Synthesis of numerical integrators for the real-time digital simulation of continuous systems

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SYNTHESIS OF NUMERICAL INTEGRATORS FOR THE
REAL-TIME DIGITAL SIMULATION OF CONTINUOUS SYSTEMS

By
Michael J. Panzitta

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UNIVERSITY OF CENTRAL FLORIDA
SYNTHESIS OF NUMERICAL INTEGRATORS FOR THE REAL-TIME DIGITAL SIMULATION OF CONTINUOUS SYSTEMS

by

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B.S.E., University of Central Florida, 1984
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by

Michael J. Panzitta
ABSTRACT

Real-time digital simulation is a powerful means for engineers and scientists in government, industry, and academia to perform research and training as well as serving as a basis for many commercial applications. Due to the special constraints imposed by digitally simulating continuous systems in real time, however, many of these systems either require costly high-speed components or are unable to provide suitable performance characteristics using affordable computers.

This dissertation describes a new technique for the synthesis of numerical integrators specifically designed for the real-time digital simulation of continuous systems. This methodology is based upon the fact that the state derivatives in a simulation model typically have a significantly limited bandwidth. This information is exploited to improve the efficiency of numerical integrators by selecting the coefficients of a general-form integrator such that it approximates an ideal integrator over the limited frequency spectrum of the state derivative.

The specific constraints and performance characteristics necessitated by the real-time environment are first identified and addressed. A method for analyzing the frequency response of individual state variables in a system is presented together with frequency-domain gain and phase error metrics which characterize the performance of the numerical integrators within limited frequency spectra. The resulting technique for synthesizing numerical integrators based upon their required performance and nominal input signal characteristics is then presented. The method consists of varying parameter values of a general-form difference equation to minimize gain and phase errors over the limited frequency band of that particular integrator. These techniques
are applied first to a simple example as a proof of concept; it is then used to improve the cost/performance ratio for the more complex case of an automobile vehicle dynamics model.

The results of this research provide a means for the evaluation and development of highly effective numerical integrators for real-time simulations. The improved efficiency of simulations using these techniques permits higher fidelity models to be implemented on lower cost platforms. This enables these simulators to be used in a broader range of applications where high fidelity is required while at the same time minimizing recurring costs.
To my dear family and friends

and in loving memory of Joseph Panzitta
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The myriad tasks involved in completing this work would not have been possible without the support and motivation of my family, friends, and coworkers. I would like to thank my numerous friends and colleagues at Evans & Sutherland Computer Corporation for the opportunity to work first-hand with state-of-the-art driving simulators and at Daimler-Benz AG in Berlin, Germany for the insight that they have provided from the world's highest fidelity driving simulator. In particular, I would like to extend my thanks to Mike Bartholomew, Dave Jolley, Hr. Wilfried Käding, Hr. Volkhard Schill, Hr. Hannsjörg Schmieder, and Hr. Joachim Stritzke.

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INTRODUCTION

Real-time digital simulation of continuous systems is a powerful tool which is utilized in a broad spectrum of applications, including simulators and training systems. Numerical integration algorithms—a key component of real-time simulators—must be accurate and able to operate in real time to provide high-fidelity simulations of complex systems. As simulators become more widely employed in academic, commercial, and government sectors, cost-effective implementations are necessary to allow higher fidelity simulations widespread entry into these markets.

Unlike the general-purpose integration techniques used in numerical analysis, operating in a real-time environment imposes many additional constraints. These important considerations must be recognized when implementing a simulation required to operate in real-time.

Since many integration techniques which have become standard practice in numerical analysis cannot be used in real-time simulation applications, new methods must be developed which address these specific limitations and requirements. A class of integrators can be defined which meet the requirements of real-time simulation, and established techniques can be used to analyze the stability and accuracy of these synthesized numerical integration algorithms.

The discussion of a new technique for synthesizing real-time numerical integrators begins by presenting a simple continuous system. This system is stimulated with inputs which define the boundaries of the system's operating envelope, and the frequency spectrum of the simulation's state derivatives are analyzed. It is shown that the various integrators in a system are subjected to different bandwidths at their inputs.
This information allows the integrator to be synthesized to best meet the contextual performance requirements.

A class of real-time numerical integrators is then defined, along with criteria for measuring frequency-domain errors. These metrics form the basis for a set of parameters which can be used to specify the desired characteristics of a numerical integrator over its potentially limited input bandwidth. Using the simple continuous system introduced earlier, the error criteria for the general forms of these integrators are then compared with commonly used reference integrators of first and second order. The results show favorable correspondence between the analytical solution and the simulated response using synthesized integrators, with the second-order synthesized integrator showing a significant performance improvement over both reference integrators.

The vehicle dynamics model used by the University of Central Florida Driving Simulator is then presented. The current simulation implementation is used as a baseline from which the model's integrator cutoff frequencies are determined through the use of four test cases. Error metrics are then derived from the baseline simulation, and these are used as specifications for the synthesis of new integrators with the intent to improve the integration step size of the new simulation without degrading its performance. The new simulation is then compared to the baseline by analyzing their time responses to the four test cases developed earlier. The mixed results reflect the potential for the techniques developed here to be used in many applications while highlighting several changes which would improve the general utility of these methods.

Finally, the overall results of the paper are summarized and analyzed. Both advantages and disadvantages of the methods developed here are discussed, as well as several possible improvements to these techniques. A number of topics are addressed as candidates for further research in this field.
A set of appendices provide more detailed derivations of first- and second-order synthesized integrators as well as that of their corresponding reference integrators. Additionally, the missile test case developed earlier is simulated using a simple multirate method to illustrate the applicability of this methodology to the work developed in this paper.
CHAPTER 1

REQUIREMENTS OF REAL-TIME SIMULATION

The development and deployment of simulations which are designed to operate in a real-time environment pose many unique problems and issues in addition to those which are normally encountered in the modeling of continuous systems. These issues are addressed by first examining the wide variety of applications which employ real-time simulation. After this introduction, the specific limitations imposed by the requirement to operate in real-time are identified. Finally, a general form of numerical integrator is defined based upon these real-time criteria.

Real-Time Simulation Applications

Real-time simulation is employed in many applications, spanning the academic, commercial, and government sectors. Some of its most prevalent applications are in engineering, research, and training simulators. Such simulators have been developed for commercial and military aircraft, armored vehicles, automobiles, nuclear power plants, ships and submarines, space vehicles, and countless other complex systems.

Simulators such as these may be employed for a variety of practical reasons. In the case of engineering simulators, these resources provide the ability to analyze systems which have not yet been constructed or which cannot otherwise be readily examined. An example of this use is the F-22 Pilot Training System Device, which will be used both as a trainer for the operational weapon system as well as an engineering tool to support the design, validation, and flight testing of the actual aircraft (Baldwin and Landry 1992).
Training simulators also offer advantages which cannot be easily achieved by using the device being simulated. The use of a simulator allows hazardous conditions to be avoided when training dangerous operations, such as crash avoidance. Simulators allow instructors to exercise fine control of the training environment in order to achieve a high degree of training effectiveness. In most circumstances, simulators offer a much lower cost for training than does the actual device. Particularly in the case of military applications, simulators allow training to occur without undue disturbance of the environment, which has become an increasingly controversial topic.

From the first flight trainer developed by Edwin Link in the late 1920's to the modern simulators in international military and civil arenas, flight simulation has been one of the most widespread forms of the technology. These training devices have become a firm part of international aircrew training methodologies, having received approval from the Federal Aviation Administration (FAA) in the United States and corresponding foreign agencies (Rolfe and Staples 1986).

Real-time simulators are also used to train nuclear power plant operators. These devices emulate the behavior of pressurized and boiling water reactors and have been in use on an international scale since the 1950's. Because of the effectiveness of these trainers, the Nuclear Regulatory Commission (NRC) has incorporated simulator training in operator licensing and renewal procedures (Chen 1979).

As with flight simulators, driving trainers may be found in both military and civilian programs. Military trainers have been focused on tank driver trainers since these devices are more cost effective than the actual vehicles and avoid the extensive environmental impact that has occurred in the past. While cost is always a factor in procuring these simulators, the degree of fidelity required to carry out the desired training tasks imposes costs in the millions of dollars per seat.
In contrast, the cost of automobile simulators in the public and commercial sectors is considerably more sensitive and, in many cases, prohibitive. Fairly realistic simulators are used to train passenger car drivers in many public and private schools. Until recently, film-based and other electromechanical simulators have been used for the majority of these systems; however, as simulation technologies have advanced, the use of all-digital simulation has allowed these trainers to achieve higher fidelity at lower cost. Such realism in the simulation is not only desirable for public driver training programs but is a necessity for use in law enforcement and long-haul transportation.

It is towards this end that this research is directed. Even as the cost of computer hardware has been steadily decreasing, computationally efficient methods for implementing high-fidelity, real-time models are necessary to help bring simulators to broader public and commercial markets. In this way, a greater number of people will be able to benefit from the same powerful technologies presently in use in laboratories and military installations.

**Limitations Within the Real-Time Digital Environment**

These technically diverse applications share another mutual thread; each of these continuous systems are being simulated in real time by a digital computer. By virtue of this common requirement, these simulations have a variety of imposed limitations in common.

Digital simulations which operate in real time must obey two fundamental rules. First, the simulation must not rely upon an external input until it is actually available from the appropriate transducer. Second, the simulation must execute a complete step within the frame time allotted.
To illustrate these limitations, consider the simple simulation system shown in Figure 1. The operator controls a single input to the simulator,

\[ u_k = u(kT) \]

and receives a single output from the simulator

\[ y_k = y(kT) \]

where \( T \) is the frame time of the simulator. If the simulator is characterized by the state equation

\[ \frac{d}{dt} y(t) = F(y(t), u(t)) \]

or

\[ y(t) = \int_0^t F(y(\tau), u(\tau)) d\tau \]

then it can be approximated in a discrete representation using explicit Euler integration, which results in the difference equation

\[ y_k = y_{k-1} + TF(y_{k-1}, u_{k-1}) \]

Note that the current state \( y_k \) relies only upon the previous state \( y_{k-1} \) and the previous input \( u_{k-1} \). If Equation (5) is able to execute in the digital hardware within the frame time \( T \)—including any overhead required by the simulation—then this system is able to operate in real time. However, if execution requires a time in excess of \( T \), then it cannot operate in real time; to do so would require the frame rate of the

![Figure 1. A simple simulation system](image)
simulation to be reduced sufficiently (without causing instability), a more powerful (and generally more expensive) computer to host the simulation, or a more efficient means of implementing the simulation on the existing host computer.

A Class of Real-Time Numerical Integrators

In order to address the limitations imposed by the real-time environment, a general form for a real-time numerical integrator will be derived. In addition to these constraints, a number of assumptions will also be applied to limit the scope to a reasonable level. The resulting general-form numerical integrator will then be compared to several reference integrators currently used in real-time applications.

Assumptions

For the purpose of limiting the scope of this research to methods which are applicable to the real-time environment, several assumptions for integrators to be synthesized have been made. These assumptions not only will ensure that the resulting numerical integration methods will be appropriate for real-time operation but provides a framework from which a customized integration algorithm may be generated from a set of specified performance parameters.

The first and foremost assumption is that the integrator is capable of operating in a real-time environment. As stated earlier, this implies that the current integrator output is dependent only upon a linear combination of previous integrator inputs and outputs. However, integrators in which the output is dependent upon the current input value will also be considered, as these implicit integrators are commonly used in combination with an explicit integrator in tandem configurations as a technique employed to reduce phase lag (Harbor 1988 and Howe 1990).

The integrator is assumed to have a fixed step size $T$. Algorithms which utilize variable-length integration steps will not be considered because of the increased
complexity that they impose upon the entire simulation. In addition, it is assumed that all integrators in a simulation will operate at the same fixed integration step; thus, multirate simulations will not be considered.

It is assumed that the integrator will make only one derivative evaluation per step. This is because computationally efficient integration methods are being sought, and the derivative evaluation is often the most computationally complex operation performed at each step in the numerical integration. Thus, multiple-pass algorithms such as Runge-Kutta integrators will not be considered.

Finally, only algorithms which implement a single integration will be considered. This limitation ensures that individual numerical integrators are synthesized based upon their specific input bandwidth and accuracy criteria.

The General Form of a Real-Time Numerical Integrator

The general form of an nth-order numerical integrator\(^1\) which conforms to the aforementioned assumptions may be represented by the difference equation

\[
y_k = \sum_{i=1}^{n} a_i y_{k-i} + \sum_{i=0}^{n} b_i T u_{k-i} , \tag{6}
\]

where \(y_k\) and \(u_k\) are the values at the output and input of the integrator at time \(kT\), respectively. The z-domain transfer function representation of this integrator is

\[
W(z) = \frac{Y(z)}{U(z)} = \frac{T \sum_{i=0}^{n} b_i z^{n-i}}{z^n - \sum_{i=1}^{n} a_i z^{n-i}} . \tag{7}
\]

---

1. The order of a numerical integrator, as used within this paper, is not dependent upon the order of the difference equation used to describe it. Instead, it is a measure of the highest order of accuracy possible using that form of the difference equation. Thus, a first-order difference equation with two \(b\) terms can be defined to have global truncation errors at best proportional to \(T^n\) (Benyon, 1968); hence, this form of the difference equation is a more general form for a second-order integrator.
Equations (6) and (7) represent the implicit form of the numerical integrator; the explicit form is the same except that

\[ b_0 = 0 \quad (8) \]

This is based upon the assumption that \( y_k \) is directly dependent upon the value of \( u_k \), which if expanded would result in \( y_k \) being present on both sides of the equation. It is this circumstance which is identified as an implicit equation; thus, an integrator which causes this condition (also known as an algebraic loop) is called an implicit integrator.

This general form of numerical integrator is the same one used by Euler integrators (both implicit and explicit), trapezoidal integrators, and Adams-Bashforth integrators. These integration methods meet the requirements previously stated for use in real-time simulation environments. The specific forms of these integration methods, which will be used as references for comparison, are identified below.

The explicit Euler integrator, which is also known as rectangular or forward rectangular integration, is defined by the difference equation

\[ y_k = y_{k-1} + Tu_{k-1} \quad (9) \]

and has a z-domain transfer function of

\[ W(z) = \frac{Y(z)}{U(z)} = \frac{T}{z-1} \quad (10) \]

Implicit Euler integration, also known as backward rectangular integration, is defined by the difference equation

\[ y_k = y_{k-1} + Tu_k \quad (11) \]

and has a z-domain transfer function of

\[ W(z) = \frac{Y(z)}{U(z)} = \frac{Tz}{z-1} \quad (12) \]
Second-order Adams-Bashforth integration is defined by the difference equation

\[ y_k = y_{k-1} + \frac{T}{2} (3u_{k-1} - u_{k-2}) \]  

and has a z-domain transfer function of

\[ W(z) = \frac{Y(z)}{U(z)} = \frac{T}{2} \frac{3z - 1}{z^2 - 1}. \]  

Trapezoidal integration, which is the implicit equivalent of second-order Adams-Bashforth integration\(^1\), is defined by the difference equation

\[ y_k = y_{k-1} + \frac{T}{2} (u_k + u_{k-1}) \]  

and has a z-domain transfer function of

\[ W(z) = \frac{Y(z)}{U(z)} = \frac{Tz + 1}{2z - 1}. \]  

The four reference integration methods are summarized in Table 1 below. This illustrates the coefficients from the general form of the real-time integrator described by Equation (6) and Equation (7). The following analyses will be limited to numerical integrators of orders one and two in order to limit the scope of this research to a

<table>
<thead>
<tr>
<th>Integrator</th>
<th>Order</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(b_0)</th>
<th>(b_1)</th>
<th>(b_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit Euler</td>
<td>1</td>
<td>1</td>
<td></td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Implicit Euler</td>
<td>1</td>
<td>1</td>
<td></td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Second-order Adams Bashforth</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>Trapezoidal</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
</tr>
</tbody>
</table>

\(^1\) Using the graphical interpretation of the Adams-Bashforth and trapezoidal integrators.
manageable level. However, the techniques developed here can be used with integrators of any order.
CHAPTER 2
SPECTRAL ANALYSIS OF INTEGRATOR INPUTS

A fundamental premise of this paper is that the input bandwidth of integrators in a simulation model is significantly limited under nominal operating conditions. In order to verify this postulation, a simple, linear continuous system will be analyzed and the frequency spectra of the integrator inputs examined. Once this is done, a method to quantify this limited bandwidth will be developed.

An Example Continuous System

Harbor (1988) describes a simplified single-plane model for a Sidewinder missile at a fixed flight condition. The Sidewinder missile has a stable airframe transfer function and no autopilot. The missile is modeled as an open loop system consisting of a control actuation system (CAS), representing the single-plane fin actuator, which receives an acceleration command and generates an acceleration actuation to the airframe model, which generates the resulting acceleration of the missile.

A block diagram representation of the system is illustrated in Figure 2. Note that both the CAS and the airframe are modeled as second-order linear subsystems.

![Figure 2. Block diagram of a Sidewinder missile model](image-url)
The acceleration command which drives the CAS is of the form

\[ a_{cmd} = A_{cmd} \left( 1 - e^{-t/\tau_{cmd}} \right), \]

where \( A_{cmd} \) is the amplitude of the acceleration command, and \( \tau_{cmd} \) is the time constant of the analog-to-digital converter from which the signal is generated. The values for the parameters of the Sidewinder missile model are listed in Table 2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta_{cas} )</td>
<td>Damping ratio of the control actuation system.</td>
<td>0.6</td>
</tr>
<tr>
<td>( \omega_{cas} )</td>
<td>Natural frequency of the control actuation system.</td>
<td>220 rad/s</td>
</tr>
<tr>
<td>( \zeta_{air} )</td>
<td>Damping ratio of the airframe.</td>
<td>0.05</td>
</tr>
<tr>
<td>( \omega_{air} )</td>
<td>Natural frequency of the airframe.</td>
<td>10 rad/s</td>
</tr>
<tr>
<td>( \tau_{cmd} )</td>
<td>Time constant of the acceleration command.</td>
<td>0.001 s</td>
</tr>
</tbody>
</table>

A baseline of the Sidewinder missile model will be taken using a unity input for \( A_{cmd} \). Since the system and its input are linear, closed-form analytical representations can be found for the state derivatives which represent inputs to each of the integrators. Using the parameter values listed in Table 2, the resulting equation for the command signal is

\[ a_{cmd}(t) = 1 - e^{-1000t}, \]

and the equations for the time response of the state derivatives and state variables are

\[ \dot{a}_{cas}(t) = -61793.2 e^{-1000t} + 61703.2 e^{-132t} \cos 176t - 29309 e^{-132t} \sin 176t, \]

\[ \dot{a}_{cas}(t) = 61.7032 e^{-1000t} - 61.7032 e^{-132t} \cos 176t + 304.309 e^{-132t} \sin 176t, \]

\[ a_{cas}(t) = 1 - 0.0617032 e^{-1000t} - 0.938297 e^{-132t} \cos 176t - 1.05431 e^{-132t} \sin 176t. \]
\[ \ddot{a}_{\text{air}}(t) = -6.17588e^{-1000t} + 100.492e^{-t/2}\cos9.98749t + 1.45211e^{-t/2}\sin9.98749t - 94.3161e^{-132t}\cos176t - 105.624e^{132t}\sin176t , \]  
(22)

\[ \dot{a}_{\text{air}}(t) = 0.00617588e^{-1000t} - 0.647489e^{-t/2}\cos9.98749t + 10.0294e^{-t/2}\sin9.98749t + 0.641314e^{-132t}\cos176t - 0.0549015e^{-132t}\sin176t , \]  
(23)

and

\[ a_{\text{air}}(t) = 1 - 6.17588 \times 10^{-6}e^{-1000t} - 0.998444e^{-t/2}\cos9.98749t - 0.114815e^{-t/2}\sin9.98749t - 0.00154939e^{-132t}\cos176t + 0.00248178e^{-132t}\sin176t . \]  
(24)

**Frequency Content of Model Variables**

Figures 3 through 9 show the time response and frequency content of each of the model variables described above. Note that the time scales for the time response plots of the different variables differ, allowing the significant characteristics to be seen: the variable \( a_{\text{cmd}} \) is plotted up to 0.01 seconds; the variables \( a_{\text{cas}}, \dot{a}_{\text{cas}}, \) and \( \ddot{a}_{\text{cas}} \) are plotted up to 0.1 seconds; and the variables \( a_{\text{air}}, \dot{a}_{\text{air}}, \) and \( \ddot{a}_{\text{air}} \) are plotted up to 20 seconds.

![Figure 3. Time response and frequency content of \( a_{\text{cmd}} \)](image-url)
Figure 4. Time response and frequency content of $\ddot{a}_{\text{cas}}$

Figure 5. Time response and frequency content of $\dot{a}_{\text{cas}}$

Figure 6. Time response and frequency content of $a_{\text{cas}}$
Figure 7. Time response and frequency content of $\ddot{a}_{\text{air}}$

Figure 8. Time response and frequency content of $\dot{a}_{\text{air}}$

Figure 9. Time response and frequency content of $a_{\text{air}}$
The magnitude frequency response plots were generated from a 1000-point discrete Fourier transform of each of the time responses over the time period stated above. The number of points and the time periods were chosen so as to minimize aliasing and provide sufficient frequency resolution to characterize the variable in the frequency domain. For the purpose of clarity, the DC components and the negative frequencies (which, because the time-domain signals are real-valued, are mirror images of the positive frequencies) have been omitted.

**Determining Cutoff Frequencies of Model Variables**

As can be seen in the magnitude frequency response plots from the previous section, the majority of the information content of the response is contained in the lower frequencies. It is desirable to determine a cutoff frequency $\omega_C$ for each of these responses such that a specified cutoff percentage $P_C$ of the overall signal content (not including the DC component) lies in the range $0 < \omega < \omega_C$, or

$$P_C \int_0^\infty |F(\omega)| \, d\omega = \int_0^{\omega_C} |F(\omega)| \, d\omega,$$

where $F(\omega)$ is the Fourier domain frequency response of the signal under examination.

Since the frequency responses are determined by the discrete Fourier transform, the cutoff frequency can only be approximated by substituting a summation of discrete magnitudes in place of the integrals of Equation (25). Additionally, the unbounded upper limit in the left-hand integral must be replaced by the maximum frequency computed by the discrete Fourier transform. Table 3 shows the approximated cutoff frequencies for each of the seven state variables in the missile model using a cutoff percentage of 75% ($P_C = 0.75$).

In general, there are several guidelines for determining the approximate cutoff frequencies of a model's variables. First, each state derivative should be analyzed...
independently of all other state derivatives in the system. The purpose is to collect information about the input of a specific integrator in order to exploit the specific qualities of the signals that it nominally processes. Second, a cutoff percentage should be determined for each of the state derivatives in the system. Next, a representative set of responses must be generated in order to determine a realistic characterization of the state derivative’s frequency content under most, if not all, operating conditions. It is of little use to determine a cutoff frequency for a state derivative which exhibits frequency response characteristics greatly different from those used to determine that cutoff frequency. At the same time, care must be taken to ensure that the inputs used to generate responses are realistic; for example, an impulse or step input may not be realistic when simulating human inputs. Finally, the cutoff frequency of the state derivative is selected as the maximum value of all of the test responses used to characterize it. These guidelines therefore provide a framework to concisely characterize the frequency response of each of the state derivatives within their normal operating envelope.

Table 3. Cutoff frequencies for Sidewinder missile model variables

<table>
<thead>
<tr>
<th>Model Variable</th>
<th>Cutoff Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{cmd}$</td>
<td>80,344.3 rad/s</td>
</tr>
<tr>
<td>$\ddot{a}_{cas}$</td>
<td>1,380.9 rad/s</td>
</tr>
<tr>
<td>$\ddot{a}_{cas}$</td>
<td>376.6 rad/s</td>
</tr>
<tr>
<td>$a_{cas}$</td>
<td>7,092.9 rad/s</td>
</tr>
<tr>
<td>$\ddot{a}_{air}$</td>
<td>49.9 rad/s</td>
</tr>
<tr>
<td>$\ddot{a}_{air}$</td>
<td>12.3 rad/s</td>
</tr>
<tr>
<td>$a_{air}$</td>
<td>43.3 rad/s</td>
</tr>
</tbody>
</table>
CHAPTER 3
PARAMETRIC SPECIFICATION OF A CLASS OF NUMERICAL INTEGRATORS

Having determined practical bounds for state derivative bandwidths, it is necessary to develop a means of evaluating and specifying the accuracy of numerical integrators under these limited bandwidth conditions. It is clear that metrics such as characteristic root errors (Howe 1986, Panzitta 1993) are inappropriate for measuring the accuracy of a numerical integrator within a limited frequency band; however, the transfer function gain and phase errors described by Howe (1986) provide a basis for metrics which are able to handle limited bandwidth analysis.

Defining Frequency-Domain Error Criteria

Gain and phase response are two characteristics which will be used to analyze the class of real-time integrators described by Equation (6). These characteristics will be measured relative to those of an ideal integrator

\[ H_1(s) = \frac{1}{s} \]  \hspace{1cm} (26)

or

\[ W_1(z) = \frac{T}{\ln z} \]  \hspace{1cm} (27)

Gain Error Metrics

The gain error of a particular numerical integrator having a transfer function \( W(z) \) is defined using the natural logarithm of the ratio of integrator gains. The analytical definition is
where \( \omega \) is the frequency at which the gain error is measured. The use of a logarithmic scale permits a balanced scale of error centered at zero. A gain error of zero indicates that the gain of the integrator in question is exactly matched with that of the ideal integrator at the specified frequency. A gain error which is greater than or less than zero indicates that the gain of the integrator in question is greater than or less than that of the ideal integrator at the specified frequency.

Three specific methods will be used to quantify the gain error over the frequency band of interest \( 0 \leq \omega \leq \omega_C \), where \( \omega_C \) is the cutoff frequency described in Chapter 2. These metrics are maximum gain error, integral absolute gain error, and integral squared gain error. The maximum absolute gain error is defined as

\[
\varepsilon_{\text{gain, max}} = \max_{0 \leq \omega T \leq \omega_C T} |\varepsilon_{\text{gain}}(\omega T)| ,
\]

the integral absolute gain error is defined as

\[
\varepsilon_{\text{gain, IA}} = \int_0^{\omega_C T} |\varepsilon_{\text{gain}}(\omega T)| d\omega T ,
\]

and the integral squared gain error is defined as

\[
\varepsilon_{\text{gain, IS}} = \int_0^{\omega_C T} \left[ |\varepsilon_{\text{gain}}(\omega T)|^2 \right] d\omega T .
\]

Similar methods as these are generally used in parameter estimation problems (Klee 1991a) and in the design of optimal digital filters (Franklin and Powell 1980).
Phase Error Metrics

The phase error of a particular numerical integrator having a transfer function $W(z)$ is defined as the difference between the phase angle of that integrator and the phase angle of the ideal integrator. This is expressed as

$$
\varepsilon_{\text{phase}}(\omega T) = \arg W(e^{j\omega T}) - \arg W_1(e^{j\omega T}) = \arg W(e^{j\omega T}) + \frac{\pi}{2}.
$$

(32)

Thus, a phase error of zero indicates that the phase of the integrator in question is exactly matched with that of the ideal integrator at the specified frequency. A phase error which is greater than or less than zero indicates that the integrator in question leads or lags the ideal integrator at the specified frequency.

The three methods to be used to quantify the phase error over the frequency band of interest correspond to the three methods used to specify gain error over the same band. The maximum absolute phase error is defined as

$$
\varepsilon_{\text{phase, max}} = \max_{0 \leq \omega T \leq \omega_c T} |\varepsilon_{\text{phase}}(\omega T)|,
$$

(33)

the integral absolute phase error is defined as

$$
\varepsilon_{\text{phase, IA}} = \int_{0}^{\omega_c T} |\varepsilon_{\text{phase}}(\omega T)| d\omega T,
$$

(34)

and the integral squared phase error is defined as

$$
\varepsilon_{\text{phase, IS}} = \int_{0}^{\omega_c T} [\varepsilon_{\text{phase}}(\omega T)]^2 d\omega T.
$$

(35)

Defining Specification Parameters

Several parameters may be used to specify the performance of a numerical integrator of the form given in Equation (6). These parameters are summarized below in Table 4.
Table 4. Numerical integrator performance parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{\omega_c T}$</td>
<td>This parameter combines the cutoff frequency with the integration step size.</td>
</tr>
<tr>
<td>$\epsilon_{\text{gain}}$</td>
<td>Zero or more of the frequency band gain error parameters ($\epsilon_{\text{gain, max}}, \epsilon_{\text{gain, IA}}, \text{or } \epsilon_{\text{gain, IS}}$).</td>
</tr>
<tr>
<td>$\epsilon_{\text{phase}}$</td>
<td>Zero or more of the frequency band phase error parameters ($\epsilon_{\text{phase, max}}, \epsilon_{\text{phase, IA}}, \text{or } \epsilon_{\text{phase, IS}}$).</td>
</tr>
<tr>
<td>$n$</td>
<td>The order of the numerical integrator.</td>
</tr>
</tbody>
</table>

It is desirable that all of these parameters be minimized; however, this is not easily accomplished without using multiple optimization techniques. Thus, in general, all but one of these parameters must have a specified maximum (or be left unspecified) while the last parameter is minimized.

The gain and phase error metrics described above are analyzed in further detail in Appendix A through Appendix D. These provide specific derivations for first and second order integrators, both explicit and implicit. Also included are gain and phase error metrics for each of the four reference integrators. These results are in agreement with error measures identified by Benyon (1968) and Howe (1986), as well as reinforcing intuitive and observed properties of these integrators.

Comparing Synthesized and Reference Numerical Integrators

The Sidewinder missile model introduced in Chapter 2 will be used to illustrate how these error metrics can be used to evaluate simulations designed using traditional integrators as well as to design a simulation which uses synthesized integrators. The requirements of this simulation are

$$\epsilon_{\text{gain, max}} \leq 0.1,$$ \hspace{1cm} (36)

or a maximum gain deviation of within approximately ±10%,

$$\epsilon_{\text{phase, IS}} \leq 0.0081 \text{ rad}^2.$$ \hspace{1cm} (37)
or an integral squared phase error of within approximately $0.464\text{deg}^2$, an integrator order $n$ of one or two, and to maximize the integration step $T$.

These specifications will be first applied to a simulation which uses explicit Euler integrators throughout the simulation and then to a simulation which uses second-order Adams-Bashforth integrators throughout the simulation. Finally, these specifications will be applied to simulations which use first- and second-order synthesized integrators to best match the specification requirements.

The time response to the command input from Chapter 2 will be generated for the defined simulation. The time responses of each state derivative and state variable will then be compared to the analytical solution.

One assumption made in Chapter 1 to facilitate the generation of synthesized integrators was to disallow multirate simulations. However, it can be seen from the results of the remainder of this chapter that some of the techniques presented here may be applied to multirate simulation. Appendix E revisits the simulations performed in this chapter using multirate simulations to further improve their performance.

Explicit Euler Integration

The equation for gain error for the explicit Euler integrator is

$$
\epsilon_{\text{gain}}(\omega T) = \ln\left(\frac{\omega T}{\sqrt{2 - 2\cos\omega T}}\right)
$$

as derived in Appendix A. This equation will be used to determine the largest value of $T$ that satisfies the maximum absolute gain error requirement for each of the four integrators in the system. A similar computation is performed using the phase error,

$$
\epsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\sin\omega T}{\cos\omega T - 1}\right),
$$
also derived in Appendix A, in satisfying the integral squared phase error requirement. The results are summarized below in Table 5.

**Table 5. Explicit Euler integrator step sizes for Sidewinder missile simulation**

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_C$</th>
<th>$T$ required by:</th>
<th>$\varepsilon_{gain,\text{max}}$</th>
<th>$\varepsilon_{\text{phase,IS}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{a}_{\text{cas}}$</td>
<td>1380.9</td>
<td></td>
<td>0.0011106</td>
<td>0.0003330</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{cas}}$</td>
<td>376.6</td>
<td></td>
<td>0.0040724</td>
<td>0.0012209</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{air}}$</td>
<td>49.9</td>
<td></td>
<td>0.0307345</td>
<td>0.0092141</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{air}}$</td>
<td>12.3</td>
<td></td>
<td>0.1246870</td>
<td>0.0373809</td>
</tr>
</tbody>
</table>

Note that the integration step sizes are linear with respect to the cutoff frequency $\omega_C$. It can be seen that the maximum absolute gain error of 0.1 requires that all four of the integrators conform to

$$\omega_C \cdot T = 1.53365 \quad (40)$$

Similarly, the integral squared phase error of 0.0081 requires that all four of the integrators conform to

$$\omega_C \cdot T = 0.459788 \quad (41)$$

It is easily seen that the integral squared phase error requirement is driving the step size of the explicit Euler integrator. Therefore, the fixed integration step size for the simulation using explicit Euler integrators is

$$T = 0.0003330 \quad (42)$$

Based upon using this value as the step size for all four integrators in the simulation, Table 6 lists the actual maximum absolute gain and integral squared phase errors.

Figures 10 through 15 show the time response of the state derivatives and state variables identified in Chapter 2. The analytical solutions are displayed as a solid trace, and the simulated response using explicit Euler integration is depicted by a
Figure 10. Time response of $\dot{a}_{\text{cas}}$ simulated with explicit Euler integrators

Figure 11. Time response of $\dot{a}_{\text{cas}}$ simulated with explicit Euler integrators

Figure 12. Time response of $a_{\text{cas}}$ simulated with explicit Euler integrators
Figure 13. Time response of $\ddot{a}_{\text{air}}$ simulated with explicit Euler integrators

Figure 14. Time response of $\ddot{a}_{\text{air}}$ simulated with explicit Euler integrators

Figure 15. Time response of $a_{\text{air}}$ simulated with explicit Euler integrators
Table 6. Error metrics for Sidewinder missile simulation using explicit Euler integration

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\varepsilon_{\text{gain,max}}$</th>
<th>$\varepsilon_{\text{phase,JS}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{a}_{\text{cas}}$</td>
<td>0.0088261</td>
<td>0.008102862</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{air}}$</td>
<td>0.0006554</td>
<td>0.000164358</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>0.0000115</td>
<td>0.000000382</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>0.0000007</td>
<td>0.000000006</td>
</tr>
</tbody>
</table>

dashed trace. Note that the simulated time response is very close to the analytical response; however, this simulation requires approximately 12,012 integrations per second to achieve this degree of accuracy.

First-Order Synthesized Explicit Integration

The general first-order explicit integrator is of the form

$$y_k = a_1 y_{k-1} + b_1 Tu_{k-1}.$$  \hfill (43)

It has a gain error of

$$\varepsilon_{\text{gain}}(\omega T) = \ln \frac{|b_1| \omega T}{\sqrt{1 - 2a_1 \cos \omega T + a_1^2}}$$ \hfill (44)

and a phase error of

$$\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} - \tan^{-1} \left( \frac{\sin \omega T}{\cos \omega T - a_1} \right).$$ \hfill (45)

Since the phase error is dependent only upon $a_1$, this constant will be determined first.

Figure 16 illustrates the maximum value of $\omega_c T$ for values of $a_1$ in the neighborhood of $a_1 = 1$ which satisfy the integral squared phase error requirement. Note the singularity that occurs at $a_1 = 1$, which results in $\omega_c T = 0.459788$, the same value obtained with the explicit Euler integrator in Equation (41). The singularity effectively requires this parameter selection for any integrator having a finite phase error requirement. The fixed integration step size is therefore
Figure 16. Parameter selection based upon phase error requirement

\[ T = 0.0003330 \, . \] (46)

The equation for gain error now becomes

\[ \varepsilon_{\text{gain}}(\omega T) = \ln \frac{|b_1| \omega T}{\sqrt{2 - 2 \cos \omega T}} \, . \] (47)

While a choice of \( b_1 = 1 \) would result in maximum absolute gain error less than the required value of 0.1 (since the integrator is now an explicit Euler integrator), we will instead seek to minimize the maximum absolute gain error for the range of \( \omega T \) to be used in the simulation.

The gain error in Equation (47) increases monotonically over the range \( 0 < \omega T \leq 0.459788 \) and has a limit of \( \ln |b_1| \) at \( \omega T = 0 \). Thus, to minimize the maximum absolute gain error for the \( \dot{a}_{\text{cas}} \) integrator, the value of \( \varepsilon_{\text{gain}} \) should have equal value and opposite sign at the limit \( \omega T = 0 \) and \( \omega T = 0.459788 \), which may be represented as

\[ -\ln |b_1| = \ln \frac{|b_1| \omega T}{\sqrt{2 - 2 \cos \omega T}} \bigg|_{\omega T = 0.459788} \, . \] (48)

When solved, the result is \( b_1 = 0.995598 \). The resulting integration step sizes are summarized in Table 7 below.
Table 7. First-order explicit synthesized integrator step sizes for Sidewinder missile simulation

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_c$</th>
<th>$T$ required by:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{a}_{\text{cas}}$</td>
<td>1380.9</td>
<td>$\varepsilon_{\text{gain,max}}$ 0.0011343, $\varepsilon_{\text{phase,IS}}$ 0.0003330</td>
</tr>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>376.6</td>
<td>$\varepsilon_{\text{gain,max}}$ 0.0041593, $\varepsilon_{\text{phase,IS}}$ 0.0012209</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{air}}$</td>
<td>49.9</td>
<td>$\varepsilon_{\text{gain,max}}$ 0.0313910, $\varepsilon_{\text{phase,IS}}$ 0.0092141</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>12.3</td>
<td>$\varepsilon_{\text{gain,max}}$ 0.1273504, $\varepsilon_{\text{phase,IS}}$ 0.0373809</td>
</tr>
</tbody>
</table>

Again, note that the integration step sizes are linear with respect to the cutoff frequency $\omega_c$. It can be seen that the maximum absolute gain error of 0.1 requires that all four of the integrators conform to

$$\omega_c T = 1.566355 \quad (49)$$

which is less than that required by the explicit Euler integrator. The integral squared phase error requirement results in the same value for $\omega_c T$ as the explicit Euler integrator and, again, is the driving requirement for the integration step size. Thus, the fixed integration step size for the simulation using first-order synthesized explicit integrators is

$$T = 0.0003330 \quad (50)$$

The result is an improved maximum gain error for all of the integrators compared to the explicit Euler integrator while maintaining the same integral squared phase errors. These values are listed in Table 8.

Table 8. Error metrics for Sidewinder missile simulation using first-order explicit synthesized integration

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\varepsilon_{\text{gain,max}}$</th>
<th>$\varepsilon_{\text{phase,IS}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{a}_{\text{cas}}$</td>
<td>0.0044144</td>
<td>0.008102862</td>
</tr>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>0.0037563</td>
<td>0.000164358</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{air}}$</td>
<td>0.0043996</td>
<td>0.000000382</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>0.0043999</td>
<td>0.000000006</td>
</tr>
</tbody>
</table>
Figures 17 through 22 show the time response of the state derivatives and state variables identified in Chapter 2. The analytical solutions are displayed as a solid trace, and the simulated response using first-order explicit synthesized integration is depicted by a dashed trace. Because the fixed integration step size is the same as that of the Euler integration, this simulation also requires approximately 12,012 integrations per second. These time responses are not noticeably different than that of the simulation using explicit Euler integration.

Figure 17. Time response of $\dot{a}_{\text{cas}}$ simulated with first-order explicit synthesized integrators

Figure 18. Time response of $\dot{a}_{\text{cas}}$ simulated with first-order explicit synthesized integrators
Figure 19. Time response of $a_{\text{cas}}$ simulated with first-order explicit synthesized integrators

Figure 20. Time response of $\ddot{a}_{\text{air}}$ simulated with first-order explicit synthesized integrators

Figure 21. Time response of $\ddot{a}_{\text{air}}$ simulated with first-order explicit synthesized integrators
Figure 22. Time response of $a_{air}$ simulated with first-order explicit synthesized integrators

Second-Order Adams-Bashforth Integration

The equation for gain error for the second-order Adams-Bashforth integrator is

$$
\varepsilon_{gain}(\omega T) = \ln\left(\frac{\omega T}{2} \sqrt{\frac{5 - 3 \cos \omega T}{1 - \cos \omega T}}\right)
$$

as derived in Appendix C. It is used to determine the largest value of $T$ that satisfies the maximum absolute gain error requirement of for each of the four integrators in the system. A similar computation is performed using the phase error

$$
\varepsilon_{phase}(\omega T) = \frac{\pi}{2} + \tan^{-1}\left(\frac{4 \sin \omega T - \sin 2 \omega T}{3 - 4 \cos \omega T + \cos 2 \omega T}\right)
$$

also derived in Appendix C, in satisfying the integral squared phase error requirement. The results are summarized in Table 9 below.

Table 9. Second-order Adams-Bashforth integrator step sizes for Sidewinder missile simulation

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_c$</th>
<th>$T$ required by:</th>
<th>$\varepsilon_{gain,max}$</th>
<th>$\varepsilon_{phase,JS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{a}_{cas}$</td>
<td>1380.9</td>
<td>0.0003730</td>
<td>0.0007898</td>
<td></td>
</tr>
<tr>
<td>$\dot{a}_{cas}$</td>
<td>376.6</td>
<td>0.0013678</td>
<td>0.0028962</td>
<td></td>
</tr>
<tr>
<td>$\dot{a}_{air}$</td>
<td>49.9</td>
<td>0.0103226</td>
<td>0.0218577</td>
<td></td>
</tr>
<tr>
<td>$\dot{a}_{air}$</td>
<td>12.3</td>
<td>0.0418780</td>
<td>0.0886743</td>
<td></td>
</tr>
</tbody>
</table>
Note that the integration step sizes are linear with respect to the cutoff frequency $\omega_C$. It can be seen that the maximum absolute gain error of 0.1 requires that all four of the integrators conform to

$$\omega_C T = 0.5151 .$$

(53)

Similarly, the integral squared phase error of 0.0081 requires that all four of the integrators conform to

$$\omega_C T = 1.0907 .$$

(54)

In contrast to the explicit Euler integrator, the absolute maximum gain error requirement drives the step size for the second-order Adams-Bashforth integrator. Thus, the fixed integration step size for the simulation using second-order Adams-Bashforth integrators is

$$T = 0.0003730 .$$

(55)

Note that this value is not appreciably better than when using the explicit Euler integrator. Based upon using this value as the step size for all four integrators in the simulation, Table 10 lists the actual maximum absolute gain and integral squared phase errors.

**Table 10. Error metrics for Sidewinder missile simulation using second-order Adams-Bashforth integration**

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\epsilon_{\text{gain, max}}$</th>
<th>$\epsilon_{\text{phase, IS}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dddot{a}_{\text{cas}}$</td>
<td>0.0999916</td>
<td>0.000070900</td>
</tr>
<tr>
<td>$\dddot{a}_{\text{cas}}$</td>
<td>0.0081557</td>
<td>0.000000009</td>
</tr>
<tr>
<td>$\dddot{a}_{\text{air}}$</td>
<td>0.0001443</td>
<td>$6.907 \times 10^{-15}$</td>
</tr>
<tr>
<td>$\dddot{a}_{\text{air}}$</td>
<td>0.0000088</td>
<td>$3.820 \times 10^{-19}$</td>
</tr>
</tbody>
</table>

Figures 23 through 28 show the time response of the state derivatives and state variables identified in Chapter 2. The analytical solutions are displayed as a solid
Figure 23. Time response of $\ddot{a}_{\text{cas}}$ simulated with second-order Adams-Bashforth integrators

Figure 24. Time response of $\dot{a}_{\text{cas}}$ simulated with second-order Adams-Bashforth integrators

Figure 25. Time response of $a_{\text{cas}}$ simulated with second-order Adams-Bashforth integrators
Figure 26. Time response of $\dot{a}_{\text{air}}$ simulated with second-order Adams-Bashforth integrators

Figure 27. Time response of $\ddot{a}_{\text{air}}$ simulated with second-order Adams-Bashforth integrators

Figure 28. Time response of $a_{\text{air}}$ simulated with second-order Adams-Bashforth integrators
trace, and the simulated response using second-order Adams-Bashforth integration is depicted by a dashed trace. Even though the second-order Adams-Bashforth integrator is visibly more accurate than the explicit Euler integrator, this simulation still requires approximately 10,724 integrations per second, which is not significantly less than the simulations using the explicit Euler or the first-order synthesized integrators.

Second-Order Synthesized Explicit Integration

The general second-order explicit integrator is of the form

\[ y_k = a_1 y_{k-1} + a_2 y_{k-2} + b_1 u_{k-1} + b_2 u_{k-2} \]  \hspace{1cm} (56)

It has a gain error of

\[ \varepsilon_{\text{gain}}(\omega T) = \ln(\omega T) \left( \frac{b_1^2 + 2b_1 b_2 \cos \omega T + b_2^2}{1 + a_1^2 + a_2^2 + 2a_1 (a_2 - 1) \cos \omega T - 2a_2 \cos 2\omega T} \right) \]  \hspace{1cm} (57)

and a phase error of

\[ \varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} - \tan^{-1}\left(\frac{(a_1 b_2 - a_2 b_1 - b_1) \sin \omega T - b_2 \sin 2\omega T}{a_1 b_1 + a_2 b_2 + (a_1 b_2 + a_2 b_1 - b_1) \cos \omega T - b_2 \cos 2\omega T}\right) \]  \hspace{1cm} (58)

It is noted that the maximum absolute gain error must be equal to or greater than the limit of the absolute gain error at \( \omega T = 0 \), since the maximum applies over the range \( 0 < \omega T \leq \omega_c T \). It can be shown that

\[ \lim_{\omega T \to 0} \varepsilon_{\text{gain}}(\omega T) = \ln \left| \frac{b_1 + b_2}{a_2 + 1} \right| \]  \hspace{1cm} (59)

provided that

\[ a_1 = 1 - a_2 \]  \hspace{1cm} (60)

Using the parameter values for the second-order Adams-Bashforth integrator as a starting point, the parameters are adjusted to improve (i.e. increase) the value of \( \omega_c T \) which satisfies the design requirements. Figure 29 illustrates how the maximum value
Figure 29. Variance in maximum integration step size in the neighborhood of the second-order Adams-Bashforth parameter values

of $\omega T$ increases for both the gain and phase requirements as the parameter $a_1$ is decreased. Note the abrupt drop in the gain graph at the value $a_1 = 0.89484$; this is the point at which $\epsilon_{\text{gain}}(0) = 0.1$, which is the required value. If $a_1$ is decreased further, then the gain error at $\omega T = 0$ will exceed the required value, and the maximum absolute gain error can never be met. In order to preserve this condition, Equation (59) implies that

$$b_1 + b_2 = 1^{1}$$

Next, the parameters $b_1$ and $b_2$ are adjusted in order to further maximize $\omega_c T$. This is performed using Mathematica (Wolfram 1991) to find the values of $\omega_c T$ for the gain and phase requirements using symbolic and numerical integrations as well as root-finding techniques embedded in the program. Because of the extremely complex, nonlinear, and discontinuous nature of these equations, the variation of the parameters $b_1$ and $b_2$ must be performed manually using Mathematica to determine objective function values along the axis described by Equation (61). The value of $\omega_c T$ is found to be maximized when the remaining parameters are $b_1 = 1.28086$ and $b_2 = -0.28086$. Table 11 summarizes the progression of parameter values and the

1. This condition is not strictly required; however, it simplifies the search for a local maximum without significantly affecting the search process.
Table 11. Progressive selection of parameter values for second-order synthesized explicit integrator

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$\omega_c T (\varepsilon_{\text{gain}})$</th>
<th>$\omega_c T (\varepsilon_{\text{phase}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>1.5</td>
<td>-0.5</td>
<td>0.5151</td>
<td>1.0967</td>
</tr>
<tr>
<td>0.89483</td>
<td>0.10517</td>
<td>1.5</td>
<td>-0.5</td>
<td>0.716295</td>
<td>1.30858</td>
</tr>
<tr>
<td>0.89483</td>
<td>0.10517</td>
<td>1.28086</td>
<td>-0.28086</td>
<td>0.931969</td>
<td>0.931973</td>
</tr>
</tbody>
</table>

corresponding values of $\omega_c T$ implied by the gain and phase error constraints. The second-order synthesized integrator can now be expressed as

$$y_k = 0.89483y_{k-1} + 0.10517y_{k-2} + 1.28086Tu_{k-1} - 0.28086Tu_{k-2} .$$  (62)

Figures 30 and 31 illustrate the gain and phase errors for this integrator, respectively.

![Figure 30. Gain error for second-order synthesized explicit integrator](image)

![Figure 31. Phase error for second-order synthesized explicit integrator](image)
The required step sizes for each of the four integrators in the Sidewinder missile simulation are summarized in Table 12. It can be seen that the fixed integration step size for the simulation using the second-order synthesized explicit integrator is

$$T = 0.0006749.$$  \hfill (63)

Based upon using this value as the step size for all four integrators in the simulation, Table 13 lists the actual maximum absolute gain and integral squared phase errors.

Table 12. Second-order synthesized explicit integrator step sizes for Sidewinder missile simulation

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_C$</th>
<th>$T$ required by:</th>
<th>$\varepsilon_{\text{gain, max}}$</th>
<th>$\varepsilon_{\text{phase, IS}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>1380.9</td>
<td></td>
<td>0.0006749</td>
<td>0.0006749</td>
</tr>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>376.6</td>
<td></td>
<td>0.0024747</td>
<td>0.0024747</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{air}}$</td>
<td>49.9</td>
<td></td>
<td>0.0186767</td>
<td>0.0186767</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>12.3</td>
<td></td>
<td>0.0757698</td>
<td>0.0757698</td>
</tr>
</tbody>
</table>

Table 13. Error metrics for Sidewinder missile simulation using explicit Euler integration

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\varepsilon_{\text{gain, max}}$</th>
<th>$\varepsilon_{\text{phase, IS}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>0.0999992</td>
<td>0.008099874</td>
</tr>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>0.0999992</td>
<td>0.000089709</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{air}}$</td>
<td>0.0999992</td>
<td>0.000000196</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>0.0999992</td>
<td>0.000000003</td>
</tr>
</tbody>
</table>

Figure 32 illustrates the integrator's stability region for $\lambda T$ using a standard first-order system having a root at $z = \lambda$.

Figures 33 through 38 show the time response of the state variables identified in Chapter 2. The analytical solutions are displayed as a solid trace, and the simulated response using second-order synthesized explicit integration is depicted by a dashed
Figure 32. Stability region for the second-order synthesized explicit integrator

Figure 33. Time response of $\ddot{a}_{\text{cas}}$ simulated with second-order synthesized explicit integrators

Figure 34. Time response of $\dot{a}_{\text{cas}}$ simulated with second-order synthesized explicit integrators
Figure 35. Time response of $a_{\text{cas}}$ simulated with second-order synthesized explicit integrators

Figure 36. Time response of $a_{\text{air}}$ simulated with second-order synthesized explicit integrators

Figure 37. Time response of $\ddot{a}_{\text{air}}$ simulated with second-order synthesized explicit integrators
Figure 38. Time response of $a_{air}$ simulated with second-order synthesized explicit integrators

trace. Note that the simulated time response is close to the analytical response but can readily be distinguished from the simulations using other integration methods. However, this simulation requires only approximately 5,927 integrations per second to achieve this degree of accuracy.
CHAPTER 4

IMPROVING THE PERFORMANCE OF AN AUTOMOBILE SIMULATOR

In order to validate the usefulness of the techniques presented heretofore, these methods will be used in a practical example. The case study to be used is the vehicle dynamics model employed by the University of Central Florida Driving Simulator. This simulator has been developed as a low-cost trainer with application in driver education, alcohol-related driving behavior studies, emergency vehicle operation, and driving skills assessment for the mentally and physically handicapped (Klee 1990).

The baseline vehicle dynamics model used by the driving simulator is first presented. Next, a pseudo-continuous implementation of the baseline model is exercised through a number of test cases in order to estimate the cutoff frequencies for each of the model's integrators. With these cutoff frequencies, the gain and phase error metrics can be computed for the baseline model implementation, which uses Euler integrators (explicit and implicit) operating at a 20 Hz frame rate. Synthesized integrators are then derived based upon the error metrics determined above in an effort to reduce the frame rate while maintaining acceptable gain and phase performance. Finally, time-domain responses of the implementation using the synthesized integrators are compared to those generated by the baseline implementation.

The Vehicle Dynamics Model

The simulator utilizes a simplified three degree of freedom vehicle dynamics model, derived from the model described by ENSCO (1988). Because routine driving
requirements consist primarily of maintaining longitudinal and lateral control of the
vehicle, a three degree-of-freedom model is suitable for many of the intended
applications of the simulator (Klee 1990). Figure 39 illustrates the inputs and outputs

![Diagram](image)

**Figure 39. Block diagram of UCF driving simulator vehicle dynamics model**

do the vehicle dynamics model. The model consists of five interdependent passive
subsystems and seven integrators as illustrated in Figure 40. The external inputs to the
system include the throttle (accelerator) deflection, the steering wheel deflection, the
brake pedal force, and the gear shift position; these are listed in Table 14. The model’s

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Range</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{\text{SW}}$</td>
<td>N/A(^a)</td>
<td>Steering wheel deflection</td>
</tr>
<tr>
<td>$\delta_{\text{throttle}}$</td>
<td>0 to 1</td>
<td>Normalized throttle (accelerator) deflection</td>
</tr>
<tr>
<td>$F_{\text{BP}}$</td>
<td>0 to 40 lb.</td>
<td>Brake pedal force</td>
</tr>
<tr>
<td>$P_{\text{gear shift}}$</td>
<td>[Park, Reverse, Neutral, 3, 2, 1]</td>
<td>Automatic transmission gearshift position</td>
</tr>
</tbody>
</table>

\(^a\). The steering wheel has no physical stops; instead it is monitored by a rotary encoder and
 driven by a torque motor to provide tactile feedback.

state variables, three of which were identified as external outputs of the model, are
listed in Table 15. Finally, the parameters of the vehicle dynamics model, along with
some physical constants used by the model, are listed in Table 16.
Because this is a three degree of freedom model, it is assumed that the terrain is level and homogeneous wherever the vehicle travels. This is reflected in the assumption that the position of the vehicle is defined by its Cartesian plane coordinate location \((x, y)\) and its heading \(\theta\); the elevation of the vehicle is assumed to be at mean
Table 15. Vehicle dynamics model state variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>Yaw rate</td>
</tr>
<tr>
<td>$R$</td>
<td>Automatic transmission gear ratio</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Heading</td>
</tr>
<tr>
<td>$u$</td>
<td>Longitudinal body velocity</td>
</tr>
<tr>
<td>$v$</td>
<td>Lateral body velocity</td>
</tr>
<tr>
<td>$x$</td>
<td>Inertial $x$ position</td>
</tr>
<tr>
<td>$y$</td>
<td>Inertial $y$ position</td>
</tr>
</tbody>
</table>

Table 16. Vehicle dynamics model parameters and constants

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>3.29ft</td>
<td>Distance from vehicle center of gravity to front axle</td>
</tr>
<tr>
<td>$A$</td>
<td>17.5ft$^2$</td>
<td>Vehicle reference area</td>
</tr>
<tr>
<td>$b$</td>
<td>4.58ft</td>
<td>Distance from vehicle center of gravity to rear axle</td>
</tr>
<tr>
<td>$B_{TC}$</td>
<td>0.0077ft-lb-s$^2$/rad$^2$</td>
<td>Torque converter input torque coefficient</td>
</tr>
<tr>
<td>$C_D$</td>
<td>0.4</td>
<td>Aerodynamic drag coefficient</td>
</tr>
<tr>
<td>$C_{N, \beta}$</td>
<td>$-0.676$rad$^{-1}$</td>
<td>Aerodynamic yawing moment coefficient</td>
</tr>
<tr>
<td>$C_{Y, \beta}$</td>
<td>$-2.68$rad$^{-1}$</td>
<td>Aerodynamic side force coefficient</td>
</tr>
<tr>
<td>$\Delta u_{1\rightarrow 2}$</td>
<td>41.1ft/s</td>
<td>Slope of speed at which to shift from first to second gear with respect to throttle position</td>
</tr>
<tr>
<td>$\Delta u_{2\rightarrow 1}$</td>
<td>41.1ft/s</td>
<td>Slope of speed at which to shift from second to first gear with respect to throttle position</td>
</tr>
<tr>
<td>$\Delta u_{2\rightarrow 3}$</td>
<td>41.1ft/s</td>
<td>Slope of speed at which to shift from second to third gear with respect to throttle position</td>
</tr>
<tr>
<td>$\Delta u_{3\rightarrow 2}$</td>
<td>41.1ft/s</td>
<td>Slope of speed at which to shift from third to second gear with respect to throttle position</td>
</tr>
<tr>
<td>$F_{BP, \text{max}}$</td>
<td>401b</td>
<td>Maximum brake pedal force</td>
</tr>
<tr>
<td>$F_{\text{brake, max}}$</td>
<td>1500.0lb</td>
<td>Maximum braking force</td>
</tr>
<tr>
<td>$F_{\text{front, max}}$</td>
<td>820.0lb</td>
<td>Maximum side force of front tires</td>
</tr>
<tr>
<td>$F_{\text{rear, max}}$</td>
<td>1180.0lb</td>
<td>Maximum side force of rear tires</td>
</tr>
<tr>
<td>$F_{\text{roll}}$</td>
<td>50.0lb</td>
<td>Rolling resistance</td>
</tr>
<tr>
<td>$g$</td>
<td>32.2ft/s$^2$</td>
<td>Gravitational constant</td>
</tr>
<tr>
<td>$I_{ZZ}$</td>
<td>1165.0slug-ft$^2$</td>
<td>Vehicle yaw moment of inertia</td>
</tr>
<tr>
<td>$K_{\text{loss}}$</td>
<td>0.179ft-lb-s/rad</td>
<td>Coefficient for viscous friction and pumping losses</td>
</tr>
<tr>
<td>$K_{\text{SR}}$</td>
<td>20.0</td>
<td>Steering ratio</td>
</tr>
<tr>
<td>$K_{\text{throttle}}$</td>
<td>153.0ft-lb</td>
<td>Engine torque for maximum throttle</td>
</tr>
<tr>
<td>$m$</td>
<td>78.0slug</td>
<td>Vehicle mass</td>
</tr>
<tr>
<td>$Q_0$</td>
<td>15.1ft-lb</td>
<td>Engine idle torque</td>
</tr>
</tbody>
</table>
It is further assumed that no external disturbances such as wind shall be applied to the vehicle.

**Lateral Force Model**

The front wheel deflection,

\[
\delta_{\text{front}} = \frac{\delta_{\text{SW}}}{K_{\text{SR}}} \cdot \frac{u}{u_{\text{limited}}} ,
\]

is computed by attenuating the steering wheel deflection by the steering ratio. The result is limited in order to compensate for reduced sensitivity at low speeds, such that

\[
u_{\text{limited}} = \begin{cases} 
    u & , u > T\dot{u}_{\text{min}} \\
    \dot{u}_{\text{min}} & , u \leq T\dot{u}_{\text{min}}
\end{cases}
\]
The lateral force acting on the front wheels is

\[
F_{\text{front}} = \begin{cases} 
-F_{\text{front}, \text{max}}, & 2Y_{\text{front}} \left( \delta_{\text{front}} - \frac{v + ar}{u_{\text{limited}}} \right) < -F_{\text{front}, \text{max}} \\
2Y_{\text{front}} \left( \delta_{\text{front}} - \frac{v + ar}{u_{\text{limited}}} \right), & \left| 2Y_{\text{front}} \left( \delta_{\text{front}} - \frac{v + ar}{u_{\text{limited}}} \right) \right| \leq F_{\text{front}, \text{max}} \\
F_{\text{front}, \text{max}}, & 2Y_{\text{front}} \left( \delta_{\text{front}} - \frac{v + ar}{u_{\text{limited}}} \right) > F_{\text{front}, \text{max}} 
\end{cases}
\]

which is limited by the maximum side force which can be applied to the front tires. Similarly, the lateral force acting on the rear wheels is

\[
F_{\text{rear}} = \begin{cases} 
-F_{\text{rear}, \text{max}}, & -2Y_{\text{rear}} \frac{v - br}{u_{\text{limited}}} < -F_{\text{rear}, \text{max}} \\
-2Y_{\text{rear}} \frac{v - br}{u_{\text{limited}}}, & \left| -2Y_{\text{rear}} \frac{v - br}{u_{\text{limited}}} \right| \leq F_{\text{rear}, \text{max}} \\
F_{\text{rear}, \text{max}}, & -2Y_{\text{rear}} \frac{v - br}{u_{\text{limited}}} > F_{\text{rear}, \text{max}} 
\end{cases}
\]

Finally, the aerodynamic side force,

\[
F_A = \frac{\rho A |u| v C_{Y, \beta}}{2}
\]

is computed using an approximation for side-slip angles. Therefore, the vehicle’s lateral acceleration is

\[
\dot{v} = \frac{F_{\text{front}} + F_{\text{rear}} + F_A}{m} - ur
\]

**Yaw Moment Model**

The aerodynamic yawing moment,

\[
N_{\beta} = \frac{\rho A |u| v (a + b) C_{N, \beta}}{2}
\]

is also computed using the approximation for small side-slip angles. The yaw acceleration is therefore
\[
\dot{r} = \frac{N_\beta + aF_{\text{front}} - bF_{\text{rear}}}{l_{ZZ}}.
\]

(71)

**Engine/Transmission Model**

The angular velocity of the torque converter is

\[
\omega_{\text{TC}} = \frac{uR}{r_{\text{wheel}}}.
\]

(72)

The engine angular speed is modeled as

\[
\omega_{\text{engine}} = \frac{\sqrt{K_{\text{loss}}^2 + 4B_{\text{TC}} (Q_0 + K_{\text{throttle}} \delta_{\text{throttle}} + B_{\text{TC}} \omega_{\text{TC}}^2)} - K_{\text{loss}}}{2B_{\text{TC}}},
\]

(73)

the torque converter input torque is

\[
Q_{\text{TC,in}} = B_{\text{TC}} (\omega_{\text{engine}}^2 - \omega_{\text{TC}}^2),
\]

(74)

and the torque converter output torque is

\[
Q_{\text{TC,out}} = \begin{cases} 
Q_{\text{TC,in}}, & \omega_{\text{TC}} > \omega_{\text{engine}} \\
Q_{\text{TC,in}} \left(2 - \frac{\omega_{\text{TC}}}{\omega_{\text{engine}}}\right), & \omega_{\text{TC}} \leq \omega_{\text{engine}} 
\end{cases}
\]

(75)

The gear ratio is modeled with a first-order lag from the raw gear ratio selected by the position of the gear shift. The rate of change of the gear ratio is

\[
\dot{R} = \frac{R_{\text{gear}} - R}{\tau_{\text{shift}}},
\]

(76)

and the current gear \(P_{\text{gear}}\) and raw gear ratio \(R_{\text{gear}}\) are determined by the state table shown in Table 17. Note that changing the gear shift position \(P_{\text{gear shift}}\) to Park or Reverse while the vehicle is moving shall result in a sudden halt since the longitudinal velocity and acceleration are immediately zeroed.
Table 17. State change table for automatic transmission model

<table>
<thead>
<tr>
<th>Condition</th>
<th>Result</th>
<th>$P_{\text{gear, in}}$</th>
<th>$P_{\text{gear, out}}$</th>
<th>$u$</th>
<th>$u$ and $\dot{u}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutral</td>
<td>Reverse</td>
<td>$R_{\text{reverse}}$</td>
<td>$R_1$</td>
<td>1</td>
<td>$u = \dot{u} = 0$</td>
</tr>
<tr>
<td>Park</td>
<td>Neutral</td>
<td>0</td>
<td>$u = \dot{u} = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reverse</td>
<td>Reverse</td>
<td>$R_{\text{reverse}}$</td>
<td>$R_2$</td>
<td>2</td>
<td>$u = \dot{u} = 0$</td>
</tr>
<tr>
<td>Neutral</td>
<td>Neutral</td>
<td>0</td>
<td>$u = \dot{u} = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 or 3</td>
<td>$u &gt; u_1 \rightarrow_2 + \Delta u_1 \rightarrow_2 \delta_{\text{throttle}}$</td>
<td>3</td>
<td>$R_3$</td>
<td>3</td>
<td>$u = \dot{u} = 0$</td>
</tr>
<tr>
<td>Park</td>
<td>Neutral</td>
<td>0</td>
<td>$u = \dot{u} = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reverse</td>
<td>Reverse</td>
<td>$R_{\text{reverse}}$</td>
<td>$R_1$</td>
<td>1</td>
<td>$u = \dot{u} = 0$</td>
</tr>
<tr>
<td>Neutral</td>
<td>Neutral</td>
<td>0</td>
<td>$u = \dot{u} = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$u &lt; u_2 \rightarrow_3 + \Delta u_2 \rightarrow_3 \delta_{\text{throttle}}$</td>
<td>2</td>
<td>$R_2$</td>
<td>2</td>
<td>$u = \dot{u} = 0$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$R_2$</td>
<td>$u = \dot{u} = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$R_1$</td>
<td>$u = \dot{u} = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reverse</td>
<td>Neutral</td>
<td>0</td>
<td>$u = \dot{u} = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1, 2, or 3</td>
<td>1</td>
<td>$R_1$</td>
<td>$u = \dot{u} = 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Longitudinal Force Model**

The tractive force induced by the torque converter is

$$ F_{\text{tractive}} = \frac{RQ_{\text{TC, out}}}{r_{\text{wheel}}} \quad (77) $$

the braking force—limited by its maximum value—is

$$ F_{\text{brake}} = \begin{cases} \frac{F_{\text{BP}}}{F_{\text{BP, max}}} F_{\text{brake, max}}, & F_{\text{BP}} < F_{\text{BP, max}} \\ F_{\text{brake, max}}, & F_{\text{BP}} \geq F_{\text{BP, max}} \end{cases} \quad (78) $$

and the aerodynamic drag using the small side-slip angle approximation is
Hence, the longitudinal acceleration is

\[
\dot{u} = \begin{cases} 
rv + \frac{F_{\text{tractive}} - F_{\text{roll}} - D - F_{\text{brake}}}{m} , & P_{\text{gear}} \in \{\text{Neutral, 1, 2, 3}\} \\
rv + \frac{F_{\text{tractive}} + F_{\text{roll}} + D + F_{\text{brake}}}{m} , & P_{\text{gear}} \in \{\text{Reverse}\} 
\end{cases}
\]

**Body to Inertial Coordinate Transformation**

The inertial coordinate system velocities are determined using the simple linear transformations of the body coordinate system velocities:

\[
\dot{x} = u \cos \theta - v \sin \theta
\]  
(81)

and

\[
\dot{y} = u \sin \theta + v \cos \theta
\]  
(82)

**State Variable Update**

The state variables are updated by performing the following integrations:

\[
u = \begin{cases} 
-u_{\max} , & \int_{0}^{t} \dot{u} dt \leq -u_{\max} \\
\int_{0}^{t} \dot{u} dt , & \int_{0}^{t} \dot{u} dt < u_{\max} \\
u_{\max} , & \int_{0}^{t} \dot{u} dt \geq u_{\max}
\end{cases}
\]  
(83)

\[
v = \int_{0}^{t} \dot{v} dt
\]  
(84)

\[
r = \int_{0}^{t} \dot{r} dt
\]  
(85)

\[
\theta = \int_{0}^{t} \dot{r} dt
\]  
(86)
In the actual simulation, these integrals are implemented using a combination of explicit and implicit Euler integrators, each operating at a 20 Hz fixed frame rate. The integrators which have $\dot{u}$, $\dot{v}$, $\dot{r}$, and $\dot{R}$ as inputs are all explicit Euler integrators; the remaining integrators use the implicit Euler method. The integrators are executed in the order that the equations appear above, which helps reduce phase lag in the $x$ and $y$ outputs; this can be seen by examining Figure 40.

**Estimation of Integrator Cutoff Frequencies**

The vehicle dynamics model contains seven integrators as identified in Equations (83) through (89). In order to make a reasonable estimation of the cutoff frequency, $\omega_C$, for each of these integrators, a set of test cases must be constructed such that each integrator is exercised at or near its operating limit. Using a cutoff percentage of $P_C = 0.85$ for all seven integrators, the cutoff frequency of each integrator is determined for each of the test cases. The maximum value of the cutoff frequency for each integrator over all of the test cases is used as the final estimated value. This emphasizes the need for a comprehensive set of test cases which cause the model to exhibit the most extreme behavior in order to accurately estimate the input bandwidth to each integrator.

The first test case is a simple straight acceleration at maximum throttle. The other three test cases are a series of “S”-turns executed at one-eighth, one-fourth, and
one-half throttle. It is expected that these test cases sufficiently exercise the vehicle dynamics model integrators to obtain good estimates of the cutoff frequency for each integrator.

Two additional test cases were considered but were not deemed to have reasonable results. The first of these is the case of straight deceleration, which is the logical extension of the straight acceleration test case. In this case, the vehicle is decelerated from maximum velocity at zero throttle and maximum braking. Unfortunately, the vehicle dynamics model is not accurate for low or negative velocities, and the longitudinal velocity drops past zero and continues until it is limited at the maximum negative velocity. Obviously, this is not a desired condition, and it should not be used to determine the envelope of normal behavior for the system’s integrators.

The second test case that was rejected is one in which a turn is sustained with a constant throttle and steering wheel deflection. This case, which was successfully simulated, only exercised the yaw velocity \( (r) \) and \( (x, y) \) velocity integrators, since all other integrator inputs are zero. This case turns out to be redundant with the other four test cases chosen above.

All of the test cases utilize the baseline vehicle dynamics model implementation (e.g. Euler integrators) running at a frame rate of 1,000 Hz, which was determined by Dumas (1993) to offer adequate pseudo-continuous behavior. The simulations are run for a duration necessary to achieve equilibrium in its final state and to minimize aliasing in the frequency domain. Wherever possible, the simulations are started from an equilibrium state. Finally, all of the test cases utilize a 1,000-point discrete Fourier transform from which the cutoff frequency is determined.
Test Case 1: Straight Line Acceleration and Maximum Throttle

This test case starts the vehicle at its first gear idle velocity and accelerates at maximum throttle to its maximum velocity. During this time, there is no steering wheel, brake pedal, or gear shift input other than their initial conditions, which are zero steering wheel deflection, zero brake pedal force, and the gear shift in the third gear (drive) position.

This simulation exercises the $\dot{u}$, $\dot{R}$, and $\dot{x}$ integrators; however, since this test case is independent of initial heading, the results of the $\dot{x}$ integrator can be applied to the $\dot{y}$ integrator. Because of the nature of this test case, the other state derivatives are not exercised and remain zero throughout the duration of the simulation.

Figures 41 through 43 show the time response and frequency content of the state variables $\dot{u}$, $\dot{R}$, and $\dot{x}$, respectively. The cutoff frequencies for these integrators (plus the inferred cutoff frequency for the $\dot{y}$ integrator) which were determined by this test case are summarized in Table 18.

Test Case 2: “S”-Turns at One-Eighth Throttle

This test case starts the vehicle with an initial longitudinal velocity of 29.5 ft/s, which is approximately the steady-state velocity of the vehicle while engaged in the

![Figure 41. Time response and frequency content of $\dot{u}$ for test case 1](image-url)
Table 18. Vehicle dynamics integrator cutoff frequencies from test case 1

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_c$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{u}$</td>
<td>62.33</td>
</tr>
<tr>
<td>$\dot{R}$</td>
<td>70.37</td>
</tr>
<tr>
<td>$\dot{x}$</td>
<td>55.79</td>
</tr>
<tr>
<td>$\dot{y}$</td>
<td>55.79</td>
</tr>
</tbody>
</table>

“S”-turns at one-eighth throttle with no brake pedal force applied. The gear shift is maintained in the third gear (drive) position; the transmission gear starts and remains in third gear, and the transmission gear ratio starts and remains at 3.0. The steering wheel input is a sinusoid with a magnitude of 1.2 radians and a frequency of 1 Hz. This
input was determined by Dumas (1993) to be representative of the upper limit of a driver's input spectrum. The trajectory of the vehicle is illustrated in Figure 44.

This simulation exercises the \( \dot{u}, \dot{v}, \dot{r}, r, \dot{x}, \) and \( \dot{y} \) integrators. Since the vehicle is maintained in the same range of velocities, \( \dot{R} \) is zero throughout the simulation.

Figures 45 through 50 show the time response and frequency content of the state variables \( \dot{u}, \dot{v}, \dot{r}, r, \dot{x}, \) and \( \dot{y} \), respectively. The cutoff frequencies for these integrators which were determined by this test case are summarized in Table 19.

![Figure 44. Vehicle trajectory for test case 2](image1)

![Figure 45. Time response and frequency content of \( \dot{u} \) for test case 2](image2)
Figure 46. Time response and frequency content of $\dot{v}$ for test case 2

Figure 47. Time response and frequency content of $\dot{r}$ for test case 2

Figure 48. Time response and frequency content of $r$ for test case 2
Figure 49. Time response and frequency content of $\dot{x}$ for test case 2

Figure 50. Time response and frequency content of $\dot{y}$ for test case 2

Table 19. Vehicle dynamics integrator cutoff frequencies from test case 2

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_C$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{u}$</td>
<td>13.79</td>
</tr>
<tr>
<td>$\dot{v}$</td>
<td>65.18</td>
</tr>
<tr>
<td>$\dot{r}$</td>
<td>58.29</td>
</tr>
<tr>
<td>$r$</td>
<td>31.96</td>
</tr>
<tr>
<td>$\dot{x}$</td>
<td>31.96</td>
</tr>
<tr>
<td>$\dot{y}$</td>
<td>6.89</td>
</tr>
</tbody>
</table>
Test Case 3: "S"-Turns at One-Fourth Throttle

This test case starts the vehicle with an initial longitudinal velocity of 52.4 ft/s, which is approximately the steady-state velocity of the vehicle while engaged in the "S"-turns at one-fourth throttle with no brake pedal force applied. All other simulation parameters are the same as described above in test case 2. The trajectory of the vehicle is illustrated in Figure 51.

Figures 52 through 50 show the time response and frequency content of the state derivatives $\dot{u}$, $\dot{v}$, $\dot{r}$, $r$, $\dot{x}$, and $\dot{y}$, respectively. The cutoff frequencies for these integrators which were determined by this test case are summarized in Table 20.

![Figure 51. Vehicle trajectory for test case 3](image)

![Figure 52. Time response and frequency content of $\dot{u}$ for test case 3](image)
Figure 53. Time response and frequency content of $\dot{\gamma}$ for test case 3

Figure 54. Time response and frequency content of $\dot{r}$ for test case 3

Figure 55. Time response and frequency content of $r$ for test case 3
Figure 56. Time response and frequency content of $\dot{x}$ for test case 3

Figure 57. Time response and frequency content of $\dot{y}$ for test case 3

Table 20. Vehicle dynamics integrator cutoff frequencies from test case 3

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_c$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{u}$</td>
<td>41.99</td>
</tr>
<tr>
<td>$\dot{v}$</td>
<td>47.01</td>
</tr>
<tr>
<td>$\dot{r}$</td>
<td>52.65</td>
</tr>
<tr>
<td>$\dot{z}$</td>
<td>40.11</td>
</tr>
<tr>
<td>$\dot{x}$</td>
<td>31.96</td>
</tr>
<tr>
<td>$\dot{y}$</td>
<td>6.89</td>
</tr>
</tbody>
</table>
Test Case 4: "S"-Turns at One-Half Throttle

This test case starts the vehicle with an initial longitudinal velocity of 75 ft/s, which is approximately the steady-state velocity of the vehicle while engaged in the "S"-turns at one-half throttle with no brake pedal force applied. All other simulation parameters are the same as described above in test case 2. The trajectory of the vehicle is illustrated in Figure 58.

Figures 59 through 64 show the time response and frequency content of the state derivatives $\dot{u}$, $\dot{v}$, $\dot{r}$, $r$, $\dot{x}$, and $\dot{y}$, respectively. The cutoff frequencies for these integrators which were determined by this test case are summarized in Table 21.

![Figure 58. Vehicle trajectory for test case 4](image)

![Figure 59. Time response and frequency content of $\dot{u}$ for test case 4](image)
Figure 60. Time response and frequency content of $\dot{v}$ for test case 4

Figure 61. Time response and frequency content of $\dot{r}$ for test case 4

Figure 62. Time response and frequency content of $r$ for test case 4
Figure 63. Time response and frequency content of $\dot{x}$ for test case 4

Figure 64. Time response and frequency content of $\dot{y}$ for test case 4

Table 21. Vehicle dynamics integrator cutoff frequencies from test case 4

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_c$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{u}$</td>
<td>62.68</td>
</tr>
<tr>
<td>$\dot{v}$</td>
<td>35.72</td>
</tr>
<tr>
<td>$\dot{r}$</td>
<td>48.26</td>
</tr>
<tr>
<td>$\dot{r}$</td>
<td>44.50</td>
</tr>
<tr>
<td>$\ddot{x}$</td>
<td>31.96</td>
</tr>
<tr>
<td>$\ddot{y}$</td>
<td>6.89</td>
</tr>
</tbody>
</table>
**Integrator Cutoff Frequency Estimates**

Based upon the four test cases above, the cutoff frequencies of the seven integrators in the vehicle dynamics model can be determined. These are listed below in Table 22.

**Table 22. Summary of vehicle dynamics integrator cutoff frequencies**

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_c$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{u}$</td>
<td>62.68</td>
</tr>
<tr>
<td>$\dot{v}$</td>
<td>65.18</td>
</tr>
<tr>
<td>$\dot{r}$</td>
<td>58.29</td>
</tr>
<tr>
<td>$\dot{\theta}$</td>
<td>44.50</td>
</tr>
<tr>
<td>$\dot{\phi}$</td>
<td>70.37</td>
</tr>
<tr>
<td>$\dot{\chi}$</td>
<td>55.79</td>
</tr>
<tr>
<td>$\dot{\psi}$</td>
<td>55.79</td>
</tr>
</tbody>
</table>

**Determining Baseline Simulation Error Metrics**

The two frequency domain metrics that will be used to evaluate the vehicle dynamics model are the maximum absolute gain error and the maximum absolute phase error. These values will be determined for each integrator in the baseline simulation; these values will then be used to specify performance for integrators to be synthesized in the next section.

Appendix A and Appendix B describe the gain and phase error metric equations for the explicit and implicit Euler integrators, respectively. The maximum absolute gain error is

$$
\varepsilon_{\text{gain,max}}(\omega T) = \ln \left( \frac{\omega T}{\sqrt{2 - 2 \cos \omega T}} \right)
$$

for both the explicit and implicit Euler integrators, and the maximum absolute phase error is
for both the explicit and implicit Euler integrators. Table 23 lists these error metrics

\[ e_{\text{phase, max}}(\omega T) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\sin \omega T}{1 - \cos \omega T}\right) \]  \hspace{1cm} (91)

for both the explicit and implicit Euler integrators. Table 23 lists these error metrics

**Table 23. Baseline vehicle dynamics simulation error metrics**

<table>
<thead>
<tr>
<th>Integrator</th>
<th>( \omega_c T )</th>
<th>( e_{\text{gain, max}}(\omega_c T) )</th>
<th>( e_{\text{phase, max}}(\omega_c T) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{u} )</td>
<td>3.3140</td>
<td>0.5087</td>
<td>1.6570</td>
</tr>
<tr>
<td>( \dot{v} )</td>
<td>3.2590</td>
<td>0.4900</td>
<td>1.6303</td>
</tr>
<tr>
<td>( \dot{r} )</td>
<td>2.9145</td>
<td>0.3830</td>
<td>1.4573</td>
</tr>
<tr>
<td>( \dot{r} )</td>
<td>2.2250</td>
<td>0.2155</td>
<td>1.1125</td>
</tr>
<tr>
<td>( \dot{\dot{r}} )</td>
<td>3.5185</td>
<td>0.5828</td>
<td>1.7593</td>
</tr>
<tr>
<td>( \dot{x} )</td>
<td>2.7895</td>
<td>0.3483</td>
<td>1.3948</td>
</tr>
<tr>
<td>( \dot{y} )</td>
<td>2.7895</td>
<td>0.3483</td>
<td>1.3948</td>
</tr>
</tbody>
</table>

for each of the seven integrators in the model.

**Synthesizing Integrators for the Vehicle Dynamics Simulation**

For each of the seven integrators in the vehicle dynamics model, a new integrator will be synthesized to meet both the maximum absolute gain and phase error criteria determined in the previous section. The goal is to be able to operate the simulation with new integrators that can be executed at a lower integration rate. At the same time, the new integrators must not exceed the maximum absolute gain and phase errors established by the Euler integrators over the range of frequencies typically dominated by nominal operation of the simulator.

Based upon the findings described in Chapter 3, a first-order synthesized integrator would be unable to improve the integration step due to the phase error constraint. Therefore, second-order integrators will be generated. These integrators will be explicit since the maximum absolute gain and phase errors are the same for both implicit and explicit Euler integrators.\(^1\) The synthesized integrators will be derived in
Longitudinal Acceleration Integrator

Starting with the parameters used by the second-order Adams-Bashforth integrator, the parameters are perturbed in directions which seek to maximize the value of \( \omega_c T \) for the given maximum absolute gain and phase error constraints. A summary of steps used in adjusting the parameters is shown in Table 24.

Table 24. Progressive selection of parameter values for synthesized longitudinal acceleration integrator

<table>
<thead>
<tr>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( \omega_c T (\varepsilon_{\text{gain}}) )</th>
<th>( \omega_c T (\varepsilon_{\text{phase}}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.44751</td>
<td>3.25656</td>
</tr>
<tr>
<td>0.3370</td>
<td>0.6630</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.70938</td>
<td>3.17337</td>
</tr>
<tr>
<td>0.3370</td>
<td>0.6630</td>
<td>0.5677</td>
<td>0.4323</td>
<td>4.45729</td>
<td>3.67131</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.699</td>
<td>0.301</td>
<td>4.31002</td>
<td>4.20227</td>
</tr>
</tbody>
</table>

The resulting synthesized integrator is described by the difference equation

\[
y_k = y_{k-1} + 0.699u_{k-1} + 0.301u_{k-2}
\]  

Figures 65 and 66 illustrate the gain and phase errors for this integrator, respectively. The maximum integration step time required by this integrator is therefore

\[
T = 0.06340
\]

which results in a 26.8% improvement over the Euler integrator.

Lateral Acceleration Integrator

Starting with the parameters used by the second-order Adams-Bashforth integrator, the parameters are perturbed in directions which seek to maximize the value

1. This does not apply to tandem integrators which are used to perform a double integration. In fact, by examining the phase error graphs for explicit and implicit Euler integrators in the Appendices, it can be seen that their phase errors should cancel exactly. However, the output of the first integrator is used for other calculations, so the pair of integrators cannot be considered as a single process performing a double integration with no phase error. This subject will be discussed in further detail in the Conclusions.
of $\omega_c T$ for the given maximum absolute gain and phase error constraints. A summary of steps used in adjusting the parameters is shown in Table 25.

### Table 25. Progressive selection of parameter values for synthesized lateral acceleration integrator

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$\omega_c T (\varepsilon_{\text{gain}})$</th>
<th>$\omega_c T (\varepsilon_{\text{phase}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.40577</td>
<td>3.22094</td>
</tr>
<tr>
<td>0.3677</td>
<td>0.6323</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.6757</td>
<td>3.16571</td>
</tr>
<tr>
<td>0.3677</td>
<td>0.6323</td>
<td>0.577</td>
<td>0.423</td>
<td>4.40437</td>
<td>3.62466</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.705</td>
<td>0.295</td>
<td>4.28366</td>
<td>4.10943</td>
</tr>
</tbody>
</table>

The resulting synthesized integrator is described by the difference equation
\[ y_k = y_{k-1} + 0.705u_{k-1} + 0.295u_{k-2} \] \hspace{1cm} (94)

Figures 67 and 68 illustrate the gain and phase errors for this integrator, respectively. The maximum integration step time required by this integrator is therefore

\[ T = 0.06305 \] \hspace{1cm} (95)

which results in a 26.1\% improvement over the Euler integrator.

\textbf{Yaw Acceleration Integrator}

Starting with the parameters used by the second-order Adams-Bashforth integrator, the parameters are perturbed in directions which seek to maximize the value

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure67}
\caption{Gain error for synthesized lateral acceleration integrator}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure68}
\caption{Phase error for synthesized lateral acceleration integrator}
\end{figure}
of $\omega_c T$ for the given maximum absolute gain and phase error constraints. A summary of steps used in adjusting the parameters is shown in Table 26.

**Table 26. Progressive selection of parameter values for synthesized yaw acceleration integrator**

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$\omega_c T (\varepsilon_{\text{gain}})$</th>
<th>$\omega_c T (\varepsilon_{\text{phase}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.17108</td>
<td>2.99019</td>
</tr>
<tr>
<td>0.5334</td>
<td>0.4666</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.46988</td>
<td>3.07158</td>
</tr>
<tr>
<td>0.5334</td>
<td>0.4666</td>
<td>0.7607</td>
<td>0.2393</td>
<td>3.01721</td>
<td>3.01654</td>
</tr>
</tbody>
</table>

The resulting synthesized integrator is described by the difference equation

$$y_k = 0.5334y_{k-1} + 0.4666y_{k-2} + 0.7607u_{k-1} + 0.2393u_{k-2}.$$  \hspace{1cm} (96)

Figures 69 and 70 illustrate the gain and phase errors for this integrator, respectively. The maximum integration step time required by this integrator is therefore

$$T = 0.05175,$$ \hspace{1cm} (97)

which results in a modest 3.5% improvement over the Euler integrator.

**Yaw Rate Integrator**

Starting with the parameters used by the second-order Adams-Bashforth integrator, the parameters are perturbed in directions which seek to maximize the value
Figure 70. Phase error for synthesized yaw acceleration integrator

of $\omega_c T$ for the given maximum absolute gain and phase error constraints. A summary of steps used in adjusting the parameters is shown in Table 27.

Table 27. Progressive selection of parameter values for synthesized yaw rate integrator

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$\omega_c T(\varepsilon_{\text{gain}})$</th>
<th>$\omega_c T(\varepsilon_{\text{phase}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>1.5</td>
<td>-0.5</td>
<td>0.802758</td>
<td>2.52554</td>
</tr>
<tr>
<td>0.7596</td>
<td>0.2404</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.07939</td>
<td>2.70207</td>
</tr>
<tr>
<td>0.7596</td>
<td>0.2404</td>
<td>0.8845</td>
<td>0.1155</td>
<td>2.41736</td>
<td>2.4175</td>
</tr>
</tbody>
</table>

The resulting synthesized integrator is described by the difference equation

$$y_k = 0.7596y_{k-1} + 0.2404y_{k-2} + 0.8845u_{k-1} + 0.1155u_{k-2}$$  \hspace{1cm} (98)

Figures 71 and 72 illustrate the gain and phase errors for this integrator, respectively. The maximum integration step time required by this integrator is therefore

$$T = 0.05433$$  \hspace{1cm} (99)

which results in an 8.7% improvement over the Euler integrator.

Transmission Gear Ratio Rate of Change Integrator

Starting with the parameters used by the second-order Adams-Bashforth integrator, the parameters are perturbed in directions which seek to maximize the value
of $\omega_cT$ for the given maximum absolute gain and phase error constraints. A summary of steps used in adjusting the parameters is shown in Table 28.

Table 28. Progressive selection of parameter values for transmission gear ratio rate of change integrator

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$\omega_cT(\varepsilon_{\text{gain}})$</th>
<th>$\omega_cT(\varepsilon_{\text{phase}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.61599</td>
<td>3.39327</td>
</tr>
<tr>
<td>0.2090</td>
<td>0.7910</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.83862</td>
<td>3.18358</td>
</tr>
<tr>
<td>0.2090</td>
<td>0.7910</td>
<td>0.5372</td>
<td>0.4628</td>
<td>4.66443</td>
<td>3.76106</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.683</td>
<td>0.317</td>
<td>4.39929</td>
<td>4.45933</td>
</tr>
</tbody>
</table>

The resulting synthesized integrator is described by the difference equation
\[ y_k = y_{k-1} + 0.683u_{k-1} + 0.317u_{k-2} \]  \hspace{1cm} (100)

Figures 73 and 74 illustrate the gain and phase errors for this integrator, respectively. The maximum integration step time required by this integrator is therefore

\[ T = 0.06252 \]  \hspace{1cm} (101)

which results in a 25% improvement over the Euler integrator.

**X- and Y-Velocity Integrators**

Starting with the parameters used by the second-order Adams-Bashforth integrator, the parameters are perturbed in directions which seek to maximize the value

![Figure 73. Gain error for synthesized transmission gear ratio rate of change integrator](image1)

![Figure 74. Phase error for synthesized transmission gear ratio rate of change integrator](image2)
of $\omega_c T$ for the given maximum absolute gain and phase error constraints. A summary of steps used in adjusting the parameters is shown in Table 29.

### Table 29. Progressive selection of parameter values for synthesized x- and y-velocity integrators

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$\omega_c T(e_{\text{gain}})$</th>
<th>$\omega_c T(e_{\text{phase}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.09578</td>
<td>2.90666</td>
</tr>
<tr>
<td>0.5834</td>
<td>0.4166</td>
<td>1.5</td>
<td>-0.5</td>
<td>1.39721</td>
<td>3.02078</td>
</tr>
<tr>
<td>0.5834</td>
<td>0.4166</td>
<td>0.7850</td>
<td>0.2150</td>
<td>2.92812</td>
<td>2.92744</td>
</tr>
</tbody>
</table>

The resulting synthesized integrator is described by the difference equation

$$y_k = 0.5834y_{k-1} + 0.4166y_{k-2} + 0.7850u_{k-1} + 0.2150u_{k-2}.$$  \(102\)

Figures 75 and 76 illustrate the gain and phase errors for this integrator, respectively. The maximum integration step time required by this integrator is therefore

$$T = 0.05247,$$  \(103\)

which results in a 4.9% improvement over the Euler integrators.

### Summary of Integrator Synthesis Results

By examining the results of the synthesis process for each of the vehicle dynamics model integrators, it can be seen that the yaw acceleration integrator provides
the driving requirement for a fixed integration step size for the simulation. Although some integrators exhibited significant improvements in integration step times, the integration step size of the simulation is governed by the smallest step size of all of the model’s integrators, resulting in an overall improvement of 3.5%.

During the synthesis of the integrators, it became clear that it becomes increasingly difficult to improve phase performance as the value of $\omega_c T$ for the integrator approaches $\pi$. This is because the phase error for the second-order synthesized explicit integrator is $\pi/2$ with little regard for the parameter values\(^1\). For this reason, it is expected that better results could be obtained by using one of the other phase metrics instead of maximum absolute phase error.

Comparing Baseline and Synthesized Simulations

The integrators which were developed in the previous section are now implemented in the vehicle dynamics simulation and compared with the baseline simulation results. The test cases that were originally run to determine required integrator bandwidth are now run on the modified simulation, which execute at a

---

1. Phase error is always $\pi/2$ unless $b_1 = b_2 = 0.5$, assuming that $a_1 + a_2 = 1$ and $b_1 + b_2 = 1$. This case, however, causes infinite gain error at this value of $\omega T$. 

---

Figure 76. Phase error for synthesized x- and y- velocity integrators
0.05175 second integration step time (19.32 Hz). The baseline simulation is executed at a 0.05 second integration step time (20 Hz) for the purpose of comparison. For each of the four test cases, the time response of each integrator output will be compared; the vehicle trajectory is also compared for each of the three “S”-turn cases.

**Test Case 1: Straight Line Acceleration and Maximum Throttle**

As with the first test case performed earlier, this simulation is based upon a full-throttle acceleration from an initial velocity of 5.75 ft/s. Figures 77 through 79 compare the baseline simulation (depicted by a solid trace) with the one using the

![Figure 77](image1.png)

**Figure 77.** Comparison of baseline and synthesized simulations for longitudinal velocity in test case 1

![Figure 78](image2.png)

**Figure 78.** Comparison of baseline and synthesized simulations for transmission gear ratio in test case 1
Figure 79. Comparison of baseline and synthesized simulations for x-direction position in test case 1

synthesized integrators (depicted by a dashed trace) for longitudinal velocity, transmission gear ratio, and x-direction position, respectively. Note that for the x-position state variable, the simulation using synthesized integrators exhibits a significant lag from the baseline simulation, which is closely representative of the pseudo-continuous response illustrated in Chapter 3. It is likely that this is due to the non-zero DC gain error, which is due to the parameter configuration in which $a_1 \neq 1$ and $a_2 \neq 0$. This is necessary because the value of $\omega_c T$ is less than $\pi$, which limits the ability of parameter adjustment to improve upon this value based upon maximum absolute gain and phase error constraints.

Test Case 2: “S”-Turns at One-Eighth Throttle

Like its counterpart presented earlier, this test case starts the vehicle at an initial longitudinal velocity and engages in 1 Hz “S”-turns at one-eighth throttle. Figures 80 through 86 compare the baseline simulation (depicted by a solid trace) with the one using the synthesized integrators (depicted by a dashed trace) for vehicle trajectory, longitudinal velocity, lateral velocity, yaw rate, heading, and x- and y-direction positions, respectively. Note that there are significant differences in the vehicle trajectory and x- and y-position results. It is once again likely that these inaccuracies
Figure 80. Comparison of baseline and synthesized simulations for vehicle trajectory in test case 2

Figure 81. Comparison of baseline and synthesized simulations for longitudinal velocity in test case 2

Figure 82. Comparison of baseline and synthesized simulations for lateral velocity in test case 2
Figure 83. Comparison of baseline and synthesized simulations for yaw rate in test case 2

Figure 84. Comparison of baseline and synthesized simulations for heading in test case 2

Figure 85. Comparison of baseline and synthesized simulations for x-direction position in test case 2
are due to the non-zero DC gain error, which is due to the parameter configuration in which \( a_1 \neq 1 \) and \( a_2 \neq 0 \).

Test Case 3: "S"-Turns at One-Fourth Throttle

Like its counterpart presented earlier, this test case starts the vehicle at an initial longitudinal velocity and engages in 1 Hz "S"-turns at one-fourth throttle. Figures 87 through 93 compare the baseline simulation (depicted by a solid trace) with the one using the synthesized integrators (depicted by a dashed trace) for vehicle trajectory, longitudinal velocity, lateral velocity, yaw rate, heading, and x- and y-direction.

![Figure 86. Comparison of baseline and synthesized simulations for y-direction position in test case 2](image)

Figure 86. Comparison of baseline and synthesized simulations for y-direction position in test case 2

![Figure 87. Comparison of baseline and synthesized simulations for vehicle trajectory in test case 3](image)

Figure 87. Comparison of baseline and synthesized simulations for vehicle trajectory in test case 3
Figure 88. Comparison of baseline and synthesized simulations for longitudinal velocity in test case 3

Figure 89. Comparison of baseline and synthesized simulations for lateral velocity in test case 3

Figure 90. Comparison of baseline and synthesized simulations for yaw rate in test case 3
Figure 91. Comparison of baseline and synthesized simulations for heading in test case 3

Figure 92. Comparison of baseline and synthesized simulations for x-direction position in test case 3

Figure 93. Comparison of baseline and synthesized simulations for y-direction position in test case 3
positions, respectively. Note that there are once again significant differences in the vehicle trajectory and x- and y-position results. It is again likely that these inaccuracies are due to the non-zero DC gain error, which is due to the parameter configuration in which $a_1 \neq 1$ and $a_2 \neq 0$.

**Test Case 4: “S”-Turns at One-Half Throttle**

Like its counterpart presented earlier, this test case starts the vehicle at an initial longitudinal velocity and engages in 1 Hz “S”-turns at one-half throttle. Figures 94 through 100 compare the baseline simulation (depicted by a solid trace) with the one using the synthesized integrators (depicted by a dashed trace) for vehicle trajectory.

![Figure 94. Comparison of baseline and synthesized simulations for vehicle trajectory in test case 4](image)

**Figure 94. Comparison of baseline and synthesized simulations for vehicle trajectory in test case 4**

![Figure 95. Comparison of baseline and synthesized simulations for longitudinal velocity in test case 4](image)

**Figure 95. Comparison of baseline and synthesized simulations for longitudinal velocity in test case 4**
Figure 96. Comparison of baseline and synthesized simulations for lateral velocity in test case 4

Figure 97. Comparison of baseline and synthesized simulations for yaw rate in test case 4

Figure 98. Comparison of baseline and synthesized simulations for heading in test case 4
longitudinal velocity, lateral velocity, yaw rate, heading, and x- and y-direction positions, respectively. Note that there are again significant differences in the vehicle trajectory and x- and y-position results. Again, it is likely that these inaccuracies are due to the non-zero DC gain error, which is due to the parameter configuration in which $a_1 \neq 1$ and $a_2 \neq 0$.

**Analysis of Test Case Results**

It is apparent that the results of the four test cases were not as expected. However, a clear pattern is shown in that the integrators constrained to $\omega_c T < \pi$

![Figure 99. Comparison of baseline and synthesized simulations for x-direction position in test case 4](image)

**Figure 99. Comparison of baseline and synthesized simulations for x-direction position in test case 4**

![Figure 100. Comparison of baseline and synthesized simulations for y-direction position in test case 4](image)

**Figure 100. Comparison of baseline and synthesized simulations for y-direction position in test case 4**
exhibited significant errors in their outputs. Tracing this problem back leads to the fact that all of these integrators have non-zero gain errors at $\omega T = 0$, which can be caused by $a_1 \neq 1$ and $a_2 \neq 0$ and also by $b_1 + b_2 \neq 1$; the inaccuracies seen here were incurred by the former case.

Tracing this further leads to the realization that the use of maximum absolute gain and phase errors as constraining metrics is not well suited for the specification of synthesized integrators. These specifications cause parameters to be adjusted in such a way that the gain and phase error curves are displaced greatly for minimal improvements in the integration step size. In fact, when $\omega_c T < \pi$ for a particular integrator, the improvements due to the maximum absolute phase error constraint are limited by the fact that $\epsilon_{\text{phase}}(\pi) = \pi/2$. Moreover, no improvement can be attained when $\omega_c T = \pi$ since the phase error at this point is practically invariant, only changing with parameter values that cause gain error to become unbounded.

Serious consideration should be given to adding the additional constraints of having zero gain and phase errors at DC. By doing so, the resulting integrator’s gain and phase errors will approach that of the ideal integrator as the integration step size approaches zero. Another reason for including these constraints is that the DC component is not used in estimating cutoff frequencies since the DC component is usually of a much greater magnitude than any of the frequency components.
CONCLUSIONS

This paper has documented the study of improving the performance of numerical integrators for application in the real-time simulation of continuous systems. The following sections summarize the results which have been obtained as a result of this research. In addition, topics which deserve further study but are outside the scope of this study are identified.

Summary and Interpretation of Results

The techniques described in this paper are based upon the premise that the input bandwidth of integrators is typically limited under nominal operating conditions. The test cases examined in this report clearly validate this thesis. Analysis of the inputs to the simulation’s integrators has illustrated that the bulk of the signal energy dominates the DC end of the spectra. The remainder of the paper has sought to exploit this information to produce numerical integrators which can operate at lower frame rates without significant loss of fidelity.

Several steps have been taken in pursuit of this goal. First, a means has been identified to quantify the significant portion of the subject integrator’s input frequency spectrum. This is accomplished by driving the simulation with inputs that cause the model to operate at or near its operational envelope while at the same time monitoring the inputs of the integrators in situ. These time-domain input sequences are then transformed into the frequency domain via a discrete Fourier transform. Excluding the DC term and the mirrored (i.e. negative frequency) terms, the overall energy in the frequency spectrum is computed and a cutoff frequency is determined based upon a specified percentage of total energy to be contained between DC and the cutoff
frequency. In more complex models, multiple simulations must be run with different inputs in order to achieve results which reasonably represent behavior at the extremes of the models envelope; in such cases, the largest cutoff frequency of all runs is selected for each of the integrators under study.

Next, metrics have been established which can be used to quantify integrator performance within a limited bandwidth. Errors in the subject integrator’s gain and phase over the bounded frequency range are used in order to account for analysis within a limited bandwidth as well as to easily accommodate the analysis of nonlinear systems. Plots of these gain and phase error metrics for the reference integrators (explicit Euler, implicit Euler, second-order Adams-Bashforth, and trapezoidal) are consistent with expectations and correlate well with common observations and other metrics.

Finally, a design process for numerical integrators has been defined to meet specifications based upon the aforementioned metrics. A general-form difference equation provides the starting point for the synthesis of a numerical integrator. The unspecified coefficients of the difference equation are then tailored in a minimization process in order to meet or exceed the specified requirements. The coefficients of one of the reference integrators is typically used as a starting point for the minimization.

Having thus been established, these techniques were then applied to two different test cases: a linear, open-loop missile model, and a more complex, nonlinear vehicle dynamics model. Both cases demonstrated that the integrator inputs are significantly band limited for the range of control inputs applied. The use of second-order synthesized explicit integrators in the missile model provided approximately 45% improvement in integration rate over the second-order Adams-Bashforth implementation and over 50% improvement in integration rate over the explicit Euler implementation. The vehicle dynamics model fared marginally, realizing only a 3.5%
improvement in integration rate from combined implicit/explicit Euler integrators to second-order synthesized explicit integrators; it was determined, however, that this result may have been significantly skewed due to the use of maximum absolute error metrics to specify performance.

Benefits of the Integrator Synthesis Technique

This process of synthesizing numerical integrators to match specified error metrics and exploit band-limited characteristics of the model has proven to be effective in reducing the required integration rate for some simulations. While these techniques are by no means panacean, they have provided valuable insight into new methods for specifying, evaluating, and improving the performance of numerical integration algorithms.

The validation of the premise that state derivatives are generally band-limited sets the stage for improving numerical integrator efficiency by matching frequency characteristics of an ideal integrator within a smaller frequency band. Thus, a numerical integrator may be contextually tailored to the simulation model, resulting in either increased accuracy or a decreased integration step time.

The frequency-domain error metrics which were derived earlier provide both a means of measuring the accuracy of common numerical integrators and a means of doing so within a limited frequency band. Therefore, these metrics can be used to evaluate the performance of integrators as well as to stipulate requirements on integrators to be synthesized. These metrics not only confirm well-known attributes—such as the phase-cancelling properties of the cascaded explicit/implicit Euler integrators—but also impart additional insight into the relative behavior of different algorithms.
Finally, the process of synthesizing numerical integrators for application within a particular simulation model has proven effective in increasing the efficiency of the simulation. As demonstrated here, this technique has application within both fixed- and variable-rate simulations. This method offers a significant technical milestone towards producing higher fidelity simulations with lower cost computer systems.

Limitations and Areas Needing Improvement

As noted earlier in Chapters 3 and 4, the methods used to perform multiparameter, multivariate minimization require improvement. Because of the complexity and nonlinearity, automated methods such as those provided by Mathematica (Wolfram, 1991) have great difficulty in even locating local extrema. Instead, only a single parameter was minimized at a time, and parameter values—other than those which could be otherwise constrained—were started from those of the appropriate reference integrator. While this procedure was successful in obtaining improvements with respect to the reference integrators, the process was extremely tedious, labor-intensive, and subject to a great deal of intuition in selecting and adjusting parameter values; hence, the end results are likely not to have been minimized over the whole valid range of parameter values. For these reasons, a more automated and efficient minimization process would be highly desirable, especially if numerical integrators of order three or greater are to be explored.

A second area which could benefit from further improvement is the validation of the vehicle dynamics test case. A more credible validation could have been obtained through a more rigorous testing procedure, including the implementation of the synthesized numerical integrators into the simulator and the statistical analysis of actual driver response between the baseline and synthesized simulations. The driving simulator vehicle dynamics case study was not tested to this degree due to time and
logistical considerations; however, conducting such further tests would have made an even stronger case for the effectiveness of these techniques.

Finally, it was noted in Chapter 4 that the absolute maximum error criteria were not well suited for the specification and synthesis of numerical integration algorithms. A more detailed analysis of the error metrics developed here (i.e. absolute maximum, integral absolute, and integral squared gain and phase errors), together with a listing of relative advantages and disadvantages, may have provided additional insight as to the appropriateness of each metric for a particular application or desired goal.

**Recommended Improvements to the Vehicle Dynamics Model**

In Chapter 4, the vehicle dynamics model for the University of Central Florida Driving Simulator was described. While this model has proven sufficient in its current use, there are several simple modifications which can be made to improve the behavior of the model under nominal operating conditions.

The longitudinal force model determines longitudinal acceleration based upon the sum of forces as follows:

\[
\dot{u} = \begin{cases} 
rv + \frac{F_{\text{tractive}} - F_{\text{roll}} - D - F_{\text{brake}}}{m}, & P_{\text{gear}} \in \{\text{Neutral}, 1, 2, 3\} \\
rv + \frac{F_{\text{tractive}} + F_{\text{roll}} + D + F_{\text{brake}}}{m}, & P_{\text{gear}} \in \{\text{Reverse}\} 
\end{cases}
\]  

Since the rolling force \( F_{\text{roll}} \) is a constant, the longitudinal acceleration will never be in equilibrium when the car is not moving and no tractive or braking force is applied. This violates the Newtonian (and intuitive) concept that a body at rest will tend to remain at rest. It is recommended that the rolling force be made proportional to the longitudinal velocity, perhaps with some limiting value; this would be a closer approximation to the physical behavior of an automobile and should result in a more realistic behavior from the driver’s perspective.
Similarly, the braking force $F_{\text{brake}}$ is dependent only upon the force applied to the brake pedal and not upon the longitudinal velocity. Once again, this appears contrary to the observed behavior of a real-world automobile, since there is no longitudinal force applied to the vehicle when the vehicle is stopped and brake pedal is depressed. Therefore, it is recommended that the braking force also be dependent upon longitudinal velocity as well as the brake pedal force.

**Topics for Further Study and Research**

There are numerous topics that this research has only touched upon briefly but which deserve further attention. These issues, while not central to the focus of this paper, are promising candidates for research which could further improve the techniques described here. The subjects below are presented in order of their relation to the appropriate sections of this paper as they appear.

**Applications in Multiple Integrations**

In defining the assumptions by which the general form of a real-time numerical integrator would be described, it was assumed that only single integrations would be considered. This assumption helps to simplify the scope of the research, but it also ignores the fact that most dynamics models rely upon multiple integrations in order to obtain positional information from accelerations. A technique that is often used is to place two integrators in tandem, particularly the explicit and implicit Euler integrators since their phase errors at the output of the second integrator are canceled (Harbor 1988, Howe 1990). The errors from the first integrator—whose output is used elsewhere in the model—are not canceled, and these errors are propagated back into the model.

One method of using the techniques developed in this paper towards a solution to this problem is to make the double integration parallel redundant as illustrated in
Figure 101. Now, the single integration and the double integration can be optimized independently without having to compromise one for the other. This comes at the expense of performing the first integration redundantly, although the double integration can be combined into a single, more efficient computation since the intermediate result is no longer needed.

Application in Multirate Simulations

Another assumption made with regard to the general form of a numerical integrator was that only fixed-rate simulations would be considered. While this assumption has been maintained throughout the body of this paper, it was noted in Chapter 3 that some of the techniques that were developed may be highly relevant to multirate simulations. This was pursued briefly in Appendix E using the missile simulation as an example.

This topic is one which, if explored further, could lead to significant improvements in simulator efficiency. Since this subject was only discussed briefly in order to identify its feasibility, there are many aspects which are worthy of further study. One of these aspects is the manner in which the integrator outputs are extrapolated between updates, since other integrators may require the former integrator's value for derivative evaluations between updates. The example in
Appendix E used only zero-order extrapolation (sample and hold), which produced mixed results. Additional study in this area could lead to improved simulations using multirate methods.

Selection of Model Stimuli

In order to determine cutoff frequencies of individual integrators, it is necessary to exercise the model at or near the envelope of its operational capacity. This is done by driving the system with inputs which cause such extreme behavior in the model. Determining what kinds of inputs will cause this behavior is not always obvious, especially with regard to complex, nonlinear systems.

Exploration into the selection of input stimuli which induce extreme behavior in the model would be important in making these techniques practical for use with sophisticated simulation models. Such research would help ensure that a cutoff frequency is not selected so low as to omit high frequency modes resulting in inaccurate or even unstable behavior by the improperly designed integrator.

Selection of DFT Resolution and Cutoff Percentage

Two parameters which have a significant effect on the determination of the cutoff frequency of an integrator are the number of points in the discrete Fourier transform (DFT) and the cutoff percentage $P_C$. In this paper, the value of $P_C$ was chosen arbitrarily so as to include a relatively large portion of the energy spectra. The number of points in the DFT was chosen to be large enough so as to reduce aliasing to a reasonably low level.

Both of these parameters, while having a significant bearing on the accuracy and effectiveness of the simulation, have been selected by rather subjective methods. A more analytical and objective approach to the selection of these values would help improve the repeatability and consistency of the integrator specification process. This
is because the integrator cutoff frequency, which these parameters greatly influence, has as much bearing upon the design of the synthesized integrator as the specification of its error metrics. Once again, further research into a logical selection process for these values can help avoid problems with accuracy and stability during operational use of the simulator.

**Summary**

The techniques developed in this report have been reasonably demonstrated to provide improvements in the performance and efficiency of numerical integrators used in the real-time digital simulation of continuous systems. These improvements have achieved the goal of this research, which was to allow complex, realistic models to be simulated without the need for a more expensive implementation. It is hoped that the results of this research help to allow simulation to be used in a wider variety of applications and with greater fidelity and lower cost than would otherwise be possible.
APPENDICES
APPENDIX A  
DERIVATION OF FIRST-ORDER EXPLICIT INTEGRATOR  
ERROR METRICS

From Equation (6), the difference equation for a general first-order explicit integrator is

\[ y_k = a_1 y_{k-1} + b_1 T u_{k-1}, \]  

(105)

and the z-domain transfer function is

\[ W(z) = \frac{b_1}{z - a_1}. \]  

(106)

Gain Error

The gain error for the general first-order explicit integrator is determined by substituting Equation (106) into the definition for gain error described by Equation (28) yielding

\[ \varepsilon_{\text{gain}}(\omega T) = \ln \omega T \frac{b_1}{|e^{j\omega T} - a_1|}. \]  

(107)

In order to represent this in a more palatable form, the complex exponential is first converted to Cartesian form, yielding

\[ \varepsilon_{\text{gain}}(\omega T) = \ln \omega T \frac{b_1}{|\cos \omega T - a_1 + j\sin \omega T|}. \]  

(108)

Computing the absolute value by taking the square root of the sum of the squares of the real and imaginary parts and reducing yields the gain error for the general first-order explicit integrator in its final form,
Phase Error

The phase error for the general first-order explicit integrator is determined by substituting Equation (106) into the definition for phase error described by Equation (32) yielding

$$\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( T \frac{b_1}{e^{j\omega T} - a_1} \right). \quad (110)$$

Once again, the complex exponential is first converted to Cartesian form, yielding

$$\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( T \frac{b_1}{\cos \omega T - a_1 + j\sin \omega T} \right). \quad (111)$$

The imaginary term is moved from the denominator to the numerator, and extraneous constants are removed. The resulting equation is now

$$\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( \cos \omega T - a_1 - j\sin \omega T \right). \quad (112)$$

The final form of phase error for the general first-order explicit integrator can now be expressed as

$$\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} - \tan^{-1} \left( \frac{\sin \omega T}{\cos \omega T - a_1} \right). \quad (113)$$

Explicit Euler Integration

The explicit Euler integrator is a special case of the general first-order explicit integrator as described in Chapter 1. Thus, its gain error is

$$\varepsilon_{\text{gain}}(\omega T) = \ln \left( \frac{\omega T}{\sqrt{2 - 2\cos \omega T}} \right), \quad (114)$$

and its phase error is
\[ \varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\sin \omega T}{\cos \omega T - 1}\right). \]  

The gain error defined by Equation (114) is illustrated in Figure 102, and the phase error defined by Equation (115) is illustrated in Figure 103. The following sections graphically depict the maximum absolute, integral absolute, and integral squared error metrics for gain and phase errors of the explicit Euler integrator.

**Gain Error Metrics**

Note that the gain error shown in Figure 102 is nonnegative and monotonically increasing with respect to \( \omega T \); therefore, the maximum absolute gain error has the same shape as the gain error curve shown above. This is illustrated below in Figure 102.

![Figure 102. Explicit Euler integrator gain error](image)

**Figure 102. Explicit Euler integrator gain error**

![Figure 103. Explicit Euler integrator phase error](image)

**Figure 103. Explicit Euler integrator phase error**
104. Integral absolute and integral squared gain errors are shown below in Figure 105 and Figure 106, respectively.

**Phase Error Metrics**

Figure 107 illustrates the maximum absolute phase error for the explicit Euler integrator. Note that because the phase error is nonpositive and monotonically decreasing, the maximum absolute phase error is merely the absolute value of the phase error, which merely inverts the graph. Figure 108 and Figure 109 illustrate the integral absolute and integral squared phase errors.

![Figure 104. Explicit Euler integrator maximum absolute gain error](image)

**Figure 104. Explicit Euler integrator maximum absolute gain error**

![Figure 105. Explicit Euler integrator integral absolute gain error](image)

**Figure 105. Explicit Euler integrator integral absolute gain error**
Figure 108. Explicit Euler integral absolute phase error

Figure 107. Explicit Euler integral maximum absolute phase error

Figure 106. Explicit Euler integral squared gain error
Figure 109. Explicit Euler integrator integral squared phase error
APPENDIX B
DERIVATION OF FIRST-ORDER IMPLICIT INTEGRATOR ERROR METRICS

From Equation (6), the difference equation for a general first-order implicit integrator is

\[ y_k = a_1 y_{k-1} + b_0 T u_k + b_1 T u_{k-1} , \]

and the z-domain transfer function is

\[ W(z) = \frac{b_0 z + b_1}{z - a_1} . \]

Gain Error

The gain error for the general first-order implicit integrator is determined by substituting Equation (117) into the definition for gain error described by Equation (28) yielding

\[ \varepsilon_{\text{gain}}(\omega T) = \ln \omega T \left| \frac{b_0 e^{j\omega T} + b_1}{e^{j\omega T} - a_1} \right| . \]

In order to represent this in a more palatable form, the complex exponentials are first converted to Cartesian form, yielding

\[ \varepsilon_{\text{gain}}(\omega T) = \ln \omega T \left| \frac{b_0 \cos \omega T + b_1}{\cos \omega T - a_1} + j \frac{b_0 \sin \omega T}{\cos \omega T + j \sin \omega T} \right| . \]

Computing the absolute value by taking the square root of the sum of the squares of the real and imaginary parts and reducing yields the gain error for the general first-order implicit integrator in its final form,
The phase error for the general first-order explicit integrator is determined by substituting Equation (117) into the definition for phase error described by Equation (32) yielding

\[
\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( \frac{b_0 e^{j\omega T} + b_1}{e^{j\omega T} - a_1} \right). \tag{121}
\]

Once again, the complex exponentials are first converted to Cartesian form, yielding

\[
\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( \frac{b_0 \cos \omega T + b_1 + j b_0 \sin \omega T}{\cos \omega T - a_1 + j \sin \omega T} \right). \tag{122}
\]

The imaginary term is moved from the denominator to the numerator, and extraneous constants are removed. The resulting equation is now

\[
\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( \frac{b_0 \cos \omega T + b_1 + j (a_1 b_0 + b_1) \sin \omega T}{\cos \omega T - a_1 + j \sin \omega T} \right). \tag{123}
\]

The final form of phase error for the general first-order explicit integrator can now be expressed as

\[
\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} - \tan^{-1} \left( \frac{(a_1 b_0 + b_1) \sin \omega T}{b_0 - a_1 b_1 + (b_1 - a_1 b_0) \cos \omega T} \right). \tag{124}
\]

Implicit Euler Integration

The implicit Euler integrator is a special case of the general first-order implicit integrator as described in Chapter 1. Thus, its gain error is

\[
\varepsilon_{\text{gain}}(\omega T) = \ln \left( \frac{\omega T}{\sqrt{2 - 2 \cos \omega T}} \right). \tag{125}
\]
and its phase error is

\[ e_{\text{phase}}(\omega T) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\sin \omega T}{1 - \cos \omega T}\right). \] (126)

The gain error defined by Equation (125) is the same as that of the explicit Euler integrator described by Equation (114) in Appendix A; it is illustrated in Figure 102. The phase error defined by Equation (126) is illustrated in Figure 110.

Because the gain error for the implicit Euler integrator is the same as that of the explicit Euler integrator, the maximum absolute, integral absolute, and integral squared errors are also the same. These are illustrated in Appendix A.

Similarly, the phase error for the implicit Euler integrator described by Equation (126) and illustrated in Figure 110 is the same as that of the explicit Euler integrator derived in Appendix A except that one is the negative of the other. Because the maximum absolute, integral absolute, and integral squared metrics rely upon the absolute value of the phase error, these metrics will be identical.

---

**Figure 110. Implicit Euler integrator phase error**
APPENDIX C

DERIVATION OF SECOND-ORDER EXPLICIT INTEGRATOR ERROR METRICS

From Equation (6), the difference equation for a general second-order explicit integrator is

\[ y_k = a_1y_{k-1} + a_2y_{k-2} + b_1 T u_{k-1} + b_2 T u_{k-2}, \]  

(127)

and the z-domain transfer function is

\[ W(z) = T \frac{b_1 z + b_2}{z^2 - a_1 z - a_2}. \]  

(128)

**Gain Error**

The gain error for the general second-order explicit integrator is determined by substituting Equation (128) into the definition for gain error described by Equation (28) yielding

\[ \varepsilon_{\text{gain}}(\omega T) = \ln \omega T \left| \frac{b_1 e^{j\omega T} + b_2}{e^{2j\omega T} - a_1 e^{j\omega T} - a_2} \right|. \]  

(129)

In order to represent this in a more palatable form, the complex exponentials are first converted to Cartesian form, yielding

\[ \varepsilon_{\text{gain}}(\omega T) = \ln \omega T \left| \frac{b_1 \cos \omega T + b_2 + j b_1 \sin \omega T}{\cos 2\omega T - a_1 \cos \omega T - a_2 + j \sin 2\omega T - ja_1 \sin \omega T} \right|. \]  

(130)

Computing the absolute value by taking the square root of the sum of the squares of the real and imaginary parts and reducing yields the gain error for the general second-order explicit integrator in its final form,
The phase error for the general second-order explicit integrator is determined by substituting Equation (128) into the definition for phase error described by Equation (32) yielding

$$\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( T \frac{b_1 e^{j\omega T} + b_2}{e^{2j\omega T} - a_1 e^{j\omega T} - a_2} \right). \quad (132)$$

Once again, the complex exponentials are first converted to Cartesian form, yielding

$$\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( T \frac{b_1 \cos \omega T + b_2 + j b_1 \sin \omega T}{\cos 2\omega T - a_1 \cos \omega T - a_2 + j \sin 2\omega T - j a_1 \sin \omega T} \right). \quad (133)$$

The imaginary terms are moved from the denominator to the numerator, and extraneous constants are removed. The resulting equation is now

$$\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left\{ \frac{b_1 \cos \omega T + b_2 \cos 2\omega T +}{-a_1 b_1 - a_2 b_2 + (b_1 - a_2 b_1 - a_1 b_2) \cos \omega T + b_2 \cos 2\omega T + \right. \left. j \left[ (a_1 b_2 - a_2 b_1 - b_1) \sin \omega T - b_2 \sin 2\omega T \right] \right\}. \quad (134)$$

The final form of phase error for the general second-order explicit integrator can now be expressed as

$$\varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} - \tan^{-1} \left( \frac{(a_1 b_2 - a_2 b_1 - b_1) \sin \omega T - b_2 \sin 2\omega T}{a_1 b_1 + a_2 b_2 + (a_1 b_2 + a_2 b_1 - b_1) \cos \omega T - b_2 \cos 2\omega T} \right). \quad (135)$$

**Second-Order Adams-Bashforth Integration**

The second-order Adams-Bashforth integrator is a special case of the general second-order explicit integrator as described in Chapter 1. Thus, its gain error is
\[ \varepsilon_{\text{gain}}(\omega T) = \ln \left( \frac{\omega T}{2} \sqrt{\frac{5 - 3 \cos \omega T}{1 - \cos \omega T}} \right), \quad (136) \]

and its phase error is

\[ \varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \tan^{-1}(\frac{4 \sin \omega T - \sin 2 \omega T}{3 - 4 \cos \omega T + \cos 2 \omega T}), \quad (137) \]

The gain error defined by Equation (136) is illustrated in Figure 111, and the phase error defined by Equation (137) is illustrated in Figure 112. The following sections graphically depict the maximum absolute, integral absolute, and integral squared error metrics for gain and phase errors of the second-order Adams-Bashforth integrator.

**Figure 111.** Second-order Adams-Bashforth integrator gain error

**Figure 112.** Second-order Adams-Bashforth integrator phase error
Gain Error Metrics

Note that the gain error shown in Figure 111 is nonnegative and monotonically increasing with respect to $\omega T$; therefore, the maximum absolute gain error has the same shape as the gain error curve shown above. This is illustrated below in Figure 113. Integral absolute and integral squared gain errors are shown below in Figure 114 and Figure 115, respectively.

Phase Error Metrics

Figure 116 illustrates the maximum absolute phase error for the explicit Euler integrator. Note that because the phase error is nonpositive and monotonically increasing with respect to $\omega T$.
decreasing, the maximum absolute phase error is merely the absolute value of the phase error, which merely inverts the graph. Figure 117 and Figure 118 illustrate the integral absolute and integral squared phase errors.

Figure 115. Second-order Adams-Bashforth integrator integral squared gain error

Figure 116. Second-order Adams-Bashforth integrator maximum absolute phase error
Figure 117. Second-order Adams-Bashforth integrator integral absolute phase error

Figure 118. Second-order Adams-Bashforth integrator integral squared phase error
APPENDIX D
DERIVATION OF SECOND-ORDER IMPLICIT INTEGRATOR ERROR METRICS

From Equation (6), the difference equation for a general second-order implicit integrator is

\[ y_k = a_1 y_{k-1} + a_2 y_{k-2} + b_0 T u_k + b_1 T y_{k-1} + b_2 T u_{k-2} \]  \hspace{1cm} (138)

and the z-domain transfer function is

\[ W(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2} \] \hspace{1cm} (139)

**Gain Error**

The gain error for the general second-order implicit integrator is determined by substituting Equation (139) into the definition for gain error described by Equation (28) yielding

\[ \varepsilon_{\text{gain}}(\omega T) = \ln \omega T \left| \frac{b_0 e^{2j\omega T} + b_1 e^{j\omega T} + b_2}{e^{2j\omega T} - a_1 e^{j\omega T} - a_2} \right| \] \hspace{1cm} (140)

In order to represent this in a more palatable form, the complex exponentials are first converted to Cartesian form, yielding

\[ \varepsilon_{\text{gain}}(\omega T) = \ln \omega T \left| \frac{b_0 \cos 2\omega T + b_1 \cos \omega T + b_2 + j b_0 \sin 2\omega T + j b_1 \sin \omega T}{\cos 2\omega T - a_1 \cos \omega T - a_2 + j \sin 2\omega T - j a_1 \sin \omega T} \right| \] \hspace{1cm} (141)

Computing the absolute value by taking the square root of the sum of the squares of the real and imaginary parts and reducing yields the gain error for the general second-order implicit integrator in its final form,
The phase error for the general second-order implicit integrator is determined by substituting Equation (139) into the definition for phase error described by Equation (32) yielding

\[ \varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( \frac{T \left( b_0 e^{2j\omega T} + b_1 e^{j\omega T} + b_2 \right)}{e^{2j\omega T} - a_1 e^{j\omega T} - a_2} \right) . \]  

(143)

Once again, the complex exponentials are first converted to Cartesian form, yielding

\[ \varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left( \frac{b_0 \cos 2\omega T + b_1 \cos \omega T + b_2 + j b_0 \sin 2\omega T + j b_1 \sin \omega T}{\cos 2\omega T - a_1 \cos \omega T - a_2 + j \sin 2\omega T - j a_1 \sin \omega T} \right) . \]  

(144)

The imaginary term is moved from the denominator to the numerator, and extraneous constants are removed. The resulting equation is now

\[ \varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \arg \left\{ a_2 b_2 - a_1 b_1 + b_0 - (a_2 b_1 + a_1 b_2) \cos \omega T - 
(a_2 b_0 - b_2) \cos 2\omega T - (a_1 b_0 - b_1) \cos \omega T \cos 2\omega T + 
j[ (-a_1 b_0 + a_1 b_2 - a_2 b_1 - b_1) \sin \omega T - (a_2 b_0 - b_2) \sin 2\omega T - 
(a_1 b_0 - b_1) \sin \omega T \sin 2\omega T] \right\} . \]  

(145)

The final form of phase error for the general second-order implicit integrator can now be expressed as

\[ \varepsilon_{\text{phase}}(\omega T) = \frac{\pi}{2} + \tan^{-1} \left( \frac{(a_1 b_0 - a_1 b_2 + a_2 b_1 + b_1) \sin \omega T + (a_2 b_0 - b_2) \sin 2\omega T + (a_1 b_0 - b_1) \sin \omega T \sin 2\omega T}{a_1 b_1 + a_2 b_2 - b_0 + (a_2 b_1 + a_1 b_2) \cos \omega T + (a_2 b_0 - b_2) \cos 2\omega T + (a_1 b_0 - b_1) \cos \omega T \cos 2\omega T} \right) . \]  

(146)
Trapezoidal Integration

The trapezoidal integrator is a special case of the general second-order implicit integrator as described in Chapter 1. Thus, its gain error is

$$\varepsilon_{\text{gain}}(\omega T) = \ln\left(\frac{\omega T}{2} \left| \frac{\cos \omega T}{\sin \omega T} \right| \right),$$  \hspace{1cm} (147)

and its phase error is

$$\varepsilon_{\text{phase}}(\omega T) = \begin{cases} 0, & 0 < \omega T < \pi \\ \pi, & \pi < \omega T < 2\pi \end{cases}.$$  \hspace{1cm} (148)

The gain error defined by Equation (147) is illustrated in Figure 119. Because the equation for phase error is trivial, it is not illustrated. The following sections graphically depict the maximum absolute, integral absolute, and integral squared error metrics for gain errors of the trapezoidal integrator.

Gain Error Metrics

Note that the gain error shown in Figure 119 is nonpositive and monotonically decreasing with respect to $\omega T$ in the region $0 < \omega T < \pi$; therefore, the maximum absolute gain error is merely the absolute value of the gain error in this region. At $\omega T = \pi$, the gain error becomes infinite. The absolute maximum gain error illustrated in Figure

![Figure 119. Trapezoidal integrator gain error](image-url)
119 therefore only depicts the range $0 < \omega T < \pi$. Integral absolute and integral squared gain errors are shown below in Figure 121 and Figure 122, respectively.

![Figure 120](image1.png)

**Figure 120. Trapezoidal integrator maximum absolute gain error**

![Figure 121](image2.png)

**Figure 121. Trapezoidal integrator integral absolute gain error**
Figure 122. Trapezoidal integrator integral squared gain error
One of the assumptions made in Chapter 1 was that multirate simulations would not be considered. While this topic is not the focus of this paper, it is important to recognize that the techniques presented here are applicable to multirate simulations. Thus, the example Sidewinder missile simulations presented in Chapter 3 will be repeated here using multirate simulation techniques.

The method used for the multirate simulation is to step each integrator at an independent rate. Integrator outputs are extrapolated using a zero-order hold, which maintains the last value generated by the integrator until that integrator is stepped again.

**Explicit Euler Integration**

Recall from Chapter 3 that the integration step size was driven by the integral squared phase error requirement, which stipulated that

\[ \omega_C T = 0.459788 \] \hspace{1cm} (149)

Relaxing the assumption prohibiting multirate simulations, the step sizes for each of the four integrators are summarized below in Table 30. Based upon these step sizes, the maximum absolute gain error is

\[ \varepsilon_{\text{gain}, \text{max}} = 0.0088261 \] \hspace{1cm} (150)

for all four integrators. Similarly, the integral squared phase error is

\[ \varepsilon_{\text{phase, IS}} = 0.0081029 \] \hspace{1cm} (151)
Table 30. Explicit Euler integrator step sizes for multirate Sidewinder missile simulation

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_C$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>1380.9</td>
<td>0.0003330</td>
</tr>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>376.6</td>
<td>0.0012209</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>49.9</td>
<td>0.0092141</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>12.3</td>
<td>0.0373809</td>
</tr>
</tbody>
</table>

for all four integrators.

Figures 123 through 128 show the time response of the state derivatives and state variables identified in Chapter 2. The analytical solutions are displayed as a solid trace, and the simulated response is depicted by a dashed trace. In this multirate

![Figure 123](image1.png)

**Figure 123.** Multirate simulation time response of $\dot{a}_{\text{cas}}$ simulated with explicit Euler integrators

![Figure 124](image2.png)

**Figure 124.** Multirate simulation time response of $\dot{a}_{\text{cas}}$ simulated with explicit Euler integrators
simulation, only approximately 3,958 integrations per second are required in order to meet the specified requirements. Note that the CAS state variables exhibit more error than that of the fixed-rate simulation, particularly $\dot{a}_{\text{cas}}$. It is likely that the error is caused by a combination of the slower integration step of the $\dot{a}_{\text{cas}}$ integrator (which feeds back into $\dot{a}_{\text{cas}}$) together with the zero-order hold at the integrator outputs. The airframe output, however, closely tracks the analytical solution.

![Figure 125. Multirate simulation time response of $a_{\text{cas}}$ simulated with explicit Euler integrators](image)

![Figure 126. Multirate simulation time response of $\ddot{a}_{\text{air}}$ simulated with explicit Euler integrators](image)
First-Order Synthesized Explicit Integration

The integration step size for the first-order synthesized explicit integrator derived in Chapter 3 was, like the explicit Euler integrator, driven by the integral squared phase error requirement, which stipulated that

$$\omega_C T = 0.459788$$  \hspace{1cm} (152)

The step sizes for each of the four integrators in a multirate simulation are summarized below in Table 31. Based upon these step sizes, the maximum absolute gain error is

$$\varepsilon_{\text{gain, max}} = 0.0088261$$  \hspace{1cm} (153)

Figure 127. Multirate simulation time response of $a_{\text{air}}$ simulated with explicit Euler integrators

Figure 128. Multirate simulation time response of $a_{\text{air}}$ simulated with explicit Euler integrators
Table 31. First-order synthesized explicit integrator step sizes for multirate Sidewinder missile simulation

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_c$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{a}_{\text{cas}}$</td>
<td>1.3809</td>
<td>0.0003330</td>
</tr>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>0.3766</td>
<td>0.0012209</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{air}}$</td>
<td>0.0499</td>
<td>0.0092141</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>0.0123</td>
<td>0.0373809</td>
</tr>
</tbody>
</table>

for all four integrators, and the integral squared phase error is

$$E_{\text{phase,IS}} = 0.00810029$$

for all four integrators.

Figures 129 through 134 show the time response of the model variables identified in Chapter 2. The analytical solutions are displayed as a solid trace, and the simulated response is depicted by a dashed trace. Like the explicit Euler integrator case, this multirate simulation requires approximately 3,958 integrations per second to meet the specified requirements. This simulation also exhibits the same type of errors in the CAS model variables that were identified in the explicit Euler multirate simulation.

Figure 129. Multirate simulation time response of $\ddot{a}_{\text{cas}}$ simulated with first-order synthesized explicit integrators
Figure 130. Multirate simulation time response of $\dot{a}_{\text{cas}}$ simulated with first-order synthesized explicit integrators.

Figure 131. Multirate simulation time response of $a_{\text{cas}}$ simulated with first-order synthesized explicit integrators.

Figure 132. Multirate simulation time response of $\ddot{a}_{\text{air}}$ simulated with first-order synthesized explicit integrators.
Second-Order Adams-Bashforth Integration

In Chapter 3, it was determined that the integration step size for the second-order Adams-Bashforth integration was driven by the maximum absolute gain error requirement, which stipulated that

\[ \omega_c T = 0.5151 \]  \hspace{1cm} (155)

The integration step sizes for each of the four integrators in a multirate simulation are summarized below in Table 32. Based upon these step sizes, the maximum absolute gain error is

\[ \hat{a}_{\text{air}} \]

\[ a_{\text{air}} \]

\[ \text{Time (s)} \]

\[ 0 \quad 5 \quad 10 \quad 15 \quad 20 \]

\[ 7.5 \quad 5 \quad 2.5 \quad 0 \quad -2.5 \quad -7.5 \]

\[ \text{Time (s)} \]

\[ 0 \quad 5 \quad 10 \quad 15 \quad 20 \]

\[ 1.75 \quad 1.5 \quad 1.25 \quad 1 \quad 0.75 \quad 0.5 \quad 0.25 \]

**Figure 133.** Multirate simulation time response of \( \hat{a}_{\text{air}} \) simulated with first-order synthesized explicit integrators

**Figure 134.** Multirate simulation time response of \( a_{\text{air}} \) simulated with first-order synthesized explicit integrators
Table 32. Second-order Adams-Bashforth integrator step sizes for multirate Sidewinder missile simulation

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_c$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{a}_{\text{cas}}$</td>
<td>1380.9</td>
<td>0.0003730</td>
</tr>
<tr>
<td>$\dot{a}_{\text{cas}}$</td>
<td>376.6</td>
<td>0.0013678</td>
</tr>
<tr>
<td>$\ddot{a}_{\text{air}}$</td>
<td>49.9</td>
<td>0.0103226</td>
</tr>
<tr>
<td>$\dot{a}_{\text{air}}$</td>
<td>12.3</td>
<td>0.0418780</td>
</tr>
</tbody>
</table>

$$\varepsilon_{\text{gain,max}} = 0.0999916$$

(156)

for all four integrators, and the integral squared phase error is

$$\varepsilon_{\text{phase.IS}} = 0.0000709$$

(157)

for all four integrators.

Figures 135 through 140 show the time response of the model variables identified in Chapter 2. The analytical solutions are displayed as a solid trace, and the simulated response is depicted by a dashed trace. This multirate simulation requires approximately 3,533 integrations per second to meet the specified requirements. However, the errors in all of the model variables are significant, and it is unlikely that this simulation would be usable.

Figure 135. Multirate simulation time response of $\ddot{a}_{\text{cas}}$ simulated with second-order Adams-Bashforth integrators
Second-Order Synthesized Explicit Integration

In Chapter 3, the integration step size for the second-order synthesized explicit integrator was balanced between both the absolute maximum gain error and the integral squared phase error requirements, which stipulated that

$$\omega_c T = 0.93197$$ \hspace{1cm} (158)

The step sizes for each of the four integrators in a multirate simulation are summarized below in Table 33. Based upon these step sizes, the maximum absolute gain error is

$$\epsilon_{\text{gain,max}} = 0.0999992$$ \hspace{1cm} (159)

![Figure 136. Multirate simulation time response of $\dot{a}_{\text{cas}}$ simulated with second-order Adams-Bashforth integrators](image)

![Figure 137. Multirate simulation time response of $a_{\text{cas}}$ simulated with second-order Adams-Bashforth integrators](image)
Figure 138. Multirate simulation time response of $\ddot{a}_{\text{air}}$ simulated with second-order Adams-Bashforth integrators

Figure 139. Multirate simulation time response of $\dot{a}_{\text{air}}$ simulated with second-order Adams-Bashforth integrators

Figure 140. Multirate simulation time response of $a_{\text{air}}$ simulated with second-order Adams-Bashforth integrators
Table 33. Second-order synthesized explicit integrator step sizes for multirate Sidewinder missile simulation

<table>
<thead>
<tr>
<th>Integrator</th>
<th>$\omega_c$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{a}_{\text{cas}}$</td>
<td>1380.9</td>
<td>0.0006749</td>
</tr>
<tr>
<td>$\tilde{a}_{\text{cas}}$</td>
<td>376.6</td>
<td>0.0024747</td>
</tr>
<tr>
<td>$\tilde{a}_{\text{air}}$</td>
<td>49.9</td>
<td>0.0186767</td>
</tr>
<tr>
<td>$\tilde{a}_{\text{air}}$</td>
<td>12.3</td>
<td>0.0757698</td>
</tr>
</tbody>
</table>

for all four integrators, and the integral squared phase error is

$$\varepsilon_{\text{phase.IS}} = 0.0080999$$  \hspace{1cm} (160) \hspace{1cm}

for all four integrators.

Figures 141 through 146 show the time response of the state variables identified in Chapter 2. The analytical solutions are displayed as a solid trace, and the simulated response is depicted by a dashed trace. In this multirate simulation, only approximately 1,953 integrations per second are required to meet the specified requirements. However, this simulation exhibits the same poor characteristics of the multirate Adams-Bashforth simulation, with simulated time response significantly deviating from the analytical time response.

Figure 141. Multirate simulation time response of $\ddot{a}_{\text{cas}}$ simulated with second-order synthesized explicit integrators
Figure 142. Multirate simulation time response of $\dot{a}_{\text{cas}}$ simulated with second-order synthesized explicit integrators

Figure 143. Multirate simulation time response of $a_{\text{cas}}$ simulated with second-order synthesized explicit integrators

Figure 144. Multirate simulation time response of $\ddot{a}_{\text{air}}$ simulated with second-order synthesized explicit integrators
Figure 145. Multirate simulation time response of \( \dot{a}_{\text{air}} \) simulated with second-order synthesized explicit integrators

Figure 146. Multirate simulation time response of \( a_{\text{air}} \) simulated with second-order synthesized explicit integrators
REFERENCES


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<td>MAR 16 1995</td>
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