

# Pseudoquotients: Construction, Applications, And Their Fourier Transform

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PSEUDOQUOTIENTS: CONSTRUCTION, APPLICATIONS, AND  
THEIR FOURIER TRANSFORM

by

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## ABSTRACT

A space of pseudoquotients can be described as a space of either single term quotients (the injective case) or the quotient of sequences (the non-injective case) where the parent sets for the numerator and the denominator satisfy particular conditions. The first part of this project is concerned with the minimal conditions required to have a well-defined set of pseudoquotients. We continue by adding more structure to our sets and discuss the effect on the resultant pseudoquotient.

Pseudoquotients can be thought of as extensions of the parent set for the numerator since they include a natural embedding of that set. We answer some questions about the extension properties. One family of these questions involves assuming a structure (algebraic or topological) on a set and asking if the set of pseudoquotients generated has the same structure. A second family of questions looks at maps between two sets and asks if there is an extension of that map between the corresponding pseudoquotients? And if so, do the properties of the original map survive the extension?

The result of our investigations on the abstract setting will be compared with some well-known spaces of pseudoquotients and Boehmians (a particular case of non-injective pseudoquotients). We will show that the conditions discussed in the first part are satisfied and we will use that to reach conclusions about our extension spaces and the extension maps. The Fourier transform is one of the maps that we will continuously revisit and discuss.

Many spaces of pseudoquotients, Boehmians in particular, have been introduced where the initial set is a particular class of functions on either locally compact groups  $\mathbb{R}$ , [12],

and  $\mathbb{R}^{\mathbb{N}}$ , [14], or a compact group such as a sphere, [16]. The natural question is, can we generalize the construction to any locally compact group. In [18] and [13] such construction is discussed however here we go further. We use *characters* and define the Fourier transform of integrable and square integrable Boehmians on locally compact group. Then we discuss the properties of such transform. We also use the Fourier transform of a particular space of pseudoquotients to characterize Lévy measures.

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## CHAPTER ONE: INTRODUCTION

The construction of pseudoquotients is described in [10] and since then many such spaces have been considered and discussed in various papers. Our goal here is to discuss the construction of pseudoquotients with minimal conditions and study the properties by adding more structure to the basic form. The construction of pseudoquotients requires a set pair  $(X, G)$  which satisfies some conditions. The particular class of these set pairs that can produce pseudoquotients will be called  $\Sigma$ -pairs and we will discuss them in CHAPTER TWO. One of the main requirements for having a  $\Sigma$ -pair is for  $G$  to act on  $X$ . Depending on a property of this action, construction of pseudoquotients parts in two types: The first type is the injective case which is the construction resulted from a  $\Sigma$ -pair  $(X, G)$  where the action of  $G$  is injective on  $X$ . We will discuss this type first and use that to build intuition. The next case, an example for which is the construction of Boehmians, is the non-injective or the general case. In this case  $G$  does not act injectively on  $X$ .

In CHAPTER TWO we will discuss the construction of both injective and non-injective case. In each case, we will start with the case where  $X$  is just a set and gradually we add more structure to  $X$ . We first discuss the algebraic properties and then the topological properties. The repeating theme is the properties of the extensions, and the extension maps; for a  $\Sigma$ -pair  $(X, G)$ , we can think of the resultant set of pseudoquotients as an extension of  $X$  since we will show there is a natural embedding of  $X$  into that set. One set of questions we would like answered is, if  $X$  has particular algebraic or topological properties, will the pseudoquotients also have those properties. We will show that most of our fundamental properties move



nically over the extension. Second group of questions involve the properties of the extension map: Suppose  $(X, G)$  and  $(Y, H)$  are  $\Sigma$ -pairs. If we have a map between  $X$  and  $Y$ , can we extend it to a map between the corresponding extensions? If so, what properties of the maps will survive the extension and under what conditions? For the algebraic properties, for example, If  $X$  and  $Y$  are groups, will their extension also be a group? And if so, will a homomorphism between  $X$  and  $Y$  extend to a homomorphism between the extensions?

In CHAPTER THREE we will look at three examples of the injective case: One with the functions on the positive real line, one with the functions defined on all of  $\mathbb{R}$ , and one with a particular class of functions (*positive definite* functions) defined on a locally compact group. For the first example we will consider the extension of the Laplace transform whereas in the other two examples we will look at the extension of the Fourier transform.

In CHAPTER FOUR we will look at four examples of non-injective pseudoquotients. The first two sections of that chapter will focus on the construction of the space of integrable and square integrable Boehmians on a locally compact group. We will define the Fourier transform on these spaces as the extension map and study its properties in each case. In the third section of CHAPTER FOUR we will describe the construction of pseudoquotients on involutive algebras and discuss the Fourier transform of such objects. And finally in the last section, we will discuss the space of pseudoquotients that extends the space of positive definite functions on a locally compact group.

The inspiration for the generalizations specially those involving locally compact groups and the Fourier transform has come from the recent work of P. Mikusiński and D. Atanasiu, most notably those in [4], [5], and [6].

# CHAPTER TWO: THE CONSTRUCTION OF PSEUDOQUOTIENTS

## 2.1 $\Sigma$ -Pairs

To discuss the construction of pseudoquotients we first define a class of set pairs  $(X, G)$  which we call  $\Sigma$ -pairs. These are the main ingredient for construction of pseudoquotients.

**Definition 1.** We say the pair  $(X, G)$  is a  $\Sigma$ -pair if  $X$  is a nonempty set and  $G$  is a commutative semigroup *acting* on  $X$ . Here, by action of  $G$  on  $X$  we mean

1. For every  $x \in X$  and  $g \in G$ ,  $gx \in X$
2. For every  $f, g \in G$ ,  $(fg)x = f(gx)$  for all  $x \in X$ .

Condition (2) simply states that the operation in  $G$  is actually composition of functions when acting on  $X$ .

The pair  $(\mathbb{Z}, \mathbb{N})$  is an example of a  $\Sigma$ -pair if we think of elements of the semigroup  $(\mathbb{N}, \cdot)$  of natural numbers acting on the set of integers,  $\mathbb{Z}$ , by multiplication.

Having the objects  $\Sigma$ -pairs, we can define a mapping between two such objects.

**Definition 2** ( $\Sigma$ -morphism). A map  $(\mu, \alpha) : X \times G \rightarrow Y \times H$ , defined by  $(\mu, \alpha)(x, g) = (\mu x, \alpha g)$  is a  $\Sigma$ -morphism if  $\mu : X \rightarrow Y$  is a set map,  $\alpha : G \rightarrow H$  is a semigroup homomorphism, and the consistency condition  $\alpha(f)\mu(x) = \mu(fx)$  is satisfied.

What the consistency condition implies is that the following diagram commutes:

$$\begin{array}{ccc}
 X & \xrightarrow{\mu} & Y \\
 \downarrow f & & \downarrow \alpha(f) \\
 X & \xrightarrow{\mu} & Y
 \end{array}$$

Although we will not get too deep into the categorical approach to pseudoquotients, it is tempting to introduce the collection of  $\Sigma$ -pairs along with  $\Sigma$ -morphisms as a category. For that we need to check that the composition of two  $\Sigma$ -morphisms is also a  $\Sigma$ -morphism, check the existence of an identity morphism, and verify the associativity of the morphisms: For the composition, suppose  $(\mu, \alpha) : (X, G) \rightarrow (Y, H)$  and  $(\nu, \beta) : (Y, H) \rightarrow (Z, K)$  are  $\Sigma$ -morphisms. We need to check the consistency condition for  $(\nu, \beta) \circ (\mu, \alpha) : (X, G) \rightarrow (Z, K)$ . Let  $x \in X$  and  $f \in G$ . Notice  $(\nu \circ \mu)(fx) = \nu(\mu(fx))$  which implies  $\nu(\alpha(f)\mu(x)) = \beta(\alpha(f))\nu(\mu(x))$  because  $(\mu, \alpha)$  and  $(\nu, \beta)$  are  $\Sigma$ -morphisms. And of course,  $\beta(\alpha(f))\nu(\mu(x)) = (\beta \circ \alpha)(f)(\nu \circ \mu)(x)$ .

For the identity morphism, notice  $(\text{id}_X, \text{id}_G)$  is a  $\Sigma$ -morphism. Finally it is a simple verification that the associativity property is satisfied for  $\Sigma$ -morphisms.

We will use  $\Sigma$ -pairs to construct *pseudoquotients*.

## 2.2 Pseudoquotients: The Injective Case (Generalized Quotients)

In this section we will only consider injective  $\Sigma$ -pairs; we say a  $\Sigma$ -pair  $(X, G)$  is an injective one if the elements of  $G$  act injectively on  $X$ . For example the pair  $(\mathbb{N}, \mathbb{N})$  (defined similar to  $(\mathbb{Z}, \mathbb{N})$  mentioned earlier) is an injective  $\Sigma$ -pair. Notice the difference between the two  $\mathbb{N}$ 's in this pair: The first  $\mathbb{N}$  is a set and the second  $\mathbb{N}$  is a semigroup acting where the action is

defined by the multiplication of natural number.

In order to stress this difference, we denote the action associated with every  $n \in \mathbb{N}$  by  $M_n$ . Hence instead of writing  $nm$  we write  $M_n(m)$ . Our  $\Sigma$ -pair then can be written as  $(\mathbb{N}, M_{\mathbb{N}})$ , where  $M_{\mathbb{N}}$  is the semigroup of actions associated with natural numbers. The construction of the space of *pseudoquotients* from an injective  $\Sigma$ -pair is very similar to that of positive rational number from  $(\mathbb{N}, M_{\mathbb{N}})$ . We discuss this construction to build some intuition: Positive rational numbers can be described as the quotient space  $\mathbb{N} \times M_{\mathbb{N}} / \sim$  where  $(m, M_n) \sim (m', M_{n'})$  if and only if  $M_n(m') = M_{n'}(m)$  or simply put  $nm' = n'm$ . This construction extends the semigroup  $(\mathbb{N}, \cdot)$  of natural numbers to the group of positive rational number with multiplication.

We look at some other examples to have a better feel for the construction: Consider the  $\Sigma$ -pair  $(\mathbb{N}, P_{\mathbb{N}})$  where  $P_{\mathbb{N}}$  is the semigroup of actions on  $\mathbb{N}$  defined by exponentiation, i.e.  $P_n(m) = m^n$  for all  $n, m \in \mathbb{N}$ . With this action, the equivalence  $\sim$  in  $\mathbb{N} \times P_{\mathbb{N}}$  can be defined by  $(m, P_n) \sim (m', P_{n'})$  if and only if  $P_n(m') = P_{n'}(m)$  or  $m^{n'} = m'^n$ . The resulting quotient set  $\mathbb{N} \times P_{\mathbb{N}} / \sim$  is the set of natural numbers with all their natural roots.

It may be clearer if we look at a less familiar operation on natural numbers: define the action on  $\mathbb{N}$  by  $F_n(m) = m!^n$ , where we define  $!^n$  inductively by  $m!^n = (m!)!^{n-1}$ . (For example,  $3!^2 = (3!)! = 6! = 720$ .) Further, we define  $F_0$  to be the identity operator. Notice for every pair  $(n, F_m) \in \mathbb{N} \times F_{\mathbb{N}}$ , where  $F_{\mathbb{N}}$  is the semigroup of  $F_n$ 's for  $n \in \{0, 1, 2, \dots\}$ , the two components are different objects and the action of  $F_m$  on  $n$  is not a direct operation between  $n$  and  $m$ . We define  $\sim$  by  $(m, F_n) \sim (m', F_{n'})$  if and only if  $F_n(m') = F_{n'}(m)$  or  $m!^{n'} = m'!^n$ . The space of generalized quotients in this case is the quotient space  $\mathbb{N} \times F_{\mathbb{N}} / \sim$ .

An example of an element in this space is the equivalence class of  $\frac{720}{F_2}$  which also includes  $\frac{6}{F_1}$  and  $\frac{3}{F_0}$ . To get a full extension we should use a generalized definition of factorial but that is really not the purpose of this example.

The basic idea of the above examples is the same: We start with an injective  $\Sigma$ -pair  $(X, G)$  and form the product space  $X \times G$ . The extension is the quotient space  $X \times G / \sim$  where  $\sim$  is an equivalence that relies on the action of  $G$  on  $X$ . As in the aforementioned examples, we define  $\sim$  by

$$(x, g) \sim (y, h) \Leftrightarrow hx = gy.$$

Notice we always have for every  $f \in G$ ,  $(x, g) \sim (fx, fg)$  since  $fgx = gfx$  by commutativity of  $G$ . We can check that  $\sim$  defined in this way is an equivalence relation: It is clear that  $\sim$  is reflexive and symmetric. Hence all we need to check is that it is transitive. Let  $(x, f), (y, g)$ , and  $(z, h)$  be in  $(X \times G)$  with  $(x, f) \sim (y, g)$  and  $(y, g) \sim (z, h)$ . Since  $(x, f) \sim (y, g)$  then  $gx = fy$  and  $hgx = hfy$  which by commutativity of  $G$  implies  $ghx = fhy$ . Since  $(y, g) \sim (z, h)$ , we have  $ghx = fgz$  or  $hx = fz$  which implies  $(x, f) \sim (z, h)$ .

**Definition 3.** The quotient space  $(X \times G) / \sim$  is the space of generalized quotients denoted by  $\mathcal{B}(X, G)$ .

Similar to the representation of rational numbers, we will use quotients to denote elements of  $\mathcal{B}(X, G)$ . Hence instead of  $[(x, g)]$  we write  $\frac{x}{g}$  or instead of  $[(x, g)] = [(fx, fg)]$  we write  $\frac{x}{g} = \frac{fx}{fg}$ .

There is a natural embedding of  $X$  into  $\mathcal{B}(X, G)$  defined by  $x \mapsto \frac{fx}{f}$ , for any  $f \in G$ .

**Lemma 1.** *The mapping  $\iota : X \rightarrow \mathcal{B}(X, G)$  defined by  $\iota(x) = \frac{fx}{f}$  is independent of the choice of  $f$  and is injective.*

*Proof.* For any other element  $g$  of  $G$ ,  $\frac{fx}{f} = \frac{gx}{g}$  since  $gfx = fgx$ . The injectivity follows the injectivity of elements of  $G$  acting on  $X$ .  $\square$

**Theorem 1.** *Action of  $G$  can be extended to a bijective action on  $\mathcal{B}(X, G)$ , defined by*

$$g\left(\frac{x}{f}\right) = \frac{gx}{f}.$$

for every  $g \in G$ .

*Proof.* Let  $g \in G$ . Suppose  $\frac{x}{f} = \frac{y}{h}$ . We want to show  $g\left(\frac{x}{f}\right) = g\left(\frac{y}{h}\right)$  or equivalently  $\frac{gx}{f} = \frac{gy}{h}$ . The latter is true because  $hgx = fgy$  by our hypothesis and the commutativity and injectivity of  $G$ . For surjectivity, the  $g$ -preimage of every  $\frac{x}{f} \in \mathcal{B}(X, G)$  is  $\frac{x}{gf} \in \mathcal{B}(X, G)$  since  $g\left(\frac{x}{gf}\right) = \frac{gx}{gf} = \frac{x}{f}$ . To check the compatibility of action of  $G$  on  $X$  and its action on  $\mathcal{B}(X, G)$  note that for every  $g \in G$ ,

$$g(\iota(x)) = g\left(\frac{fx}{f}\right) = \frac{g(fx)}{f} = \frac{f(gx)}{f} = \iota(gx).$$

Finally for the injectivity, suppose  $\frac{gx}{f} = \frac{gx'}{f'}$ . Then  $f'(gx) = f(gx')$  or  $g(f'x) = g(fx')$ . But since  $g$  acts injectively on  $X$ ,  $f'x = fx'$  or  $\frac{x}{f} = \frac{x'}{f'}$ .  $\square$

Elements of  $G$  being bijections from  $\mathcal{B}(X, G)$  to  $\mathcal{B}(X, G)$  have inverses. Notice for every  $f \in G$ ,  $f^{-1}\frac{x}{g} = \frac{x}{fg}$ . Indeed, for every  $\frac{x}{g}$ ,

$$f\left(f^{-1}\left(\frac{x}{g}\right)\right) = f\left(\frac{x}{fg}\right) = \frac{fx}{fg} = \frac{x}{g},$$

and similarly

$$f^{-1} \left( f \left( \frac{x}{g} \right) \right) = f^{-1} \left( \frac{fx}{g} \right) = \frac{fx}{fg} = \frac{x}{g}.$$

This shows that not only  $(\mathcal{B}(X, G), G)$  is a  $\Sigma$ -pair, but it is in a sense nicer than  $(X, G)$ : Elements of  $G$  act bijectively on  $\mathcal{B}(X, G)$ .

$G$  and  $G^{-1} = \{f^{-1} : f \in G\}$  are both subsets of a group of bijections on  $\mathcal{B}(X, G)$ . There exists a smallest group  $\overline{G}$  containing  $G \cup G^{-1}$  and we have  $\overline{\overline{G}} = \overline{G}$ .

It is worth noting that since every element  $x$  of  $X$  can be thought of as the element  $\frac{gx}{g}$  of  $\mathcal{B}(X, G)$ , we have  $g^{-1}x = g^{-1} \left( \frac{gx}{g} \right) = \frac{g^{-1}gx}{g} = \frac{x}{g}$ , which justifies using the quotient notation. It is natural to ask if we can obtain more information about the embedding in Lemma 1, if  $G$  is a group. Our next result answers that question.

**Proposition 1.** *If  $G$  is a group then the embedding  $\iota$  in Lemma 1 is a bijection.*

*Proof.* If  $G$  is a group, then for any  $\frac{x}{h} \in \mathcal{B}(X)$ ,  $h^{-1}x$  is its preimage with respect to  $\iota$  since

$$\iota(h^{-1}x) = \frac{h(h^{-1}x)}{h} = \frac{x}{h}.$$

□

To add to our terminology, we say  $(X, G)$  is a  $\Gamma$ -pair if  $G$  is a group and  $(X, G)$  is a  $\Sigma$ -pair.

**Corollary 1.** *There is a bijection between  $\mathcal{B}(X, G)$  and  $\mathcal{B}(\mathcal{B}(X, G), \overline{G})$ .*

Borrowing from the category theory language, we can say that there is a functor from injective  $\Sigma$ -pairs  $(X, G)$  to injective  $\Gamma$ -pairs  $(\mathcal{B}(X, G), \overline{G})$ . Also notice that if  $(X, G)$  and

$(X, \hat{G})$  are  $\Sigma$ -pairs with  $G \subset \hat{G}$  then  $\mathcal{B}(X, G) \subseteq \mathcal{B}(X, \hat{G})$ . Hence we have the following important theorem:

**Theorem 2.**  $\mathcal{B}(X, G)$  is isomorphic to  $\mathcal{B}(\mathcal{B}(X, G), G)$ .

*Proof.* Using Corollary 1 and Theorem 1 we have

$$\mathcal{B}(X, G) \hookrightarrow \mathcal{B}(\mathcal{B}(X, G), G) \subseteq \mathcal{B}(\mathcal{B}(X, G), \overline{G}) = \mathcal{B}(X, G) \quad (2.1)$$

where the equal sign denotes a bijection. □

This theorem shows that re-applying the extension method of pseudoquotients with the same  $G$  does not generate a new space. For example, we showed that  $\mathcal{B}(\mathbb{N}, M_{\mathbb{N}})$  is the space of positive rational numbers  $\mathbb{Q}^+$ . Re-application of the extension produces the space  $\mathcal{B}(\mathbb{Q}^+, M_{\mathbb{N}})$ .

We can now extend the construction of  $\mathcal{B}$  to product spaces.

**Theorem 3.** If  $(X, G)$  and  $(Y, H)$  are  $\Sigma$ -pairs then there is a bijection between  $\mathcal{B}(X \times Y, G \times H)$  and  $\mathcal{B}(X, G) \times \mathcal{B}(Y, H)$ .

*Proof.* We define the composition of two elements of  $G \times H$  by  $(g, h)(g', h') = (gg', hh')$ . Clearly this operation is associative and commutative. Also, if  $G$  and  $H$  are semigroups, so is  $G \times H$ . The action of  $G \times H$  on  $X \times Y$  is defined by  $(g, h)(x, y) = (gx, hy)$ . We see that the elements of  $G \times H$  are well-defined injections from  $X \times Y$  to  $X \times Y$ . We claim the mapping  $\varphi$  defined by  $\frac{(x, y)}{(g, h)} \mapsto \left( \frac{x}{g}, \frac{y}{h} \right)$  is a bijection: Let  $((x, y), (g, h)) \sim ((x', y'), (g', h'))$ , which is the same thing as saying  $\frac{(x, y)}{(g, h)} = \frac{(x', y')}{(g', h')}$ . Then we have  $(gx', hy') = (g'x, h'y)$ . In



other words,  $gx' = g'x$  and  $hy' = h'y$ , or equivalently  $(x, g) \sim (x', g')$  and  $(y, h) \sim (y', h')$ .

Putting these new equalities in quotients notation, we get

$$\frac{x}{g} = \frac{x'}{g'} \text{ and } \frac{y}{h} = \frac{y'}{h'}$$

or

$$\left( \frac{x}{g}, \frac{y}{h} \right) = \left( \frac{x'}{g'}, \frac{y'}{h'} \right)$$

which implies

$$\varphi \frac{(x, y)}{(g, h)} = \varphi \frac{(x', y')}{(g', h')}$$

and thus we get  $\varphi$  is a well-defined map. Similarly we can show if  $\varphi \frac{(x, y)}{(g, h)} = \varphi \frac{(x', y')}{(g', h')}$  then we have  $\frac{(x, y)}{(g, h)} = \frac{(x', y')}{(g', h')}$  which implies  $\varphi$  is injective. Surjectivity of  $\varphi$  is clear.  $\square$

In Theorem 3, we showed that there is a bijection between two spaces  $\mathcal{B}(X \times Y, G \times H)$  and  $\mathcal{B}(X, G) \times \mathcal{B}(Y, H)$ . If we show there is a  $\Sigma$ -morphism from  $(\mathcal{B}(X \times Y, G \times H), G \times H)$  to  $(\mathcal{B}(X, G) \times \mathcal{B}(Y, H), G \times H)$ , that would imply that the following diagram commutes:

$$\begin{array}{ccc} (X, G), (Y, H) & \xrightarrow{\beta} & (\mathcal{B}(X, G), G), (\mathcal{B}(Y, H), H) \\ \downarrow \times & & \downarrow \times \\ (X \times Y, G \times H) & \xrightarrow{\beta} & (\mathcal{B}(X \times Y, G \times H), G \times H) \simeq (\mathcal{B}(X, G) \times \mathcal{B}(Y, H), G \times H) \end{array}$$

Where  $\beta : \Sigma\text{-pairs} \rightarrow \Sigma\text{-pairs}$  is defined by  $(X, G) \mapsto (\mathcal{B}(X, G), G)$  and  $\times$  as the cross-product of  $\Sigma$ -pairs is defined by  $((X, G), (Y, H)) \mapsto (X \times Y, G \times H)$ .

**Theorem 4.**  $(\mathcal{B}(X, G) \times \mathcal{B}(Y, H), G \times H)$  is a  $\Sigma$ -pair and  $(\varphi, \text{id}) : (\mathcal{B}(X \times Y, G \times H), G \times H) \rightarrow (\mathcal{B}(X, G) \times \mathcal{B}(Y, H), G \times H)$  is a  $\Sigma$ -morphism, where  $\varphi \frac{(x, y)}{(g, h)} = \left(\frac{x}{g}, \frac{y}{h}\right)$  for all  $(x, g) \in (X, G)$  and  $(y, h) \in (Y, H)$ .

*Proof.* We define the operation of  $(g', h') \in G \times H$  on  $\left(\frac{x}{g}, \frac{y}{h}\right) \in \mathcal{B}(X, G) \times \mathcal{B}(Y, H)$  by

$$(g', h') \left(\frac{x}{g}, \frac{y}{h}\right) = \left(g' \left(\frac{x}{g}\right), h' \left(\frac{y}{h}\right)\right) = \left(\frac{g'x}{g}, \frac{h'y}{h}\right).$$

The closure, injectivity, and commutativity of this action follows from those of elements of  $G$  and  $H$  acting on  $X$  and  $Y$ , respectively. Next we show  $(\varphi, \text{id})$  is a  $\Sigma$ -morphism:

$$\text{id}(g', h') \varphi \frac{(x, y)}{(g, h)} = (g', h') \left(\frac{x}{g}, \frac{y}{h}\right) = \left(\frac{g'x}{g}, \frac{h'y}{h}\right) = \varphi \frac{(g', h')(x, y)}{(g, h)}.$$

So the consistency condition holds and  $(\varphi, \text{id})$  is a  $\Sigma$ -morphism. □

The following theorem is an important result regarding the extension map.

**Theorem 5.** Suppose  $(\mu, \alpha) : (X, G) \rightarrow (Y, H)$  is a  $\Sigma$ -morphism. Then we have

1. The map  $\tilde{\mu} : \mathcal{B}(X, G) \rightarrow \mathcal{B}(Y, H)$  defined by  $\tilde{\mu} \left(\frac{x}{f}\right) = \frac{\mu(x)}{\alpha(f)}$  is a unique extension of  $\mu$  such that  $(\tilde{\mu}, \alpha) : (\mathcal{B}(X, G), G) \rightarrow (\mathcal{B}(Y, H), H)$  is a  $\Sigma$ -morphism.
2. There exists a  $\bar{\alpha} : \bar{G} \rightarrow \bar{H}$  such that  $(\tilde{\mu}, \bar{\alpha}) : (\mathcal{B}(X, G), \bar{G}) \rightarrow (\mathcal{B}(Y, H), \bar{H})$  is a  $\Sigma$ -morphism.
3. If  $\mu$  is an injection, then so is  $\tilde{\mu}$ .
4. If  $\mu$  and  $\alpha$  are surjective, then so is  $\tilde{\mu}$ .

*Proof.* 1. First we need to show this map is well-defined. Suppose  $\frac{x}{f} = \frac{y}{g}$  in  $\mathcal{B}(X, G)$ .

Then  $fy = gx$  and  $\mu(fy) = \mu(gx)$ . Since  $(\mu, \alpha)$  is a  $\Sigma$ -morphism and hence satisfies the consistency condition we have  $\alpha(f)\mu(y) = \alpha(g)\mu(x)$ . From this last equality we have  $\frac{\mu(x)}{\alpha(f)} = \frac{\mu(y)}{\alpha(g)}$ . The consistency condition for  $(\tilde{\mu}, \alpha)$  holds because  $\alpha(f)\tilde{\mu}\left(\frac{x}{g}\right) = \alpha(f)\frac{\mu(x)}{\alpha(g)} = \frac{\alpha(f)\mu(x)}{\alpha(g)}$ . Because  $(\mu, \alpha)$  is a  $\Sigma$ -morphism, this last quotient can be re-written as  $\frac{\mu(fx)}{\alpha(g)} = \tilde{\mu}\left(\frac{fx}{g}\right)$ .

Next we show that  $\tilde{\mu}$  is an extension of  $\mu$ . That is,  $\tilde{\mu}\left(\frac{fx}{f}\right) = \mu(x)$ . Notice by the definition of  $\tilde{\mu}$  we have  $\tilde{\mu}\left(\frac{fx}{f}\right) = \frac{\mu(fx)}{\alpha(f)}$ . We can change the numerator by the consistency condition and we get  $\frac{\alpha(f)\mu(x)}{\alpha(f)} = \mu(x)$ . For the uniqueness, suppose there is another extension map  $\bar{\mu} : \mathcal{B}(X, G) \rightarrow \mathcal{B}(Y, H)$  for  $\mu$  such that  $(\bar{\mu}, \alpha)$  is a  $\Sigma$ -morphism. Then we have for every  $\frac{x}{f} \in \mathcal{B}(X, G)$ ,

$$\alpha(f)\bar{\mu}\left(\frac{x}{f}\right) = \bar{\mu}(fx)$$

by the consistency condition and the fact that  $(\bar{\mu}, \alpha)$  is a  $\Sigma$ -morphism. But since  $\bar{\mu}$  is an extension of  $\mu$ , that is equal to

$$\mu(x) = \tilde{\mu}(x) = \tilde{\mu}\left(\frac{fx}{f}\right) = \alpha(f)\tilde{\mu}\left(\frac{x}{f}\right).$$

Thus we have

$$\alpha(f)\bar{\mu}\left(\frac{x}{f}\right) = \alpha(f)\tilde{\mu}\left(\frac{x}{f}\right).$$

Because  $\alpha(f) \in H$  which acts injectively on  $\mathcal{B}(Y, H)$ , we get  $\bar{\mu}\left(\frac{x}{f}\right) = \tilde{\mu}\left(\frac{x}{f}\right)$  for all  $\frac{x}{f}$ .

2. Define  $\bar{\alpha} : \bar{G} \rightarrow \bar{H}$  to be a homomorphism extension of  $\alpha$  such that  $\bar{\alpha}(f^{-1}) = (\alpha(f))^{-1}$ .

We need to show that the consistency condition is satisfied. Clearly it is true for  $f \in G$ .

It suffices to show that the condition holds for  $f^{-1}$ , where  $f \in G$ . We have

$$\bar{\alpha}(f^{-1})\tilde{\mu}\left(\frac{x}{g}\right) = (\alpha(f))^{-1}\tilde{\mu}\left(\frac{fx}{fg}\right) = (\alpha(f))^{-1}\alpha(f)\tilde{\mu}\left(\frac{x}{fg}\right) = \tilde{\mu}(f^{-1}\frac{x}{g}).$$

3. Suppose  $\tilde{\mu}\left(\frac{x}{f}\right) = \tilde{\mu}\left(\frac{y}{h}\right)$ , then by the definition of  $\tilde{\mu}$  we have  $\frac{\mu(x)}{\alpha(f)} = \frac{\mu(y)}{\alpha(h)}$ . By definition of equivalence in  $\mathcal{B}(Y, H)$   $\alpha(h)\mu(x) = \alpha(f)\mu(y)$  which by the consistency condition leads to  $\mu(hx) = \mu(fy)$ . Since  $\mu$  is an injection,  $hx = fy$  or  $\frac{x}{f} = \frac{y}{h}$ .

4. Let  $\frac{y}{h} \in \mathcal{B}(Y, H)$ . Then there exist  $x \in X$  and  $f \in G$  such that  $\mu(x) = y$  and  $\alpha(f) = h$ .

Hence  $\tilde{\mu}\left(\frac{x}{f}\right) = \frac{\mu(x)}{\alpha(f)} = \frac{y}{h}$ . So  $\tilde{\mu}$  is a surjection.

□

**Remark 1.** Simply speaking, this theorem establishes the following important extension property: Set properties such as injection and surjection for maps between sets  $X$  and  $Y$  extend to the extension maps between  $\mathcal{B}(X, G)$  to  $\mathcal{B}(Y, H)$ . As we will see this is also the case for the general pseudoquotients. In the next section we will further show that the extension map will preserve some algebraic and topological properties but for now the bijection will suffice since our sets need not have any particular properties.

We can use Theorem 5 to prove the following corollary that shows the application of the pseudoquotient construction to a set  $X$  and to  $gX$ , where  $g$  is an element of the acting set  $G$ , does not produce different spaces.

**Corollary 2.** For every  $\Sigma$ -pair  $(X, G)$ ,  $\mathcal{B}(X, G) = \mathcal{B}(gX, G)$  for all  $g \in G$ .

*Proof.* We claim  $(g, \text{id}_G) : (X, G) \rightarrow (gX, G)$  is a  $\Sigma$ -morphism. All we need to show is the

consistency condition:

$$\text{id}_G(f)g(x) = fgx = g(fx)$$

by commutativity of  $G$ . Because  $g$  is a bijection from  $X$  to  $gX$ , the extension to  $\bar{g} : \mathcal{B}(X, G) \rightarrow \mathcal{B}(gX, G)$  is a bijection.  $\square$

To better understand the construction of pseudoquotients,  $\Sigma$ -morphisms and an application of Corollary 2 we look at an example. In this example we will use the traditional definition of the Fourier transform of functions on  $\mathbb{R}$  and we denote it by  $\mathcal{F}$  or by  $\hat{\cdot}$ .

**Example 1.** Let  $X$  be the space  $L^2(\mathbb{R})$  of square integrable functions on  $\mathbb{R}$ . If we let  $\mathcal{S}(\mathbb{R})$  denote the space of rapidly decreasing functions on  $\mathbb{R}$  (i.e.  $\mathcal{S}(\mathbb{R})$  is the space of all functions  $\varphi$  such that for every  $m, n \in \mathbb{N}$ ,  $\sup_{m,n} \|x^m D^n \varphi\|_\infty < \infty$ ), then let  $G = \mathcal{S}_0(\mathbb{R}) = \{\varphi \in \mathcal{S}(\mathbb{R}), \text{supp } \hat{\varphi} = \mathbb{R}\}$ . Let  $\hat{X} = \{\hat{f} : f \in X\}$  and  $\hat{G} = \{\hat{\varphi} : \varphi \in G\}$ .  $(X, G)$  and  $(\hat{X}, \hat{G})$  are  $\Sigma$ -pairs where  $G$  acts on  $X$  by convolution and  $\hat{G}$  acts on  $\hat{X}$  by pointwise multiplication.  $(\mathcal{F}, \mathcal{F}) : (X, G) \rightarrow (\hat{X}, \hat{G})$  is a  $\Sigma$ -morphism: For every  $f \in X$  and  $\varphi \in G$ ,  $\mathcal{F}(\varphi * f) = \mathcal{F}(\varphi)\mathcal{F}(f)$ , so we have the consistency condition. By Theorem 5, there is a bijection between  $\mathcal{B}(X, G)$  and  $\mathcal{B}(\hat{X}, \hat{G})$ . Hence instead of working with convolution quotients we can work with regular fractions. (i.e. pointwise division of functions) Another interesting observation on this example is that by Corollary 2 we can say that  $\mathcal{B}(L^2(\mathbb{R}), \mathcal{S}_0(\mathbb{R})) = \mathcal{B}(\varphi * L^2(\mathbb{R}), \mathcal{S}_0(\mathbb{R}))$ . The advantage is that elements of  $\varphi * L^2(\mathbb{R})$  are in  $\mathcal{C}^\infty(\mathbb{R})$ , which gives us a much better space to work with.

**Remark 2.** In this example, we had to restrict ourselves to those elements of  $\mathcal{S}(\mathbb{R})$  with Fourier transforms that have support  $\mathbb{R}$ . We need to do this because  $\mathcal{S}(\mathbb{R})$  does not act

injectively on  $L^2(\mathbb{R})$  but  $\mathcal{S}_0(\mathbb{R})$  does. We can see that by choosing  $\varphi \in \mathcal{S}(\mathbb{R})$  and  $f \in L^2(\mathbb{R})$  such that  $\hat{\varphi}$  and  $\hat{f}$  have disjoint supports. (Clearly  $\varphi * f = \hat{\varphi}\hat{f} = 0$  without  $f$  being identically zero.) But the restriction to  $\mathcal{S}_0(\mathbb{R})$  seems too strong. It would be nice if we could use all  $\mathcal{S}(\mathbb{R})$  especially because we have a nice property that  $\hat{\mathcal{S}}(\mathbb{R}) = \mathcal{S}(\mathbb{R})$ .

We can demonstrate the same problem with a more elementary example. This example was also mentioned in [14]:

**Example 2.** Let  $X = L_{\text{loc}}(\mathbb{R})$  and let  $G = L_c^1(\mathbb{R})$ , the space of integrable functions with compact support.  $G$  acts on  $X$  by convolution  $*$  defined by

$$(\varphi * f)(x) = \int_{\mathbb{R}} \varphi(t)f(x-t)dt$$

for every  $\varphi \in L_c^1(\mathbb{R})$  and  $f \in L_{\text{loc}}(\mathbb{R})$ . Clearly this is a well-defined action. This is an interesting case and potentially a nice structure except there is a problem: The action is not injective. For example, the convolution of the locally integrable function  $\sin(x)$  with  $\chi_{[0,2\pi]}$  (the characteristic function of  $[0, 2\pi]$ ) is 0 without either one of the two functions be identically 0.

To deal with the situations where the action of  $G$  on  $X$  is not injective we define the construction of pseudoquotients more generally.

### 2.3 Pseudoquotients: The Non-injective Case

In cases where in a  $\Sigma$ -pair  $(X, G)$  the action of commutative semigroup  $G$  on  $X$  is not injective, we define a more general space of pseudoquotients.

Let  $I$  be an index set. Consider the sets  $X^I$  and  $G^I$ . We use maps  $\bar{x} : I \rightarrow X$  and  $\bar{f} : I \rightarrow G$  to represent the elements of each set, respectively. Notice that in the case of  $I = \mathbb{N}$ ,  $\bar{x}$  is a sequence:  $\bar{x} = (x_1, x_2, \dots)$ . Extending this notation to the case of other index sets we denote  $\bar{x}(i)$  by  $x_i$ . The same notation will be used for elements of  $G^I$ .

We define the following three operations:

1. For every  $\bar{g} \in G^I$  and  $\bar{x} \in X^I$ , we define  $\bar{g}\bar{x}$  by  $(\bar{g}\bar{x})(i) = g_i x_i$ .
2. For every  $g \in G$  and  $\bar{x} \in X^I$ , we define  $g\bar{x}$  by  $(g\bar{x})(i) = g x_i$ .
3. For every  $\bar{g} \in G^I$  and  $x \in X$ , we define  $\bar{g}x$  by  $(\bar{g}x)(i) = g_i x$ .

Notice that (2) and (3) are extensions of (1) where  $\bar{g}$  and  $\bar{x}$  are constant maps. Furthermore, the operation in  $G^I$  is defined by  $(\bar{f}\bar{g})(i) = f_i g_i$  for all  $\bar{f}, \bar{g} \in G^I$ . This operation is commutative because it extends the commutative operation on  $G$ . Let  $\mathcal{X} \subset X^I$  be such that  $\mathcal{X}$  contains all the constant maps and  $\bar{f}\bar{x} \in \mathcal{X}$  for all  $\bar{f} \in \Delta$  and all  $\bar{x} \in \mathcal{X}$ . Clearly such a set always exist since  $X^I$  is one such set. We will call the pair  $(\mathcal{X}, \Delta)$  a  $\Delta$ -extension of  $(X, G)$ .

More concisely we have:

**Definition 4** ( $\Delta$ -extension). For a  $\Sigma$ -pair  $(X, G)$  with index set  $I$ , we say  $(\mathcal{X}, \Delta)$  is a  $\Delta$ -extension of  $(X, G)$  if

1.  $\mathcal{X} \subset X^I$  containing all constant maps.
2.  $\Delta \subset G^I$  is a semigroup acting on  $\mathcal{X}$
3. For every  $\bar{f} \in \Delta$  and  $\bar{x} \in \mathcal{X}$ ,  $\bar{f}\bar{x} \in \mathcal{X}$ .

Clearly  $(\mathcal{X}, \Delta)$  is a  $\Sigma$ -pair. Without any extra conditions these  $\Sigma$ -pairs do not produce interesting results. Hence we impose the extra condition of  $\Delta$  acting injectively on the constant sets. Injectivity in this sense means for every  $x$  and  $y$  in  $X$ ,  $\bar{f}x = \bar{f}y$  implies  $x = y$ . Such  $\Delta$  may not exist, however a sufficient condition on  $G$  is the *totality* condition, as it was called in [15], described the following way: if  $x$  and  $y$  are distinct elements in  $X$  then there exists a  $f \in G$  such that  $fx \neq fy$ . With the injectivity condition we have the following:

**Theorem 6.** *Suppose  $(\mu, \alpha) : (X, G) \rightarrow (Y, H)$  is a  $\Sigma$ -morphism. We can extend the operation of  $\mu$  to  $\mu : \mathcal{X} \rightarrow \mathcal{Y}$  by  $(\mu\bar{x})(i) = \mu x_i$ . Similarly, we can extend the operation of  $\alpha$  to  $\alpha : \Delta_G \rightarrow \Delta_H$  by  $(\alpha\bar{f})(i) = \alpha f_i$  for all  $i \in I$ . Then the mapping  $(\mu, \alpha) : (\mathcal{X}, \Delta_G) \rightarrow (\mathcal{Y}, \Delta_H)$  is a  $\Sigma$ -morphism. (Note that implicitly we are assuming  $\alpha(\Delta_G) \subseteq \Delta_H$ .)*

*Proof.* We can verify that the consistency condition  $\alpha(\bar{f})\mu(\bar{x}) = \mu(\bar{f}\bar{x})$  is satisfied: For every  $i \in I$ ,  $(\alpha(\bar{f})\mu(\bar{x}))(i) = (\alpha(\bar{f}))(i)(\mu(\bar{x}))(i) = \alpha(f_i)\mu(x_i)$ . Because  $(\mu, \alpha)$  is a  $\Sigma$ -morphism, this last expression is equal to  $\mu(f_i x_i) = \mu((\bar{f}\bar{x})(i)) = (\mu(\bar{f}\bar{x}))(i)$ .  $\square$

What we have shown with this theorem is that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mu} & \mathcal{Y} \\ \downarrow \bar{f} & & \downarrow \alpha(\bar{f}) \\ \mathcal{X} & \xrightarrow{\mu} & \mathcal{Y} \end{array}$$

Having the  $\Sigma$ -pair  $(\mathcal{X}, \Delta)$  we can continue with the construction of pseudoquotients: Let  $\mathcal{A}$  be defined by

$$\mathcal{A} = \{(\bar{x}, \bar{f}) \in \mathcal{X} \times \Delta : f_i x_j = f_j x_i \text{ for all } i, j \in I\}.$$



The property  $f_i x_j = f_j x_i$  for all  $i, j \in I$  is called the exchange property. We define the relation  $\sim$  on  $\mathcal{A}$  by

$$(\bar{x}, \bar{f}) \sim (\bar{y}, \bar{g}) \Leftrightarrow g_j x_i = f_i y_j \text{ for all } i, j \in I.$$

We can verify that  $\sim$  is an equivalence on  $\mathcal{A}$ : By the definition of  $\mathcal{A}$  it is clear that  $\sim$  is reflexive and symmetric. We can verify the transitivity. Suppose

$$(\bar{x}, \bar{f}) \sim (\bar{y}, \bar{g}) \text{ and } (\bar{y}, \bar{g}) \sim (\bar{z}, \bar{h})$$

Then

$$g_i x_j = f_j y_i \text{ and } g_k z_l = h_l y_k$$

for all  $i, j, k, l \in I$ . For every  $k \in I$  we have

$$g_k z_i = h_i y_k$$

or

$$f_j g_k z_i = f_j h_i y_k,$$

where  $j \in I$ . Using the commutativity of  $G$  we get

$$f_j g_k z_i = h_i f_j y_k$$

or

$$f_j g_k z_i = h_i g_k x_j.$$

Once again using the commutativity of  $G$  and the fact that  $\Delta$  is injective on  $X$  we get

$$f_j z_i = h_i x_j.$$

We define the space of pseudoquotients  $\mathcal{B}(\mathcal{X}, \Delta)$  to be the quotient space  $\mathcal{A}/\sim$ . A typical element of this space is denoted by  $\frac{\bar{x}}{\bar{f}}$  which represents the equivalence class  $[(\bar{x}, \bar{f})]$  in  $\mathcal{A}$ . Similar to the injective case of pseudoquotients, we have the embedding  $\iota : X \rightarrow \mathcal{B}(\mathcal{X}, \Delta)$  defined by  $\iota(x) = \frac{\bar{f}x}{\bar{f}}$ , for some  $\bar{f} \in \Delta$ . To see that  $\iota$  is independent of the choice of  $\bar{f}$ , note that for any other  $\bar{g} \in \Delta$ ,  $\frac{\bar{f}x}{\bar{f}} = \frac{\bar{g}x}{\bar{g}}$  since  $g_j(f_i x) = f_i(g_j x)$  for all  $i, j \in I$  by the commutativity of  $G$ . We have the following property of the embedding  $\iota$ :

**Lemma 2.** *If the embedding  $\iota : X \rightarrow \mathcal{B}(\mathcal{X}, \Delta)$  is injective.*

*Proof.* Suppose  $\frac{\bar{f}x}{\bar{f}} = \frac{\bar{f}y}{\bar{f}}$ , then we have for every  $i, j \in I$ ,  $f_j(f_i x) = f_i(f_j y) = f_j(f_i y)$ . Since  $\Delta$  is injective on  $X$ ,  $x = y$ . □

**Example 3** (The space of  $\mathcal{B}(\mathbb{R})$ ). The space of Boehmians on  $\mathbb{R}$  for the first time was introduced in [10]. Here we describe this construction: Let  $X = \mathcal{C}(\mathbb{R})$  and let  $G = \mathcal{C}_c^+(\mathbb{R})$  (continuous non-negative functions with compact support over the real line). Action of  $G$  on  $X$  is convolution  $*$ . Note that  $(G, *)$  is a commutative semigroup. Next we define  $\Delta$  be the set of *delta-sequences*; we say a sequence  $\{\varphi_n\}$  of functions in  $\mathcal{C}_c^+(\mathbb{R})$  is a delta-sequence if it satisfies the following conditions:

1.  $\int_{\mathbb{R}} \varphi_n d\mu = 1$ ,

2. For every neighborhood  $U$  of 0 there exists  $n_0 \in \mathbb{N}$  such that for every  $n \geq n_0$ ,

$$\int_{U^c} \varphi_n d\mu = 0.$$

Using the language introduced in the construction of pseudoquotients, we have

$$\mathcal{A} = \{(\bar{f}, \bar{\varphi}) : f_n \in X, \varphi_n \in G, \varphi_n * f_m = \varphi_m * f_n, \forall m, n \in \mathbb{N}\}$$

and the equivalence  $\sim$  on  $\mathcal{A}$  is defined by

$$(\bar{f}, \bar{\varphi}) \sim (\bar{g}, \bar{\psi}) \Leftrightarrow \psi_n * f_m = \varphi_m * g_n \quad \forall m, n \in \mathbb{N}.$$

The space of Boehmians on  $\mathbb{R}$  denoted by  $\mathcal{B}(\mathcal{C}(\mathbb{R}), G)$ , or by just  $\mathcal{B}(\mathbb{R})$ , is the quotient space  $\mathcal{A}/\sim$ .

Taking a closer look at the construction of pseudoquotient in this general sense, we notice a not so subtle discrepancy with the injective case. For the construction of  $\mathcal{B}(X, G)$ , we start with a  $\Sigma$ -pair  $(X, G)$  where  $G$  acts on  $X$  injectively. However, in the general case the  $\Delta$  of the  $\Sigma$ -pair  $(\mathcal{X}, \Delta)$  does not act injectively on all of  $\mathcal{X}$  and we only require it to act injectively on constant sequences (i.e. on  $X$ ). The reasoning for this not so natural restriction is that in the former case, if we force  $\Delta$  be injective as actions on  $\mathcal{X}$ , then the construction reduces to the injective case. On the other hand if we take away the injection requirement the relation  $\sim$  will not be an equivalence (the transitivity will fail). Bringing up this point about our definition of general pseudoquotients is not to say it is a handicap. In fact this definition fits many of our examples nicely. However the discrepancy is a technical point that is worth noting.

Comparing the two structure of pseudoquotients, the injective and the general case, we see that the injective case is a special case of the general case where  $I = \{1\}$ . The natural question here would be, how many of the set properties we showed for the injective case will generalize.

One of the properties we are interested in is the one involving the re-application of the construction with the same  $\Delta$ . We first show that this process can be done. The

operation of  $G : X \rightarrow X$  can be extended to  $G : \mathcal{B}(\mathcal{X}, \Delta) \rightarrow \mathcal{B}(\mathcal{X}, \Delta)$ : For every  $g \in G$  we defined  $g\left(\frac{\bar{x}}{\bar{f}}\right) = \frac{g\bar{x}}{\bar{f}}$ . Note that  $\frac{g\bar{x}}{\bar{f}} \in \mathcal{B}(\mathcal{X}, \Delta)$  since for every  $i, j \in I$ , we have  $f_j(gx_i) = g(f_jx_i) = g(f_ix_j) = f_i(gx_j)$ . We can also verify that this action is well-defined: suppose  $\frac{\bar{x}}{\bar{f}} = \frac{\bar{y}}{\bar{h}}$  which means for every  $i, j \in I$ ,  $h_jx_i = f_iy_j$  which implies  $gh_jx_i = gf_iy_j$  or  $\frac{g\bar{x}}{\bar{f}} = \frac{g\bar{y}}{\bar{h}}$ . Using this extension of  $G$  we can say  $\Delta$  acts on  $(\mathcal{B}(\mathcal{X}, \Delta))^I$ . We should verify that  $\Delta$  is injective on  $\mathcal{B}(\mathcal{X}, \Delta)$ . Suppose for every  $k \in I$  we have  $\frac{g_k\bar{x}}{\bar{f}} = \frac{g_k\bar{y}}{\bar{h}}$ . Fix  $k$ , then from the definition of equivalence in  $\mathcal{B}(\mathcal{X}, \Delta)$  we have  $g_k h_j x_i = g_k f_i y_j$  for every  $i, j \in I$ . We can say this for every  $k \in I$  and since  $h_j x_i$  and  $f_i y_j$  are in  $X$ , and  $\Delta$  acts injectively on  $X$ , we have  $h_j x_i = f_i y_j$  or  $\frac{\bar{x}}{\bar{f}} = \frac{\bar{y}}{\bar{h}}$ . As before we let

$$\mathcal{A} = \{(\bar{F}, \bar{f}) \in (\mathcal{B}(\mathcal{X}, \Delta))^I \times \Delta : f_i F_j = f_j F_i \forall i, j \in I\}.$$

We define the equivalence in  $\mathcal{A}$  by:  $(\bar{F}, \bar{f}) \sim (\bar{G}, \bar{g})$  if and only if for every  $i, j \in I$ ,  $g_i F_j = f_j G_i$ . Clearly this relation is reflexive and symmetric. For transitivity, suppose

$$(\bar{F}, \bar{f}) = (\bar{G}, \bar{g}) \text{ and } (\bar{G}, \bar{g}) = (\bar{H}, \bar{h})$$

note that by the second equality we have for all  $i, j \in I$ ,  $h_j G_i = g_i H_j$ . Fix some  $i$  and  $j$ .

Then we have for every  $k \in I$

$$f_k h_j G_i = f_k g_i H_j.$$

Using the first equality we get

$$h_j g_i F_k = f_k g_i H_j.$$

This is true for every  $i$  and  $\Delta$  acts on  $X$  injectively so we get

$$h_j F_k = f_k H_j$$

or

$$(\bar{F}, \bar{f}) = (\bar{H}, \bar{h})$$

Define  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta) = \mathcal{A} / \sim$ . To avoid unnecessary complications with the notation we drop the power and instead of  $\mathcal{B}((\mathcal{B}(\mathcal{X}, \Delta))^I, \Delta)$  we just simply write  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$ . Clearly  $\mathcal{B}(\mathcal{X}, \Delta)$  can be embedded in  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$  by an extension of the map  $\iota$  discussed in the construction of pseudoquotients.

As a demonstration of how we work with elements of  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$  we show that the action of  $G$  on  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$ , where  $g \left( \frac{\bar{F}}{\bar{f}} \right) = \frac{g\bar{F}}{\bar{f}}$  for all  $g \in G$ , is associative: Suppose  $\frac{\bar{F}}{\bar{f}} = \frac{\bar{F}'}{\bar{f}'}$  which means  $f'_j F_i = f_i F'_j$  or  $gf'_j F_i = gf_i F'_j$ . Using commutativity of  $G$ , we get  $\frac{g\bar{F}}{\bar{f}} = \frac{g\bar{F}'}{\bar{f}'}$ . Finally, for every  $g, h \in G$ ,

$$g\left(h \frac{\bar{F}}{\bar{f}}\right) = g\left(\frac{h\bar{F}}{\bar{f}}\right) = \frac{g(h\bar{F})}{\bar{f}}$$

which by associativity of  $G$  on  $\mathcal{B}(\mathcal{X}, \Delta)$  is equal to  $\frac{(gh)\bar{F}}{\bar{f}} = (gh) \frac{\bar{F}}{\bar{f}}$ .

A reasonable question is, by re-application of the construction of pseudoquotients, do we get a bigger set or can we say there is a bijection between  $\mathcal{B}(\mathcal{X}, \Delta)$  and  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$ . In general we cannot say much. However, if  $\Delta$  satisfies certain conditions we can show the bijection. One of these conditions is  $\Delta$  having the *diagonal property*:

**Definition 5** (The diagonal property). We say a set  $\Delta$  has the diagonal property if for every set  $\{\bar{\alpha}_i\}$  of elements in  $\Delta$  there exist an injection  $p : I \rightarrow I$  such that  $\bar{\gamma}$ , defined by  $\gamma_i = \bar{\alpha}_{p(i)}$  for all  $i \in I$ , is in  $\Delta$ .

**Theorem 7.** *If  $\Delta$  has the diagonal property, then  $\mathcal{B}(\mathcal{X}, \Delta)$  and  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$  are isomorphic.*

*Proof.* We showed that we have the injection  $\iota : \mathcal{B}(\mathcal{X}, \Delta) \rightarrow \mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$ . Suppose  $F \in \mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$  then there exist a  $\{f_i\} \in \Delta$  such that for every  $i \in I$ ,  $f_i F \in \mathcal{B}(\mathcal{X}, \Delta)$ . Similarly, for each  $i \in I$  there exists  $\{g_{i,j}\}$  such that for every  $j \in I$ ,  $g_{i,j}(f_i F) \in X$ . Since  $\Delta$  has the diagonal property, there exists an injection  $p : I \rightarrow I$  such that  $\{g_{p(i),i}\} \in \Delta$ . Let  $\bar{g}$  be such that  $g_i = g_{p(i),i}$ . We have for every  $i \in I$ ,  $(g_i f_i)F \in X$ . Let  $h_i = g_i f_i$ . There is  $x_i \in X$  such that  $h_i F = x_i$ . We claim the mapping  $\kappa$  defined by  $F \mapsto \frac{\bar{x}}{\bar{h}}$  is the inverse of  $\iota$ . Hence we show  $\kappa$  is an injection from  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$  to  $\mathcal{B}(\mathcal{X}, \Delta)$  such that  $\iota \circ \kappa = \text{id}$ . We first check  $\frac{\bar{x}}{\bar{h}} \in \mathcal{B}(\mathcal{X}, \Delta)$ , to verify that we need to show for every  $i, j \in I$ ,  $h_i x_j = h_j x_i$ ;

Notice

$$h_i x_j = h_i(h_j F) = h_j(h_i F) = h_j x_i.$$

Next we show the well-definedness: Suppose  $F = G$  in  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$ . If  $\frac{\bar{x}}{\bar{h}} = \kappa(F)$  and  $\frac{\bar{x}'}{\bar{h}'} = \kappa(G)$  we want to show  $h'_j x_i = h_i x'_j$  for all  $i, j \in I$ . We have

$$h'_j x_i = h'_j(h_i F) = h_i(h'_j F) = h_i(h'_j G) = h_i x'_j.$$

And finally to check that it is an injection, suppose  $\frac{\bar{x}}{\bar{h}}$  and  $\frac{\bar{x}'}{\bar{h}'}$  are as before and  $\frac{\bar{x}}{\bar{h}} = \frac{\bar{x}'}{\bar{h}'}$  which is the same thing as saying  $h'_j x_i = h_i x'_j$  or  $h'_j(h_i F) = h_i(h'_j G)$  for all  $i, j \in I$ . Since this is true for all  $i \in I$  and  $\Delta$  acts on  $\mathcal{B}(\mathcal{X}, \Delta)$  injectively,  $h'_j F = h'_j G$  for all  $j$  and since  $\Delta$  acts on  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$  injectively, we have  $F = G$ . Finally to show  $\iota \circ \kappa = \text{id}$  notice  $\iota(\kappa F) = \iota\left(\frac{\bar{x}}{\bar{h}}\right)$  and since  $x_i = h_i F$ , we get

$$\iota(\kappa F) = \iota\left(\frac{\bar{x}}{\bar{h}}\right) = \iota\left(\frac{\bar{x}}{\bar{g}\bar{f}}\right) = \frac{\bar{g}\bar{x}}{\bar{g}} = \frac{\bar{g}\bar{x}}{\bar{g}\bar{f}} = \frac{\bar{x}}{\bar{g}} = \frac{\bar{h}F}{\bar{g}} = \frac{\bar{g}\bar{f}F}{\bar{g}} = \frac{\bar{g}F}{\bar{g}} = F.$$

□

**Remark 3.** The diagonal property may seem like a very strong requirement, but in our basic example  $\mathcal{B}(\mathbb{R})$ ,  $\Delta$  which is the set of delta-sequences has this property. Notice, if we have a sequence  $\bar{\delta}_n$  in  $\Delta$ , then we can construct a sequence  $\bar{\delta}'$  the following way: Let  $\delta'_1 = \delta_{11}$ . For every natural number  $n > 1$ , let  $m$  be such that  $\text{supp } \delta_{nm} \subset \text{supp } \delta'_{n-1}$  and let  $\delta'_n = \delta_{nm}$ . Clearly  $\bar{\delta}'$  is a delta-sequence.

The semigroup of delta-sequences used in the construction of Boehmians is not the only example. In fact we have the following theorem:

**Theorem 8.** *If  $s : G \rightarrow [0, \infty)$  has the property  $s(fg) \leq s(f) + s(g)$  for all  $f, g \in G$  and  $\Delta \subset G^{\mathbb{N}}$  is defined to be all sequences  $(f_n)$  with the property  $s(f_n) \rightarrow 0$ , then  $\Delta$  has the diagonal property.*

*Proof.* Suppose we have  $(\bar{f}_n)$  a sequence of elements of  $\Delta$ . To construct the diagonal that is also in  $\Delta$ , let  $g_1 = f_{11}$ . For every  $n > 1$ , let  $g_n = f_{nm}$ , where  $m$  is a large enough number so that we have  $s(f_{nm}) < s(g_{n-1})$ . We can do this since for each  $n$ ,  $s(f_{nm}) \rightarrow 0$  as  $m \rightarrow \infty$ . Clearly  $(g_n)$  is in  $\Delta$ . □

In the case of the delta-sequences,  $s$  is the support number of the functions in the sequence. (For a function  $\varphi \in \mathcal{C}_c(\mathbb{R})$ , a positive real number  $a$  is called the support number of  $\varphi$  if  $a$  is the smallest number such that  $\text{supp } (\varphi) \subset [-a, a]$ .)

Another condition under which the reapplication of the construction of pseudoquotients does not produce a new set is the *common denominator* property:

**Definition 6** (The common denominator property). We say  $(\mathcal{B}(\mathcal{X}, \Delta))^I$  has the common denominator property if for every  $\bar{F} \in (\mathcal{B}(\mathcal{X}, \Delta))^I$  there exist a  $\bar{f} \in \Delta$  such that for every

$$i, j \in I, f_i F_j \in X$$

**Theorem 9.** *If  $(\mathcal{B}(\mathcal{X}, \Delta))^I$  has the common denominator property, then there is a bijection between  $\mathcal{B}(\mathcal{X}, \Delta)$  and  $\mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$ .*

*Proof.* The proof follows similar outline as the proof of Theorem 7. All we need to show is that there exists injective  $\kappa : \mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta) \rightarrow \mathcal{B}(\mathcal{X}, \Delta)$  such that  $\iota \circ \kappa = \text{id}$ . Let  $F \in \mathcal{B}(\mathcal{B}(\mathcal{X}, \Delta), \Delta)$ . Then there exists  $\bar{f} \in \Delta$  such that for every  $i$ ,  $f_i F \in \mathcal{B}(\mathcal{X}, \Delta)$ . As an element of  $(\mathcal{B}(\mathcal{X}, \Delta))^I$ ,  $\bar{f}F$  can be written with the same denominator, say  $\bar{g} \in \Delta$ . That is we have for every  $i, j \in I$ ,  $g_i(f_j F) \in X$ . Let  $h_i = g_i f_i$  and  $x_i = h_i F$ . The rest of the proof is identical to that of Theorem 7.  $\square$

At this point let us revisit Theorem 5 and prove a similar result for the general case.

**Theorem 10.** *Suppose  $(\mu, \alpha) : (X, G) \rightarrow (Y, H)$  is a  $\Sigma$ -morphism. Then*

1. *The map  $\tilde{\mu} : \mathcal{B}(\mathcal{X}, \Delta_G) \rightarrow \mathcal{B}(\mathcal{Y}, \Delta_H)$  defined by  $\tilde{\mu}\left(\frac{\bar{x}}{f}\right) = \frac{\mu(\bar{x})}{\alpha(f)}$  is a unique extension of  $\mu$  to a  $\Sigma$ -morphism  $(\tilde{\mu}, \alpha) : (\mathcal{B}(\mathcal{X}, \Delta_G), G) \rightarrow (\mathcal{B}(\mathcal{Y}, \Delta_H), H)$ .*
2. *If  $\mu$  is an injection, then so is  $\tilde{\mu}$ .*

*Proof.* 1. We can verify that  $\tilde{\mu}$  is well-defined: Suppose  $\frac{\bar{x}}{f} = \frac{\bar{y}}{g}$  which means for every  $i, j \in I$ , we have

$$f_i y_j = g_j x_i.$$

Applying  $\mu$  to both sides we get

$$\mu(f_i y_j) = \mu(g_j x_i)$$



which, using the consistency condition, can be written as

$$\alpha(f_i)\mu(y_j) = \alpha(g_j)\mu(x_i)$$

or

$$\frac{\mu(\bar{x})}{\alpha(\bar{f})} = \frac{\mu(\bar{y})}{\alpha(\bar{g})} \text{ or } \tilde{\mu}\left(\frac{\bar{x}}{\bar{f}}\right) = \tilde{\mu}\left(\frac{\bar{y}}{\bar{g}}\right).$$

In order for  $\tilde{\mu}$  to be an extension of  $\mu$  we should have  $\mu(x) = \tilde{\mu}\left(\frac{\bar{f}x}{\bar{f}}\right)$ , for every  $x \in X$  and  $\bar{f} \in \Delta$ . For all  $x \in X$ , we have  $\tilde{\mu}\left(\frac{\bar{f}x}{\bar{f}}\right) = \frac{\mu(\bar{f}x)}{\alpha(\bar{f})}$  which by the consistency condition is equal to  $\frac{\alpha(\bar{f})\mu(x)}{\alpha(\bar{f})} = \mu(x)$ . To show  $(\tilde{\mu}, \alpha)$  is a  $\Sigma$ -morphism, notice

$$\alpha(g)\tilde{\mu}\left(\frac{\bar{x}}{\bar{f}}\right) = \alpha(g)\left(\frac{\mu(\bar{x})}{\alpha(\bar{f})}\right) = \frac{\alpha(g)\mu(\bar{x})}{\alpha(\bar{f})}.$$

Using the consistency condition since  $(\mu, \alpha)$  is a  $\Sigma$ -morphism, we have

$$\frac{\alpha(g)\mu(\bar{x})}{\alpha(\bar{f})} = \frac{\mu(g\bar{x})}{\alpha(\bar{f})} = \tilde{\mu}\left(\frac{g\bar{x}}{\bar{f}}\right) = \tilde{\mu}\left(g\frac{\bar{x}}{\bar{f}}\right).$$

For uniqueness, suppose  $\bar{\mu}$  is an extension of  $\mu$  such that  $(\bar{\mu}, \alpha)$  is a  $\Sigma$ -morphism from  $(\mathcal{B}(\mathcal{X}, \Delta_G), G)$  to  $(\mathcal{B}(\mathcal{Y}, \Delta_H), H)$ . Let  $\frac{\bar{x}}{\bar{f}}$  be an element of  $\mathcal{B}(\mathcal{X}, \Delta_G)$ . Then  $\bar{\mu}\left(\frac{\bar{x}}{\bar{f}}\right) = \frac{\bar{y}}{\bar{h}}$  for some  $\frac{\bar{y}}{\bar{h}}$  in  $\mathcal{B}(\mathcal{Y}, \Delta_H)$ . Because  $\bar{\mu}$  is an extension of  $\mu$  we have for every  $j \in I$ ,

$$\mu(x_j) = \bar{\mu}(x_j) = \bar{\mu}\left(f_j\frac{\bar{x}}{\bar{f}}\right) = \alpha(f_j)\bar{\mu}\left(\frac{\bar{x}}{\bar{f}}\right) = \alpha(f_j)\frac{\bar{y}}{\bar{h}}.$$

What these equalities show is that for every  $j \in I$ ,

$$\mu(x_j) = \alpha(f_j)\frac{\bar{y}}{\bar{h}}$$

which implies

$$\bar{\mu}\left(\frac{\bar{x}}{\bar{f}}\right) = \frac{\bar{y}}{\bar{h}} = \frac{\mu(\bar{x})}{\alpha(\bar{f})} = \tilde{\mu}\left(\frac{\bar{x}}{\bar{f}}\right).$$

2. Suppose  $\mu$  is injective and

$$\frac{\mu(\bar{x})}{\alpha(\bar{f})} = \frac{\mu(\bar{y})}{\alpha(\bar{g})}.$$

This equivalence means for every  $i, j \in I$ ,

$$\alpha(g_j)\mu(x_i) = \alpha(f_i)\mu(y_j)$$

which by the consistency condition implies

$$\mu(g_j x_i) = \mu(f_i y_j).$$

Because  $\mu$  is injective we have

$$g_j x_i = f_i y_j$$

for all  $i, j$  which is the same as saying  $\frac{\bar{x}}{\bar{f}} = \frac{\bar{y}}{\bar{g}}$ .

□

It is a rather interesting point that even though there is no  $\Sigma$ -morphisms from  $(X, G)$  to  $(\mathcal{X}, \Delta)$  or from  $(\mathcal{X}, \Delta)$  to  $(\mathcal{B}(\mathcal{X}, \Delta), G)$ , we do have a  $\Sigma$ -morphism from  $(X, G)$  to  $(\mathcal{B}(\mathcal{X}, \Delta), G)$ .

## 2.4 Algebraic Properties

We start this section by considering  $\Sigma$ -pairs  $(X, G)$  in which  $(X, \oplus)$  is a group ( $\oplus$  not necessarily a commutative operation) and  $G$  is a commutative semigroup of homomorphisms acting on  $X$ . We define the  $\oplus$  operation on  $X^I$  by  $(\bar{x} \oplus \bar{y})(i) = x_i \oplus y_i$  for all  $i \in I$ . We investigate the algebraic properties of the space of pseudoquotients and the extension map

$\tilde{\mu}$  defined in Theorem 10. As we mentioned in the last section, the injective case is a special case of general pseudoquotients so all the demonstrations will be for the general case. Let  $\mathcal{X}$  and  $\Delta$  be as described in the last section with additional condition that  $(\mathcal{X}, \oplus)$  is a group. Further, we can define  $\oplus : \mathcal{B}(\mathcal{X}, \Delta) \times \mathcal{B}(\mathcal{X}, \Delta) \rightarrow \mathcal{B}(\mathcal{X}, \Delta)$  by

$$\frac{\bar{x}}{\bar{f}} \oplus \frac{\bar{y}}{\bar{g}} = \frac{(\bar{g}\bar{x}) \oplus (\bar{f}\bar{y})}{\bar{f}\bar{g}}. \quad (2.2)$$

Since it is usually clear where the operation is taking place, we use  $\oplus$  to denote the operation on both  $X$  and  $\mathcal{B}(\mathcal{X}, \Delta)$ . We can verify that  $\oplus$  is well-defined: Suppose  $\frac{\bar{x}}{\bar{f}} = \frac{\bar{x}'}{\bar{f}'}$ . Then we have

$$\begin{aligned} \frac{\bar{x}}{\bar{f}} \oplus \frac{\bar{y}}{\bar{g}} &= \frac{\bar{g}\bar{x} \oplus \bar{f}\bar{y}}{\bar{f}\bar{g}} = \frac{\bar{f}'\bar{g}\bar{x} \oplus \bar{f}'\bar{f}\bar{y}}{\bar{f}'\bar{f}\bar{g}} = \frac{\bar{g}\bar{f}'\bar{x} \oplus \bar{f}\bar{f}'\bar{y}}{\bar{f}'\bar{f}\bar{g}} \\ &= \frac{\bar{g}\bar{f}'\bar{x}' \oplus \bar{f}\bar{f}'\bar{y}}{\bar{f}'\bar{f}\bar{g}} \\ &= \frac{\bar{x}'}{\bar{f}'} \oplus \frac{\bar{y}}{\bar{g}} \end{aligned}$$

**Theorem 11.** *If  $(X, \oplus)$  and  $(\mathcal{X}, \oplus)$  are groups then so is  $\mathcal{B}(\mathcal{X}, \Delta)$  with  $\oplus$  defined as in (2.2).*

*Proof.* The closure under  $\oplus$  is clear. For associativity let  $\frac{\bar{x}}{\bar{f}}, \frac{\bar{y}}{\bar{g}}, \frac{\bar{z}}{\bar{h}} \in \mathcal{B}(\mathcal{X}, \Delta)$  then we have

$$\frac{\bar{x}}{\bar{f}} \oplus \left( \frac{\bar{y}}{\bar{g}} \oplus \frac{\bar{z}}{\bar{h}} \right) = \frac{\bar{x}}{\bar{f}} \oplus \left( \frac{\bar{h}\bar{y} \oplus \bar{g}\bar{z}}{\bar{g}\bar{h}} \right) = \frac{\bar{g}\bar{h}\bar{x} \oplus \bar{f}\bar{h}\bar{y} \oplus \bar{f}\bar{g}\bar{z}}{\bar{f}\bar{g}\bar{h}} = \frac{\bar{h}(\bar{g}\bar{x} \oplus \bar{f}\bar{y}) \oplus \bar{f}\bar{g}\bar{z}}{\bar{f}\bar{g}\bar{h}} = \left( \frac{\bar{x}}{\bar{f}} \oplus \frac{\bar{y}}{\bar{g}} \right) \oplus \frac{\bar{z}}{\bar{h}}.$$

For the identity let  $e$  be the identity in  $X$ . We claim the embedding of  $e$  in  $\mathcal{B}(\mathcal{X}, \Delta)$  is the identity. Note that this element will have the form  $\frac{e}{\bar{f}}$  for some  $\bar{f} \in \Delta$  and we also have, for any other  $\bar{g} \in \Delta$ ,  $\frac{e}{\bar{f}} = \frac{e}{\bar{g}}$ . Let  $\frac{\bar{x}}{\bar{g}} \in \mathcal{B}(\mathcal{X}, \Delta)$ . Then

$$\frac{e}{\bar{f}} \oplus \frac{\bar{x}}{\bar{g}} = \frac{\bar{g}e \oplus \bar{f}\bar{x}}{\bar{f}\bar{g}} = \frac{\bar{f}\bar{x}}{\bar{f}\bar{g}} = \frac{\bar{x}}{\bar{g}}.$$

Similarly we can show  $\frac{\bar{x}}{\bar{g}} \oplus \frac{e}{\bar{f}} = \frac{\bar{x}}{\bar{g}}$ . Finally the inverse of each  $\frac{\bar{x}}{\bar{g}}$  is  $\frac{\bar{y}}{\bar{g}}$  where  $y_i$  is the inverse of  $x_i$  in  $X$  for all  $i \in I$ .  $\square$

In the above theorem, if  $X$  is commutative, then so is  $\mathcal{B}(\mathcal{X}, \Delta)$ .

**Theorem 12.** *Let  $(X, G), (Y, H)$  be  $\Sigma$ -pairs with the  $\Sigma$ -morphism  $(\mu, \alpha) : (X, G) \rightarrow (Y, H)$ . Suppose  $X$  and  $Y$  are groups. Then if  $\mu$  is a group homomorphism then so is  $\tilde{\mu} : \mathcal{B}(\mathcal{X}, \Delta_G) \rightarrow \mathcal{B}(\mathcal{Y}, \Delta_H)$  defined by*

$$\tilde{\mu}\left(\frac{\bar{x}}{\bar{f}}\right) = \frac{\mu(\bar{x})}{\alpha(\bar{f})}.$$

*Proof.* To show that  $\tilde{\mu}$  is a group homomorphism, let  $\frac{\bar{x}}{\bar{f}}, \frac{\bar{y}}{\bar{h}} \in \mathcal{B}(\mathcal{X}, \Delta)$ , then we have

$$\tilde{\mu}\left(\frac{\bar{x}}{\bar{f}} \oplus \frac{\bar{y}}{\bar{g}}\right) = \tilde{\mu}\left(\frac{\bar{g}\bar{x} \oplus \bar{f}\bar{y}}{\bar{f}\bar{g}}\right) = \frac{\mu(\bar{g}\bar{x} \oplus \bar{f}\bar{y})}{\alpha(\bar{f}\bar{g})} = \frac{\mu(\bar{g}\bar{x}) \oplus \mu(\bar{f}\bar{y})}{\alpha(\bar{f})\alpha(\bar{g})}$$

since  $\mu$  is a homomorphisms and by the definition of  $\Sigma$ -morphism we know  $\alpha$  is a semigroup homomorphism. By the consistency condition this last expression is equal to

$$\frac{\alpha(\bar{g})\mu(\bar{x}) \oplus \alpha(\bar{f})\mu(\bar{y})}{\alpha(\bar{f})\alpha(\bar{g})} = \frac{\mu(\bar{x})}{\alpha(\bar{f})} \oplus \frac{\mu(\bar{y})}{\alpha(\bar{g})} = \tilde{\mu}\left(\frac{\bar{x}}{\bar{f}}\right) \oplus \tilde{\mu}\left(\frac{\bar{y}}{\bar{g}}\right).$$

$\square$

Next we give  $X$  more structure: Let  $X$  be a vector space over a field  $F$ . We can define the scalar multiplication on  $X^I$  by  $(\lambda\bar{x})(i) = \lambda x_i$  for all  $i \in I$  and  $\lambda \in F$ . Furthermore, we say  $G$  acts linearly on  $X$  if for every  $f \in G$ ,  $x, y \in X$ , and  $\lambda, \lambda' \in F$ ,  $f(\lambda x + \lambda' y) = \lambda(fx) + \lambda'(fy)$ .

**Theorem 13.** *If  $X$  and  $\mathcal{X}$  are vector spaces and  $G$  acts linearly on  $X$ , then  $\mathcal{B}(\mathcal{X}, \Delta)$  is also a vector space.*

*Proof.* The group properties extend to  $\mathcal{B}(\mathcal{X}, \Delta)$ . The only missing part is the scalar multiplication which we can define by

$$\lambda \frac{\bar{x}}{\bar{f}} = \frac{\lambda \bar{x}}{\bar{f}}$$

where  $\lambda \in F$ . To see this operation is well-defined, let  $\frac{\bar{x}}{\bar{f}} = \frac{\bar{x}'}{\bar{f}'}$  in  $\mathcal{B}(\mathcal{X}, \Delta)$ . Then we have  $\lambda(\bar{f}'\bar{x}) = \lambda(\bar{f}\bar{x}')$  or  $\bar{f}'(\lambda\bar{x}) = \bar{f}(\lambda\bar{x}')$  which is the same thing as  $\frac{\lambda\bar{x}}{\bar{f}} = \frac{\lambda\bar{x}'}{\bar{f}'}$ .  $\square$

**Theorem 14.** *If  $X$  and  $\mathcal{X}$  are vector spaces over a field  $F$ , with  $G$  acting linearly on  $X$ , then the embedding  $\iota : X \rightarrow \mathcal{B}(\mathcal{X}, \Delta)$  is linear.*

*Proof.* For every  $x, y \in X$  and  $\lambda \in F$  we have

$$\iota(x + y) = \frac{\bar{f}(x + y)}{\bar{f}} = \frac{\bar{f}x + \bar{f}y}{\bar{f}} = \frac{\bar{f}(\bar{f}x + \bar{f}y)}{\bar{f}\bar{f}} = \frac{\bar{f}x}{\bar{f}} + \frac{\bar{f}y}{\bar{f}} = \iota(x) + \iota(y)$$

and

$$\iota(\lambda x) = \frac{\bar{f}(\lambda x)}{\bar{f}} = \frac{\lambda \bar{f}x}{\bar{f}} = \lambda \left( \frac{\bar{f}x}{\bar{f}} \right) = \lambda \iota(x).$$

$\square$

Let  $(X, +, \cdot)$  be a ring and  $G$  a semigroup of ring homomorphisms. As in the case of  $\oplus$  we can extend  $+$  to  $X^I$ . We can also extend the  $\cdot$  operation to  $X^I$  and define it for every  $\bar{x}, \bar{y} \in X^I$  by  $(\bar{x} \cdot \bar{y})(i) = x_i \cdot y_i$  for all  $i \in I$ . Let  $\mathcal{X}$  be such that  $(\mathcal{X}, +, \cdot)$  is a ring. We like to see if  $\mathcal{B}(\mathcal{X}, \Delta)$  is also a ring with  $+$  and some extension of  $\cdot$ . We have two choices for defining  $\cdot : \mathcal{B}(\mathcal{X}, \Delta) \rightarrow \mathcal{B}(\mathcal{X}, \Delta)$ . One way is to use similar definition as for  $\oplus$ : For every  $\frac{x}{f}, \frac{y}{g} \in \mathcal{B}(\mathcal{X}, \Delta)$  we define:

$$\frac{x}{f} \cdot \frac{y}{g} = \frac{gx \cdot fy}{fg}.$$

Showing  $\cdot$  is well-defined and  $(\mathcal{B}(\mathcal{X}, \Delta), +, \cdot)$  is a ring are straightforward calculations. (The process follows almost exact steps as in for  $\oplus$ .) The other choice for  $\cdot$  is similar to multiplying fractions. The following theorem describes all the necessary details.

**Theorem 15.** *If  $(X, +, \cdot)$  is a ring and for every  $\bar{f} \in \Delta$  and  $\bar{x}, \bar{y} \in \mathcal{X}$ ,*

$$\bar{f}(\bar{x} \cdot \bar{y}) = (\bar{f}\bar{x}) \cdot \bar{y},$$

*then  $\mathcal{B}(\mathcal{X}, \Delta)$  is also a ring with operations  $+$  defined in (2.2) and  $\cdot$  defined by:*

$$\frac{\bar{x}}{\bar{f}} \cdot \frac{\bar{y}}{\bar{g}} = \frac{\bar{x} \cdot \bar{y}}{\bar{f}\bar{g}}. \quad (2.3)$$

*Proof.* By Theorem 11 we know that  $\mathcal{B}(\mathcal{X}, \Delta)$  is a group. We first verify that  $\cdot$  is a well-defined operation: Suppose  $\frac{\bar{x}}{\bar{f}} = \frac{\bar{x}'}{\bar{f}'}$ . Then we have

$$\frac{\bar{x}}{\bar{f}} \cdot \frac{\bar{y}}{\bar{g}} = \frac{\bar{x} \cdot \bar{y}}{\bar{f}\bar{g}} = \frac{\bar{f}'(\bar{x} \cdot \bar{y})}{\bar{f}'\bar{f}\bar{g}} = \frac{(\bar{f}'\bar{x}) \cdot \bar{y}}{\bar{f}'\bar{f}\bar{g}} = \frac{(\bar{f}'\bar{x}') \cdot \bar{y}}{\bar{f}'\bar{f}\bar{g}} = \frac{\bar{f}'(\bar{x}' \cdot \bar{y})}{\bar{f}'\bar{f}\bar{g}} = \frac{\bar{x}' \cdot \bar{y}}{\bar{f}'\bar{g}} = \frac{\bar{x}'}{\bar{f}'} \cdot \frac{\bar{y}}{\bar{g}}.$$

The associativity of  $\cdot$  follows those of  $\cdot$  on  $X$  and the operation in  $G$ . The verification of distributivity of  $\cdot$  over addition is just an elementary calculation. The identity in  $\mathcal{B}(\mathcal{X}, \Delta)$  with respect to  $\cdot$  is the embedding of 1, the  $\cdot$ -identity in  $X$ , hence has the form  $\frac{\bar{f}}{\bar{f}}$ .  $\square$

**Corollary 3.** *Let  $X$  be an algebra over a field  $F$ , and  $\mathcal{X}$  be a subalgebra of  $X^I$ . If elements of  $G$  are algebra homomorphisms on  $X$  then  $\mathcal{B}(\mathcal{X}, \Delta)$  is also an algebra.*

*Proof.* The proof is the direct consequence of Theorem 14 and Theorem 15.  $\square$

## 2.5 Topological Properties

### 2.5.1 The Injective Case

To discuss the topological properties of either the injective case or the general case of pseudoquotients we consider the  $\Sigma$ -pair  $(X, G)$  in which  $X$  and  $G$  are topological spaces. We start with the injective case and in the next section we will get inspiration from these results to show similar topological properties for the general case.

The first topology we will consider is the quotient topology, denoted by  $\tau_Q$ , where our quotient map  $q : X \times G \rightarrow \mathcal{B}(X, G)$  is defined by  $q(x, f) = \frac{fx}{f}$  and the topology on  $X \times G$  is the product topology.

**Theorem 16.** *The embedding  $\iota : X \rightarrow \mathcal{B}(X, G)$ , where  $\iota(x) = \frac{fx}{f}$  is as described in the construction of generalized quotients and elements of  $G$  are continuous maps, is  $\tau_Q$ -continuous.*

*Proof.* Notice that for every  $x \in X$ , the map  $x \mapsto fx$  is continuous on  $X$  and the map  $fx \mapsto (fx, f)$  is continuous from  $X$  to  $X \times G$  where the topology on  $X \times G$  is the product topology. Finally, the quotient map which takes  $(fx, f) \mapsto \frac{fx}{f}$  is continuous. Thus  $\iota$  being a composition of these three maps is continuous.  $\square$

As we discussed in the previous section, if we have a  $\Sigma$ -morphism  $(\mu, \alpha) : (X, G) \rightarrow (Y, H)$ , then we can uniquely extend  $\mu$  to  $\tilde{\mu} : \mathcal{B}(X, G) \rightarrow \mathcal{B}(Y, H)$ . Under some simple conditions which we discussed earlier,  $\tilde{\mu}$  inherits some of the set map properties of  $\mu$  such as injection and surjection, and some algebraic properties such as being a group homomorphism. A natural question would be, does  $\tilde{\mu}$  inherit continuity if  $\mathcal{B}(X, G)$  and  $\mathcal{B}(Y, H)$  are equipped

with the quotient topology? To show this we need the following theorem which can be found in [17].

**Theorem 17.** *Let  $q : X \rightarrow Y$  be a quotient map. Let  $Z$  be a space and let  $g : X \rightarrow Z$  be a map that is constant on each set  $q^{-1}(\{y\})$ , for  $y \in Y$ . Then  $g$  induces a map  $f : Y \rightarrow Z$  such that  $f \circ q = g$ . The induced map  $f$  is continuous if and only if  $g$  is continuous.*

Now our main theorem:

**Theorem 18.** *Suppose  $(X, G)$  and  $(Y, H)$  are  $\Sigma$ -pairs and let  $\mathcal{B}(X, G)$  and  $\mathcal{B}(Y, H)$  have the quotient topology. If  $(\mu, \alpha) : (X, G) \rightarrow (Y, H)$  is a  $\Sigma$ -morphism with  $\mu : X \rightarrow Y$  and  $\alpha : G \rightarrow H$  continuous and  $\alpha$  a surjection, then  $\tilde{\mu} : \mathcal{B}(X, G) \rightarrow \mathcal{B}(Y, H)$ , defined by  $\tilde{\mu}(\frac{x}{g}) = \frac{\mu(x)}{\alpha(g)}$  is continuous.*

*Proof.* Let  $q : (X, G) \rightarrow \mathcal{B}(X, G)$  and  $q' : (Y, H) \rightarrow \mathcal{B}(Y, H)$  be quotient maps. To use Theorem 17 we first need to show that  $q' \circ (\mu, \alpha)$  is constant on each set  $q^{-1}(\frac{x}{g})$ , for  $\frac{x}{g} \in \mathcal{B}(X, G)$ . Let  $\frac{x}{g} \in \mathcal{B}(X, G)$ .  $q^{-1}(\frac{x}{g}) = [(x, g)] \in (X, G)$ . If  $(x, g) \sim (x', g')$  then we have  $q' \circ (\mu, \alpha)(x, g) = q'(\mu(x), \alpha(g)) = \frac{\mu(x)}{\alpha(g)}$  and similarly  $q' \circ (\mu, \alpha)(x', g') = \frac{\mu(x')}{\alpha(g')}$ . Finally we have  $\frac{\mu(x)}{\alpha(g)} = \frac{\mu(x')}{\alpha(g')}$ . Next we need to show that  $q' \circ (\mu, \alpha)$  is continuous: Let  $U \subset \mathcal{B}(Y, H)$  be an open set and  $(x, g) \in (X, G)$  be such that  $q' \circ (\mu, \alpha)(x, g) \in U$ . If there is no such  $(x, g)$  then that implies the preimage is the empty set which is open. Otherwise,  $(q')^{-1}U = \bigcup_i U_i \times H_i$  where  $U_i \subset Y$  and  $H_i \subset H$  are open. Now applying  $(\mu, \alpha)^{-1}$  to  $\bigcup_i U_i \times H_i$  we get  $\bigcup_i \mu^{-1}U_i \times \alpha^{-1}H_i$  which is open in  $X \times G$  because  $\mu$  and  $\alpha$  are continuous maps, and for some  $n$ ,  $(x, g) \in \mu^{-1}U_i \times \alpha^{-1}H_i$ . □

While the quotient topology is the topology usually associated with quotient spaces, it



has the drawback of making convergence difficult to characterize. For this reason, we wish to consider an alternate topology on  $\mathcal{B}(X, G)$  in which convergent sequences have a simple characterization. This can be done by first defining a convergence and then inducing a topology from it. Here is the convergence we will use:

**Definition 7.** Let  $F_n, F \in \mathcal{B}(X, G)$ . We define  $P$ -convergence to be:  $F_n \xrightarrow{P} F$  if there exist  $f \in G$  such that  $fF_n, fF \in X$ , and  $fF_n \rightarrow fF$  in  $X$ .

In what follows, we will write  $x_n : \in A$  to indicate that the sequence  $(x_n)$  is eventually in  $A$ , i.e., there exists an  $N_0$  such that  $x_n \in A$  for all  $n > N_0$ .

$P$ -convergence is similar to type I convergence discussed in [7]. The difference is that type I convergence is defined for the field of Mikusiński operators. Speaking with quotient notation in mind, this convergence means, a sequence of quotients converges if all quotients can be written with the same “denominators” and the “numerators” converge. The  $P$ -convergence is not topological however we can use it to define a topology we call the *natural topology*.

**Definition 8.** We say  $U \subset \mathcal{B}(X, G)$  is open in the natural topology (or  $U \in \tau_N$ ) if  $F_n \xrightarrow{P} F \in U$  implies  $F_n : \in U$ .

Topology  $\tau_N$  defines a convergence which we call the *natural* (or  $N$ -) convergence and denote it by  $\xrightarrow{N}$ . In general this convergence is not the same as the  $P$ -convergence. However we have the following:

**Theorem 19.** Let  $F_n, F \in \mathcal{B}(X, G)$  for all  $n \in \mathbb{N}$ . If every subsequence of  $F_n$  has a subsequence which is  $P$ -convergent to  $F$  then  $F_n \xrightarrow{N} F$ .

*Proof.* Suppose every subsequence of  $F_n$  has a subsequence  $P$ -convergent to  $F$ . Let  $U \subset \mathcal{B}(X, G)$  be  $\tau_N$  open containing  $F$ . If  $F_n$  is not eventually in  $U$ , then there is a subsequence of  $F_n$  with all the elements outside of  $U$ . But that subsequence has a subsequence  $P$ -converging to  $F$  with all the elements outside of  $U$  which is a contradiction since  $U$  is a  $\tau_N$ -open set and every  $P$ -convergent sequence must eventually be in  $U$ . □

Even though  $\tau_N$  makes defining convergence easier than  $\tau_Q$ , defining elements of  $\tau_N$  is not intuitive.

In the following theorems we compare the two topologies. We will see that under a mild assumption they are equal. We also present an example to show that in general this is not necessarily true.

**Theorem 20.**  $\tau_N$  is a finer topology than  $\tau_Q$ . That is,  $\tau_Q \subset \tau_N$ .

*Proof.* Let  $U \in \tau_Q$ . To show that  $U \in \tau_N$ , we need to show that

$$F_n \xrightarrow{P} F \in U \Rightarrow F_n : \in U,$$

Suppose  $F_n \xrightarrow{P} F \in U$ . Then we can write  $F_n = \frac{x_n}{f}$ ,  $F = \frac{x}{f}$ , where  $x_n \rightarrow x \in X$ . Then  $(x, f) \in q^{-1}(U)$ . In fact,  $(x, f) \in U_x \times V_f$ , for open sets  $U_x \subset X$  and  $V_f \subset G$ , which is an open subset of  $q^{-1}(U)$ . Because of the  $P$ -convergence,  $(x_n, f) : \in U_x \times V_f \subset q^{-1}(U)$ . Thus  $F_n : \in U$  which makes  $U$  a  $\tau_N$  open set. □

**Theorem 21.** If  $X$  first-countable, then  $\tau_Q$  is finer than  $\tau_N$ .

*Proof.* Let  $U \in \tau_N$ , and suppose  $U \notin \tau_Q$ . Then  $q^{-1}(U)$  is not open in  $X \times G$ . But  $X$  is first countable, so we can construct a sequence  $x_n$  in  $X$  such that  $(x, f) \in q^{-1}(U)$ , for some  $f \in G$  with  $(x_n, f) \rightarrow (x, f)$  and  $(x_n, f) \notin q^{-1}(U)$ . This implies  $\frac{x_n}{f}$   $P$ -converges to  $\frac{x}{f}$  but every  $\frac{x_n}{f}$  is outside  $U$  which is a contradiction.  $\square$

**Corollary 4.** *If  $X$  first-countable, then  $\tau_Q = \tau_N$ .*

The following example demonstrates a case where  $\tau_Q$  and  $\tau_N$  are not equal: Let  $X$  be the real line with the countable complement topology, and let  $G$  consists of the identity map. Then  $\mathcal{B}(X, G) = X$ , and the quotient topology is the same as the topology on  $X$ . Since there are only trivial convergent sequences in  $X$ , the natural topology on  $\mathcal{B}(X, G)$  is discrete and thus  $\tau_N \neq \tau_Q$ .

### 2.5.2 The General Case

In this section we discuss different possible topologies for the space of pseudoquotients in the general case. The standard way of defining a topology on a space  $\mathcal{B}(\mathcal{X}, \Delta)$  is by inducing a topology on  $\mathcal{B}(\mathcal{X}, \Delta)$  from the topologies of  $X$  and  $G$  the following way: Let  $X$  and  $G$  be topological spaces with  $G$  possibly having the discrete topology, then the topologies of  $X$  and  $G$  induce product topologies on  $X^I$  and  $G^I$  and we can restrict those topologies to  $\mathcal{X}$  and  $\Delta$ . The space  $\mathcal{X} \times \Delta$  will have the product topology and finally  $\mathcal{X} \times \Delta / \sim$  or  $\mathcal{B}(\mathcal{X}, \Delta)$  will have the quotient topology. With this topology we have the following property for the embedding  $\iota : X \rightarrow \mathcal{B}(\mathcal{X}, \Delta)$  defined by  $\iota(x) = \frac{\tilde{f}x}{f}$  (as we described in the construction section).

**Theorem 22.** *The embedding  $\iota : X \rightarrow \mathcal{B}(\mathcal{X}, \Delta)$  is continuous with respect to the quotient topology provided elements of  $G$  are continuous on  $X$ .*

*Proof.* Notice the mapping defined by  $x \mapsto \bar{f}x$  is continuous since elements of  $G$  are continuous. Also the mapping defined by  $\bar{f}x \mapsto (\bar{f}x, \bar{f})$  is obviously continuous. And finally the quotient map taking  $(\bar{f}x, \bar{f})$  to  $\frac{\bar{f}x}{\bar{f}}$  is continuous. Thus  $\iota$  being the composition of these three continuous maps is continuous.  $\square$

**Theorem 23.** *Let  $(\mu, \alpha) : X \times G \rightarrow Y \times H$  be a  $\Sigma$ -morphism with  $\alpha : G \rightarrow H$  a continuous surjection. If  $\mu$  is continuous, then the extension map  $\tilde{\mu} : \mathcal{B}(\mathcal{X}, \Delta_G) \rightarrow \mathcal{B}(\mathcal{Y}, \Delta_H)$ , defined by  $\tilde{\mu}\left(\frac{\bar{x}}{\bar{f}}\right) = \frac{\mu(\bar{x})}{\alpha(\bar{f})}$ , is continuous with respect to the quotient topology on  $\mathcal{B}(\mathcal{X}, \Delta_G)$  and  $\mathcal{B}(\mathcal{Y}, \Delta_H)$ .*

*Proof.* Once again we use Theorem 17. Suppose  $q : X^I \times G^I \rightarrow \mathcal{B}(\mathcal{X}, \Delta_G)$  and  $q' : Y^I \times H^I \rightarrow \mathcal{B}(\mathcal{Y}, \Delta_H)$  be quotient maps. Let the  $g$  in Theorem 17 be the composition map  $g = q' \circ (\mu, \alpha)$ . To check that  $g$  is constant on  $q^{-1}\left(\frac{\bar{x}}{\bar{f}}\right)$  for every  $\frac{\bar{x}}{\bar{f}}$ , note that  $q^{-1}\left(\frac{\bar{x}}{\bar{f}}\right) = [(\bar{x}, \bar{f})]$  and if  $(\bar{x}, \bar{f}) \sim (\bar{x}', \bar{f}')$  in  $\mathcal{B}(\mathcal{X}, \Delta_G)$ ,  $q' \circ (\mu, \alpha)(\bar{x}, \bar{f}) = q'(\mu(\bar{x}), \alpha(\bar{f})) = \frac{\mu(\bar{x})}{\alpha(\bar{f})}$  and  $q' \circ (\mu, \alpha)(\bar{x}', \bar{f}') = q'(\mu(\bar{x}'), \alpha(\bar{f}')) = \frac{\mu(\bar{x}')}{\alpha(\bar{f}'')}$ . Since for every  $n, m \in I$  we have  $\alpha(f_m)\mu(x'_n) = \mu(f_m x'_n) = \mu(f'_n x_m) = \alpha(f'_n)\mu(x_m)$  the quotients  $\frac{\mu(\bar{x}')}{\alpha(\bar{f}'')}$  and  $\frac{\mu(\bar{x})}{\alpha(\bar{f})}$  are equivalent. And to show the continuity of  $g$  we need to show the continuity of the general form of  $(\mu, \alpha) : X^I \times G^I \rightarrow Y^I \times H^I$ . For an open set  $U \subset Y^I$  we have  $U = \Pi_i U_i$  where  $U_i = Y$  for but finitely many  $i$ 's and open sets in  $Y$  for the rest. Since  $\mu$  is a continuous map,  $\mu$ -preimage of  $U$  is the product of copies of  $X$  and finitely many open sets which is open in  $X^I$ . Similarly we can show the preimage of an open set  $V$  in  $H^I$  is open in  $G^I$  using continuity of  $\alpha$ . Hence  $(\mu, \alpha)^{-1}U \times V$  is open in  $X^I \times G^I$ . Thus  $g$  being the composition of two continuous functions is continuous

and that implies  $\tilde{\mu}$  is continuous. □

Although this way of defining a topology on the quotient space  $\mathcal{B}(\mathcal{X}, \Delta)$  is intuitive, as in the injective case it is not easy to define convergence using this topology. Hence we take similar steps as for the generalized quotients and first define a convergence and then using the convergence we define a topology. In [11] P. Mikusiński described two choices for defining convergence on  $\mathcal{B}(\mathcal{X}, \Delta)$ . We will not use these in future sections but for the sake of completeness, we will mention them here.

We start by describing  $\delta$ -convergence in  $\mathcal{B}(\mathcal{X}, \Delta)$ .

**Definition 9.** Suppose  $F_n, F \in \mathcal{B}(\mathcal{X}, \Delta)$  for all  $n \in \mathbb{N}$ . We say  $F_n \xrightarrow{\delta} F$  if there exist  $\bar{f} \in \Delta$  such that for every  $k \in I$ ,  $f_k F_n \rightarrow f_k F$  in  $X$  as  $n \rightarrow \infty$ .

For example consider the space of Boehmians  $\mathcal{B}(\mathbb{R})$  described in the previous section. In this example,  $X = \mathcal{C}(\mathbb{R})$ ,  $G = \mathcal{C}_c^+(\mathbb{R})$ , and the index set is  $\mathbb{N}$ . Here the action of  $G$  on  $X$  is defined by convolution. Additionally,  $\Delta$  in this case is the set of delta sequences. We say a sequence of Boehmians  $F_n$  converges to Bohmian  $F$  if there exists a delta-sequence  $(\delta_n)$  such that for every  $k \in \mathbb{N}$ ,  $\delta_k * F_n \rightarrow \delta_k * F$  in  $\mathcal{C}(\mathbb{R})$ .

Recalling the common denominator property described in the construction of pseudoquotients and using Theorem 2.6 of [11] we can conclude that a set of  $\delta$ -convergent pseudoquotients has the common denominator property.  $\delta$ -convergence is not a topological convergence however we can use this convergence to define  $\Delta$ -convergence which we will see is topological.

**Definition 10.** Suppose  $F_n, F \in \mathcal{B}(\mathcal{X}, \Delta)$  for all  $n \in \mathbb{N}$ . We say  $F_n \xrightarrow{\Delta} F$  if each subsequence of  $(F_n)$  contains a subsequence which is  $\delta$ -convergent to  $F$ .

Remembering the *Urysohn's condition* we know that under some mild conditions (i.e. the uniqueness of the limits, convergence of constant sequences, and convergent subsequences converging to the same limit), we have a convergence is topological if every subsequence of a convergent sequence has a subsequence that is also convergent to the same limit. Hence we have  $\Delta$ -convergence is a topological convergence and the topology induced is called  $\Delta$ -topology and is denoted by  $\tau_\Delta$ .

## CHAPTER THREE: SOME EXAMPLES OF GENERALIZED QUOTIENTS

### 3.1 The Space $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$ and The Laplace Transform

In [19], K. Yosida introduces a space of generalized quotients over the positive real line. Although the process may seem elementary, this construction is a very good example for injective pseudoquotients we described in CHAPTER TWO. Not to repeat what has already been done in [19], we will use a different, less trivial function to generate the semigroup  $G$ . In the next section we will generalize this construction to the whole real line.

The space of continuous Laplace transformable functions defined on  $[0, \infty)$ , denoted by  $\mathcal{C}_{\mathcal{L}}$ , with pointwise addition and the convolution operation defined by

$$g * f(t) = \int_0^t g(t-u)f(u)du$$

is a commutative ring. Let  $g \in \mathcal{C}_{\mathcal{L}}$  be the function  $e^{-\frac{1}{t}}$  on  $(0, \infty)$  and 0 at 0. The action of  $g$  on  $f$  is  $g * f$  which is

$$g * f(t) = \int_0^t e^{\frac{1}{u-t}} f(u)du.$$

Let  $G = \{g^n, n \in \mathbb{N}\}$ . Here the power is the convolution power, i.e.  $g^2 = g * g$  and  $g^n = g * g^{n-1}$ . So far we have a commutative semigroup  $(G, *)$  acting on a set  $\mathcal{C}_{\mathcal{L}}$ . We need to check  $G$  acts injectively on  $\mathcal{C}$ .

**Lemma 3.** *The action of  $G$  on  $\mathcal{C}_{\mathcal{L}}$  is injective.*

*Proof.* For this we use Laplace transform: First we show  $g * f = 0$  implies  $f = 0$ . If  $g * f = 0$  then  $\mathcal{L}\{g * f\} = 0$  which implies  $\mathcal{L}\{g\}\mathcal{L}\{f\} = 0$ . But notice

$$\mathcal{L}\{g\}(s) = \int_0^\infty e^{-st} e^{-\frac{1}{t}} dt = \int_0^\infty e^{-st - \frac{1}{t}} dt > 0.$$

Thus  $\mathcal{L}\{f\} = 0$  which implies  $f = 0$ . In general if we have  $g^n * f = 0$  then  $g * (g^{n-1} * f) = 0$  which implies  $g^{n-1} * f = 0$ . Using inductive argument we get  $f = 0$ .  $\square$

On  $\mathcal{C}_{\mathcal{L}} \times G$  we can introduce the relation  $\sim$  by

$$(f, k) \sim (\tilde{f}, \tilde{k}) \text{ if and only if } \tilde{k} * f = k * \tilde{f}.$$

From CHAPTER TWO we have such relation is an equivalence, since  $G$  is a commutative semigroup of injections on  $\mathcal{C}_{\mathcal{L}}$ . We define the space of generalized quotients  $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$  to be the quotient space  $\mathcal{C}_{\mathcal{L}} \times G / \sim$ . We will denote elements of  $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$  using the quotient notation and we use “=” for equivalence.  $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$  is a ring with  $+$  and  $*$  operations defined by:

$$\frac{f}{k} + \frac{\tilde{f}}{\tilde{k}} = \frac{\tilde{k} * f + k * \tilde{f}}{k * \tilde{k}}$$

and

$$\frac{f}{k} * \frac{\tilde{f}}{\tilde{k}} = \frac{f * \tilde{f}}{k * \tilde{k}}.$$

We have shown that  $+$  defined this way is well-defined and since for every  $g \in G$  and  $f, \tilde{f} \in \mathcal{C}_{\mathcal{L}}$ ,  $g * (f * \tilde{f}) = (g * f) * \tilde{f}$ , the  $*$  operation is well-defined on  $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$ . Furthermore  $\frac{g}{g}$ , where  $g$  can be replaced with any  $k \in G$ , acts as an identity on  $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$  since for any  $\frac{f}{k} \in \mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$ ,

$$\frac{g}{g} * \frac{f}{k} = \frac{g * f}{g * k} = \frac{f}{k},$$



where the last equality follows from  $k * (g * f) = (g * k) * f$ , by the commutativity of convolution on  $\mathcal{C}_{\mathcal{L}}$ . We will denote  $\frac{g}{g}$  by  $\delta$ .

We can define scalar multiplication on  $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$  by

$$\alpha \cdot \frac{f}{k} = \frac{\alpha f}{k},$$

where  $\alpha \in \mathbb{C}$ . Hence  $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$  is an algebra. We define the embedding  $\iota : \mathcal{C}_{\mathcal{L}} \rightarrow \mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$  by  $\iota(f) = \frac{g*f}{g}$  for all  $f \in \mathcal{C}_{\mathcal{L}}$ . As we saw in the general construction chapter, this embedding is well-defined and independent of  $g$ . Hence we will identify every  $f$  with  $\frac{g*f}{g}$  if needed.

### 3.1.1 The Generalized Derivative

We first define the derivative of elements of  $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$ :

**Definition 11.** For every  $\frac{f}{k} \in \mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$ , we define the derivative of  $\frac{f}{k}$  to be  $\frac{g'}{g} * \frac{f}{k}$ .

It can be verified that  $\frac{g'}{g}$  is well-defined. For a differentiable function  $f$  in  $\mathcal{C}_{\mathcal{L}}$  we have

$$\begin{aligned} \frac{(g' * f)(t)}{g(t)} &= \frac{\int_0^t g'(t-x)f(x)dx}{g(t)} \\ &= \frac{f(0)g(t) + (g * f')(t)}{g(t)} \\ &= f(0)\delta + \frac{(g * f')(t)}{g(t)}. \end{aligned}$$

In the case where  $f(0) = 0$  the general form reduces to the classical definition of derivatives.

### 3.1.2 The Laplace Transform

We will first define the Laplace transform for the functions in  $\mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$  and then demonstrate its consistency.

**Definition 12.** For a  $\frac{f}{k} \in \mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$ , we define the Laplace transform of  $\frac{f}{k}$  to be  $\mathcal{L}\{\frac{f}{k}\} = \frac{\mathcal{L}\{f\}}{\mathcal{L}\{k\}}$ .

Because for every  $k \in G$  we have  $k = g^n$  for some  $n \in \mathbb{N}$  and the exponent is convolution power, then the denominator is reduced to  $\mathcal{L}\{k\} = \mathcal{L}\{g^n\} = (\mathcal{L}\{g\})^n$ , where the exponent  $n$  here is a multiplicative power.

**Theorem 24.** *The Laplace transform is well-defined.*

*Proof.* Suppose  $B \in \mathcal{B}(\mathcal{C}_{\mathcal{L}}, G)$  has two representation  $\frac{f}{g^n}$  and  $\frac{\tilde{f}}{g^m}$ . Since these two representations are equivalent we have

$$g^m * f = g^n * \tilde{f}$$

We can take the Laplace of both sides in the traditional sense

$$(\mathcal{L}\{g\})^m \mathcal{L}\{f\} = (\mathcal{L}\{g\})^n \mathcal{L}\{\tilde{f}\}$$

which implies

$$\frac{\mathcal{L}\{f\}}{(\mathcal{L}\{g\})^n} = \frac{\mathcal{L}\{\tilde{f}\}}{(\mathcal{L}\{g\})^m}.$$

□

### 3.2 The Space $\mathcal{B}(\mathcal{C}_{exp}(\mathbb{R}), \mathcal{E})$ and The Fourier Transform

Here we will extend results presented above to a particular subset of continuous functions.

Let  $E(t) = e^{-|t|}$  and let  $\mathcal{C}_{exp}(\mathbb{R})$  be the set of all continuous functions  $f$  such that  $fE$  is integrable. On this set we define convolution of two functions  $f$  and  $g$  by

$$f * g(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx$$

whenever the integral exists. Let  $\mathcal{E}$  be defined by

$$\mathcal{E} = \{E^n : n = 1, 2, \dots\}.$$

where the power here is a convolutive power (i.e.  $E^2 = E * E$ , and  $E^n = E * E^{n-1}$ ).  $(\mathcal{E}, *)$  is a commutative semigroup. Further, we can show that  $\mathcal{E}$  acts on  $\mathcal{C}_{exp}(\mathbb{R})$  by convolution.

We can show this action injective:

**Theorem 25.** *For all  $n \in \mathbb{N}$ , if  $E^n * f = 0$  for  $f \in \mathcal{C}_{exp}(\mathbb{R})$  then  $f$  is the zero function.*

*Proof.* We prove by induction on  $n$ . Suppose  $E * f = 0$ . Then we have

$$\int_{-\infty}^t e^{x-t} f(x) dx + \int_t^{\infty} e^{t-x} f(x) dx = 0$$

or

$$e^{-t} \int_{-\infty}^t e^x f(x) dx + e^t \int_t^{\infty} e^{-x} f(x) dx = 0$$

Multiplying both sides by  $e^t$  we get

$$\int_{-\infty}^t e^x f(x) dx + e^{2t} \int_t^{\infty} e^{-x} f(x) dx = 0$$

After differentiating both sides, by the fundamental theorem of calculus we get:

$$e^t f(t) + 2e^{2t} \int_t^{\infty} e^{-x} f(x) dx - e^{2t} e^{-t} f(t) = 0$$

Canceling the first and third term we get

$$2e^{2t} \int_t^{\infty} e^{-x} f(x) dx = 0$$

or

$$\int_t^{\infty} e^{-x} f(x) dx = 0.$$

Hence

$$-e^{-t}f(t) = 0$$

which implies  $f \equiv 0$ . Now if we have  $E^2 * f = 0$ , that is the same thing as saying  $E*(E*f) = 0$

which implies  $E * f \equiv 0$  or  $f \equiv 0$ . By induction we get  $E^n * f = 0$  implies  $f \equiv 0$ .  $\square$

Continuing with the construction of quotients we say  $(f_1, k_1) \sim (f_2, k_2)$  or  $\frac{f_1}{k_1} = \frac{f_2}{k_2}$ , where  $f_1, f_2 \in \mathcal{C}_{exp}(\mathbb{R})$  and  $k_1, k_2 \in \mathcal{E}$ , if and only if  $k_2 * f_1 = k_1 * f_2$ . Since  $\mathcal{E}$  acts injectively,  $\sim$  is

an equivalence. Let  $\mathcal{B}(\mathcal{C}_{exp}(\mathbb{R}), \mathcal{E}) = \mathcal{C}_{exp}(\mathbb{R}) \times \mathcal{E} / \sim$ .  $\mathcal{B}(\mathcal{C}_{exp}(\mathbb{R}), \mathcal{E})$  is a ring with operations

$+$  and  $*$  defined by

$$\frac{f_1}{k_1} + \frac{f_2}{k_2} = \frac{k_2 * f_1 + k_1 * f_2}{k_1 * k_2}$$

and

$$\frac{f_1}{k_1} * \frac{f_2}{k_2} = \frac{f_1 * f_2}{k_1 * k_2}.$$

which are well-defined. We embed  $\mathcal{C}_{exp}(\mathbb{R})$  in  $\mathcal{B}(\mathcal{C}_{exp}(\mathbb{R}), \mathcal{E})$  by the mapping  $f \mapsto \frac{E*f}{E}$ .  $\frac{E}{E}$  is

the identity and is a representation of Dirac delta.

### 3.2.1 The Generalized Derivative

We have

$$E'(t) = \begin{cases} -e^{-t} & \text{if } t \leq 0 \\ e^t & \text{if } t > 0 \end{cases}$$

The equivalence class of  $\frac{E'}{E}$  is called the differentiation operator and we define the generalized

derivative of  $f \in \mathcal{C}_{exp}(\mathbb{R})$  by  $\frac{E'}{E} * f$  or equivalently  $\frac{E'*f}{E}$ . We can verify that the action of  $\frac{E'}{E}$

on differentiable functions amounts to the derivative of the function.

**Theorem 26.** For a differentiable function  $f \in \mathcal{C}_{exp}(\mathbb{R})$ ,  $\frac{E' * f}{E} = f'$ .

*Proof.* First we calculate the numerator  $E' * f$ .

$$\begin{aligned} E' * f(t) &= \int_{-\infty}^{\infty} E'(t-x)f(x)dx \\ &= - \int_{-\infty}^t e^{x-t} f(x)dx + \int_t^{\infty} e^{t-x} f(x)dx \\ &= -e^{-t} \int_{-\infty}^t e^x f(x)dx + e^t \int_t^{\infty} e^{-x} f(x)dx \end{aligned}$$

Integrating by parts we get

$$\begin{aligned} &= -f(x) + e^{-t} \int_{-\infty}^t e^x f'(x)dx + f(t) + e^t \int_t^{\infty} e^{-x} f'(x)dx \\ &= e^{-t} \int_{-\infty}^t e^x f'(x)dx + e^t \int_t^{\infty} e^{-x} f'(x)dx \\ &= \int_{-\infty}^t e^{x-t} f'(x)dx + \int_t^{\infty} e^{t-x} f'(x)dx \\ &= \int_{-\infty}^{\infty} e^{-|t-x|} f'(x)dx = E * f'(t). \end{aligned}$$

Thus we have  $\frac{E' * f}{E} = \frac{E * f'}{E}$  which is  $f'$ . □

Furthermore we can define the derivative of elements of  $\mathcal{C}_{exp}(\mathbb{R})$  that are not differentiable everywhere. In order to do that, we analyze the action of  $\frac{E'}{E}$  on two particular types of functions.

**Theorem 27.** Let  $f \in \mathcal{C}_{exp}(\mathbb{R})$  be positive on  $(0, \infty)$  and 0 everywhere else with possible discontinuity at 0. Then the generalized derivative of  $f$  is

$$\frac{E' * f}{E} = f(0)\delta_0 + \frac{E * f'}{E}.$$

*Proof.* We calculate the numerator first.

$$E' * f(t) = \int_{-\infty}^{\infty} E'(t-x)f(x)dx = \int_0^{\infty} E'(t-x)f(x)dx.$$

We consider the two cases  $t > 0$  and  $t < 0$ . When  $t > 0$  we get

$$\begin{aligned} &= \int_0^t -e^{x-t}f(x)dx + \int_t^{\infty} e^{t-x}f(x)dx \\ &= -e^{-t} \int_0^t e^x f(x)dx + e^t \int_t^{\infty} e^{-x} f(x)dx \end{aligned}$$

after integrating by parts we get

$$\begin{aligned} &= -f(t) + e^{-t}f(0) + e^{-t} \int_0^t e^x f'(x)dx + f(t) + e^t \int_t^{\infty} e^{-x} f'(x)dx \\ &= e^{-t}f(0) + E * f'(t) = Ef(0) + E * f'(t). \end{aligned}$$

When  $t < 0$  we get

$$\begin{aligned} E' * f(t) &= \int_0^{\infty} e^{t-x}f(x)dx \\ &= e^t \int_0^{\infty} e^{-x}f(x)dx \\ &= e^t \left( f(0) + \int_0^{\infty} e^{-x}f'(x)dx \right) \\ &= e^t f(0) + \int_0^{\infty} e^{t-x}f'(x)dx = Ef(0) + E * f'(t). \end{aligned}$$

So we have

$$\frac{E' * f}{E} = f(0)\delta_0 + \frac{E * f'}{E}.$$

□

Following similar steps as in the proof of Theorem 27 we have

**Theorem 28.** *Let  $f \in \mathcal{C}_{exp}(\mathbb{R})$  be positive on  $(-\infty, 0)$  and 0 everywhere else with possible discontinuity at 0. Then the generalized derivative of  $f$  is*

$$\frac{E' * f}{E} = -f(0)\delta_0 + \frac{E * f'}{E}.$$

### 3.2.2 The Fourier Transform

We define the Fourier transform of an element  $\frac{f}{k}$  of  $\mathcal{B}(\mathcal{C}_{exp}(\mathbb{R}), \mathcal{E})$  by

$$\mathcal{F}\left(\frac{f}{k}\right) = \frac{\hat{f}}{\hat{k}}.$$

We can verify  $\mathcal{F}$  is well-defined: Suppose  $\frac{f}{E^n} = \frac{g}{E^m}$ . Then we have  $E^n * g = E^m * f$ . By taking the Fourier transform of both sides we get  $\hat{E}^n \hat{g} = \hat{E}^m \hat{f}$  or  $\frac{\hat{f}}{\hat{E}^n} = \frac{\hat{g}}{\hat{E}^m}$ .

Through straightforward calculation we get  $\hat{E}(s) = \frac{2}{1+s^2}$ . Hence for every  $\frac{f}{E^n} \in \mathcal{B}(\mathcal{C}_{exp}(\mathbb{R}), \mathcal{E})$  we have the Fourier transform is  $\left(\frac{1+s^2}{2}\right)^n \hat{f}$ . Furthermore, remembering the differentiation property of the Fourier transform where  $\mathcal{F}(f')(s) = is\hat{f}(s)$ , we get

$$\mathcal{F}\left(\frac{E' * f}{E}\right)(s) = \frac{\hat{E}'(s)\hat{f}(s)}{\hat{E}(s)} = \frac{is\hat{E}(s)\hat{f}(s)}{\hat{E}(s)} = is\hat{f}(s).$$

### 3.3 Pseudoquotients With Positive Definite Functions

We start this section with some basic definitions:

A topological space  $\mathcal{G}$  is locally compact if every  $x \in \mathcal{G}$  has a compact neighborhood.  $(\mathcal{G}, \cdot, \tau)$  is a topological group if  $(\mathcal{G}, \cdot)$  is a group,  $\tau$  is a Hausdorff topology on  $\mathcal{G}$ , and the maps  $x \mapsto x^{-1}$  and  $(x, y) \mapsto x \cdot y$  are continuous on  $\mathcal{G}$  with respect to  $\tau$ . If we have the group structure, we can say a topological space is locally compact if it has a compact basis

at the origin. It is clear that  $\mathbb{R}$  with addition and the standard topology is a locally compact group.

A non-zero Radon measure  $\mu$  on a locally compact group  $\mathcal{G}$  is a left Haar measure if for every Borel set  $U \subset \mathcal{G}$ ,  $\mu(xU) = \mu(U)$ . Similarly we can define right Haar measure as having the property that  $\mu(Ux) = \mu(U)$ . In Proposition 2.9 in [9], it is shown that the definition of left Haar measures is equivalent to saying for every  $f \in \mathcal{C}_c(\mathcal{G})$ , and  $\alpha \in \mathcal{G}$ , we have

$$\int_{\mathcal{G}} f(\alpha x) d\mu(x) = \int_{\mathcal{G}} f(x) d\mu(x).$$

In the same proposition, we see that a necessary and sufficient condition for a measure  $\mu$  to be a left Haar measure is for  $\tilde{\mu}$  defined by  $\tilde{\mu}(U) = \mu(U^{-1})$  to be a right Haar measure. For an Abelian  $\mathcal{G}$ , we can verify that the left and right Haar measures are the same.

The following two theorems concerning the existence and uniqueness of a Haar measure on a locally compact group can be found in [9].

**Theorem 29.** *Every Abelian locally compact group  $\mathcal{G}$  possesses a Haar measure.*

**Theorem 30.** *If  $\mu$  and  $\lambda$  are Haar measures on a locally compact group  $\mathcal{G}$ , then there exists  $c \in (0, \infty)$  such that  $\mu = c\lambda$ .*

Using Theorem 30 and the definition of Haar measures we can say that if  $\mu$  is a Haar measure on a Abelian locally compact group  $\mathcal{G}$  then for every  $f \in \mathcal{C}_c(\mathcal{G})$ , and  $\alpha \in \mathcal{G}$ , we have

$$\int_{\mathcal{G}} f(\alpha + x) d\mu(x) = \int_{\mathcal{G}} f(x) d\mu(x) = \int_{\mathcal{G}} f(-x) d\mu(x)$$

where  $-x$  is the inverse of  $x$  with respect to the commutative operation  $+$  in  $\mathcal{G}$ . Simply speaking, a Haar measure is translation and reflection invariant. For example the Lebesgue



measure is a Haar measure on  $\mathbb{R}$  with addition. As another example, the measure  $\mu$  defined by  $\mu(A) = \int_A \frac{1}{t} dt$  is a Haar measure on  $\mathbb{R}^+$ , positive real numbers, with multiplication: All we need to show is that this measure is a left Haar measure, note that for  $A = (a, b)$  and  $x \in \mathbb{R}^+$  we have

$$\mu(xA) = \int_{xa}^{xb} \frac{1}{t} dt = \ln(xb) - \ln(xa) = \ln(x) + \ln(b) - \ln(x) - \ln(a) = \ln(b) - \ln(a) = \mu(A).$$

### 3.3.1 The Space of Integrable Functions on $\mathcal{G}$

Consider the set of integrable functions  $L^1(\mathcal{G})$  with the associated norm  $\|f\| = \int_{\mathcal{G}} |f(x)| d\mu(x)$ .

The convolution product of two  $L^1(\mathcal{G})$  functions  $f$  and  $g$  is defined by

$$f * g(x) = \int_{\mathcal{G}} f(y)g(y^{-1}x) d\mu(y). \quad (3.1)$$

In the case where  $\mathcal{G}$  is Abelian, we may use  $-y$  instead of  $y^{-1}$  in (3.1).  $L^1(\mathcal{G})$  is closed under  $*$  and the inequality  $\|f * g\| \leq \|f\| \|g\|$  holds.

Next, we define *characters* and use them to define the Fourier transform of an integrable function. We say a continuous homomorphism  $\omega$  is a character on a locally compact group  $\mathcal{G}$ , if it maps  $\mathcal{G}$  into the unit circle group in  $\mathbb{C}$ .  $\hat{\mathcal{G}}$  denotes the set of all characters of  $\mathcal{G}$ . We define the Fourier transform of a function  $f \in L^1(\mathcal{G})$  to be

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{\mathcal{G}} \omega(x^{-1})f(x) d\mu(x) \text{ or } \int_{\mathcal{G}} \overline{\omega(x)}f(x) d\mu(x). \quad (3.2)$$

This transform has the property that if  $(L_y f)(x) = f(xy^{-1})$  then

$$(L_y f)(\omega) = \overline{\omega(y)} \hat{f}(\omega).$$

A note-worthy result, which can be found in [9], is that the Fourier transform of a function in  $L^1(\mathcal{G})$  is in  $\mathcal{C}(\hat{\mathcal{G}})$ : Suppose  $\omega_n \rightarrow \omega$  in  $\hat{\mathcal{G}}$ , that is  $\sup_K |\omega_n - \omega| \rightarrow 0$  for all compact sets

$K \subset \mathcal{G}$ . Then we have:

$$|\hat{f}(\omega_n) - \hat{f}(\omega)| \leq \int_{\mathcal{G}} |f(x)| |\omega_n(x) - \omega(x)| d\mu(x).$$

Let  $K \subset \mathcal{G}$  be a compact set such that  $\int_{x \in K^c} |f(x)| |\omega_n(x) - \omega(x)| d\mu(x) < \frac{\epsilon}{2}$ . This is possible since  $|\omega_n(x) - \omega(x)| \leq 2$  and  $f$  is integrable. So we can write:

$$\begin{aligned} \int_{\mathcal{G}} |f(x)| |\omega_n(x) - \omega(x)| d\mu(x) &= \int_{x \in K^c} |f(x)| |\omega_n(x) - \omega(x)| d\mu(x) \\ &\quad + \int_{x \in K} |f(x)| |\omega_n(x) - \omega(x)| d\mu(x) \\ &< \frac{\epsilon}{2} + \int_{x \in K} |f(x)| |\omega_n(x) - \omega(x)| d\mu(x). \end{aligned}$$

On the other hand, since  $K$  is a compact set and  $\hat{\mathcal{G}}$  has the topology of uniform convergence on compact sets, we can find  $N$  such that for all  $n > N$  we have  $\int_{x \in K} |f(x)| |\omega_n(x) - \omega(x)| d\mu(x) < \frac{\epsilon}{2}$ .

### 3.3.2 Positive Definite functions

In this section we review some basics about *positive definite* functions. We start by defining measures of positive type.

**Definition 13.** We say a Radon measure  $\nu$  is of positive type or is positive definite if for every  $f \in \mathcal{C}_c(\mathcal{G})$  we have

$$\int_{\mathcal{G}} \check{f} * f d\nu \geq 0. \tag{3.3}$$

where  $\check{f}(x) = f(-x)$ .

We say a function  $p : \mathcal{G} \rightarrow \mathbb{C}$  is positive definite if

$$\sum_{i,j=1}^n c_i \bar{c}_j p(x_i - x_j) \geq 0 \tag{3.4}$$

for all  $c_1, \dots, c_n \in \mathbb{C}$  and  $x_1, \dots, x_n \in \mathcal{G}$  where  $n$  is an arbitrary positive integer.

We can use positive definite measures to define positive definite functions: A bounded continuous function  $p$  is positive definite if, the measure  $\nu = p \cdot \mu$  is positive definite for the Haar measure  $\mu$  on  $\mathcal{G}$ . To show this is consistent with (3.4), notice if we expand expression on the left side of (3.3) and substitute  $p \cdot \mu$  for  $\nu$  we get

$$\int_{\mathcal{G}} \int_{\mathcal{G}} f(x-s) \overline{f(-s)} p(x) d\mu(s) d\mu(x) \geq 0$$

which by change of variable  $-s$  to  $s$  and because  $\mu$  is a Haar measure we get

$$\int_{\mathcal{G}} \int_{\mathcal{G}} f(x+s) \overline{f(s)} p(x) d\mu(s) d\mu(x) \geq 0.$$

Now if we do the change of variable again but this time  $x$  to  $x-s$  we get

$$\int_{\mathcal{G}} \int_{\mathcal{G}} f(x) \overline{f(s)} p(x-s) d\mu(s) d\mu(x) \geq 0$$

which has a form similar to (3.4). Indeed, Proposition 3.35 in [9] says that a bounded continuous function  $p$  is positive definite if and only if

$$\int_{\mathcal{G}} (\overline{f} * f) p d\mu \geq 0.$$

Positive definite function on a locally compact group  $\mathcal{G}$  form a cone which we will denote by  $P(\mathcal{G})$ .

Here we mention a version of Bochner's theorem as stated in [9].

**Theorem 31** (Bochner's theorem). *If  $p$  is a positive definite function then there exists a unique positive measure  $\mu_p$  on  $\hat{\mathcal{G}}$  such that*

$$p(x) = \int_{\hat{\mathcal{G}}} \omega(x) d\mu_p(\omega).$$

Let  $\mathcal{P}(\mathcal{G})$  denote the space generated by  $P(\mathcal{G})$ , i.e. for every  $p \in \mathcal{P}(\mathcal{G})$  we have  $p = p_1 - p_2 + ip_3 - ip_4$  for  $p_1, p_2, p_3, p_4 \in P(\mathcal{G})$ . By Bochner's theorem, we have every element of  $P(\mathcal{G})$  can be associated with a bounded positive measure. Extending this theorem we can say that each element of  $\mathcal{P}(\mathcal{G})$  can be associated with an arbitrary bounded measure.

In [8], J. Dieudonné presents a proof to the theorem that shows for every  $f \in L^1(\mathcal{G})$  and  $p \in \mathcal{P}(\mathcal{G})$ ,  $f * p \in \mathcal{P}(\mathcal{G})$ . We give a slightly different presentation of the proof which is hopefully more clear to the general reader than the original: Remember the anti-Fourier transform of a function  $f \in L^1(\mathcal{G})$  is defined by

$$\overline{\mathcal{F}}(f)(\omega) = \int_{\mathcal{G}} \omega(x) f(\omega) d\mu(x)$$

The following lemma will be used in our presentation of the theorem.

**Lemma 4.** *Let  $f \in L^1(\mathcal{G})$  and  $p \in \mathcal{P}$ . Then we have*

$$\int_{\mathcal{G}} f(x) p(x) d\mu(x) = \int_{\hat{\mathcal{G}}} \overline{\mathcal{F}} f(\omega) d\mu_p(\omega). \quad (3.5)$$

*Proof.* By Bochner's theorem we have

$$p(x) = \int_{\hat{\mathcal{G}}} \omega(x) d\mu_p(\omega)$$

for a bounded positive measure  $\mu'$  on  $\hat{\mathcal{G}}$ . Then looking at the left hand side of (3.5) we get

$$\begin{aligned} \int_{\mathcal{G}} f(x) p(x) d\mu(x) &= \int_{\mathcal{G}} f(x) \int_{\hat{\mathcal{G}}} \omega(x) d\mu_p(\omega) d\mu(x) \\ &= \int_{\hat{\mathcal{G}}} \int_{\mathcal{G}} f(x) \omega(x) d\mu(x) d\mu_p(\omega) \\ &= \int_{\hat{\mathcal{G}}} \overline{\mathcal{F}} f(\omega) d\mu_p(\omega), \end{aligned}$$

by Fubini's theorem. □

We now continue with the presentation of the proof: Because the space of bounded measures contains integrable functions, and every bounded measure can be written as a linear combination of four positive definite measures, it is sufficient to show for every positive definite function  $p$  and positive definite measure  $\nu$  we have the convolution  $p * \nu$  is positive definite. That is, we want to show for every continuous function  $g$  on  $\mathcal{G}$  with compact support we have:

$$\int_{\mathcal{G}} (p * \nu) \bar{g} * g d\mu \geq 0. \quad (3.6)$$

Notice we have

$$p * \nu(x) = \int_{\mathcal{G}} p(x - s) d\nu(s). \quad (3.7)$$

By substituting from (3.7) in the left hand side of the inequality (3.6) we get:

$$\int_{\mathcal{G}} \int_{\mathcal{G}} \int_{\mathcal{G}} p(x - s) g(y) \bar{g}(x - y) d\mu(x) d\mu(y) d\nu(s).$$

Changing the variable from  $x - s$  to  $x$  we get

$$\int_{\mathcal{G}} \int_{\mathcal{G}} \int_{\mathcal{G}} p(x) g(y) \bar{g}(s + x - y) d\mu(x) d\mu(y) d\nu(s)$$

and by changing  $y$  to  $s + y$  we get

$$\int_{\mathcal{G}} \int_{\mathcal{G}} \int_{\mathcal{G}} p(x) g(s + y) \bar{g}(x - y) d\mu(x) d\mu(y) d\nu(s)$$

or

$$\int_{\mathcal{G}} p(x) \left( \int_{\mathcal{G}} g(s + y) \bar{g}(x - y) d\mu(y) \right) d\mu(x) \int_{\mathcal{G}} d\nu(s) \quad (3.8)$$

Observe that the expression in the parenthesis is equal to  $L_s g * \bar{g}$  and the function defined by  $x \mapsto (L_s g * \bar{g})(x)$  is an integrable function. Furthermore, We have:

$$\begin{aligned}
\overline{\mathcal{F}}(g(s+y) * \bar{g}(y)) &= \overline{\mathcal{F}}(g(s+y)) \overline{\mathcal{F}}(\bar{g}(y)) \\
&= \overline{\omega(s)} \overline{\mathcal{F}}(g) \int_{\mathcal{G}} \bar{g}(y) \omega(y) d\mu(y) \\
&= \overline{\omega(s)} \overline{\mathcal{F}}(g) \int_{\mathcal{G}} \overline{g(-y)} \omega(y) d\mu(y) \\
&= \overline{\omega(s)} \overline{\mathcal{F}}(g) \int_{\mathcal{G}} \overline{g(-y)} \overline{\omega(-y)} d\mu(y) \\
&= \overline{\omega(s)} \overline{\mathcal{F}}(g) \int_{\mathcal{G}} \overline{g(-y)} \omega(-y) d\mu(y) \\
&= \overline{\omega(s)} \overline{\mathcal{F}}(g) \overline{\int_{\mathcal{G}} g(y) \omega(y) d\mu(y)} \\
&= \overline{\omega(s)} \overline{\mathcal{F}}(g) \overline{\overline{\mathcal{F}}(g)} \\
&= \overline{\omega(s)} |\overline{\mathcal{F}}(g)|^2.
\end{aligned}$$

And so by Lemma 4, we can simplify (3.8) to

$$\int_{\hat{\mathcal{G}}} \left( \int_{\mathcal{G}} \overline{\omega(s)} |\overline{\mathcal{F}}(g)(\omega)|^2 d\mu'(\omega) \right) d\nu(s)$$

which by Fubini's theorem is equal to

$$\int_{\hat{\mathcal{G}}} \left( \int_{\mathcal{G}} \overline{\omega(s)} d\nu(s) \right) |\overline{\mathcal{F}}(g)(\omega)|^2 d\mu'(\omega)$$

The quantity inside the parenthesis is the Fourier transform of  $\nu$  which is a positive definite measure so its Fourier transform is a positive function. Thus the whole integral is positive and our initial  $p * \nu$  is a positive definite function.

### 3.3.3 The Construction

Having covered the basic definitions, we can now continue with the construction of  $\mathcal{B}(\mathcal{P}(\mathcal{G}), G)$ .

The result in this section are established in [3] but here we mention them with more details as an example of the construction of generalized quotients.

Let

$$G = \{\varphi \in L^1(\mathcal{G}) : \hat{\varphi}(\omega) > 0 \quad \forall \omega \in \hat{\mathcal{G}}\}.$$

We can show that as long as  $\hat{\mathcal{G}}$  is  $\sigma$ -finite,  $G$  is nonempty: From the  $\sigma$ -finiteness we get there are open sets  $A_1, A_2, \dots$  of finite measure in  $\hat{\mathcal{G}}$  whose union is  $\hat{\mathcal{G}}$  and are mutually disjoint. For every  $n \in \mathbb{N}$  let  $\varphi_n \in L^2(\mathcal{G})$  be such that  $\hat{\varphi}_n = \chi_{A_n}$  which is square integrable on  $\hat{\mathcal{G}}$ . We claim we can find a sequence  $(a_n)$  of positive real numbers such that

1.  $\sum_{n=1}^{\infty} a_n \varphi_n \in L^2(\mathcal{G})$
2.  $\sum_{n=1}^{\infty} a_n \hat{\varphi}_n \in L^1(\hat{\mathcal{G}})$

Notice  $a_n = \min\left(\frac{1}{2^n \|\varphi_n\|_2}, \frac{1}{2^n \|\hat{\varphi}_n\|_1}\right)$  is one such sequence. We can verify that

$$\varphi = \varphi_1 * \sum_{n=1}^{\infty} a_n \varphi_n$$

is in  $G$ :  $\varphi_1 \in L^2(\mathcal{G})$  and  $\sum_{n=1}^{\infty} a_n \varphi_n \in L^2(\mathcal{G})$  so the product is in  $L^1(\mathcal{G})$ . Finally to check that  $\hat{\varphi} > 0$ , note that

$$\hat{\varphi} = \hat{\varphi}_1 * \sum_{n=1}^{\infty} a_n \hat{\varphi}_n.$$

$\hat{\varphi}$  is always greater than 0 because  $\hat{\varphi}_1 \geq 0$  and  $\sum_{n=1}^{\infty} a_n \hat{\varphi}_n$  is always strictly positive. Also, because we are actually assuming that  $\mathcal{G}$  is an Abelian  $\sigma$ -compact locally compact group,  $(G, *)$  is a commutative semigroup.

**Proposition 2.** *Action of  $G$  on  $\mathcal{P}(\mathcal{G})$  defined by convolution is injective.*

We have already demonstrated a version of the proof for  $\varphi * f \in \mathcal{P}(\mathcal{G})$  for  $\varphi \in L^1(\mathcal{G})$  and  $f \in \mathcal{P}(\mathcal{G})$  from [8]. For injection, suppose  $\varphi * f = 0$  for  $\varphi \in G$  and  $f \in \mathcal{P}(\mathcal{G})$ . Then taking the Fourier transform we get  $\hat{\varphi}\hat{f} = 0$ . But since  $\varphi \in G$ , its Fourier transform is always positive so  $f = 0$ .

All the preconditions are satisfied so we define  $\mathcal{B}(\mathcal{P}(\mathcal{G}), G)$  to be the space of generalized quotients as described in CHAPTER TWO where  $X = \mathcal{P}(\mathcal{G})$  and for each  $\frac{f}{\varphi}, \frac{g}{\psi} \in \mathcal{B}(\mathcal{P}(\mathcal{G}), G)$  we have  $\frac{f}{\varphi} = \frac{g}{\psi}$  if and only if  $\psi * f = \varphi * g$ . We can extend the Fourier transform defined on  $\mathcal{P}(\mathcal{G})$  to one on  $\mathcal{B}(\mathcal{P}(\mathcal{G}), G)$ . We define the Fourier transform of  $\frac{f}{\varphi} \in \mathcal{B}(\mathcal{P}(\mathcal{G}), G)$  by

$$\mathcal{F}\left(\frac{f}{\varphi}\right) = \frac{\hat{f}}{\hat{\varphi}}.$$

This operation is well-defined since it is the extension of the Fourier transform and the consistency condition is satisfied (i.e.  $\widehat{\hat{\varphi}\hat{f}} = \widehat{\varphi * f}$ .) We have the following characterization theorem for the Fourier transform of  $\mathcal{B}(\mathcal{P}(\mathcal{G}), G)$  which simply put says that the Fourier transform is a bijection between  $\mathcal{B}(\mathcal{P}(\mathcal{G}), G)$  and the space of all Radon measures (not only bounded anymore) on  $\hat{\mathcal{G}}$ .

**Theorem 32.** *If  $\hat{\mathcal{G}}$  is  $\sigma$ -compact, then the Fourier transform is an isomorphism from  $\mathcal{B}(\mathcal{P}(\mathcal{G}), G)$  onto  $\mathcal{M}(\hat{\mathcal{G}})$  (the set of Radon measures on  $\hat{\mathcal{G}}$ ).*

*Proof.* It suffices to show that  $\mathcal{F}$  is surjective. Let  $K_1, K_2, \dots \subset \hat{\mathcal{G}}$  be a sequence of compact sets such that  $\cup_{n=1}^{\infty} K_n = \hat{\mathcal{G}}$  and  $VK_n \subset K_{n+1}$  for some neighborhood  $V$  of  $\hat{e}$  and for all  $n \in \mathbb{N}$ . Define

$$g_n = \chi_{K_n} \quad \text{and} \quad \sigma = \chi_V. \tag{3.9}$$



Note that  $\text{supp}(g_n * \sigma) \subset K_{n+1}$  for all  $n \in \mathbb{N}$ .

Let  $\mu \in \mathcal{M}(\hat{\mathcal{G}})$ . Since  $(g_n * \sigma)\mu$  is a bounded Radon measure on  $\hat{\mathcal{G}}$ , there exist positive numbers  $a_1, a_2, \dots$  such that

$$\sum_{n=1}^{\infty} a_n (g_n * \sigma)\mu$$

defines a bounded Radon measure on  $\hat{\mathcal{G}}$ . By Bochner's theorem, there exists an  $f \in \mathcal{P}(\hat{\mathcal{G}})$  such that

$$\hat{f} = \sum_{n=1}^{\infty} a_n (g_n * \sigma)\mu.$$

Without loss of generality, we can assume that the numbers  $a_1, a_2, \dots$  are chosen so that

$$\sum_{n=1}^{\infty} a_n \|\mathcal{F}^{-1}(g_n * \sigma)\|_1 < \infty.$$

Let

$$\varphi = \sum_{n=1}^{\infty} a_n \mathcal{F}^{-1}(g_n * \sigma).$$

Clearly,  $\varphi \in G$ . Moreover,  $\sum_{n=1}^{\infty} a_n (g_n * \sigma)\mu = \hat{\varphi}\mu$  and thus

$$\mathcal{F}\left(\frac{f}{\varphi}\right) = \frac{\hat{f}}{\hat{\varphi}} = \frac{\hat{\varphi}\mu}{\hat{\varphi}} = \mu.$$

□

## CHAPTER FOUR: SOME EXAMPLES OF PSEUDOQUOTIENTS

In this chapter we look at some of the examples of spaces of pseudoquotients defined on locally compact groups. The first of such spaces is that of integrable Boehmians. In what follows,  $(\mathcal{G}, \cdot, \tau)$  is a  $\sigma$ -compact locally compact group with a Haar measure  $\mu$ . Later to assure the existence of a commutative semigroup of delta-sequences we will additionally require  $\mathcal{G}$  to be first countable and Abelian.

### 4.1 Integrable Boehmians

In [12], P. Mikusiński describes the space of integrable Boehmians on the real line and in [16] a similar construction is described for Boehmians on a sphere. A natural question would be if we can generalize this construction to functions on any locally compact group. In [13] the basics of such construction is discussed. Dennis Nemzer, in his doctoral dissertation discusses this general construction in more details however he does not implement the idea of characters, or define the Fourier transform.

For the construction of Boehmians we need the commutative semigroup  $\Delta$  for which we are going to use delta-sequences. In [13] a definition for delta-sequences on a locally compact group is given. We will adopt a similar definition.

**Definition 14.** A sequence  $(\delta_n)$  of real valued functions in the center of  $(L^1(\mathcal{G}), *)$  is called a delta-sequence if

1.  $\int_{\mathcal{G}} \delta_n(x) d\mu(x) = 1$  for every  $n \in \mathbb{N}$

2.  $\delta_n \geq 0$  for all  $x \in \mathcal{G}$ ,  $n \in \mathbb{N}$

3. For every symmetric compact set  $U$  containing the identity  $e$ ,  $\int_{U^c} \delta_n(x) d\mu(x) \rightarrow 0$  as  $n \rightarrow \infty$

The existence of such delta-sequence is discussed in [13]. In that paper a  $\mathcal{B}$ -group is defined as a locally compact group that admits a delta-sequence. A sufficient condition for a first countable locally compact group  $\mathcal{G}$  to be a  $\mathcal{B}$ -group is  $\mathcal{G}$  being Abelian, like the locally compact group  $(\mathbb{R}, +)$ , or compact, like the compact group of a sphere with rotations. As discussed earlier, from this point on we will assume that  $\mathcal{G}$  is first countable and Abelian. Let  $\Delta$  be the collection of all delta-sequences. It is straightforward calculation that  $\Delta$  with the convolution operation  $*$  is a semigroup. We also have the following:

Delta-sequences have the following property. This is a standard result so we just mention the theorem.

**Theorem 33.** *For every delta-sequence  $(\delta_n)$  we have  $\hat{\delta}_n \rightarrow 1$  uniformly on compact sets.*

We define the relation  $\sim$  on  $(L^1(\mathcal{G}))^{\mathbb{N}} \times \Delta$  by  $(f_n, \delta_n) \sim (f'_n, \delta'_n)$  if and only if  $\delta_n * f'_m = \delta'_m * f_n$ . We showed in CHAPTER TWO that if  $\Delta$  is a commutative semigroup and its action is injective on, in this case,  $L^1(\mathcal{G})$ , then  $\sim$  defined this way is an equivalence relation. By the definition of delta-sequences we have the commutativity in  $\Delta$ . Also, if  $\bar{\delta} * f = 0$  for an integrable function  $f$ , then for every  $n \in \mathbb{N}$ ,  $\delta_n * f = 0$  which implies  $\hat{\delta}_n \hat{f} = 0$ . Since  $\hat{\delta}_n \rightarrow 1$ ,  $\hat{f} = 0$  which means  $f = 0$  and hence the injectivity. We define the space of integrable Boehmians on  $\mathcal{G}$  to be  $\mathcal{B}(L^1(\mathcal{G}), \Delta) = (L^1(\mathcal{G}))^{\mathbb{N}} \times \Delta / \sim$ .  $\mathcal{B}(L^1(\mathcal{G}), \Delta)$  is a convolution algebra

where scalar multiplication and  $+$  are defined by:

$$\lambda \frac{\bar{f}}{\bar{\delta}} = \frac{\lambda \bar{f}}{\bar{\delta}},$$

$$\frac{\bar{f}}{\bar{\delta}} + \frac{\bar{f}'}{\bar{\delta}'} = \frac{\bar{\delta}' * \bar{f} + \bar{\delta} * \bar{f}'}{\bar{\delta} * \bar{\delta}'}$$

Once again, since  $\Delta$  satisfies the conditions mentioned in CHAPTER TWO, (it is a commutative semigroup acting injectively on  $L^1(\mathcal{G})$ ) these operations are well-defined. There we also introduced two choices for the multiplication ( $*$  in this case) of two pseudoquotients. Since for every  $f, f', \delta \in L^1(\mathcal{G})$ ,  $\delta * (f * f') = (\delta * f) * f'$ , we define the operation  $*$  on  $\mathcal{B}(L^1(\mathcal{G}), \Delta)$  by

$$\frac{\bar{f}}{\bar{\delta}} * \frac{\bar{f}'}{\bar{\delta}'} = \frac{\bar{f} * \bar{f}'}{\bar{\delta} * \bar{\delta}'},$$

which we also showed is well-defined. We can embed  $L^1(\mathcal{G})$  into  $\mathcal{B}(L^1(\mathcal{G}), \Delta)$  continuously using  $\iota$  defined in CHAPTER TWO.

Next we extend the Fourier transform to  $\mathcal{B}(L^1(\mathcal{G}), \Delta)$ . We do this by introducing the Fourier transform of Boehmians as the extension of the Fourier transform of integrable function on  $\mathcal{G}$ . We define the Fourier transform of  $\frac{\bar{f}}{\bar{\delta}} \in \mathcal{B}(L^1(\mathcal{G}), \Delta)$  by

$$\mathcal{F} \left( \frac{\bar{f}}{\bar{\delta}} \right) = \frac{\hat{\bar{f}}}{\hat{\bar{\delta}}}.$$

To match this with our notation of the extension map from CHAPTER TWO, the Fourier transform  $\mathcal{F} : \mathcal{B}(L^1(\mathcal{G}), \Delta) \rightarrow \mathcal{B}(\mathcal{C}(\hat{\mathcal{G}}), \hat{\Delta})$  is the  $\tilde{\mu}$  and the map  $\mathcal{F} : L^1(\mathcal{G}) \rightarrow \mathcal{C}(\hat{\mathcal{G}})$  is both  $\mu$ , and  $\alpha$ . To verify that this is actually an extension, we need to check the consistency condition: Indeed  $\mathcal{F}(\delta)\mathcal{F}(f) = \mathcal{F}(\delta * f)$ .

Since the Fourier transform is a continuous map on  $L^1(\mathcal{G})$ , its extension to the integrable Boehmians is also a continuous map with respect to the quotient topology. Additionally, since the numerator of the Fourier transform of an integrable Boehmian is a continuous function and the Fourier transform of the denominator – the Fourier transform of a delta-sequence – converges to 1 uniformly, then for every open set we can find  $n$  such that  $\hat{\delta}_n \neq 0$  on that set and hence the quotient is always a continuous function. Actually we can do better than that. We can characterize that function. To do so we follow similar steps as presented in [12] for integrable Boehmians on  $\mathbb{R}$ .

**Lemma 5.** *If  $\frac{\bar{f}}{\hat{\delta}} \in \mathcal{B}(L^1(\mathcal{G}), \Delta)$ , then the sequence  $\hat{f}_n$  converges uniformly on each compact set in  $\hat{\mathcal{G}}$ .*

*Proof.* For every compact set  $K$  in  $\hat{\mathcal{G}}$ , since  $\hat{\delta}_n \rightarrow 1$  uniformly on  $K$  as  $n \rightarrow \infty$ ,  $\hat{\delta}_k \neq 0$  for almost all  $k \in \mathbb{N}$  and on  $K$  we have

$$\hat{f}_n = \hat{f}_n \frac{\hat{\delta}_k}{\hat{\delta}_k} = \frac{\hat{f}_n \hat{\delta}_k}{\hat{\delta}_k} = \frac{(f_n * \delta_k)^\wedge}{\hat{\delta}_k} = \frac{(f_k * \delta_n)^\wedge}{\hat{\delta}_k} = \frac{\hat{f}_k}{\hat{\delta}_k} \hat{\delta}_n.$$

Hence  $\hat{f}_n$  converges to  $\frac{\hat{f}_k}{\hat{\delta}_k}$  uniformly on compact sets. □

Using above lemma, for every  $\frac{\bar{f}}{\hat{\delta}} \in \mathcal{B}(L^1(\mathcal{G}), \Delta)$ , the Fourier transform can be defined by

$$\mathcal{F}\left(\frac{\bar{f}}{\hat{\delta}}\right) = \frac{\bar{f}}{\hat{\delta}} = \lim_{n \rightarrow \infty} \hat{f}_n$$

in the space of continuous functions on  $\mathcal{G}$ .

The following lemma shows that the Fourier transform on  $\mathcal{B}(L^1(\mathcal{G}), \Delta)$  is consistent with the Fourier transform of functions in  $L^1(\mathcal{G})$ .

**Lemma 6.** *For every  $f \in L^1(\mathcal{G})$ ,  $\mathcal{F}(\iota(f)) = \hat{f}$ .*

*Proof.* Let  $\bar{\delta} \in \Delta$ . Then we have

$$\mathcal{F}\left(\frac{\bar{\delta} * f}{\bar{\delta}}\right) = \lim_{n \rightarrow \infty} (\delta_n * f)^\wedge = \lim_{n \rightarrow \infty} \hat{\delta}_n \hat{f} = \hat{f}.$$

□

One of the properties of the Fourier transform concerns the effect of the shift operator. First we need to describe what we mean by shifting a Boehmian or pseudoquotients in general. Let  $L_y$ ,  $y \in \mathcal{G}$ , be the shift operator on  $L^1(\mathcal{G})$  as defined before. Obvious choices for extension of  $L_y$  to the Boehmians are to shift both the numerator and the denominator, only the numerator, or only the denominator. Notice our first option does not produce a new object. If we shift both top and bottom, we will get the exact same quotient. On the other hand if we define the shift operator so it only changes the denominator, we will lose the consistency with the shift in  $L^1(\mathcal{G})$ : On one hand we have  $L_y(f) = \frac{\bar{\delta} L_y(f)}{\bar{\delta}}$  and on the other hand, if we let  $L_y \bar{\delta}$  be the sequence  $(L_y \delta_n)$  we have  $L_y(f) = L_y\left(\frac{\bar{\delta} f}{\bar{\delta}}\right) = \frac{\bar{\delta} f}{L_y(\bar{\delta})}$  and  $\frac{\bar{\delta} L_y(f)}{\bar{\delta}} \neq \frac{\bar{\delta} f}{L_y(\bar{\delta})}$  in general. Hence the way we will define the shift operator is for the translation to act on the numerator: For a Boehmian  $\frac{\bar{f}}{\bar{\delta}}$ , we define the left shift operator by  $L_y\left(\frac{\bar{f}}{\bar{\delta}}\right) = \frac{L_y \bar{f}}{\bar{\delta}}$ . This is a well-defined operation and consistent with translation of integrable functions on  $\mathcal{G}$ . In the case of the Fourier transform of elements of  $\mathcal{B}(L^1(\mathcal{G}), \Delta)$ , we have the following:

**Theorem 34.** *For every  $\omega \in \hat{\mathcal{G}}$  and  $y \in \mathcal{G}$ , we have  $\mathcal{F}\left(L_y\left(\frac{\bar{f}}{\bar{\delta}}\right)\right)(\omega) = \overline{\omega(y)} \mathcal{F}\left(\frac{\bar{f}}{\bar{\delta}}\right)$*

*Proof.* The proof follows directly from the definition of the Fourier transform: For every

fixed  $y \in \mathcal{G}$ ,

$$\begin{aligned}
\mathcal{F} \left( L_y \left( \frac{\bar{f}}{\bar{\delta}} \right) \right) (\omega) &= \mathcal{F} \left( \frac{L_y \bar{f}}{\bar{\delta}} \right) (\omega) \\
&= \lim_{n \rightarrow \infty} (L_y f_n)^\wedge (\omega) \\
&= \lim_{n \rightarrow \infty} \overline{\omega(y)} \hat{f}_n(\omega) \\
&= \overline{\omega(y)} \lim_{n \rightarrow \infty} \hat{f}_n(\omega) \\
&= \overline{\omega(y)} \mathcal{F} \left( \frac{\bar{f}}{\bar{\delta}} \right) (\omega).
\end{aligned}$$

□

Notice that since for every integrable function  $f$ ,  $\hat{f}$  is a continuous function, the Fourier transform of elements of  $\mathcal{B}(L^1(\mathcal{G}), \Delta)$ , being uniform limit of continuous functions, are in  $\mathcal{C}(\hat{\mathcal{G}})$ . We next explore the algebraic properties of the Fourier transform.

**Theorem 35.** *The Fourier transform is a linear operator from  $\mathcal{B}(L^1(\mathcal{G}), \Delta)$  to  $\mathcal{C}(\hat{\mathcal{G}})$ .*

*Proof.* Let  $\lambda, \lambda' \in \mathbf{C}$  and  $\frac{\bar{f}}{\bar{\delta}}, \frac{\bar{f}'}{\bar{\delta}'} \in \mathcal{B}(L^1(\mathcal{G}), \Delta)$ . The result is obtained from the following sequence of equalities:

$$\begin{aligned}
\mathcal{F} \left( \lambda \frac{\bar{f}}{\bar{\delta}} + \lambda' \frac{\bar{f}'}{\bar{\delta}'} \right) &= \mathcal{F} \left( \frac{\bar{\delta}' * \lambda \bar{f} + \bar{\delta} * \lambda' \bar{f}'}{\bar{\delta} * \bar{\delta}'} \right) \\
&= \lim_{n \rightarrow \infty} (\delta'_n * \lambda f_n + \delta_n * \lambda' f'_n)^\wedge \\
&= \lim_{n \rightarrow \infty} (\delta'_n * \lambda f_n)^\wedge + \lim_{n \rightarrow \infty} (\delta_n * \lambda' f'_n)^\wedge \\
&= \lim_{n \rightarrow \infty} (\hat{\delta}'_n \lambda \hat{f}_n) + \lim_{n \rightarrow \infty} (\hat{\delta}_n \lambda' \hat{f}'_n) \\
&= \lim_{n \rightarrow \infty} (\lambda \hat{f}_n) + \lim_{n \rightarrow \infty} (\lambda' \hat{f}'_n) \\
&= \lambda \mathcal{F} \left( \frac{\bar{f}}{\bar{\delta}} \right) + \lambda' \mathcal{F} \left( \frac{\bar{f}'}{\bar{\delta}'} \right)
\end{aligned}$$

□

As a consequence of the extension property of maps, we have:

**Theorem 36.** *The Fourier transform is an injective map from  $\mathcal{B}(L^1(\mathcal{G}), \Delta)$  to  $\mathcal{C}(\hat{\mathcal{G}})$ .*

*Proof.* All we need to show is that the Fourier transform is injective on  $L^1(\mathcal{G})$ , i.e.  $\hat{f} = 0$  implies  $f = 0$  which is clearly true.

□

Using the extension property of the extension maps, since the Fourier transform is a homomorphism from  $L^1(\mathcal{G})$  to  $\mathcal{C}(\mathcal{G})$  then its extension is also a homomorphism and we have:

**Theorem 37.** *The Fourier transform is a homomorphism from  $(\mathcal{B}(L^1(\mathcal{G}), \Delta), *)$  to  $(\mathcal{C}(\hat{\mathcal{G}}), \cdot)$ .*

## 4.2 Square-integrable Boehmians

Consider the space  $L^2(\mathcal{G})$  of square-integrable functions on an Abelian first countable (to guarantee the existence of delta-sequences) locally compact group  $\mathcal{G}$ . Let convolution of two  $L^2(\mathcal{G})$  functions and the Fourier transform be defined as in (3.1) and (3.2), respectively, whenever the integrals exist. Using the Plancherel Theorem, we have that the Fourier transform can be extended to an isomorphism from the space  $L^2(\mathcal{G})$  with the convolution  $*$  to the space  $L^2(\hat{\mathcal{G}})$  with the multiplication  $\cdot$ .

In constructing the space of square-integrable Boehmians, we will use  $\Delta$ , the semigroup of delta-sequences, as described in Section 4.1. We verify the requirements for constructing pseudoquotients: By the definition of the delta-sequences, clearly  $\Delta$  is a commutative semigroup. The action of  $\Delta$  on  $L^2(\mathcal{G})$  is defined by convolutions, i.e.  $(\delta_n) * f = (\delta_n * f)$



for every  $(\delta_n) \in \Delta$  and  $f \in L^2(\mathcal{G})$ . To see that the sequence  $(\delta_n * f)$  is indeed in  $(L^2(\mathcal{G}))^{\mathbb{N}}$  note that for every  $n \in \mathbb{N}$  by Proposition 2.39 of [9] we have  $\delta_n * f \in L^2(\mathcal{G})$ . For injectivity, suppose  $\delta_n * f = 0$  for every  $n \in \mathbb{N}$ . Then we have  $\hat{\delta}_n \hat{f} = 0$  for all  $n$ . But since  $\hat{\delta}_n$  converges uniformly to 1 on compact sets, then  $\hat{f} = 0$  which implies  $f = 0$ . We continue with the construction by defining the relation  $\sim$  as before but this time on  $(L^2(\mathcal{G}))^{\mathbb{N}} \times \Delta$ : We say  $(f_n, \delta_n) \sim (f'_n, \delta'_n)$  if and only if for every  $n, m \in \mathbb{N}$ ,  $\delta_n * f'_m = \delta'_m * f_n$ . From CHAPTER TWO we have that  $\sim$  is an equivalence. We define the space of square-integrable Boehmians to be  $\mathcal{B}(L^2(\mathcal{G}), \Delta) = (L^2(\mathcal{G}))^{\mathbb{N}} \times \Delta / \sim$ .

The Fourier transform of a square-integrable Bohemian is the extension of the Fourier transform of a square-integrable functions. What that means is that for every  $\frac{\bar{f}}{\bar{\delta}} \in \mathcal{B}(L^2(\mathcal{G}), \Delta)$ ,

$$\mathcal{F} \left( \frac{\bar{f}}{\bar{\delta}} \right) = \frac{\hat{f}}{\hat{\delta}}.$$

Clearly the consistency condition is satisfied. The continuity with respect to the quotient topology and injectivity of the Fourier transform of these Boehmians follows that of the transform on  $L^2(\mathcal{G})$ . We have a nice characterization of the Fourier transform of square-integrable functions. It would be nice if we could say something about the range of the extension of this transform. Next theorem addresses this matter.

**Theorem 38.** *The Fourier transform of a square-integrable Bohemian is a locally square-integrable function.*

*Proof.* Let  $F = \frac{\bar{f}}{\bar{\delta}}$  be in  $\mathcal{B}(L^2(\mathcal{G}), \Delta)$ . Then we have  $\hat{F} = \frac{\hat{f}}{\hat{\delta}}$ . Since  $\hat{\delta}_n \rightarrow 1$  for every compact set  $K$  there exists a  $\delta_j$  such that  $\hat{\delta}_j \neq 0$  on  $K$ . Notice on  $K$ ,  $\hat{F} \hat{\delta}_j = \hat{f}_j$  and so  $\hat{F} \hat{\delta}_j$  must be a square-integrable function and  $\hat{F}$  a locally square-integrable function.  $\square$

In fact, under some mild conditions, we can actually fully characterize the range of the Fourier transform. We first define *weak delta-sequences*. We say a sequence  $(\delta'_n)$  of functions in  $L^1(\mathcal{G})$  is a weak delta-sequence if for all  $n \in \mathbb{N}$ ,  $\hat{\delta}'_n$  has a compact support and  $\hat{\delta}'_n \rightarrow 1$  uniformly on compact set in  $\hat{\mathcal{G}}$ . We denote the collections of weak delta-sequences by  $\Delta'$ . We can show  $\Delta'$  is not empty if  $\hat{\mathcal{G}}$  is  $\sigma$ -compact.

**Lemma 7.** *If  $\hat{\mathcal{G}}$  is  $\sigma$ -compact then  $\Delta'$  is not empty.*

*Proof.* Let  $V$  be a symmetric open set in  $\mathcal{G}$  with compact closure. Suppose  $f \in L^1(\mathcal{G})$  is such that  $\text{supp } f \subset V$  and

$$\int_V f(x) d\mu(x) = 1.$$

Finally consider the characteristic function  $\chi_{[K-V]}$  for some compact set  $K$ . Notice that we have

$$f * \chi_{[K-V]}(t) = \int_V f(x) \chi_{[K-V]}(t-x) d\mu(x)$$

When  $t \in K$ ,  $\chi_{[K-V]}(t-x) = 1$  and hence  $f * \chi_{[K-V]}(t) = \int_V f(x) d\mu(x) = 1$ . For all other  $t$ ,  $\chi_{[K-V]}(t-x) = 0$  and so  $f * \chi_{[K-V]}(t) = 0$ . It is clear that  $f * \chi_{[K-V]} \in L^1(\mathcal{G})$ . To construct our sequence, notice that since  $\mathcal{G}$  is first countable, we can find a sequence  $K_n$  of compact set in  $\mathcal{G}$  such that  $K_1 \subset K_2 \subset \dots$  and  $\cup_n K_n = \mathcal{G}$ . Define  $f_i := f * \chi_{[K_i-V]}$ . Let  $g, h_i \in L^2(\hat{\mathcal{G}})$  for all  $i \in \mathbb{N}$  be such that  $\hat{g} = f$  and  $\hat{h}_i = \chi_{[K_i-V]}$ . The former is possible since  $f$  is an integrable function on a bounded domain and hence is in  $L^2(\mathcal{G})$  and the latter is possible since  $\chi_{[K_i-V]}$  is in  $L^2(\mathcal{G})$ . If we let  $\varphi_i = gh_i$  then we have

$$\hat{\delta}'_i = \mathcal{F}(gh_i) = f * \chi_{[K_i-V]} = f_i \tag{4.1}$$

which converges uniformly to the constant function 1. □

**Theorem 39.** *For a locally compact group  $\mathcal{G}$ , if  $\hat{\mathcal{G}}$  is  $\sigma$ -compact then the Fourier transform is a bijection from the space of square-integrable Boehmians with weak delta-sequences in the denominator to the space of locally square integrable functions.*

*Proof.* Let  $g$  be a locally square-integrable function. We need to find  $\frac{\bar{f}}{\delta} \in \mathcal{B}(L^2(\mathcal{G}), \Delta')$  such that  $\frac{\bar{f}}{\delta} = g$ . Since  $\hat{\mathcal{G}}$  is  $\sigma$ -compact, there exist compact sets  $K_1 \subset K_2 \subset \dots$  such that  $\cup_n K_n = \hat{\mathcal{G}}$ . By Lemma 7 We can construct a weak delta-sequence  $(\delta_n)$  such that for every  $n \in \mathbb{N}$ ,  $\hat{\delta}_n \neq 0$  on  $K_n$  and  $\text{supp}(\hat{\delta}_n) \subset K_{n+1}$ . This way for every  $n$ ,  $\hat{f}_n = g\hat{\delta}_n$  is a square integrable function. Thus  $g = \frac{\bar{f}}{\delta} = \mathcal{F}\left(\frac{\bar{f}}{\delta}\right)$ . Of course, by construction,  $\frac{\bar{f}}{\delta} \in \mathcal{B}(L^2(\mathcal{G}), \Delta')$ . □

### 4.3 Pseudoquotients On Involutive Algebras

In this section we will provide a particular example for pseudoquotient with numerators being elements of an *involutive algebra*. The inspiration for this part comes from [4], but although we seem to be using a similar outline our work is noticeably different. A big difference is that [4] focuses on semigroups instead of *involutive algebras*. The results of this section are also in [2].

For a commutative algebra  $A$  with the unitary element  $e$ , and a norm  $\|\cdot\|$  satisfying  $\|xy\| \leq \|x\|\|y\|$ , we define the involution to be an automorphism defined by  $x \mapsto x^*$  having the properties:  $(x + y)^* = x^* + y^*$ ,  $(\lambda x)^* = \bar{\lambda}x^*$ ,  $(xy)^* = x^*y^*$ , and  $x^{**} = x$ , for  $\lambda \in \mathbb{C}$ , and  $x, y \in A$ . For a bounded linear functional  $f : A \rightarrow \mathbb{C}$ , we say  $f$  is of positive type if for every  $x \in A$ ,  $f(xx^*) \geq 0$ . We denote the space of bounded linear functions of positive type on  $A$  by  $\mathcal{C}_p(A)$ . A character on  $A$  is defined to be a continuous homomorphisms  $\rho : A \rightarrow \mathbb{C}$

satisfying  $\rho(e) = 1$  and  $\rho(x^*) = \overline{\rho(x)}$ . Let  $\hat{A}$  denote the set of characters on  $A$  and

$$\Gamma = \{\rho \in \hat{A} : \forall x, y \in A, |\rho(x)| \leq \|x\|, \rho(xy) = \rho(x)\rho(y)\}.$$

Notice by Tychonoff's theorem the set of continuous functionals having the magnitude at  $x$  less than or equal to  $\|x\|$  is compact. Also the set of functions preserving the multiplication ( $\rho(xy) = \rho(x)\rho(y)$ ) and addition ( $\rho(x+y) = \rho(x) + \rho(y)$ ) is a closed set with respect to the pointwise convergence. Hence,  $\Gamma$  being the intersection of a compact set and a closed set, is a compact topological space.

In [1] it is shown that  $f \in \mathcal{C}_p(A)$  if and only if

$$f(x) = \int_{\Gamma} \rho(x) d\mu_f(\rho) \tag{4.2}$$

for every  $x \in A$  and a positive Radon measure  $\mu_f$  unique to  $f$ .  $\mu_f$  is called the Fourier transform of  $f$  and is denoted by  $\hat{f}$ .

We define the translation  $E_y$  on  $\mathcal{C}_p(A)$  by  $E_y f(x) = f(xy)$ . For every  $\rho \in \Gamma$ ,  $E_y \rho(x) = \rho(xy) = \rho(x)\rho(y)$  and for any  $f \in \mathcal{C}_p(A)$ ,

$$E_y f(x) = \int_{\Gamma} \rho(xy) d\mu_f(\rho) = \int_{\Gamma} \rho(x)\rho(y) d\mu_f(\rho). \tag{4.3}$$

Let

$$G = \{yy^*, y \in A\}.$$

$G$  is a commutative semigroup with multiplication. Additionally,  $G$  acts on  $\mathcal{C}_p(A)$  by translation (i.e.  $y(f(x)) = E_y f(x)$ ). We will however write  $E_y$  instead of  $y$  whenever necessary to avoid confusion. Reflecting on (4.3) we call the map defined by  $\rho \mapsto \rho(y)$  the Fourier

transform of  $y$  and denote it by  $\hat{y}$ . In other words, for every  $\rho \in \Gamma$  and  $y \in A$ ,  $\hat{y}(\rho) = \rho(y)$ .

Hence we can rewrite (4.3) as

$$E_y f(x) = \int_{\Gamma} \rho(x) \hat{y}(\rho) d\mu_f(\rho)$$

A quick observation is that  $\widehat{xy}(\rho) = \rho(xy) = \rho(x)\rho(y) = \hat{x}\hat{y}$  for every  $\rho \in \Gamma$  and  $x, y \in A$ .

Let  $I$  be an index set and

$$\mathcal{A} = \{(\bar{f}, \bar{x}) : \forall i, j \in I, f_i \in \mathcal{C}_p(A), x_i \in G, \text{ and } E_{x_i} f_j = E_{x_j} f_i\}.$$

For any  $\bar{x} \in G^i$  we define

$$M(\bar{x}) = \{\rho \in \Gamma : \rho(x_i) = 0 \forall i \in I\}.$$

Since every  $\hat{x}_i$  is continuous,  $M(\bar{x})$  is a closed set in  $\Gamma$ . Let  $\mathcal{M}$  be all the pairs  $(\bar{f}, \bar{x})$  in  $\mathcal{A}$  with the same  $M(\bar{x}) = M$ . We define the relation  $\sim$  on  $\mathcal{M}$  the following way: For every  $(\bar{f}, \bar{x}), (\bar{g}, \bar{y}) \in \mathcal{M}$ ,  $(\bar{f}, \bar{x}) \sim (\bar{g}, \bar{y})$  if and only if for every  $i, j \in I$ ,  $E_{x_i} g_j = E_{x_j} f_i$ . We can verify that  $\sim$  is an equivalence on  $\mathcal{M}$  because the semigroup  $\{\bar{x} : (\bar{f}, \bar{x}) \in \mathcal{M}, \text{ for some } \bar{f} \in (\mathcal{C}_p(A))^I\}$  acts injectively on  $\mathcal{C}_p(A)$ .

**Theorem 40.** *For every  $(\bar{f}, \bar{x})$  and  $(\bar{g}, \bar{y})$  in  $\mathcal{M}$ ,  $(\bar{f}, \bar{x}) \sim (\bar{g}, \bar{y})$  if and only if  $\hat{x}_i \hat{g}_j = \hat{y}_j \hat{f}_i$  for all  $i, j \in I$ .*

*Proof.* From (4.2) we have for every  $i, j \in I$ , there exist positive Radon measures  $\mu_i$  and  $\nu_j$  such that

$$f_i(x) = \int_{\Gamma} \rho(x) d\mu_i(\rho)$$

and

$$g_j(x) = \int_{\Gamma} \rho(x) \nu_j(\rho).$$

Hence  $(\bar{f}, \bar{x}) \sim (\bar{g}, \bar{y})$  means for all  $i, j \in I$ ,

$$\int_{\Gamma} \rho(x) \hat{x}_i(\rho) d\nu_j(\rho) = \int_{\Gamma} \rho(x) \hat{y}_j(\rho) d\mu_i(\rho). \quad (4.4)$$

Since  $\hat{x}_i, \hat{y}_j \geq 0$ ,  $\hat{x}_i \nu_j$  and  $\hat{y}_j \mu_i$  are positive Radon measures on  $\Gamma$ . Thus because (4.4) holds for all  $x \in A$ , we can say  $\hat{x}_i \nu_j = \hat{y}_j \mu_i$  or  $\hat{x}_i \hat{g}_j = \hat{y}_j \hat{f}_i$ .

Conversely, if we have  $\hat{x}_i \hat{g}_j = \hat{y}_j \hat{f}_i$  for all  $i, j \in I$  we get

$$\begin{aligned} E_{y_j} f_i(x) &= \int_{\Gamma} \rho(x) \hat{y}_j(\rho) d\hat{f}_i(\rho) \\ &= \int_{\Gamma} \rho(x) \hat{x}_i(\rho) d\hat{g}_j(\rho) \\ &= E_{x_i} g_j(x). \end{aligned}$$

□

We define the space of pseudoquotient  $\mathcal{B}(\mathcal{C}_p(A), \mathcal{M}) = \mathcal{M} / \sim$ . Further, we can define the Fourier transform on  $\mathcal{B}(\mathcal{C}_p(A), \mathcal{M})$  by

$$\mathcal{F} \left( \frac{\bar{f}}{\bar{x}} \right) = \frac{\hat{f}}{\hat{x}}. \quad (4.5)$$

**Theorem 41.** *The Fourier transform is well-defined on  $\mathcal{B}(\mathcal{C}_p(A), \mathcal{M})$ .*

*Proof.* Suppose  $(\bar{f}, \bar{x}) \sim (\bar{g}, \bar{y})$ . Let  $U_i = \{\rho \in \Gamma : \hat{x}_i(\rho) > 0\}$  and similarly  $V_i = \{\rho \in \Gamma : \hat{y}_i(\rho) > 0\}$ . for  $i, j \in I$ . Then  $(\bar{f}, \bar{x}) \sim (\bar{g}, \bar{y})$  implies

$$\frac{\hat{f}_i}{\hat{x}_i} = \frac{\hat{g}_j}{\hat{y}_j}$$

on  $U_i \cap V_j$  for every  $i, j \in I$ . Because  $M^c = \bigcup_{i,j \in I} U_i \cap V_j$  it is clear  $\frac{\bar{f}}{\hat{x}} = \frac{\bar{g}}{\hat{y}}$ .

□

We claim for every  $\frac{\bar{f}}{\hat{x}}$ ,  $\mathcal{F}\left(\frac{\bar{f}}{\hat{x}}\right)$  is a measure. To see this, let  $\frac{\bar{f}}{\hat{x}} \in \mathcal{B}(\mathcal{C}_p(A), \mathcal{M})$  and for every  $i \in I$ ,

$$U_i = \{\rho \in \Gamma : \hat{x}_i > 0\}.$$

Clearly we have  $U_i$  is an open subset of  $M^c$ . The quotient  $\frac{\hat{f}_i}{\hat{x}_i}$  defines a measure on  $U_i$  and intersecting  $U_i$  and  $U_j$  we have

$$\frac{\hat{f}_i}{\hat{x}_i} = \frac{\hat{f}_j}{\hat{x}_j}$$

on  $U_i \cap U_j$ . We define a measure  $\lambda$  on  $M^c$  by  $\lambda = \frac{\hat{f}_i}{\hat{x}_i}$  on  $U_i$ .

**Theorem 42.** For every  $\frac{\bar{f}}{\hat{x}} \in \mathcal{B}(\mathcal{C}_p(A), \mathcal{M})$ ,  $\hat{f}_i = \hat{x}_i \lambda$  on  $M^c$ , for every  $i \in I$ .

*Proof.* All we need to show is that for every  $j \in I$ ,  $\hat{f}_i = \hat{x}_i \lambda$  on  $U_j$ . But that is true since

$$\hat{f}_i = \frac{\hat{x}_j \hat{f}_i}{\hat{x}_j} = \frac{\hat{x}_i \hat{f}_j}{\hat{x}_j} = \hat{x}_i \lambda \text{ on } U_j.$$

□

**Theorem 43.** The Fourier transform defined in (4.5) is an injection.

*Proof.* Suppose  $\frac{\bar{f}}{\hat{x}} = \frac{\bar{g}}{\hat{y}}$  we need to show  $E_{y_j} f_i = E_{x_i} g_j$ . By Theorem 42 we have

$$f_i(x) = \int_{M^c} \rho(x) \hat{x}_i(\rho) d\lambda(\rho) + \int_M \rho(x) d\mu_{f_i}(\rho) \quad (4.6)$$

Using (4.6) we get

$$\begin{aligned}
E_{y_j} f_i(x) &= \int_{M^c} \rho(x) \hat{y}_j(\rho) \hat{x}_i(\rho) d\lambda(\rho) + \int_M \rho(x) \hat{y}_j(\rho) d\mu_{f_i}(\rho) \\
&= \int_{M^c} \rho(x) \hat{y}_j(\rho) \hat{x}_i(\rho) d\lambda(\rho) \\
&= \int_{M^c} \rho(x) \hat{x}_i(\rho) \hat{y}_j(\rho) d\lambda(\rho) + \int_M \rho(x) \hat{x}_i(\rho) d\mu_{f_j}(\rho) \\
&= E_{x_i} f_j(x).
\end{aligned}$$

□

We can actually characterize the Fourier transform of  $\mathcal{B}(\mathcal{C}_p(A), \mathcal{M})$  using *Lévy measures*.

We first give the definition of these measures as stated in [4].

**Definition 15.** A positive Radon measure  $\lambda$  on  $M^c$  is called a Lévy measure associated with  $\bar{x}$  if  $\hat{x}_i \in L^1(M^c, \lambda)$  for every  $i \in I$ .

**Theorem 44.** *The Fourier transform defined by (4.5) is a bijection between  $\mathcal{B}(\mathcal{C}_p(A), \mathcal{M})$  and Lévy measures on  $M^c$ .*

*Proof.* We have shown the injection. All we need to show is the surjection. Let  $\lambda$  be a Lévy measure on  $M^c$  associated with  $\bar{x}$ . We need to find a pseudoquotient in  $\mathcal{B}(\mathcal{C}_p(A), \mathcal{M})$  whose Fourier transform is  $\lambda$ . For every  $i \in I$  let  $\mu_i$  be a Radon measure on  $M(\bar{x})$  and define  $f_i$  by

$$f_i(x) = \int_M \rho(x) d\mu_i(\rho) + \int_{M^c} \rho(x) \hat{x}_i(\rho) d\lambda(\rho).$$

Using (4.2) we see that  $f_i$  is in  $\mathcal{C}_p(A)$ . We need to check  $\frac{\bar{f}}{\bar{x}} \in \mathcal{B}(\mathcal{C}_p(A), \mathcal{M})$ . We will check that  $E_{x_j} f_i = E_{x_i} f_j$ . Notice

$$E_{x_j} f_i(x) = \int_{M^c} \rho(x) \hat{x}_j(\rho) \hat{x}_i(\rho) d\lambda(\rho) + \int_M \rho(x) \hat{x}_j(\rho) d\mu_i(\rho)$$



The second integral is zero so we can replace it with another zero integral and we get

$$E_{x_j} f_i(x) = \int_{M^c} \rho(x) \hat{x}_j(\rho) \hat{x}_i(\rho) d\lambda(\rho) + \int_M \rho(x) \hat{x}_i(\rho) d\mu_j(\rho) = E_{x_i} f_j(x).$$

□

#### 4.4 The Space $\mathcal{B}(\mathcal{P}(\mathcal{G}), \mathcal{Z})$

Our next example, as the example in Section 3.3 is mentioned in [3]. However, it provides a good example for the non-injective case and is a nice final example. In Section 3.3 we described a space of generalized quotients (the injective pseudoquotients) the Fourier transform of which was isomorphic to the space of Radon measures. In that construction we needed to find an integrable function on  $\mathcal{G}$  that had a positive Fourier transform everywhere. In this section we use the power of pseudoquotients to show similar isomorphism but with the Fourier transform of  $\mathcal{B}(\mathcal{P}(\mathcal{G}), \mathcal{Z})$ , where  $\mathcal{Z}$  denotes the semigroup of *approximate identities*.

**Definition 16.** A sequence  $(\varphi_n)$  of integrable function on  $\mathcal{G}$  is called an approximate identity if  $\hat{\varphi}_n \rightarrow 1$  uniformly on compact sets in  $\hat{\mathcal{G}}$ . We denote the semigroup of approximate identities by  $\mathcal{Z}$ .

Let  $\mathcal{P}(\mathcal{G})$  be the space of positive definite functions on  $\mathcal{G}$  as described before. it is clear that  $\mathcal{Z}$  acts injectively on  $\mathcal{P}(\mathcal{G})$ . Hence we can define the space of pseudoquotients  $\mathcal{B}(\mathcal{P}(\mathcal{G}), \mathcal{Z})$  as quotients of the form  $\frac{\bar{f}}{\bar{\varphi}}$ , with  $f_n \in \mathcal{P}(\mathcal{G})$  for every  $n \in \mathbb{N}$  and  $\bar{\varphi} \in \mathcal{Z}$ , having the exchange property.

We define the Fourier transform of elements of  $\mathcal{B}(\mathcal{P}(\mathcal{G}), \mathcal{Z})$  by

$$\mathcal{F} \left( \frac{\bar{f}}{\bar{\varphi}} \right) = \frac{\bar{\hat{f}}}{\bar{\hat{\varphi}}}.$$

$\mathcal{F}$  is clearly well-defined. Notice the Fourier transform of elements of  $\mathcal{B}(\mathcal{P}(\mathcal{G}), \mathcal{Z})$  are quotient of sequences where the sequence in the numerator is the Fourier transform of elements of  $\mathcal{P}(\mathcal{G})$ , and thus in the space  $\mathcal{M}_b(\hat{\mathcal{G}})$  of bounded Radon measures on  $\hat{\mathcal{G}}$ , and the sequence in the denominator is the Fourier transform of an element of  $\mathcal{Z}$  which therefore is a sequence of continuous functions with compact support converging to 1. Additionally just by using the properties of the Fourier transform these quotients have the exchange property. This quotients hence form a space of pseudoquotients. Notice that in this case the fraction line represents regular division. We claim this transformed space is isomorphic to the space of all Radon measures.

**Theorem 45.** *The spaces  $\mathcal{B}(\mathcal{M}_b(\hat{\mathcal{G}}), \hat{\mathcal{Z}})$  and  $\mathcal{M}(\hat{\mathcal{G}})$  of Radon measures are isomorphic.*

*Proof.* Let  $F = \frac{\hat{f}}{\hat{\varphi}} \in \mathcal{B}(\mathcal{P}(\mathcal{G}), \mathcal{Z})$ . We claim  $\hat{F}$  is a Radon measure. To do that we look at its action on elements of the space  $\mathcal{C}_c(\hat{\mathcal{G}})$  of continuous functions with compact support. Let  $\alpha \in \mathcal{C}_c(\hat{\mathcal{G}})$ . There exists an  $n \in \mathbb{N}$  such that  $\hat{\varphi}_n \neq 0$  on the support of  $\alpha$ . Then we can define

$$\langle \hat{F}, \alpha \rangle = \left\langle \frac{\hat{f}_n}{\hat{\varphi}_n}, \alpha \right\rangle = \left\langle \hat{f}_n, \frac{\alpha}{\hat{\varphi}_n} \right\rangle,$$

which since  $\hat{f}_n$  is a bounded measure and  $\hat{\varphi}_m \hat{f}_n = \hat{\varphi}_n \hat{f}_m$  for all  $m, n \in \mathbb{N}$ ,  $\langle F, \alpha \rangle$  is well-defined and the defined functional is continuous.

For surjection, suppose  $\mu \in \mathcal{M}(\hat{\mathcal{G}})$ . Let  $K_1, K_2, \dots \subset \hat{\mathcal{G}}$ ,  $g_n$ , and  $\sigma$  be as in the proof of Theorem 32. Then  $g_n, \sigma, g_n * \sigma \in L^2(\hat{\mathcal{G}})$  and thus  $\mathcal{F}^{-1}g_n, \mathcal{F}^{-1}\sigma, \mathcal{F}^{-1}(g_n * \sigma) \in L^2(X)$ . Define  $\varphi_n = \mathcal{F}^{-1}(g_n * \sigma)$ . Since  $\varphi_n = \mathcal{F}^{-1}g_n \mathcal{F}^{-1}\sigma$ , we have  $\varphi_n \in L^1(X)$ . Moreover,  $\hat{\varphi}_n \rightarrow 1$  uniformly on compact subsets of  $\hat{\mathcal{G}}$  and thus  $\bar{\varphi} \in \mathcal{Z}$ . Since, for every  $n \in \mathbb{N}$ ,  $\hat{\varphi}_n \mu \in \mathcal{M}_b(\hat{\mathcal{G}})$ , there exists  $f_n \in \mathcal{P}(\mathcal{G})$  such that  $\hat{f}_n = \hat{\varphi}_n \mu$ . Clearly,  $\frac{f_n}{\varphi_n} \in \mathcal{B}(\mathcal{P}(\mathcal{G}), \mathcal{Z})$ . Moreover, for any

$\alpha \in \mathcal{C}_c(\hat{\mathcal{G}})$ , we have

$$\left\langle \frac{\hat{f}_n}{\hat{\varphi}_n}, \alpha \right\rangle = \left\langle \hat{f}_n, \frac{\alpha}{\hat{\varphi}_n} \right\rangle = \left\langle \hat{\varphi}_n \mu, \frac{\alpha}{\hat{\varphi}_n} \right\rangle = \langle \mu, \alpha \rangle.$$

□

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