed to show that the category SL-CS is both topological and cartesian closed. It is shown in Theorem 26 below that SL-CS is also extensional. In particular, this implies that quotient maps are hereditary.

**Definition 25** *(Preuss[2002]). A topological construct $\mathcal{C}$ is called **extensional** if each $B \in |\mathcal{C}|$ can be embedded in a $\mathcal{C}$-object $B^* = B \cup \{\infty\}, \infty \notin B$, such that each $\mathcal{C}$-morphism $f : C \to B$ has the extension $f_* : A \to B^*$ whenever $C$ is a subobject of $A$, where*

$$f_*(x) = \begin{cases} f(x), & x \in C \\ \infty, & x \notin C \end{cases} \text{ is a } \mathcal{C}\text{-morphism.} \tag{4.3}$$

Given $(Y, \overline{r}) \in |\text{SL-CS}|$, denote $Y^* = Y \cup \{\infty\}$, where $\infty \notin Y$. Define

$$\overline{r}^* = (r^*_\alpha)_{\alpha \in L} \tag{4.4}$$

as follows:

(a) $\mathfrak{G} \overset{r^*_\alpha}{\to} y$ iff $\mathfrak{G}_Y$ exists and $\mathfrak{G}_Y \overset{r_\alpha}{\to} y$, or $\mathfrak{G}_Y$ fails to exist

(b) $\mathfrak{G} \overset{r^*_\alpha}{\to} \infty$, for each $\mathfrak{G} \in \mathfrak{F}_{SL}(Y^*)$.

It is straightforward to show that $(Y^*, \overline{r}^*) \in |\text{SL-CS}|$ and $(Y, \overline{r})$ is a subspace of $(Y^*, \overline{r}^*)$.

We next prove three significant results about the category SL-CS.

**Theorem 26** . *The construct SL-CS is*

(a) topological

(b) cartesian closed

(c) extensional.

**Proof.** Verification of (a) and (b) follows standard arguments. Only the proof of (c) is presented here. Let $(Z, \overline{p})$ be any subspace of $(X, \overline{q}) \in |\text{SL-CS}|$ and assume that $f : (Z, \overline{p}) \to (Y, \overline{r})$ is continuous. It must be shown that $f_* : (X, \overline{q}) \to (Y^*, \overline{r}^*)$ is also continuous, where $f_*$ and $(Y^*, \overline{r}^*) \in |\text{SL-CS}|$ are defined in (4.3) and (4.4). First, assume that $\mathfrak{F} \in \mathfrak{F}_{SL}(X)$ such that $\mathfrak{F} \xrightarrow{q_\alpha} z$, $z \in Z$. Suppose that $\mathfrak{F}_Z$ exists. If $a \in L^X$, then by Lemma 24 (b), $[\mathfrak{F}_Z](a) = \mathfrak{F}_Z(i_Z^{\leftarrow}(a)) = \mathfrak{F}((i_Z^{\leftarrow}(a))^*) \geq \mathfrak{F}(a)$. Hence $[\mathfrak{F}_Z] \geq \mathfrak{F}$ and thus $[\mathfrak{F}_Z] \xrightarrow{q_\alpha} z$. Since $(Z, \overline{p})$ is a subspace of $(X, \overline{q})$ and $f$ is continuous, $f^{\rightarrow}(\mathfrak{F}_Z) \xrightarrow{r_\alpha} f(z)$. It is shown that $(f_*^{\rightarrow}\mathfrak{F})_Y = f^{\rightarrow}(\mathfrak{F}_Z)$. Indeed, let $b \in L^Y, b^*(s) = \begin{cases} b(s), & s \in Y \\ 1, & s = \infty \end{cases}$ and according to Lemma 24 (b), $(f_*^{\rightarrow}\mathfrak{F})_Y(b) = (f_*^{\rightarrow}\mathfrak{F})(b^*) = \mathfrak{F}(b^* \circ f_*)$. Define $a = b \circ f \in L^Z$, $a^*(s) = \begin{cases} a(s), & s \in Z \\ 1, & s \in X - Z \end{cases}$ and since $b^* \circ f_* = a^*$, $\mathfrak{F}(b^* \circ f_*) = \mathfrak{F}(a^*) = \mathfrak{F}_Z(a) = \mathfrak{F}_Z(b \circ f) = (f^{\rightarrow}\mathfrak{F}_Z)(b)$. Therefore $(f_*^{\rightarrow}\mathfrak{F})_Y = f^{\rightarrow}\mathfrak{F}_Z$ and thus $f_*^{\rightarrow}\mathfrak{F} \xrightarrow{r_\alpha^*} f_*(z)$. Next, suppose that $\mathfrak{F} \xrightarrow{q_\alpha} z$ but $\mathfrak{F}_Z$ fails to exist. It is shown that $(f_*^{\rightarrow}\mathfrak{F})_Y$ also fails to exist. It follows from Proposition 3.5 [Jäger, 2001] that there exists $a \in L^X$ such that $i_Z^{\leftarrow}(a) = 1_\phi$ and yet $\mathfrak{F}(a) \neq 0$. Denote $b = f_*^{\rightarrow}(a)$ and fix $y \in Y$. Then $b(y) = \vee\{a(x) : f_*(x) = y\} = \vee\{a(z) : f(z) = y\} = 0$ and $b(\infty) = \vee\{a(x) : x \in X - Z\} \geq a(x)$, for each $x \in X - Z$. Hence $b \circ f_* \geq a$ and thus $(f_*^{\rightarrow}\mathfrak{F})(b) = \mathfrak{F}(b \circ f_*) \geq \mathfrak{F}(a) \neq 0$. Therefore $(f_*^{\rightarrow}\mathfrak{F})_Y$ fails to exist and thus by definition of $r_\alpha^*$, $f_*^{\rightarrow}\mathfrak{F} \xrightarrow{r_\alpha^*} f(z)$. Finally, assume that $\mathfrak{F} \xrightarrow{q_\alpha} x$, where $x \in X - Z$. Since all stratified $L$-filters $r_\alpha^*$-converge to $\infty$, $f_*^{\rightarrow}\mathfrak{F} \xrightarrow{r_\alpha^*} f_*(x) = \infty$ and thus $f_* : (X, \overline{q}) \to (Y^*, \overline{r}^*)$ is continuous. Therefore SL-CS is extensional. ■

## 4.2 Important Subconstructs of SL-CS

Jäger, et al. [2002] defined and investigated several important subconstructs of SL-FCS. A brief summary of some corresponding results in the construct SL-CS is given below.

**Definition 27** . *Recall that $(X, \overline{q}) \in |SL\text{-}CS|$ is called a* ***stratified** $L$**-Kent** [Kent, 1968] (**Choquet** [Kent, 1948]; **pretopological**) **convergence space** provided $\mathfrak{H} \xrightarrow{q_\alpha} x$ whenever $\mathfrak{F} \xrightarrow{q_\alpha} x$ and $\mathfrak{H} \geq \mathfrak{F} \wedge [x]$ ($\mathfrak{F} \xrightarrow{q_\alpha} x$ for each stratified $L$-ultrafilter $\mathfrak{F} \geq \mathfrak{H}$; $\mathfrak{H} \geq \mathcal{U}_{q_\alpha}(x) = \wedge\{\mathfrak{F} \in \mathfrak{F}_{SL}(X) : \mathfrak{F} \xrightarrow{q_\alpha} x\}$, respectively. Moreover, $(X, \overline{q})$ is said to be a **stratified $L$-limit space** [3] if $\mathfrak{F} \xrightarrow{q_\alpha} x$ whenever $\mathfrak{F} \geq \wedge_{i=1}^n \mathfrak{F}_i$ for some $\mathfrak{F}_i \xrightarrow{q_\alpha} x$ and $n \geq 1$.*

Let **SL-K-CS** (**SL-C-CS**, **SL-P-CS**) denote the full-subconstruct of SL-CS whose objects consist of all the stratified $L$-Kent (Choquet, pretopological) convergence spaces, respectively. Further, **SL-L-CS** defines the full-subconstruct of SL-CS possessing all the stratified $L$-limit spaces as its objects.

**Theorem 28** . *Assume that $(X, \overline{q})$ and $(Y, \overline{p})$ are objects belonging to $|SL\text{-}CS|$. Then*

(a) SL-L-CS (SL-C-CS, SL-P-CS) is bireflective in SL-CS, respectively.

(b) SL-K-CS is both bireflective and bicoreflective in SL-CS

(c) $(C(X,Y),\overline{c}) \in$ |SL-L-CS|(|SL-C-CS|, |SL-LC-CS|) provided $(Y,\overline{p}) \in$ |SL-L-CS|(|SL-C-CS|, |SL-LC-CS|), respectively.

Jäger et al. [2002] proves the corresponding results of Theorem 28 for the construct SL-FCS which, according to Theorem 16 (a), is embedded in SL-CS. However, proofs in the category SL-CS seem to be more transparent since the steps involve determining whether a stratified $L$-filter converges to an element $x \in X$ rather than having to specify its limiting $L$-fuzzy subset.

# CHAPTER 5. CONNECTIONS AMONG DIAGONAL AXIOMS FOR OBJECTS IN A SUBCONSTRUCT OF SL-CS

This chapter completes the analysis of the category SL-CS. A final comment on diagonal axioms are provided in Section 5.1 as an original theorem fully categorizing SL-CS is provided. Additionally, this section shows an object in SL-CS satisfying condition F2 is equivalent to being determined by a probabilistic convergence space that is topological. Section 5.2 provides a summary of this work, and possible areas for further research are mentioned. Two research questions are proposed related to the context of stratified lattice fuzzy topological spaces. This author states and proves two original theorems (Theorems 31 and 32), as well as Lemmas 29 and 30.

## 5.1 Diagonal Axioms and SL-CS

As we recall, $L$ denotes a linearly ordered, complete Heyting algebra. Consider the category PCS of probabilistic convergence spaces defined in section 1. Given $(X, \overline{Q}) \in |PCS|$, define $\overline{q} = (q_\alpha)_{\alpha \in L}$ as follows:

$$\mathfrak{F} \xrightarrow{q_\alpha} x \quad \text{iff} \quad \phi_{\mathfrak{F}} \xrightarrow{Q_\alpha} x, \quad \alpha \in L. \tag{5.1}$$

Then $(X, \overline{q}) \in |SL - CS|$. Recall, it is shown in Theorem 7 that $\delta : PCS \to SL - CS$ defined by

$$\delta(X, \overline{Q}) = (X, \overline{q}) \tag{5.2}$$

is a full-embedding functor. The result is still valid even though the definition of $\phi_{\mathfrak{F}}$ in (1.1) differs from that given in Flores, et al. [2006]. This section is devoted to a study of diagonal axioms in the subconstruct $\delta(PCS)$ of SL-CS. First, two fundamental lemmas are proved. Recall the definitions of $\phi_{\mathfrak{F}}$ and $\mathfrak{F}_\psi$ listed in (1.1).

**Lemma 29** . *Assume that $J$ is a set, $\Delta : J \to \mathfrak{F}(X)$, $\rho : \mathfrak{F}(X) \to \mathfrak{F}_{SL}(X)$ and $\sigma = \rho \circ \Delta$, where $\rho(\delta) = \mathfrak{F}_\delta$ for each $\delta \in \mathfrak{F}(J)$. Let $\gamma \in \mathfrak{F}(J)$; then $\phi_{G(\sigma^\to \mathfrak{F}_\gamma)} = \phi_{K(\sigma^\to \mathfrak{F}_\gamma)} = K\Delta^\to \gamma$.*

**Proof.** It follows from Lemma 18 (a) that $\phi_{G(\sigma^\to \mathfrak{F}_\gamma)} \supseteq \phi_{K(\sigma^\to \mathfrak{F}_\gamma)}$. Assume that $B \in K\Delta^\to \gamma = \cup_{A\in\gamma} \cap_{y\in A} \Delta(y)$; then there exists $A \in \gamma$ such that $B \in \Delta(y)$ for each $y \in A$. Since $\sigma(y) = \mathfrak{F}_{\mathfrak{d}(\mathfrak{y})}$, it follows from definition (1.1) that $\mathfrak{F}_{\Delta(y)}(\alpha 1_B) = 1$ for each $\alpha > 0$ and $y \in A$. Moreover, according to Lemma 3(h), $A \in \phi_{\mathfrak{F}_\gamma} = \gamma$. It follows that $K(\sigma^\to \mathfrak{F}_\gamma)(\alpha 1_B) \geq \wedge_{y\in A}\sigma(y)(\alpha 1_B) \geq \alpha$ for each $\alpha \in L$ and thus $B \in \phi_{K(\sigma^\to \mathfrak{F}_\gamma)}$. Hence $\phi_{K(\sigma^\to \mathfrak{F}_\gamma)} \supseteq K\Delta^\to \gamma$. Conversely, suppose that $B \in \phi_{G(\sigma^\to \mathfrak{F}_\gamma)}$ and let $b = \alpha 1_B$, $\alpha > 0$. Then $G(\sigma^\to \mathfrak{F}_\gamma)(b) = \mathfrak{F}_\gamma(e_b \circ \sigma) \geq \alpha$. Denote $A = \{y \in J : B \in \Delta(y)\}$ and observe that $(e_b \circ \sigma)(y) = \sigma(y)(b) = \mathfrak{F}_{\Delta(y)}(b) = 1_A(y)$. When $\alpha = 1$, $1 = \mathfrak{F}_\gamma(e_b \circ \sigma) = \mathfrak{F}_\gamma(1_A)$ and thus $A \in \gamma$. It follows that $B \in \cup_{A\in\gamma} \cap_{y\in A} \Delta(y) = K\Delta^\to \gamma$ and thus $\phi_{G(\sigma^\to \mathfrak{F}_\gamma)} \subseteq K\Delta^\to \gamma$ and hence $\phi_{G(\sigma^\to \mathfrak{F}_\gamma)} = \phi_{K(\sigma^\to \mathfrak{F}_\gamma)} = K\Delta^\to \gamma$. ■

**Lemma 30** . *Suppose that $J$ is a set, $\sigma : J \to \mathfrak{U}_{SL}(X)(\mathfrak{F}_{SL}(X)), \xi : \mathfrak{F}_{SL}(X) \to \mathfrak{F}(X)$ defined by $\xi(\mathfrak{G}) = \phi_{\mathfrak{G}}$, $\delta = \xi \circ \sigma$ and $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$; then $\phi_{G(\sigma^\to \mathfrak{F})} \supseteq \phi_{K(\sigma^\to \mathfrak{F})} = K\delta^\to \phi_{\mathfrak{F}}$ ($\phi_{K(\sigma^\to \mathfrak{F})} \supseteq K\delta^\to \phi_{\mathfrak{F}}$), respectively.*

**Proof.** Again, according to Lemma 18 (a), $\phi_{\mathfrak{G}(\sigma^\to \mathfrak{F})} \supseteq \phi_{K(\sigma^\to \mathfrak{F})}$. Assume that $B \in K\delta^\to \phi_{\mathfrak{F}} = \cup_{A\in\phi_{\mathfrak{F}}} \cap_{y\in A} \delta(y) = \cup_{A\in\phi_{\mathfrak{F}}} \cap_{y\in A} \phi_{\sigma(y)}$. Then there exists $A \in \phi_{\mathfrak{F}}$ such that $B \in \phi_{\sigma(y)}$ and thus $\sigma(y)(\alpha 1_B) \geq \alpha$ for each $y \in A$ and $\alpha \in L$. Fix $\alpha > 0$ and denote $b = \alpha 1_B$. Then $K\sigma^\to \mathfrak{F}(b) \geq \wedge_{y\in A}\sigma(y)(b) \geq \alpha$ and hence $B \in \phi_{K\sigma^\to \mathfrak{F}}$. Therefore $\phi_{K\sigma^\to \mathfrak{F}} \supseteq K\delta^\to \phi_{\mathfrak{F}}$.

Conversely, suppose that $B \in \phi_{K\sigma^{\rightarrow}\mathfrak{F}}$, $\alpha > 0$ is fixed and $b = \alpha 1_B$. Then $\alpha \leq K\sigma^{\rightarrow}\mathfrak{F}(b) = \vee_{A\in\phi_F} \wedge_{y\in A} \sigma(y)(b)$ and thus there exists $A \in \phi_{\mathfrak{F}}$ such that $\wedge_{y\in A}\sigma(y)(b) > 0$. Since $\sigma(y) \in \mathfrak{U}_{SL}(X)$, it follows from Lemma 20 that $\wedge_{y\in A}\sigma(y)(b) = 1$. Hence $B \in \phi_{\sigma(y)} = \delta(y)$ for each $y \in A$ and hence $B \in \cup_{A\in\phi_{\mathfrak{F}}} \cap_{y\in A} \delta(y) = K\delta^{\rightarrow}\phi_{\mathfrak{F}}$. Therefore $\phi_{G(\sigma^{\rightarrow}\mathfrak{F})} \supseteq K(\sigma^{\rightarrow}\mathfrak{F}) = K\delta^{\rightarrow}\phi_{\mathfrak{F}}$.

■

Connections between diagonal axioms for objects in the subconstruct $\delta(PCS)$ of SL-CS given in Lemma 30 are listed below.

**Theorem 31** . *Assume that* $(X, \overline{Q}) \in |PCS|$ *and* $(X, \overline{q}) = \delta(X, \overline{Q})$*. Then*

(a) $(X, \overline{q})$ is pretopological iff $(X, \overline{Q})$ is pretopological.

(b) The following are equivalent:

(i) $(X, \overline{q})$ obeys F2

(ii) $(X, \overline{q})$ obeys F3

(iii) $(X, \overline{q})$ is topological

(iv) $(X, \overline{Q})$ is topological (obeys F1).

(c) (i) $(X, \overline{q})$ satisfies R2 whenever it satisfies R3

(ii) $(X, \overline{q})$ satisfies R2* iff $(X, \overline{Q})$ satisfies R1.

**Proof.** (a): Assume that $(X, q_\alpha)$ is pretopological and let $\mathfrak{U}_{q_\alpha}(x)$ denote its SL-neighborhood filter at $x$. Since $\mathfrak{U}_{q_\alpha}(x) \stackrel{q_\alpha}{\rightarrow} x$, $\phi_{\mathfrak{U}_{q_\alpha}(x)} \stackrel{Q_\alpha}{\rightarrow} x$. Moreover, note that if $\gamma \stackrel{Q_\alpha}{\rightarrow} x$, then $\mathfrak{F}_\gamma \stackrel{q_\alpha}{\rightarrow} x$ since $\phi_{\mathfrak{F}_\gamma} = \gamma$. Hence $\mathfrak{F}_\gamma \geq \mathfrak{U}_{\mathfrak{q}_\alpha}(\mathfrak{x})$ and thus $\gamma = \phi_{\mathfrak{F}_\gamma} \supseteq \phi_{\mathfrak{U}_{q_\alpha}(x)}$. It follows that $\phi_{\mathfrak{U}_{q_\alpha}(x)}$ is a neighborhood filter at $x$ in $(X, Q_\alpha)$ and $(X, Q_\alpha)$ is pretopological. Conversely, assume that $(X, Q_\alpha)$ is pretopological and let $\mathfrak{V}_{Q_\alpha}(x)$ denote its neighborhood filter at $x$. Assume that $\mathfrak{F}_j \xrightarrow{q_\alpha} x$, $j \in J$. Then $\phi_{\mathfrak{F}_j} \overset{Q_\alpha}{\to} x$ and since $\phi_{\wedge_j \mathfrak{F}_j} = \cap_j \phi_{\mathfrak{F}_j} \geq \mathfrak{V}_{Q_\alpha}(x) \xrightarrow{Q_\alpha} x$, $\wedge_{j\in J}\mathfrak{F}_j \xrightarrow{q_\alpha} x$. Hence $(X, q_\alpha)$ is pretopological.

(b): Suppose that $(X, \overline{Q})$ obeys F1, $\psi : J \to X, \sigma : J \to \mathfrak{F}_{SL}(X)$ such that $\sigma(y) \overset{q_\alpha}{\to} \psi(y)$ for each $y \in J$. Let $\mathfrak{F} \in \mathfrak{F}_{SL}(J)$ satisfy $\psi^{\to}\mathfrak{F} \xrightarrow{q_\alpha} x$. Define $\xi : \mathfrak{F}_{SL}(x) \to \mathfrak{F}(X)$ by $\xi(G) = \phi_G$ and let $\delta = \xi \circ \sigma$. Then $\delta(y) = \phi_{\sigma(y)} \overset{Q_\alpha}{\to} \psi(y)$ for each $y \in J$ according to (5.1). Likewise, $\psi^{\to}\mathfrak{F} \xrightarrow{q_\alpha} x$ implies that $\phi_{\psi^{\to}\mathfrak{F}} \overset{Q_\alpha}{\to} x$ and by Lemma 3(h), $\psi^{\to}(\phi_{\mathfrak{F}}) \overset{Q_\alpha}{\to} x$. Since $(X, \overline{Q})$ obeys F1, $K\delta^{\to}\phi_{\mathfrak{F}} \overset{Q_\alpha}{\to} x$ and thus by Lemma 30, $K(\sigma^{\to}\mathfrak{F}) \xrightarrow{q_\alpha} x$ and $G(\sigma^{\to}\mathfrak{F}) \xrightarrow{q_\alpha} x$. Hence $(X, \overline{q})$ obeys F2 and F3. Conversely, assume that $(X, \overline{q})$ obeys F2 (F3), $\psi : J \to X$, $\Delta : J \to \mathfrak{F}(X)$ such that $\Delta(y) \overset{Q_\alpha}{\to} \psi(y)$ for each $y \in J$. Let $\gamma \in \mathfrak{F}(J)$ obey $\psi^{\to}\gamma \overset{Q_\alpha}{\to} x$, define $\rho : \mathfrak{F}(X) \to \mathfrak{F}_{SL}(X)$ by $\rho(\delta) = \mathfrak{F}_\delta$ for each $\delta \in \mathfrak{F}(X)$ and denote $\sigma = \rho \circ \Delta$. Note that $\sigma(y) = \mathfrak{F}_{\Delta(y)} \xrightarrow{q_\alpha} \psi(y)$ since $\phi_{\mathfrak{F}_{\mathfrak{d}(\mathfrak{y})}} = \Delta(y) \overset{Q_\alpha}{\to} \psi(y)$ for each $y \in J$. Moreover, $\phi_{\psi^{\to}\mathfrak{F}_\gamma} = \psi^{\to}(\phi_{\mathfrak{F}_\gamma}) = \psi^{\to}(\gamma) \overset{Q_\alpha}{\to} x$ implies that $\psi^{\to}\mathfrak{F}_\gamma \xrightarrow{q_\alpha} x$. Since $(X, \overline{q})$ obeys F2 (F3), $K\sigma^{\to}\mathfrak{F}_\gamma \xrightarrow{q_\alpha} x (G(\sigma^{\to}\mathfrak{F}_\gamma) \xrightarrow{q_\alpha} x)$ and thus by Lemma 29, $K\Delta^{\to}\gamma \overset{Q_\alpha}{\to} x$, respectively. Hence $(X, \overline{Q})$ obeys F1 and thus axioms F1, F2 and F3 are equivalent. The remainder of (b) follows by Theorem 22.

(c) Part (i) is proved in Theorem 19 (a). An argument employing Lemmas 29-30 shows that part (ii) is valid. ■

**Theorem 32 .** *Assume that $(X, \overline{q}) \in |SL - CS|$. Then $(X, \overline{q})$ obeys F2 iff there exists $(X, \overline{Q}) \in |PCS|$ that is topological and satisfies $\delta(X, \overline{Q}) = (X, \overline{q})$.*

**Proof.** First, suppose that $(X, \overline{Q}) \in |PCS|$ is topological and $\delta(X, \overline{Q}) = (X, \overline{q})$. Then according to Theorem 31 (b) $(X, \overline{q})$ obeys F2. Conversely, assume that $(X, \overline{q})$ obeys F2 and denote $\mathfrak{V}_\alpha(x) = \phi_{\mathfrak{U}_{q_\alpha}(x)}$ for each $x \in X$. Define $(X, \overline{P}), \overline{P} = (P_\alpha)_{\alpha\in L}$, by $\gamma \overset{P_\alpha}{\to} x$ iff

$\gamma \supseteq \mathfrak{V}_\alpha(x)$. It is easily shown that $(X, \overline{P}) \in |PCS|$. Denote $(X, \overline{p}) = \delta(X, \overline{P}) \in |SL - CS|$. Since $(X, \overline{P})$ is pretopological, it follows from Theorem 31 (a) that $(X, \overline{p}) \in |SL - P - CS|$. Note that $\mathfrak{U}_{q_\alpha}(x) \stackrel{p_\alpha}{\to} x$ since $\phi_{\mathfrak{U}_{q_\alpha}(x)} = \mathfrak{V}_\alpha(x) \stackrel{P_\alpha}{\to} x$ and thus $\overline{q} \geq \overline{p}$. Conversely, suppose that $\mathfrak{F} \stackrel{p_\alpha}{\to} x$. It is shown that $\mathfrak{F} \geq K\sigma^{\to}\mathfrak{F} \geq K\sigma^{\to}\mathfrak{U}_{q_\alpha}(x) = \mathfrak{U}_{q_\alpha}(x)$, where $\sigma(y) = \mathfrak{U}_{q_\alpha}(y)$ for each $y \in X$. Note that if $a \in L^X$ and $A \in \phi_{\mathfrak{F}}$, then $a \geq \wedge_{y \in A} a(y) \cdot 1_A$ and thus $\mathfrak{F}(a) \geq \mathfrak{F}[\wedge_{y\in A} a(y) 1_A] \geq \wedge_{y \in A}[y](a) \geq \wedge_{y\in A}\mathfrak{U}_{q_\alpha}(y)(a) = \wedge_{y \in A}\sigma(y)(a)$. It follows that $\mathfrak{F}(a) \geq \vee_{A\in\phi_F} \wedge_{y\in A} \sigma(y)(a) = K\sigma^{\to}\mathfrak{F}(a)$. Moreover, since $\mathfrak{F} \stackrel{p_\alpha}{\to} x$, $\phi_{\mathfrak{F}} \stackrel{P_\alpha}{\to} x$ and thus $\phi_{\mathfrak{F}} \supseteq \phi_{\mathfrak{U}_{q_\alpha}(x)}$. Hence $K\sigma^{\to}\mathfrak{F} = \vee_{A\in\phi_F} \wedge_{y\in A} \sigma(y) \geq \vee_{A\in\mathfrak{U}_{q_\alpha}(x)} \wedge_{y\in A} \sigma(y) = K\sigma^{\to}\mathfrak{U}_{q_\alpha}(x) = \mathfrak{U}_{q_\alpha}(x)$ since $(X, \overline{q})$ obeys F2. It follows that $\mathfrak{F} \geq \mathfrak{U}_{q_\alpha}(x)$ and thus $\mathfrak{F} \stackrel{q_\alpha}{\to} x$. Hence $\overline{q} = \overline{p}$. Since $(X, \overline{q})$ obeys F2 and $\delta(X, \overline{p}) = (X, \overline{q})$, it follows from Theorem 31 (b) that $(X, \overline{P})$ is topological. ■

Recall that $(X, \overline{q}) \in |SL - CS|$ satisfies F3 iff it is topological. Let us conclude by giving an example of $(X, \overline{q}) \in |SL - CS|$ that is topological but fails to satisfy F2.

**Example 33** . *Let $X = L = [0, 1]$ and $\alpha \in L$; define $\tau_\alpha = \{\beta 1_X, a : \beta \in L, a \in L^X$ and $a \geq (1 - \alpha)1_X\}$. Note that $(X, \tau_\alpha) \in |SL - TOP|$ for each $\alpha \in L$. Define $\mathfrak{U}_{q_\alpha}(x)(b) = \vee\{c(x) : c \in \tau, c \leq b\}$. It is easily shown that $\mathfrak{U}_{q_\alpha}(x) \in \mathfrak{F}_{SL}(X)$. Define $\mathfrak{F} \stackrel{q_\alpha}{\to} x$ iff $\mathfrak{F} \geq \mathfrak{U}_{q_\alpha}(x)$ and denote $\overline{q} = (q_\alpha)_{\alpha\in L}$. Then $(X, \overline{q}) \in |SL - P - CS|$ and by construction it is topological and according to Theorem 31 it obeys F3. It is shown that $(X, \overline{q})$ fails to satisfy F2.*

Observe that if $A \in \phi_{\mathfrak{U}_{q_\alpha}(x)}$, then $\mathfrak{U}_{q_\alpha}(x)(\beta 1_B) \geq \beta$ for each $\beta \in L$ iff $B = X$ provided $0 \leq \alpha < 1$. Hence for $0 \leq \alpha < 1$, $\phi_{\mathfrak{U}_{q_\alpha}(x)} = \dot{X}$. Denote $\sigma(y) = \mathfrak{U}_{q_\alpha}(y)$ for each $y \in X$ and let $b \in L^X$. Then for $0 \leq \alpha < 1$, $K\sigma^{\to}\mathfrak{U}_{q_\alpha}(x)(b) = \vee_{A\in\phi_{\mathfrak{U}_{q_\alpha}(x)}} \bigwedge_{y\in A} \sigma(y)(b) = \bigwedge_{y\in X}\mathfrak{U}_{q_\alpha}(y)(b) \leq \bigwedge_{y\in X} b(y) = \mathfrak{F}_0(b)$. Since $\mathfrak{F}_\circ$ is the coarsest element in $\mathfrak{F}_{SL}(X), K\sigma^{\to}\mathfrak{U}_{q_\alpha}(x) = \mathfrak{F}_\circ$ for each

$x \in X$ and $0 \leq \alpha < 1$. It follows that if $(X, \overline{q})$ obeys F2, $\mathfrak{U}_{q_\alpha}(x) = \mathfrak{F}_\circ$ for each $x \in X$ and $0 \leq \alpha < 1$. Choose $\alpha = \frac{1}{2}$ and $b = \frac{1}{2}1_{[0,\frac{1}{2}]} + 1_{(\frac{1}{2},1]}$. Then $b \in \tau_\alpha$ and thus $\mathfrak{U}_{q_\alpha}(x)(b) = b(x)$, whereas $\mathfrak{F}_\circ(b) = \frac{1}{2}$. Hence $\mathfrak{U}_{q_\alpha}(x)(b) \neq \mathfrak{F}_\circ(b)$ whenever $x \in (\frac{1}{2}, 1]$. Therefore $(X, \overline{q})$ obeys F3 but not F2. Moreover, it follows form Theorem 32 that there fails to exist an $(X, \overline{Q}) \in |PCS|$ such that $\delta(X, \overline{Q}) = (X, \overline{q})$. ■

## 5.2 Concluding Remarks and Recommendations for Further Research

The category $SL - TOP$ of stratified L-topological spaces was studied above. More generally, an L-fuzzy topological space is defined and studied by Höhle and Sostak [1999].

**Definition 34** *The pair $(x, \tau)$ is called a* ***stratified L-fuzzy topological space*** *provided $\tau : L^X \to L$ satisfies:*

(a) $\tau(1_\phi) = 1$, and $\alpha \leq \tau(\alpha 1_X)$ for each $\alpha \in L$

(b) $\tau(a) \wedge \tau(b) \leq \tau(a \wedge b)$, $a, b \in L^X$

(c) $\bigwedge_{j \in J} \tau(a_j) \leq \tau(\bigvee_{j \in J} a_j)$, $a_j \in L^X$, $j \in J$

In light of the aforementioned, this author proposes the following research questions:

**Question 1:** Can one find an appropriate diagonal condition which characterizes when a stratified L-pretopological convergence space is a stratified L-fuzzy topological space?

Jager [ 2006] has defined interior operators $I : L^X \to L^X$ which characterizes the objects in SL-FCS which are pretopological. He mentions that these interior operators do not characterize the objects in SL-CS which are pretopological.

**Question 2:** Can one define operators that characterize the objects in SL-CS which are pretopological?

It is expected that the operators should be defined on $I : L^X \text{x } L \rightarrow L^X$ in this case.

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