Optical Solitons In Periodic Structures

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by

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ABSTRACT

By nature discrete solitons represent self-trapped wavepackets in nonlinear periodic structures and result from the interplay between lattice diffraction (or dispersion) and material nonlinearity. In optics, this class of self-localized states has been successfully observed in both one-and two-dimensional nonlinear waveguide arrays. In recent years such lattice structures have been implemented or induced in a variety of material systems including those with cubic (Kerr), quadratic, photorefractive, and liquid-crystal nonlinearities. In all cases the underlying periodicity or discreteness leads to new families of optical solitons that have no counterpart whatsoever in continuous systems.

In the first part of this dissertation, a theoretical investigation of linear and nonlinear optical wave propagation in semi-infinite waveguide arrays is presented. In particular, the properties and the stability of surface solitons at the edge of Kerr (AlGaAs) and quadratic (LiNbO₃) lattices are examined. Hetero-structures of two dissimilar semi-infinite arrays are also considered. The existence of hybrid solitons in these latter types of structures is demonstrated.

Rabi-type optical transitions in z-modulated waveguide arrays are theoretically demonstrated. The corresponding coupled mode equations, that govern the energy oscillations between two different transmission bands, are derived. The results are compared with direct beam propagation simulations and are found to be in excellent agreement with coupled mode theory formulations.

In the second part of this thesis, the concept of parity-time-symmetry is introduced in the context of optics. More specifically, periodic potentials associated with $PT$-symmetric Hamiltonians are numerically explored. These new optical structures are found to exhibit surprising characteristics. These include the possibility of abrupt phase transitions, band
merging, non-orthogonality, non-reciprocity, double refraction, secondary emissions, as well as power oscillations. Even though gain/loss is present in this class of periodic potentials, the propagation eigenvalues are entirely real. This is a direct outcome of the $\mathcal{PT}$-symmetry. Finally, discrete solitons in $\mathcal{PT}$-symmetric optical lattices are examined in detail.
Dedicated to my family
ACKNOWLEDGMENTS

The great Spartan philosopher and one of the seven sages of the ancient world, Chilon, when he was asked how educated men differ from those who are illiterate, he said, "In good hopes". In this spirit, I consider it a moral obligation to express my deepest appreciation and gratitude to the people that educated me during my Ph.D study.

Firstly, I would like to thank my advisor, Professor Demetrios Christodoulides, for his teaching, overall support, guidance and patience all these years. It was an honor for me to be one of his students. He introduced me to the fascinating world of nonlinear optics and most importantly he taught how to philosophize, criticize and finally do research. He also taught me to always seek solid knowledge without arrogance and pretentious spirit. I consider him the best teacher and scientist I met in my student life. In a cynical world without imagination, it is a sign of hope that still there are people who wonder and get excited from the conceptual understanding of Physics.

I am grateful to Professor George Stegeman for his generous help and support. He taught me to respect experiments and also how the intuitive thought can give you useful insight about the nature of many optical physical phenomena. Working closely with his group offered me the invaluable opportunity to develop a better understanding about the experimental work and mentality.

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<tr>
<td>1D</td>
<td>One-dimensional</td>
</tr>
<tr>
<td>2D</td>
<td>Two-dimensional</td>
</tr>
<tr>
<td>AlGaAs</td>
<td>Aluminum Gallium Arsenide</td>
</tr>
<tr>
<td>GaAs</td>
<td>Gallium Arsenide</td>
</tr>
<tr>
<td>LiNbO₃</td>
<td>Lithium Niobate</td>
</tr>
<tr>
<td>BPM</td>
<td>Beam Propagation Method</td>
</tr>
<tr>
<td>CW</td>
<td>Continuous Wave</td>
</tr>
<tr>
<td>DNLSE</td>
<td>Discrete Nonlinear Schrödinger Equation</td>
</tr>
<tr>
<td>FWHM</td>
<td>Full Width at Half-Maximum</td>
</tr>
<tr>
<td>FW</td>
<td>Fundamental Wave</td>
</tr>
<tr>
<td>SH</td>
<td>Second Harmonic</td>
</tr>
<tr>
<td>FB</td>
<td>Floquet-Bloch</td>
</tr>
<tr>
<td>$\mathcal{PT}$</td>
<td>Parity-Time</td>
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CHAPTER ONE: INTRODUCTION

In the summer of 1953, Enrico Fermi, John Pasta, and Stanislaw Ulam conducted a series of “numerical experiments” on the newly built MANIAC I Los Alamos computer. The purpose of these numerical simulations was to closely examine the dynamics of a chain of discrete particles under the influence of nonlinear nearest-neighbor inter-coupling forces. This work was partly motivated by Fermi’s presumption that the nonlinearity itself will eventually lead to ergodicity or system “thermalization”. Yet to their surprise, no such chaotic behavior was observed. Instead this discrete dynamical model exhibited quasi-periodic evolution or recurrences. The results of this study (better known today as the FPU problem) were published in 1955 in a Los Alamos technical report, shortly after Fermi’s death [1]. The unexpected results of this seminal work remained an enigma until the early sixties, when Zabusky and Kruskal [2] realized that in the continuum limit the FPU chain can be described by a Kortweg-de Vries equation [3]. This latter equation, known then to govern shallow water waves in narrow channels (such as that observed by John Scott Russell in 1834-the great wave of translation [4]), was later found to exhibit particle-like soliton solutions [5]. A soliton is by definition a localized solution to a nonlinear dispersive wave equation that remains invariant upon propagation [6]. In fully integrable systems, these waves remain intact after collision events and in essence they behave as particle-like entities. Solitons represent self-localized or self-trapped wavepackets that owe their existence to a balance between nonlinearity and dispersion (and or diffraction) effects. It is worth noting that even though most physical systems in nature lack integrability, they can still allow
self-localized solutions—also called solitary waves. In fact, the collision and stability properties of this latter class of waves happens to be considerably richer and involved, compared to their patrician soliton cousins (see Refs. [6,7] for more information).

These early investigations [1-5] not only shed light on the FPU problem but also spawned another important area in nonlinear sciences—that of soliton physics and integrable systems [8,9]. Even more importantly these pioneering works demonstrated an intimate connection between discrete nonlinear dynamics and soliton behavior.

In later years (1970’s and early 80’s), the physics and analysis of nonlinear lattices has been the subject of intense investigation in many areas of pure and applied science. In mathematics, the first fully integrable lattice equations were identified and solved using inverse scattering methods [6,7]. Such equations include for example the Toda lattice [10], the Ablowitz-Ladik equation [11], and the Calogero-Moser N-body problem [12,13]. In solid state physics, neutral and charge soliton transport was theoretically and experimentally studied in conducting polymer chains such as polyacetylene and polythiophene based on a model proposed by Su, Schrieffer, and Heeger [14]. Over the years this soliton picture received increasing experimental support as these self-localized states were found to be involved in the electric, optical, and magnetic properties of these polymers [15]. For his contributions in the understanding of conducting polymers, Alan Heeger received the 2000 Nobel Prize in Chemistry.

Another important step in the theory and physics of nonlinear discrete systems was made in 1972 by Aleksandr Davydov when he suggested a discrete soliton model as a means to understand energy transfer in protein $\alpha$-helices [16]. Energy transfer phenomena are of paramount importance in biophysics since they are involved in a number of biological processes such as muscle contraction, enzyme catalysis, and active transport. By extending the Holstein
Hamiltonian, Davydov put forward a model that was based on a discrete nonlinear Schrödinger-like equation (DNLS) [17]. Shortly after his proposal several theoretical and experimental groups pursued this possibility [18]. Today, after years of research, the question whether Davydov solitons do exist or not in actual biological systems (at room temperatures) is still open to debate. Irrespective of this, the Davydov model provided a platform upon which the properties of DNLS were first explored [19].

Finally one may mention other discrete nonlinear dynamical systems that these days are receiving large attention. These include for example Frenkel-Kontorova models [20], breather dynamics in Josephson junction arrays [21], intrinsic localized modes in anharmonic crystals [22], and self-trapping in nonlinear circuit elements, just to mention a few.

Yet it is in the field of optics that discrete solitons have really found a fertile ground where they can be easily observed and studied [23]. The very idea of discrete optical components emerged rather gradually in the field of optics. This slow pace of development was due to several reasons. To begin with, from a classical perspective, the optical or electromagnetic field itself is a continuous function of both space and time. From a more practical point of view, an important barrier that prevented these thoughts from becoming reality was the state of fabrication technologies, especially in the first few decades after the discovery of the laser. Clearly, to discretize light behavior it will require optical elements that can confine optical energy at distinct sites. One possible scenario is to first store energy within low loss high Q-microcavities and then allow photon exchange between such components in time. This avenue however requires high-contrast dielectric elements that only became available with the advent of photonic crystal technologies after the mid 90’s [24]. Yet, there is another simpler scenario for achieving such light discretization: the one based on evanescently coupled waveguide arrays! The discrete
diffraction behavior of these arrays was first considered by Allan Jones in 1965 as a part of a larger effort towards the understanding of optical coupling processes [25]. In such periodic waveguide arrays light can be readily confined at discrete sites (in weakly guiding waveguides) whereas at the same time light exchange among channels can occur via coupling during propagation.

Optical discrete solitons in nonlinear waveguide arrays were first predicted in 1988 by Christodoulides and Joseph [26]. In this study, the primitive band structure of the waveguide array (the first band and Brillouin zone of the system) was recognized and the possibility of observing discrete self-trapped states and discrete modulational instability in Kerr arrays was suggested. In general, discrete solitons in array lattices represent collective excitations of the nonlinear chain as a whole and by their nature they have no analogue whatsoever in continuous systems. The process of optical soliton formation in such array structures can be intuitively understood as a balance between on-site nonlinearity and discrete diffraction effects arising from linear coupling among adjacent waveguides. In this case, the optical energy is nonlinearly confined in few waveguides and it can propagate undistorted free of diffraction effects.

The main goal of the work presented in this dissertation is to understand and analyze the existence and properties of self-trapped nonlinear waves in novel optical periodic structures. In particular, we study discrete surface solitons at the boundaries of semi-infinite AlGaAs and LiNbO$_3$ waveguide arrays. Rabi oscillations between two different Floquet-Bloch modes in modulated optical lattices will also be considered. In the last part we will focus on the introduction of the concept of parity-time $\mathcal{PT}$ symmetry in optical periodic systems.

Chapter 2 of this dissertation deals with the linear properties of semi-infinite waveguide arrays, in the context of coupled mode theory. It is well known that the impulse response (single
channel excitation) of an infinite array of weakly coupled waveguides has a closed form solution in terms of Bessel functions. The problem that we consider in this chapter is that of discrete diffraction in a semi-infinite lattice. By using a discrete version of the method of images, this problem can be analytically solved by applying the superposition principle [27]. The method is extended in two-dimensional geometries where wave propagation in 90-degree and 45-degree angular sectors is examined.

In Chapter 3 discrete surface solitons at the boundaries of periodic optical systems are theoretically predicted [28]. These new families of self-trapped states are in fact nonlinear surface waves existing at the interface between a periodic structure and a continuous medium. Both in-phase (at the center of the Brillouin zone) and staggered (at the edge of the Brillouin zone) surface solitons have been theoretically examined. Following these results, in phase surface solitons have been successfully observed in nonlinear Kerr AlGaAs waveguide arrays [29]. Moreover, discrete surface gap solitons in periodically poled LiNbO$_3$ waveguide arrays with quadratic nonlinearity have also been studied [30]. By tuning the phase mismatch relation of the second harmonic, discrete quadratic surface solitons with in phase or out of phase adjacent field components have been observed. In all the above cases, surface solitons exist only when their power exceeds a critical power threshold, as a result of the broken symmetry of the semi-infinite lattice. This is in contrast with what happens in an infinite array, where broad solitons are possible even at very low powers.

Even though the theoretical predictions based on coupled mode theory (valid only for the first band of the array) are in good agreement with the experimental results, we extend the analysis from the discrete to the continuous domain, in Chapter 4. This is necessary in order to account for effects involving higher order transmission bands. For this reason we follow the
formalism of Floquet-Bloch modes used in solid state physics. In particular, we consider a lattice consisting of two different semi-infinite Kerr waveguide arrays. This asymmetry leads to an effective mismatch between the propagation constants in the two regions. Consequently, the first bands of the right and left array are relatively shifted. If this “energy” shift is large enough, solitons with a propagation constant lying within the resulting gap (between the two bands), are possible. In that case, the common soliton eigenvalue is located at the top (semi-infinite band gap) of the one band and at the bottom (first band gap) of the other band. Therefore the surface soliton in one array will be in-phase whereas in the other will be staggered (π out of phase). In this case, the two surface waves can propagate locked together as a composite self-trapped state, thus forming a hybrid soliton [31]. Two-dimensional surface solitons confined at the corners and the edges of optical lattices are also theoretically investigated.

Chapter 5 is dedicated to inter-band optical transitions in periodically modulated array structures (modulated along z). These Rabi-like direct transitions occur among the bands of a periodic potential, under appropriate phase-matching conditions. In particular, the array is modulated in space with period Λ, so that the “energy difference” $2\pi/\Lambda$ spans the k-difference between the allowed bands. Coupled mode equations that describe the dynamic energy exchange between two FB modes are derived. These results are in excellent agreement with direct beam propagation (BPM) simulations. Note that this type of transitions should be possible in periodically modulated single-mode array AlGaAs lattices. Another exciting possibility is to examine whether nonlinear energy exchange among lattice solitons can also take place in such systems. This is all together a nonlinear process. Unlike linear transitions where dispersion or diffraction leads to wavepacket broadening, here the energy oscillates as a self-trapped wave between “energy” states. Given the fact that the propagation eigenvalues of lattice solitons reside
in the forbidden band gaps of the corresponding linear potential, these excitations are expected to take place between gap to gap.

In Chapter 6 the concept of parity-time ($\mathcal{PT}$) symmetry is introduced within the context of wave optics. In 1998, Carl Bender et al have demonstrated [32], that it is in fact possible even for non-Hermitian Hamiltonians to exhibit entirely real eigenvalue spectra as long as they respect parity-time requirements or $\mathcal{PT}$ symmetry [33, 34]. This fascinating result appears to be counter-intuitive since it implies that all the eigenmodes of a pseudo-Hermitian Hamiltonian are only associated with real eigen-energies. We show that $\mathcal{PT}$ symmetric systems can exhibit a host of intriguing characteristics such as non-reciprocal Bloch modes, band-merging, double refraction, secondary emissions and power oscillations [35].

Chapter 7 focuses on the theoretical prospect of self-trapped waves in $\mathcal{PT}$-symmetric optical potentials. The interplay of self-focusing nonlinearity and $\mathcal{PT}$-symmetry is examined for the first time [36]. Even enough a $\mathcal{PT}$-complex potential involves gain/loss, it is still possible to find nonlinear eigenmodes that exhibit real propagation eigenvalues. The existence and stability properties of lattice solitons in $\mathcal{PT}$-periodic structures are systematically studied by using linear stability analysis and beam propagation methods. Two-dimensional $\mathcal{PT}$-solitons are also reported. In all the above cases the power flow of the obtained solutions is considered.

The main results and conclusions of this dissertation are presented in the final Chapter 8.
References


CHAPTER TWO: DISCRETE DIFFRACTION IN SEMI-INFINITE ARRAYS

2.1 Introduction

The discrete coupling or tunneling process between periodically arranged potential wells is a fundamental topic that has been extensively investigated in many branches of physics. In optics, arrays of weakly coupled waveguides and resonators are prime examples of such systems where the coupling dynamics can be directly observed and investigated [1, 2]. Periodic array structures are typically comprised from single-mode waveguides that are coupled to each other through the evanescent tails of adjacent guided fields [2]. Likewise, the transport dynamics in periodic chains of micro-cavities (coupled resonator waveguides or CROWs) follow similar rules [3]. In these configurations, linear mode coupling leads to energy redistribution among the elements of the array, a mechanism better known as discrete diffraction. The problem of discrete diffraction in infinite optical arrays was first analytically solved by Jones in 1965 [1]. This was done by explicitly obtaining the impulse response of the infinite chain in terms of Bessel functions. This behavior was subsequently observed in 1D AlGaAs waveguide arrays by several experimental groups [4, 5]. Lately, diffraction in two-dimensional discrete systems has also been observed in femtosecond laser induced waveguide arrays [6] and in optically induced photorefractive lattices [7].
Under nonlinear conditions (when the material is nonlinear and at high excitation powers), a self-localized nonlinear entity can also be supported by the periodic potential of the array. This nonlinear wave or optical discrete soliton [8] remains invariant during propagation through a balance of diffraction and nonlinearity. Again the discreteness offers a rich spectrum of properties and possibilities that do not exist in the bulk. Modulation instability [9] and discrete/lattice solitons in cubic [5, 7-10], photorefractive [11], and quadratic waveguide arrays [12, 13] have been examined theoretically, and observed experimentally. In addition, discrete solitons in systems that exhibit non-local nonlinearities, such as semiconductor amplifiers and nematic liquid crystals, have been considered [14, 15].

Boundaries and surfaces can also introduce new physical features due to the break of the translational symmetry. For example, electromagnetic surface waves are possible along the boundary between two different media (continuous or periodic). Linear and nonlinear surface waves in both bulk and periodic environments have been considered during the past few years in various fields of science [16-22].

Clearly the presence of boundaries can considerably complicate the wave dynamics even in linear array networks. Unlike infinite discrete systems whose diffraction characteristics have been known for some time [1, 4], the corresponding behavior in semi-infinite or finite arrays remains to be explored.

In this chapter, we demonstrate that optical wave propagation in discrete boundary geometries can be analyzed using the method of images. This is done by introducing fictitious sources outside the region of interest, in a way similar to the method of images used in other fields such as electrostatics [23], mechanics [24], electrodynamics [25], and solid state physics [26]. The proposed method offers several advantages in terms of studying this broad class of
problems. More specifically, for certain lattice topologies the use of images leads to closed form solutions whereas for finite array geometries it greatly simplifies the analysis. This method is elucidated by providing pertinent examples.

2.2 Semi-infinite waveguide arrays

Let us consider a linear semi-infinite array of weakly coupled waveguides, like the one depicted in Figure 2.1. Within the context of coupled mode theory, wave propagation in such a structure is described by the following equations:

\[ i \frac{dE_0}{dz} + \beta E_0 + kE_1 = 0 \]  \hspace{1cm} (2.1a)

\[ i \frac{dE_n}{dz} + \beta E_n + k(E_{n+1} + E_{n-1}) = 0, \text{ for } n \geq 1 \]  \hspace{1cm} (2.1b)

where \( E_n [V/m] \) is the electric modal field in the \( n^{th} \) waveguide, \( \beta [m^{-1}] \) is the propagation constant, and \( k [m^{-1}] \) the coupling coefficient.

\[ n = 0 \hspace{0.5cm} n = 1 \hspace{0.5cm} n = 2 \hspace{0.5cm} \ldots \]

Figure 2.1: Semi-infinite waveguide array

We note that similar equations are applied to describe CROW microcavities in the temporal domain [3]. By utilizing the transformation \( E_n = a_n \exp(i\beta z) \) and by normalizing the propagation distance with respect to the coupling length, i.e., \( Z = \kappa z \), Eqs.(2.1) are re-written as:
\[ i \frac{da_0}{dZ} + a_i = 0 \quad (2.2a) \]

\[ i \frac{da_n}{dZ} + a_{n+1} + a_{n-1} = 0 \quad , \text{for } n \geq 1 \quad (2.2b) \]

where \( a_n \) represents the normalized modal amplitude in the \( n^{th} \) waveguide. In order to study the diffraction properties of this semi-infinite array it is first important to derive its impulse response. The impulse response of this structure is in fact the solution of Eqs. (2.2) under single site excitation, i.e., \( a_n = A_0 \delta_{nm} \) (if for example site \( m \) is initially excited).

Following the argument presented below, the diffraction problems in infinite and semi-infinite arrays are related to each other through the method of images. The goal here is to identify an equivalent infinite system in which the half region is governed by Eqs.(2.2). This can be done by demanding that the field at the site \( a_{-1} \) is always zero during propagation \( (a_{-1}(Z) = 0) \). This last requirement can only be satisfied if a fictitious source or image with a relative phase difference \( \pi \) (with respect to the actual source) is positioned symmetrically around the \( n = -1 \) site. This is because the anti-symmetric conditions used at the input guarantee that \( a_{-1}(Z) = 0 \) for all values of propagation distance \( Z \). As a result, the two semi-infinite sections of the equivalent infinite array are decoupled and thus Eqs.(2.2) hold true in the region of interest \( (n \geq 0) \). Therefore the study of the semi-infinite array (Figure 2.2 (a)) can be carried out by considering the diffraction dynamics in an infinite lattice under appropriate initial conditions (Figure 2.2 (b)). In this case the superposition of the fields emanating from the actual source and the image provide the impulse response of the semi-infinite array. By using the already known impulse response of an infinite array, that is, \( a_n(Z) = A_0 i^{\alpha} J_{\alpha}(2Z) \) when site \( m \) is excited [1], the diffraction problem can then be directly solved. Given that the excitation site is at \( m \), then its image (with
respect to the $n = -1$ waveguide) is positioned at the $-(m + 2)$ channel, (see Figure 2.2 (b)). Thus the diffracted field resulting from the actual excitation site is given by $i^{n-m} J_{n-m}(2Z)$, whereas that originating from its image is described by $(-1)i^{n+m+2} J_{n+m+2}(2Z)$. Hence the corresponding impulse response of the semi-infinite array (analytical solution to Eq.(2.2)) is given in closed form by:

$$a_n(Z) = A_0 \left[ i^{n-m} J_{n-m}(2Z) + i^{n+m} J_{n+m+2}(2Z) \right] \quad (2.3)$$

The validity of the method of images in dealing with optical discrete systems can also be formally justified. In particular, by applying Z-transform techniques [10, 27], one can analytically show, that an anti-symmetric initial condition at $Z = 0$, $a_{-1}(0) = 0$, $a_n(0) = -a_{-n-2}(0)$, remains anti-symmetric during propagation. This implies that $a_{-1}(Z) = 0$, and $a_n(Z) = -a_{-n-2}(Z)$ for every value of $Z$. Under such initial conditions the wave propagation in an infinite waveguide array in the region of interest $n \geq 0$, is governed by Eqs.(2.2) of the semi-infinite array.
Figure 2.2: (a) A semi-infinite waveguide array under single site excitation at position $m$, and (b) the equivalent infinite array with the image positioned at $-(m+2)$ site. The field in the $n = -1$ waveguide is always zero, because of the mirror symmetry.

Since the two structures are described by the same equations and since the solution of the system of ordinary differential equations happens to be unique (following Picard’s theorem [28]), we then conclude that the two array systems are mathematically equivalent for $n \geq 0$. The discrete diffraction pattern of a semi-infinite array when the channel $n = 0$ (and $n = 2$) is initially excited is depicted in Figure 2.3.

Figure 2.3: Diffraction pattern in a semi-infinite array under single channel excitation: (left) first waveguide is excited, and (right) third waveguide is excited. The inset in the left figure depicts a semi-infinite waveguide array.

The method of images can also be employed to study the diffraction in a finite array of $N$ waveguides. Obtaining the impulse response of a finite lattice is more complicated since this type of structure involves two boundaries. In this case, the problem can be mapped to that of an
infinite waveguide array where the field in the two channels (denoted L and R) located (left and right) at the fictitious edges of the system ($N+1$ sites away from each other) remains always zero. Between these two virtual boundaries, the infinite and the finite array exhibit the same behavior. To find the impulse response of this structure, let us consider the case where a single waveguide site is excited between the L and R channels. In order for the field in these waveguides to be always zero, the corresponding images must be appropriately situated in the equivalent infinite array. In particular, a negative image is positioned symmetrically with respect to the virtual site L (see Figure 2.4) and another one (negative) with respect to R. These images act as secondary excitation sites and in turn lead to two new positive images. One of these two new images results from the reflection of the secondary image located at the left of L with respect to the R site and similarly the other from a reflection at L (see Figure 2.4). This process continues indefinitely and the result is an infinite number of positive and negative pairs of images. For illustration purposes, Figure 2.4 shows the positions of these pairs of images when $N = 2$.

Figure 2.4: Equivalent infinite array configuration for the case of a finite array of $N$ elements. For simplicity here $N=2$. The region of interest lies between the L and R waveguide sites. The index $r$ denotes the image pair.
Once more, by applying the superposition principle, a closed form expression for the impulse response of an array of $N$ waveguides, can be found. When the $m^{th}$ site of the finite array is excited, the field at the $n^{th}$ site is given by the expression:

$$a_n(Z) = A_0 \sum_{r=-\infty}^{+\infty} \left\{ i^{-(2N+2)r} \left[ i^{n-m} J_{n-m-(2N+2)r} (2Z) - i^{n+m+2} J_{n+m+2-(2N+2)r} (2Z) \right] \right\}$$

(2.4)

where $r$ is the image index pair, and $2N+2$ is the period between the positive or negative images.

Figure 2.5: Diffraction pattern in an array of five elements, by using the equivalent infinite array with nine images. The region that corresponds to the finite array is located between the two white dotted lines. The site $n = 1$ was initially excited. Here only five images are shown.

The use of the method of images in analyzing finite arrays offers several advantages over other schemes especially when the number of elements $N$ is relatively large. In principle the impulse response of a finite array can be obtained by considering the projection of the input vector over the supermode-eigenvectors of the array [29]. Yet, this latter approach requires
summing up \( N \) contributions, something that is impractical when \( N \) is large. On the other hand, for finite distances, the method of images can provide a solution to this problem, by only keeping a finite number of terms in the Bessel function expansion of Eq. (2.4). This approximate description will accurately follow the wave dynamics in a finite array as long as the consecutive reflections from the boundary walls correspond to the image pair of images (accounted in the truncated expression). Figure 2.5, shows the discrete diffraction in a \( N=5 \) array, when the \( n = 1 \) site is excited up to a normalized distance of \( Z = 5 \). These results were obtained using only 4 pairs of images and are in excellent agreement with the actual response of the system.

\[ 2.3 \text{ Method of images-Formalism} \]

Let us consider the normalized wave propagation equation in a infinite waveguide array,

\[
\frac{d a_n}{dZ} + a_{n+1} + a_{n-1} = 0 \quad (2.5)
\]

By using the \( Z \)-transform we will here show that the mirror symmetry assumed at the input of the array \( (Z=0) \) is preserved during propagation. In other words, if \( a_n(0) = -a_{-n}(0), a_0(0) = 0 \), then \( a_n(Z) = -a_{-n}(Z), a_0(Z) = 0 \) for every propagation distance \( Z \). For convenience, we assume that the zero symmetry line is at \( n = 0 \).

The \( Z \)-transform \( A(Z) \) of the sequence \( \{a_n\} \) and its corresponding inverse transform \( a_n(Z) \) are defined in the complex domain as [10, 27]

\[
A(Z) = \sum_{n=-\infty}^{\infty} a_n x^n \quad (2.6) \quad , \quad a_n(Z) = \frac{1}{2\pi i} \oint A(Z)x^{-n-1} \, dx \quad (2.7)
\]

The \( Z \)-transform of Eq. (2.5), is:

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\[ A(Z, x) = A(0, x) \exp[i(x + x^{-1})Z] \]  

(2.8)

From (2.6) and the imposed initial conditions at \( Z = 0 \), the \( A(0, x) \) term can be found. More specifically, we get:

\[
A(0, x) = \sum_{n=-\infty}^{+\infty} a_n(0) x^n , \quad \text{therefore} \quad A(0, x) = \sum_{n=1}^{+\infty} a_n(0) \left(x^n - x^{-n}\right). \quad \text{This provides } A(Z, x),
\]

which is:

\[
A(Z, x) = \sum_{n=1}^{+\infty} a_n(0) \left(x^n - x^{-n}\right) \exp[i(x + x^{-1})Z] \quad (2.9)
\]

Now the inverse \( Z \)-transform can be applied to the sequences \( \{a_n\} \) and \( \{a_{-n}\} \). The sum of these two inverse transforms is:

\[
a_n(Z) + a_{-n}(Z) = \sum_{n=1}^{+\infty} a_n(0) I_n , \quad \text{where the complex integral } I_n \text{ is given by}
\]

\[
I_n = \frac{1}{2\pi i} \oint_{|z|=1} \left( z^{2n} - z^{-2n}\right) \exp[i(1 + x^{-1})Z] \frac{dz}{z}. \quad \text{By switching to polar coordinates and by using the integral representation of the Bessel function [30] we obtain,}
\]

\[
\frac{1}{2\pi i} \oint_{|z|=1} z^{-n} \exp[i(1 + x^{-1})Z] \frac{dz}{z} = i^n J_n(2Z). \quad \text{Thus} \quad I_n = i^{-2n} J_{-2n}(2Z) - i^{2n} J_{2n}(2Z) =
\]

\[= i^{-2n} (-1)^{2n} J_{-2n}(2Z) - i^{2n} J_{2n}(2Z) = 0. \quad \text{Finally, we get as a result the following relation,}
\]

\[a_n(Z) + a_{-n}(Z) = 0. \quad \text{Given that } i \frac{da_0}{dZ} + a_1 + a_{-1} = 0, \text{ then a direct integration of this relation along with the fact that } a_1(Z) + a_{-1}(Z) = 0, \text{ leads us to the conclusion } a_0(Z) = 0 \text{ for every } Z.
\]
2.4 Method of images in 2D geometries

The method of images can also be extended to analyze semi-infinite two dimensional periodic structures. In general, wave propagation in an infinite two dimensional waveguide array is governed by the normalized coupled mode equation

\[ i \frac{d a_{n,m}}{dZ} + a_{n,m+1} + a_{n,m-1} + a_{n+1,m} + a_{n-1,m} = 0 \]  \hspace{1cm} (2.10)

where \( a_{n,m} \) is the modal amplitude at the \((n,m)\) site, and \( Z \) the normalized propagation distance.

In deriving Eq.(2.10) we have assumed negligible diagonal coupling effects and we have considered only nearest-neighbor interactions. If only one channel is excited at the \((p,q)\) site, i.e., \( a_{n,m}(0) = A_0 \delta_{np} \delta_{mq} \), the discrete diffraction is described by

\[ a_{n,m}(Z) = A_0 i^{(n-p)} i^{(m-q)} J_{n-p} (2Z) J_{m-q} (2Z) \].

Since the impulse response of an infinite 2D lattice is known, closed form solutions can be obtained by applying the method of images in the case of diffraction problems involving boundaries.

A discrete arrangement in a semi-infinite plane is shown in Figure 2.6 (a). The source in this lattice is positioned at the \((p,q)\) site. Following the rationale of the previous section, the image (with a \( \pi \) phase shift) is located at \((-p-2,q)\) and the virtual zero line is at \( p = -1 \) (see Figure 2.6 (b)).
Figure 2.6: (a) A semi-infinite two dimensional lattice where the source is at the \((p,q)\) site and, (b) the equivalent infinite lattice with the source and the corresponding anti-phase image located at \((-p-2,q)\) waveguide channel. The axis of symmetry is illustrated as dotted line.

In this case, the field distribution at the \((n,m)\) site when initially \(a_{nm}(0) = A_0 \delta_{np} \delta_{mq}\) is given by

\[
a_{n,m}(Z) = A_0 \left\{ i^{(a-p)} J_{n-p}(2Z) J_{m-q}(2Z) + i^{(a+p)} J_{n-p+2}(2Z) J_{m-q}(2Z) \right\}
\]

Figure 2.7, demonstrates the intensity pattern in such a semi-infinite lattice topology when the site \((2,0)\) is initially excited.
Figure 2.7: Discrete diffraction in a semi-infinite two dimensional waveguide array for a normalized distance of $Z = 3.3$, when only the $(2,0)$ site was initially excited.

In this section we will use the images method to study 2D array angular sectors. A $90^\circ$ corner is shown in Figure 2.8(a). The discrete diffraction resulting from a single excited site can be obtained by considering the equivalent 2D infinite array under the appropriate initial conditions. In a way similar to that used in electrostatics, in order for the method of images to work, the field along the two axes of symmetry $p = -1$, and $q = -1$ must be always zero. More specifically, if the excitation occurs at the $(p,q)$ channel in the positive quadrant, the related images are located at the three symmetric positions with respect to the center of the lattice $(-1,-1)$. As depicted in Figure 2.8 (b), the two negative images A and C are situated at $(-p-2,q)$, $(p,-q-2)$ respectively, and the positive image B at $(-p-2,-q-2)$. 
Figure 2.8: (a) A two dimensional lattice angular sector of 90° degrees, where the source is at the \((p, q)\) site and, (b) the equivalent infinite two dimensional lattice with the source and the corresponding three images A, B, C appropriately positioned. The axes of symmetry are shown with dotted lines.

The two pairs (actual source, image A) and images (B,C) keep the \(p = -1\) axis at zero, while the axis \(q = -1\) is at zero because of the other two pairs (A,B) and (excitation, C). By superposing the fields from the actual source and the three images, we obtain the analytical solution for the diffracted field at the \((n, m)\) site (in the positive quadrant) when \(a_{nm}(0) = A_0 \delta_{np} \delta_{mq}\), that is:

\[
a_{n,m}(Z) = A_0 \left[ i^{(n-p)+(m-q)} J_{n-p} (2Z) J_{m-q} (2Z) + i^{(n-p)+(m+q)} J_{n-p} (2Z) J_{m+q+2} (2Z) + \\
i^{(n+p)+(m-q)} J_{n+p+2} (2Z) J_{m-q} (2Z) + i^{(n+p)+(m+q)} J_{n+p+2} (2Z) J_{m+q+2} (2Z) \right]
\]

(2.12)

The intensity distribution after a normalized distance of \(Z = 4\) where only the \((0,1)\) site is initially excited is depicted in figure 2.9.
Figure 2.9: Intensity pattern in a two dimensional 90° degree waveguide array corner for a normalized distance of $Z = 4$, under a $(0,1)$ waveguide site excitation.

Another interesting lattice configuration that can be analytically treated using this technique is that of a 2D 45° array corner. This lattice sector is contained between the $+45°$ degree axis and the $0°$ degree axis of symmetry (see Figure 2.10 (a)). In this case, seven images are required in order to keep the four axes $p = -2, q = -1, +45°$, and $-45°$ degree always at zero (see Figure 2.10 (b)).

Figure 2.10: (a) A two dimensional lattice angular sector of 45° degrees, where the source is at $(p,q)$ site and, (b) the equivalent infinite two dimensional lattice under the excitation of the
source and the corresponding seven images. The axes of symmetry are depicted with dotted lines.

When the excited site is located at \((p, q)\) (with \(q < p\)), then the field at the \((n,m)\) channel is given by the following relation:

\[
a_{n,m}(Z) = A_0 \left\{ J_{n-p}^{(n-p)}(2Z) J_{m-q}^{(m-q)}(2Z) - J_{n-q+1}^{(n-q+1)}(2Z) J_{m-p-1}^{(m-p-1)}(2Z) \right\} + \\
J_{n-q+1}^{(n-q+1)}(2Z) J_{m+p+3}^{(m+p+3)}(2Z) - J_{n+p+4}^{(n+p+4)}(2Z) J_{m-q}^{(m-q)}(2Z) \right\} + \\
J_{n+q+3}^{(n+q+3)}(2Z) J_{m+p+3}^{(m+p+3)}(2Z) - J_{n+p+4}^{(n+p+4)}(2Z) J_{m+q+2}^{(m+q+2)}(2Z) \right\} + \\
J_{n-p}^{(n-p)}(2Z) J_{m+q+2}^{(m+q+2)}(2Z) - J_{n+q+3}^{(n+q+3)}(2Z) J_{m-p-1}^{(m-p-1)}(2Z) \right\}\]

\[ (2.13) \]

Figure 2.11 shows the intensity pattern resulting from the discrete diffraction at \(Z = 3\), when the site \((0,0)\) has been initially excited.

Figure 2.11: Diffraction evolution in two dimensional 45° degree waveguide array angular sector for a normalized distance of \(Z = 3\). Channel \((0,0)\) has been excited.
References


CHAPTER THREE: DISCRETE SURFACE SOLITONS

3.1 Introduction

Surfaces waves are known to display properties that have no analogue in the bulk. Over the years they have been the subject of intense study in diverse areas of physics, chemistry and biology [1]. In the linear optical domain, such surface waves can exist at metal-dielectric interfaces (plasmon waves) [1], at the boundary of semi-infinite periodic multi-layer dielectric media [2], as well as at the interfaces of anisotropic materials [3]. In addition to these linear waves, optical surface waves are also possible as a result of nonlinearity [4]. More specifically, nonlinear TE, TM, as well as mixed polarized surface waves along single dielectric interfaces were theoretically predicted and analyzed by several authors [5-8]. These waves are purely a nonlinear phenomenon with no counterpart in the linear limit. Nonlinear surface waves were also studied in thin dielectric films [9], in photorefractive interfaces [10, 11] and in diffusive Kerr media. Yet, thus far most of the activity in this area has remained theoretical in nature and there is little, if any, experimental evidence regarding the existence of such nonlinear surface waves [12]. This is partly due to difficulties in exciting these states and, on many occasions, their high power requirements.

Recently nonlinear wave propagation in discrete systems, like waveguide arrays, has also been systematically investigated [13-17]. These discrete or lattice configurations are known to exhibit novel properties in both the linear and nonlinear regime [18]. The question therefore
arises as to whether discrete nonlinear surface waves (surface solitons) can exist at the edge of a semi-infinite waveguide array. If so, how and to what extent this new class of waves relates to their continuous counterparts?

In this chapter we demonstrate that nonlinear discrete surface-waves are possible in waveguide lattices. To some extent, these surface lattice solitons can be considered as the nonlinear analogues of the so-called Tamm states in solid state physics [19]. Our analysis indicates that, as opposed to discrete solitons in infinite arrays, this new family of self-trapped waves exhibits an interesting power threshold behavior. The stability of this class of surface solitons along with their propagation dynamics is examined in detail. Our results may pave the way toward the first observation of optical nonlinear surface waves.

### 3.2 Surface solitons in AlGaAs arrays

To analyze this problem, let us consider a semi-infinite nonlinear lattice consisting of weakly-coupled waveguides. In this system, the normalized modal field amplitudes obey a discrete nonlinear Schrödinger-like equation [13], that is,

\[ i \frac{da_0}{dZ} + a_i + \sigma |a_0|^2 a_0 = 0 , \]  

\[ i \frac{da_n}{dZ} + (a_{n+1} + a_{n-1}) + \sigma |a_n|^2 a_n = 0 \]  

where Eq.(3.1a) describes the field at the edge of the array \( n = 0 \) waveguide site and Eq. (3.1b) applies at every other site \( n \geq 1 \). The dimensionless amplitudes \( a_n \) are related to the actual
electric fields through the relation \( E_n = \left(2\kappa\lambda_0\eta_0/(\pi n\hat{n}_2) \right)^{1/2} a_n \), where \( \lambda_0 \) is the free space wavelength, \( \eta_0 \) is the free space impedance, \( \hat{n}_2 \) is the nonlinear Kerr coefficient, and \( n \) refers to the linear refractive index of the array. Note also that, for self-focusing nonlinearities \( \sigma = 1 \), whereas for defocusing \( \sigma = -1 \).

Let us first consider self-trapped states in self-focusing semi-infinite arrays with \( \sigma = 1 \). Such nonlinear surface waves can be obtained by assuming the following stationary solution \( a_n = \phi_n \exp(i\mu Z) \) in Eqs. (3.1), where \( \mu \) represents their corresponding nonlinear propagation eigenvalue and all the fields \( \phi_n \) are taken to be positive (in-phase solution). In physical units the normalized eigenvalue \( \mu \) results to a wavevector difference of \( \Delta\beta = \mu \cdot \kappa \left[ m^{-1} \right] \). In terms of refractive index difference this \( \Delta\beta \) can be also expressed as \( \Delta\beta = \frac{2\pi}{\lambda} \Delta n \left[ m^{-1} \right] \). For example, if \( \lambda_0 = 1.55 \mu m \) and \( \kappa = 700 m^{-1} \), then for \( \mu = 4 \) we get a wavevector difference of \( \Delta\beta = 2800 \left[ m^{-1} \right] \), which corresponds to a refractive index difference of \( \Delta n \approx 7 \cdot 10^{-4} \). This value represents the nonlinearly induced refractive index difference in materials such as AlGaAs. For this reason the eigenvalue \( \mu \) is called sometimes nonlinear wavevector shift or soliton eigenvalue.

In the system under consideration the eigenvalue \( \mu \) lies in the range \( \mu \geq 2 \). Solutions are found numerically [20] by using relaxation methods (continuation method based on Newton-Raphson algorithm). The total power \( P = \sum |a_n|^2 \) associated with these surface soliton is then plotted against the eigenvalue \( \mu \) as shown in Figure 3.1(a).
Figure 3.1: Normalized power versus eigenvalue $\mu$ for an in-phase soliton at the (a) edge and (b) middle of a waveguide array.

A typical intensity profile of such a nonlinear surface wave is shown for example in the inset of Figure 3.1(a). Figure 3.1(a) indicates that the $P-\mu$ curve exhibits a minimum at approximately $\mu = 2.998 \approx 3$. This in turn implies that discrete nonlinear surface waves can only exist above a certain power threshold which for this case is $P_{th} \approx 3.27$. Below this power level no surface waves can be supported. This is in contrast with what happens in an infinite array, where broad solitons are possible even at very low powers. The $P-\mu$ curve for the case of an in-phase discrete soliton in an infinite array is shown in Figure 3.1(b). As we can see no power threshold exists. As the soliton eigenvalue $\mu$ approaches the value $\mu = 2$, the power of the soliton approaches zero.

In addition to the surface waves already explored in this work, other solutions also exist. These correspond to solutions whose maxima are located close to the boundary at $n \geq 1$. The intensity profile of such a solution whose peak occurs at $n = 1$ is shown in Figure 3.2 (a) and the related $P-\mu$ curve of this $n = 1$ family is depicted in Figure 3.2 (b).
Figure 3.2: (a) Normalized intensity profile of a discrete nonlinear surface wave when the field maximum occurs at the second waveguide, and (b) the related power-eigenvalue diagram.

Before closing this paragraph it is worth mentioning, that staggered (fields are $\pi$ out of phase) surface discrete solitons exist in semi-infinite arrays of defocusing nonlinearity $\sigma = -1$. By applying the same continuation techniques, the nonlinear eigenmodes of the system can be numerically identified. In this case the soliton eigenvalues reside at the first gap of the array. The power thresholds and the stability of this family of surface solitons follow similar behavior with the in-phase solutions. The corresponding power-eigenvalue diagrams, as well as a typical field profile of a wide soliton, are depicted in the following Figure 3.3.

Figure 3.3: (a) Power-eigenvalue diagram of staggered surface solitons, and (b) field profile for eigenvalue $\mu = -2.93$.
3.3 Linear Stability analysis

The main idea of the linear stability analysis method is to examine the evolution over the propagation distance $Z$, of a perturbed solution of equation (3.1). More specifically, if an exact (numerical) solution of Eq. (3.1) is $a_n = \phi_n \exp(i\mu Z)$ (3.2), then the perturbed solution can be written in the form $a_n = (\phi_n + \epsilon \cdot W_n(Z))\exp(i\mu Z)$ (3.3), where $\epsilon << 1$ and $W_n(Z)$ is the spatial profile of the perturbation. In the general case this profile has the following form:

$$W_n(Z) = (A_n + B_n) \cdot \exp(i\omega Z) + (A_n - B_n) \cdot \exp(-i\omega^* Z) \quad (3.4)$$

where $A_n, B_n$ are real numbers and $\omega$ is the spatial modulation frequency of the perturbation. It is easy to see that if $\omega$ is real the solution (3.4) is stable over $Z$, while is unstable for complex values of $\omega$. By substituting (3.3) into (3.1b) we get the following equation for $W_n(Z)$:

$$iW_n' - \mu W_n + W_{n-1} + W_{n+1} + (W_n^* + 2W_n)\phi_n^2 = 0 \quad (3.5)$$

By combining (3.4) with (3.5) and by assuming without loss of generality that $\omega$ takes real values, we get a system of two equations:

$$\omega (A_n + B_n) + \mu (A_n + B_n) - A_{n-1} - B_{n-1} - A_{n+1} - B_{n+1} - \phi_n^2 (A_n - B_n) - 2\phi_n^2 (A_n + B_n) = 0$$

$$\omega (A_n - B_n) - \mu (A_n - B_n) + A_{n-1} - B_{n-1} + A_{n+1} - B_{n+1} + \phi_n^2 (A_n + B_n) + 2\phi_n^2 (A_n - B_n) = 0$$

After trivial algebraic manipulations the above equations can be expressed in matrix form as follows:
\[
\omega \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \tilde{M} \cdot \begin{bmatrix} A_n \\ B_n \end{bmatrix}
\]
where the matrix \( \tilde{M} \) is defined as
\[
\tilde{M} = \begin{bmatrix} 0 & L_1 \\ L_2 & 0 \end{bmatrix}
\]
and the operators \( L_1, L_2 \) are defined by the next two relations:
\[
L_1 B_n = (\phi_n^2 - \mu) B_n + B_{n+1} + B_{n-1} \quad \quad L_2 A_n = (3\phi_n^2 - \mu) A_n + A_{n+1} + A_{n-1}
\]

As we can understand from the above relations, the eigenvalues of the matrix \( \tilde{M} \) are equal to the spatial modulation frequency of the perturbation \( \omega \). Therefore the numerical calculation of the eigenvalues of \( \tilde{M} \) determines the stability of the solutions. More specifically, if the eigenvalues are real, the solution is stable, and when they are complex instabilities appear.

By applying the above analysis we can study the stability of the numerically obtained discrete surface solitons. More specifically, for all the points on the right of the minimum of the \( P - \mu \) curve of Figure 3.1 (a) (for \( \mu > 3 \) where \( dP/d\mu > 0 \)), the surface wave solutions are stable whereas for \( \mu < 3 \) they are unstable. This behavior is to great extent similar to that encountered in the continuum limit (dielectric interfaces) where the Vakhitov-Kolokolov criterion is applicable [7]. Figure 3.4(a) shows the intensity distribution of a discrete nonlinear surface wave at \( \mu = 3.2 \) and Figure 3.4(b) is the corresponding stability diagram. As can be seen all the perturbation eigenvalues lie on the real axis and thus the solution of Figure 3.4(a) is stable. On the other hand, Figure 3.4(c) depicts the intensity of a lattice surface wave when \( \mu = 2.92 \) and Figure 3.4(d) its stability diagram. In this case the solution is unstable since two of its eigenvalues have a non-zero imaginary part. We also note that even though the two solutions of Figures 3.4(a) and 3.4(c) were obtained at the same power level, their structure is different, i.e., that of 3.4(c) is broader than that of 3.4(a). The left side of the \( P - \mu \) curve
associated with unstable solutions grows abruptly to infinity because of cut-off conditions and hence only its first part is shown here. The right side of the curve (for $\mu > 3$ where $dP/d\mu > 0$) describes stable solutions and as $\mu$ increases the solutions become even more confined around the boundary. In this regime ($\mu >> 3$) these latter highly-confined solutions are approximately given by $a_n = A\exp(-np + i\mu Z)$ where $\mu = A^2 + A^{-2}$ and $p = 2\ln A$. The stability of these surface solitons was also tested dynamically by using a fourth order Runge-Kutta integration algorithm. Figure 3.5(a) shows the stable propagation of the surface wave of Figure 3.4(a) while Figure 3.5(b) illustrates that the surface soliton of Figure 3.4(c) is indeed unstable.

![Figure 3.4: (a), (c) Intensity profiles associated with discrete surface solitons for $\mu = 3.2$ and $\mu = 2.92$, respectively, and (b), (d) their corresponding stability diagrams.](image-url)
Figure 3.5: (a) Stable and (b) unstable evolution of the nonlinear surface waves shown in Figures 3.4 (a), and 3.4 (c), respectively.

3.4 Dynamic excitation

We will now illustrate the possibility of observing these nonlinear surface waves by means of pertinent examples. Let us consider an AlGaAs waveguide array similar to that used in previous experimental observations of discrete solitons used at $\lambda_0 = 1.55 \mu m$. In this system, $\hat{n}_z = 1.5 \times 10^{-13} cm^2/W$, $n = 3.5$, $\kappa = 700 m^{-1}$ and the effective area of each waveguide in this array is taken here to be $A_{\text{eff}} = 4.7 \mu m^2$. For this particular example, every normalized unit of power $P$ in Figure 3.1 (a) corresponds to 108 W. Therefore the power threshold in this case is around 353 W. The dynamic excitation of a nonlinear surface wave is also examined when only the first waveguide site is excited. Figure 3.6 (a) shows this dynamics when the input power is 390 W and Figure 3.6 (b) the corresponding evolution at 732 W. Evidently, at 732 W self-trapping occurs at the boundary and the discrete surface soliton is dynamically established. This threshold of 732 W is higher than the theoretically predicted value of 353 W. This is anticipated since we excite only the first channel of the array and not the nonlinear eigenmode of the system.
Figure 3.6: Propagation dynamics when only the first waveguide site is excited with input power (a) 390 W and (b) 732 W.

3.5 Twisted solitons

In this section we examine the relation between the surface solitons in semi-infinite arrays and anti-symmetric solitons or twisted solitons in infinite arrays. The method of images can be useful in such a comparison, even though the superposition principle is no longer valid (because of the nonlinearity). More specifically, discrete surface soliton solutions can be numerically obtained from the equations of an infinite lattice by assuming an anti-symmetric or twisted \( a_n = -a_{n-2} \) field profile. The inset of Figure 3.7 depicts a twisted soliton solution [21, 22] as obtained in an infinite array. Note that the assumed anti-symmetry makes this problem directly relevant to the semi-infinite case. The solution of Figure 3.7, exhibits a \( P - \mu \) diagram which is in fact a scaled version (by a factor of two in terms of power) of that in Figure 3.1 (a). This should have been expected since the surface soliton solution of the semi-infinite array (inset of Figure3.1 (a)) is just the right-hand part of that of the infinite system (inset of Figure3.7). Yet, the stability properties of these two states are very different. The twisted mode of the infinite array happens
to be stable for $\mu \geq 4.31$ as opposed to the surface state which is stable for $\mu \geq 2.99$. This is because these solutions correspond in reality to altogether different physical problems. The former is a surface state whereas the latter a twisted self-trapped mode.

Figure 3.7: Normalized discrete soliton power $P$ versus the normalized eigenvalue $\mu$ of a twisted soliton in an infinite array. The intensity distribution of a twisted soliton with $\mu = 2.92$ is shown in the inset.

We note that quite recently we successfully used this nonlinear version of the method of images in order to simulate the spatio-temporal dynamics of optical pulses in semi-infinite $AlGaAs$ and $LiNbO_3$ waveguide arrays [23]. This was done by using an “anti-symmetric” excitation in an equivalent infinite system, which conserves the condition $a_{-1}(Z) = 0$ necessary for this method to work.
3.6 Basic equations of quadratic surface solitons

Discrete quadratic solitons have been previously demonstrated inside arrays governed by the “cascading” quadratic nonlinearity [24]. One of the unique features of this nonlinearity is that it can change from effectively self-focusing to defocusing depending on the wavevector mismatch conditions. Thus both signs of the nonlinearity are accessible in the same sample just by, for example, changing the temperature. This property has been used to demonstrate both in-phase and staggered (adjacent fields are $\pi$ out of phase with each other) spatial solitons in these arrays [24]. In this chapter we show theoretically and experimentally that both types of quadratic surface discrete solitons exist for both signs of the cascading nonlinearity.

The system shown in Figure 3.8 was modeled by employing a coupled mode formulation for quadratic nonlinear media [24, 25].

![Figure 3.8: (upper) Model for the discrete array-continuous medium interface. (lower) Actual sample structure showing the index distribution, the FW and the SH. The index distribution and fields are overlapped in space but have been separated for clarity.](image-url)
In our system, the adjacent waveguides comprising the array are weakly coupled by their evanescent fields. Given the fact that the second harmonic (SH) TM$_{00}$-modes are strongly confined, the coupling process between the SH fields is negligible. Therefore, here we only consider coupling between the modal fields of the fundamental wave (FW). In physical units, the pertinent coupled mode equations describing the wave dynamics in a semi-infinite array are given by the following:

\[
\begin{align*}
    i \frac{\partial u_0}{\partial z} + cu_1 + \gamma u_0^* v_0 &= 0, \quad \text{for} \quad n = 0 \\
    i \frac{\partial u_n}{\partial z} + c(u_{n+1} + u_{n-1}) + \gamma u_n^* v_n &= 0, \quad \text{for} \quad n \geq 1 \\
    i \frac{\partial v_n}{\partial z} - \Delta \beta v_n + \gamma u_n^2 &= 0, \quad \text{for} \quad n \geq 0
\end{align*}
\]

(3.6)

where $u_n$ and $v_n$ (dimensionless) are the FW and SH modal amplitudes in the $n^{th}$ waveguide respectively, $c\left[\text{m}^{-1}\right]$ is the linear coupling constant and $\gamma\left[\text{m}^{-1}\right]$ is the effective quadratic nonlinear coefficient, where $\gamma = \frac{\omega}{2P} \varepsilon_0 d_{\text{eff}}^{(2)} \Phi$, with $\omega\left[\text{s}^{-1}\right]$ the angular frequency of light, $\varepsilon_0\left[\text{F/m}\right]$ the dielectric permittivity of the vacuum, $P$ an arbitrary power unit of 1 Watt, $d_{\text{eff}}^{(2)}\left[\text{m/V}\right]$ the quadratic nonlinear coefficient, and $\Phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_{\omega}^2(x,y) E_{2\omega}(x,y) dx dy$ the overlap integral between the fundamental and the second harmonic electric fields $E_{\omega}, E_{2\omega}\left[\text{V/m}\right]$, respectively. Furthermore, $\Delta \beta\left[\text{m}^{-1}\right] = 2\beta(\omega) - \beta(2\omega)$ is the wavevector mismatch between the FW and SH.
3.7 Surface solitons in LiNbO$_3$ arrays

Stationary solutions of the form $u_n = f_n \exp(\sqrt{\mu} z)$ for the FW and $v_n = s_n \exp(2\sqrt{\mu} z)$ for the SH were numerically determined by applying Newtonian relaxation techniques. Here $\mu$ is the soliton eigenvalue and is related to a nonlinear change in the propagation constant $\Delta k^{NL} = c \mu$. In-phase solitons are possible when $2\Delta k^{NL} + \Delta \beta > 0$, while staggered solitons exist for $2\Delta k^{NL} + \Delta \beta < 0$ [25].

Figure 3.9: (a) Surface soliton existence curves for in-phase solitons for $36\pi$ (red curve) and $-15.5\pi$ (blue curve). (b), (c) Intensity profiles for low, and high powers for FW (blue) and SH (red), in the case of positive mismatch $36\pi$, respectively, and (d) Intensity profiles for high
powers for FH (blue) and SH (red), in the case of negative mismatch -15.5π. The SH powers of both solitons are overlapped for large nonlinear wavevector shifts.

The power versus nonlinear wavevector shift diagrams for both the in-phase and staggered surface soliton families obtained are shown in Figure 3.9 and 3.10, respectively, along with the corresponding typical intensity profiles. Throughout this study we use the parameters typical of the experiments. More specifically, the coupling length in this array is taken to be 25 mm and the quadratic nonlinear coefficient is 18 pm/V [24].

Figure 3.10: (a) Surface soliton existence curves for staggered solitons for a mismatch of –15.5π (red curve) and 36π (blue curve). (b), (c) Intensity profiles for low, and high powers for FH (blue) and SH (red), in the case of negative mismatch -15.5π, respectively, and (d) Intensity
profiles for high powers for FH (blue) and SH (red), in the case of positive mismatch $36\pi$. The SH powers of both solitons are overlapped for large nonlinear wavevector shifts.

A number of interesting features are predicted for these quadratic surface solitons. Different from the infinite arrays case, these surface self-trapped states exist only when their power exceeds a critical level - a direct consequence of the semi-infinite geometry of the lattice. This is a feature common to surface solitons at the interface between continuous media, also found recently for surface solitons propagating due to self-focusing and self-defocusing nonlinearities in Kerr media [20, 26]. As the soliton power increases the fields become progressively more confined in the $n=0$ channel. The fraction of power carried by the SH is decreased as $\Delta k^{NL}$ increases.

Furthermore, just as found for discrete solitons in infinite 1D media, the solitons consist of coupled FW and SH fields. In addition to the expected staggered solutions, in-phase solitons were also found under negative phase mismatch conditions for $2\Delta k^{NL} + \Delta \beta > 0$, i.e. with self-focusing nonlinearities. See the blue curves in Figure 3.9(a) for the existence curves and the field distributions in Figure 3.9(d). Note that this family of solitons can only be excited if the SH is considerably stronger than the FW. Similarly in regions of positive phase-mismatch, both stable in-phase and staggered (for $2\Delta k^{NL} + \Delta \beta < 0$, i.e. a self-defocusing nonlinearity for the blue curves in Fig 3.10(a) and the fields in Figure 3.10(d)) surface solitons are predicted to exist. This mirrors the case predicted for infinite quadratically nonlinear 1D arrays [25]. We emphasize that in all cases the branch associated with the SH wave in the existence curves (see Figures 3.9 (a) and 3.10 (a)) does not depend on the value of phase-mismatch $\Delta \beta$. This can be formally proved based on the fact that the waveguides are uncoupled for the SH wave.
Finally, we note that stability analysis of Eqs. (3.6) indicates that the predicted surface solitons are stable in the regions where the slope of the curve is positive, in accordance with the Vakhitov-Kolokolov criterion [7].

An important problem is to verify which field distributions were generated, staggered or in-phase for each sign of the cascading nonlinearity. Theory has shown the ratio of the FW to SH powers are very different in the two cases. In order to compare experiment approximately with theory, the assumed hyperbolic secant temporal profile was decomposed into cw temporal slices and the pulse response was simulated by adding the slices together. A fourth order Runge-Kutta method was then used to propagate the fields under the influence of Eq. (3.6). Comparing the measured and the calculated ratios of the FW to SH powers at the output, clearly the observed surface solitons were the staggered ones for negative and the in-phase ones for positive mismatch since the experimentally measured power ratio FW/SH was much bigger than unity. It would be necessary to also input the appropriate SH field in order to excite the other surface solitons.

Using the same numerical approach, the output intensity distributions across the array were calculated versus input peak power for both positive (+36π) and negative (-15.5π) phase mismatches for in-phase and staggered solitons respectively. A sampling of these results, along with the corresponding experimental data is shown in figures 3.11 and 3.12.
Figure 3.11: Measured (left-hand-side) and calculated (right-hand-side) output field distributions for single channel excitation for two input power levels corresponding to partial collapse into a surface soliton for FH (first row) and full collapse into a surface soliton for FH (second row) and SH (last row). Phase mismatch $= +36\pi$ (self-focusing nonlinearity). The red curves represent theoretical results (at 426 W and 433 W) and the blue experimental data (at 430 W, and 600W) FW input powers.
Figure 3.12: Measured (left-hand-side) and calculated (right-hand-side) output field distributions for single channel excitation for two input power levels corresponding to partial collapse into a surface soliton for FH (first row) and full collapse into a surface soliton for FH (second row) and SH (last row). Phase mismatch = -15.5\pi (self-defocusing nonlinearity). The red curves represent theoretical results (at 308 W and 435 W) and the blue experimental data (at 420 W, and 580W).
In fact there is good qualitative agreement between experiment and theory considering the non-ideally hyperbolic secant temporal profile of the input beam and the coupling efficiency estimated from low power throughput experiments [27, 28]. If a coupling efficiency of 50% is assumed into the input channel, the resulting quantitative agreement is also good.
References


CHAPTER FOUR: SEMI-INFINITE OPTICAL LATTICES

4.1 Basic equations

So far we have considered wave propagation in waveguide arrays within the context of coupled mode theory [1]. This discrete model is describing with good accuracy first order effects (associated with the first band only) for relatively small refractive index differences. Since we are interested in examining higher order effects (from the second and third bands), we have to use the complete continuum model of paraxial equation of diffraction, based on which surface solitons from higher bands can be systematically examined.

Let us now give a brief sketch of the derivation of such an equation. The electrical field $E$ in one dimensional structure with refractive index distribution $n(x)$ is governed by the Helmholtz equation

$$\frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial x^2} + k_0^2 n^2(x) E = 0.$$ 

We are looking for plane wave solutions of the form $E = U(x, z) \exp(ikz)$, where $U$ is a slowly varying function of $z$. By employing the slowly varying envelope approximation (SVEA) $|2ikU_z| >> |U_z|$, we get

$$i \frac{\partial U}{\partial z} + \frac{1}{2k} \frac{\partial^2 U}{\partial x^2} + \frac{1}{2k} \left[ k_0^2 n^2(x) - k^2 \right] U = 0,$$

where $k = k_0 n_{cl}$, and $n_{cl}$ is the background refractive index (for AlGaAs $n_{cl} = 3.28$), and $k_0 = 2\pi/\lambda_0$. The $n(x)$ of the whole structure can be written as $n(x) = n_{cl} + \delta V(x)$, $\delta \ll 1$, where $V(x)$ represents the normalized periodic potential, and
\( \delta \) the index difference between core and substrate (for waveguide arrays \( \delta \sim 10^{-4} - 10^{-3} \)). The final result is the paraxial equation of diffraction:

\[
\frac{i}{\partial z} U + \frac{1}{2k} \frac{\partial^2 U}{\partial x^2} + k_0 \delta V(x) U = 0, \quad (4.1)
\]

where \( U \) is the envelope of the optical field, and \( x \) is the transverse axis. In the most general case of a nonlinear Kerr medium, the underlying nonlinear Schrödinger equation that describes this one-dimensional AlGaAs system (in physical units) is,

\[
\frac{i}{\partial z} U + \frac{1}{2k} \frac{\partial^2 U}{\partial x^2} + k_0 \delta V(x) U + k_0 n_2 |U|^2 U = 0, \quad \text{where } n_2 = \hat{n}_2 n_{el}/(2 \eta_0), \quad \text{and}
\]

\( \hat{n}_2 = 1.5 \times 10^{-13} \left[ \text{cm}^2 / \text{W} \right] \) is a typical value for the Kerr nonlinear coefficient in AlGaAs structures. By using the following normalizations \( \xi = z/(2k x_0^2), \eta = x/x_0, \ U = \sqrt{\delta/n_2} \cdot u(\eta, \xi), \)

\( x_0 = \left[ \frac{\lambda^2}{8 \pi^2 n_{el} \delta} \right]^{1/2} \), we get the normalized nonlinear Schrödinger equation:

\[
\frac{i}{\partial \xi} u + \frac{\partial^2 u}{\partial \eta^2} + V(\eta) u + \sigma |u|^2 u = 0, \quad (4.2)
\]

where \( V(\eta) \) represents the periodic optical potential of period \( D \), \( V(\eta) = V(\eta + D) \) and the coefficient \( \sigma \) is \( \sigma = +1(-1) \) for self-focusing (defocusing) nonlinearities.

### 4.2 Band structure-Tamm states

Before we discuss in detail the properties of surface solitons, it is beneficial to review some basic facts regarding Floquet-Bloch analysis in an infinite optical lattice [2]. This problem can be effectively analyzed by the paraxial equation of diffraction.
It is well known that due to periodicity, bands and band gaps in the k-space characterize this type of systems. In particular, the dispersion relation between the propagation eigenvalue $\beta$ and the transverse Bloch momentum $k$ gives the transmission bands associated with the periodic potential. A typical band structure of an optical lattice for index difference $\delta = 2.1 \cdot 10^{-3}$, cell’s width $a = 4.4 \mu m$, and period of $D = 10 \mu m$ is shown in Figure 4.1.

Figure 4.1: Band structure of an infinite waveguide array (first band/red line and second band/blue line). The gray area represents the first forbidden band gap of the structure.

In general the solution of equation (4.3) is given in terms of Floquet-Bloch (FB) modes $\phi_{kn}(x)$, where $n$ is the number of the band and $k$ the corresponding Bloch wavenumber [3]. Every mode can be written as $\phi_{kn}(\eta) = g_{kn}(\eta) \exp(i k \eta)$, where $g_{kn}(\eta) = g_{kn}(\eta + D)$. Since the superposition principle is valid (linear system), any arbitrary field profile can be decomposed as

$$u(\eta, \xi) = \int_{-\pi/D}^{\pi/D} \sum_{n=1}^{\infty} c_n(k) \phi_{kn}(\eta) \exp(i \beta_{kn} \xi) dk,$$

where $c_n(k)$ is the occupancy coefficient of the
corresponding FB mode of the n^{th} band. Note that this expansion is possible because of the orthogonality of FB modes in the whole lattice\[\int \phi^*_{k,m}(\eta) \phi_{k,m}(\eta) d\eta = \delta_{n,m} \delta(k-k').\]

Now the question that naturally arises is if the above are true for a semi-infinite lattice, like the one depicted in Figure 4.2.

![Semi-infinite optical lattice](image)

Figure 4.2: Semi-infinite optical lattice

In a semi-infinite potential Floquet-Bloch theorem is not valid, due to the breaking of the translational symmetry (the potential is not periodic in the whole space any more). By following the direct matching procedure in a semi-infinite Kronig-Penney model, Tamm [4] formally proved the existence of surface waves within the framework of solid state physics. In particular, he showed that surface waves are localized defect modes that propagate along the interface, with a complex Bloch wavenumber, \( k = i\mu + m\pi/D, m = 0,1,2,... \). The associated real propagation constant lies on the forbidden band gaps of the bandstructure of the corresponding infinite potential. These waves are called Tamm states [5]. As an example, let us consider such defect modes or Tamm states in a semi-infinite optical lattice. In Figure 4.3 (a) the refractive index in the continuum region is higher (by 0.84 \( \cdot \) 10^{-3}) than the background index inside the array (\( \delta = 2.1 \cdot 10^{-3} \)). As a result the channels close to the interface have effectively higher refractive index than the others. Therefore, a localized in-phase surface state exists. On the other hand,
when there is lower index in the continuum (by $1 \cdot 10^{-2}$), an out of phase defect mode appears (Figure 4.3(b)). Its eigenvalue is located at the first band gap at the edge of the Brillouin zone.

![Figure 4.3: Normalized field profile of a Tamm state: (a) in-phase, and (b) out of phase. The dashed lines represent the semi-infinite optical potential.](image)

We emphasize that in the case when the background refractive index is the same along both sides of the interface, no Tamm state exists.

### 4.3 Surface solitons in 1D lattices

Armed with this knowledge, one can move on to understand how a surface soliton forms at the edge of semi-infinite nonlinear lattice. As we discussed before, the system in this case is described by the nonlinear Schrödinger Eq. (4.2). It is important to note that these surface solitons are the direct outcome of the nonlinearity, since the specific semi-infinite waveguide array does not support linear defect modes. In other words, this new class of self-trapped waves can be considered as the nonlinear analog of the Tamm states of solid state physics. In all cases the lattice surface solitons are numerically found by using relaxation schemes based on the self-consistent method [6]. This is done by assuming stationary solutions of the
form $u(\eta, \xi) = \bar{u}(\eta) \exp(i\lambda\xi)$, where $\bar{u}(\eta)$ is the field profile, and $\lambda$ is the nonlinear correction to the propagation constant or the soliton eigenvalue. The power-eigenvalue diagram for such solutions is depicted in Figure 4.4 (a) (blue curve-solid line). In agreement with the discrete approximation examined before (chapter 3), surface solitons for the first $n = 0$ channel exist only if their power exceeds a critical threshold. The stability follows the Vakhitov-Kolokolov criterion as direct BPM simulations show.

![Figure 4.4](image)

Figure 4.4: (a) Lattice surface soliton power $P$ versus the normalized eigenvalues $\lambda$, where every line corresponds to a different soliton solution. The maximum of the field occurs at the $n=0$ waveguide site (solid blue line), $n=1$ waveguide site (dashed red line), and $n=2$ waveguide site (dash-dot green line). (b) Power threshold for a soliton localized at $n = 0$ channel versus ridge width $d$ for a fixed channel separation $D = 10\mu m$. (c), (d) and (e) show the normalized field profiles when the maximum is at $n=0$, $n=1$, and $n=2$ waveguide sites, respectively.
The dependence of the power threshold as a function of the waveguide’s width is also plotted in Figure 4.4 (b). This curve can be intuitively understood if we take into account that the coupling length has a similar dependence on the channel’s width. More specifically, the larger is the coupling length (weaker coupling), the higher the threshold is, since more power required for redistributing the power among the channels. Moreover, as shown in Figure 4.4 (a) similar results can be obtained for surface soliton fields localized at the $n=1$ and 2 channels (red, green curves, respectively) [7]. The field is asymmetric around the maximum, and this asymmetry vanishes as the localization site $n$ of the solution moves inside the array, away from the interface ($n=1$). This is anticipated, since for large values of $n$, the well-known symmetric discrete soliton is obtained. Furthermore, the power threshold characteristic of surface solitons goes to zero as $n$ (the site where the soliton peak resides) increases. Moreover, numerical simulations reveal that unstable surface solitons that reside in the $n$ channel eventually drop into the $n=1$ site due to instabilities. Intuitively, this should have been expected since for every unstable solution at the $n$ site, there always exists a stable solution of a lower power at the $n=1$ waveguide [7].

Given the fact that our experiments [7] utilized ultra-short pulses and that the waveguides are not only dispersive but also exhibit three-photon absorption, we have simulated the beam dynamics in both space and time. The underlying nonlinear Schrödinger equation that describes this one-dimensional AlGaAs system is,

$$
\begin{align*}
  & \frac{i}{\mathcal{D}} \frac{\partial U}{\partial z} + \frac{1}{2k} \frac{\partial^2 U}{\partial x^2} - \frac{k''}{2} \frac{\partial^2 U}{\partial T^2} + k_0 \delta f(x)U + k_0 n_z |U|^2 U + i a |U|^4 U = 0, \\
  & \quad \text{(4.4)}
\end{align*}
$$

where $U$ is the envelope of the optical field, $x$ is the transverse axis, and $T$ is a time coordinate moving at the group velocity of the wave. The second term in Eq. (4.4) describes the spatial
diffraction process, the third is associated with dispersion effects, while the fifth one accounts for the Kerr nonlinearity. The term associated with the normalized periodic potential \( f(x) \) arises from the periodicity of the array. In addition, \( k_0 = 2\pi / \lambda_0, k = k_0 n \) (where the refractive index of AlGaAs \( n = 3.28 \)) is the propagation wavevector, \( k'' = 1.3 \times 10^{-24} \text{m}^{-1} \text{s}^{-2} \) is the normal dispersive coefficient of the material, \( \delta = 1.5 \times 10^{-3} \) is the index difference between the core and the cladding regions in the array, and \( n_z = \hat{n}_z n / 2\eta_0 \) where \( \hat{n}_z = 1.5 \times 10^{-13} \text{cm}^2 / W \) is the Kerr nonlinearity. Finally, three photon absorption has been included in the last term of Eq. (4.4), where \( a = \alpha_3 n^2 / 8\eta_0^2 \), and the three photon absorption coefficient \( \alpha_3 \) was taken (approximately) to be \( \alpha_3 = 0.03 \text{cm}^3 / \text{GW}^2 \).

The first set of experiments [7] and simulations dealing with optimum excitation of the n=0 channel are depicted in Figure 4.5. Results at three different excitation powers are shown. The lowest power corresponds to linear diffraction (Figures 4.5 (a),(d)), the second to intermediate power levels (Figures 4.5 (b),(e)) (partial collapse of this diffraction pattern towards a discrete surface soliton), and the third (Figures 4.5 (c),(f)) to the intensity distribution of a discrete surface soliton. Even though the experiments were carried out with pulses, a rapid collapse of the output pattern into a discrete soliton was found to occur after 1.7 kW, thus supporting the existence of a power threshold. The theoretical figures 4.5 (d),(e),(f) depict integrated intensities (\( \propto \int |U|^2 dT \)) since the photodiode response depends only on photon energy. Overall the agreement between experiment and theory is very good. The fields decay exponentially into the continuous region. In the second and third cases the long “tails” trailing into the array from the n=0 channel are a consequence of the temporal pulse excitation. For example, for the highest power case, the intermediate instantaneous powers associated with the
pulse lead to only partial collapse while the low power tails produce the equivalent of the linear discrete diffraction pattern. In Figure 4.5 one can also observe some weak, diffracting radiation in the continuous region due to imperfect excitation of the first n=0 channel. In fact, when the center of the incident beam is moved to partially overlap the n=0 channel and the continuum, sufficient power is radiated into the continuous slab to also initiate beam collapse into a spatial soliton there.
Figure 4.5: Intensity patterns observed at the output of the AlGaAs array for single channel excitation at three different peak input power levels injected into channel n=0. Left-hand-side experimental results for (a) P= 450W; (b) P=1300W; (c) P=2100W. Right-hand-side numerical calculation results for (d) P= 280W; (e) P=1260W; (f) P=2200W. The inset shows the actual sample geometry.

### 4.4 2D Surface solitons

Surface solitons can also exist at the boundaries of two dimensional lattices [8]. More specifically, we have examined a nonlinear Kerr semi-infinite square lattice of waveguides. The linear refractive index between the core and the cladding is taken here to be $3 \times 10^{-3}$ and the distance between the single-mode waveguides is 6 $\mu$m in both orthogonal directions. The wave propagation in this two-dimensional self-focusing optical lattice is described by the normalized nonlinear Schrödinger equation:

$$
\begin{align*}
&\frac{i}{\partial \xi} \frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \zeta^2} + V(\eta, \zeta) u + |u|^2 u = 0
\end{align*}
$$

(4.5)

where $V(\eta, \zeta)$ is the semi-infinite index potential (see Figure 4.6 (a)). Our analysis leads to new soliton solutions existing at the corner and at the edge of the 2D lattice. The power associated with these surface solitons is plotted in Fig 4.6 (b) as a function of the corresponding eigenvalue $\lambda$. Our analysis shows that both these soliton solutions are possible only when their power exceeds a critical threshold. The threshold of the edge surface state is slightly higher than that of the corner soliton, which is physically anticipated since the latter self-trapped state is confined in fewer sites. In both cases, the propagation constants of these surface solitons are located at the
semi-infinite band gap of the lattice. Typical intensities profiles for the corner soliton and the edge soliton are depicted in Figure 4.7(a), and Figure 4.7 (b), respectively.

Before closing we would like to mention that these two-dimensional surface solitons can be directly observed at the boundaries (edges or corners) of a finite optically induced photonic lattice [9]. To create a 2D waveguide lattice, we use the optical induction method as used in discrete soliton experiments carried out in an infinite uniform lattice [10-15]. Both in-phase and gap nonlinear surface self-trapped states were observed under single-site excitation conditions. In Figures 4.8 and 4.9, the theoretical and experimental results (respectively) are presented. As we can see, the experimental data are in good agreement with the theoretical predictions.

![Figure 4.6: (a) Semi-infinite two-dimensional optical lattice. (b) Power – eigenvalues diagrams for the corner (red line) and edge (blue line) surface lattice solitons.](image.png)
Figure 4.7: Intensities of surface solitons in a semi-infinite square lattice located at (a) the corner, and (b) at the edge of the 90° degree angular sector lattice.

Figure 4.8: Numerical results of in-phase (a)–(c) surface solitons and out-of-phase (d)–(f) surface gap solitons. First column shows the soliton pattern; second column shows the interference pattern between the soliton beam and a tilted plane wave; third column shows the corresponding spatial spectra of the solitons.
Figure 4.9: Experimental results of in-phase (top) surface solitons and out-of-phase (bottom) surface gap solitons. First column shows the soliton intensity pattern; second column is the interference pattern between the soliton beam and a tilted plane wave; third column shows the corresponding spatial spectra (the added squares mark the edge of the first Brillouin Zone).

4.5 Optical Heterostructures-Hybrid solitons

In this paragraph we study nonlinear surface waves in one- and two-dimensional optical lattices, and predict surface lattice solitons propagating along the hetero-interface of two different semi-infinite waveguide arrays. A unique characteristic of this new family of solitons is that two different semi-infinite field profiles can form a composite entity, namely a hybrid surface soliton. We investigate several generic examples of hybrid surface solitons residing in different band gaps of the composite hetero-structure, and study their stability. Finally, we
predict surface lattice solitons occurring at the boundaries of a two-dimensional (2D) lattice (at a corner or at an edge), when their power level is above a critical threshold. Note that heterostructures composed of linear and nonlinear semi-infinite arrays of semiconductor multilayers have been considered before, for controlling small refractive index differences in interfaces [16].

Let us consider a lattice consisting of two different semi-infinite waveguide arrays, as shown in Figure 4.10.

![Figure 4.10: Two semi-infinite waveguide arrays joint together.](image)

Every waveguide is designed to be single-moded and the assumed nonlinearity is of the self-focusing or defocusing Kerr type. Since the difference in the refractive indices (with respect to the cladding) is small, a scalar approach is applicable for the particular problem. In this case, the system is governed by the paraxial scalar nonlinear Schrödinger Eq. (4.2), capable of describing higher-order band dynamics. For demonstration purposes, let both regions in Figure 4.10 have a linear refractive index difference of $4 \times 10^{-3}$, with equally spaced sites (10 $\mu$m center to center). Note that the difference between the width of the channels leads to an effective mismatch in the propagation constant in the two regions. As a result, the bands of the right and left array are
relatively shifted, as it can be seen in Figure 4.11. If this shift is large enough, solitons whose propagation eigenvalues reside inside a forbidden gap are possible. It can be formally shown that the band structure of the entire hetero-interface involves the individual band structures of each of the two semi-infinite optical lattices, which in turn are related to the band-diagrams of the corresponding infinite arrays. Figure 4.11 also demonstrates that for the specific design parameters used here, there is a band overlap between the second and the third band of the two different semi-infinite arrays. Therefore, soliton solutions can only be obtained in the resulting three complete band-gaps. More specifically, one can identify surface solitons with propagation constants in the semi-infinite gap, in the first gap between the two first bands and in the second gap between the first band of the left array and the second band of the right array (points A, B, and C in Figure 4.11 respectively). It is important to note that the hybrid solitons are a direct outcome of the nonlinearity, since the hetero-structure does not support linear defect modes, and does not lead to pinned states due to inhomogeneities. In all cases, the lattice surface solitons are numerically found using numerical relaxation schemes based on the self-consistent method [6].

Figure 4.12 (a) depicts the field profile of a surface soliton existing at the nonlinear hetero-interface of Fig 4.10. This soliton state corresponds to the eigenvalue A of Figure 4.11, which is located in the semi-infinite band-gap of both lattices. As a result, the two components comprising this soliton are in phase. The power-eigenvalue stability diagram ($P - \lambda$ curve) associated with this solution is shown in Figure 4.12 (b). This curve terminates close to $\lambda \approx 0.77$ which is close to the top edge of the first band of the right array. In this region the solutions start to become unstable as one may also anticipate from the Vakhitov-Kolokolov criterion. For higher eigenvalues these solutions are stable. Their stability has been tested using beam propagation methods.
Another interesting case arises when the soliton eigenvalue is located at point B of Figure 4.11. This implies that B is at the top of the first band (in the semi-infinite band gap) of the left array and at the bottom of the first band (first band gap) of the right lattice. Therefore the one part of the surface soliton field in the left array will be in-phase whereas the other part on the right array will be staggered (the field lobes are $\pi$ out of phase). As a result, the two components can propagate locked together as a composite self-trapped state, thus forming a hybrid surface soliton. A typical field profile of this type of hybrid soliton is shown in Figure 4.13 (a). The power of these hybrid solutions with respect to the corresponding soliton eigenvalue $\lambda$ is plotted in Fig 4.13 (c). As the eigenvalue $\lambda$ approaches the edge of the first band (of the right array) the staggered component of the solution becomes wider while the in-phase component is getting more localized. The converse occurs when the eigenvalues $\lambda$ is close to the edge of the second band. The stability of this solution was investigated using beam propagation methods. We found this kind of solitons are stable when $\lambda$ is close to the bottom edge of the first band of the right array and they become unstable when $\lambda$ approaches the first band of the left lattice. Another solution can be found in the third complete gap, (eigenvalue C in Figure 4.11) when the nonlinearity is of the de-focusing type ($\sigma = -1$). The field profile of this surface soliton as well as the associated $P - \lambda$ are shown in Figures 4.13 (b) and (d) respectively. In this latter case, both components at the interface are of the staggered type.
Figure 4.11: The band structure of the two coupled semi-infinite waveguide arrays. The dotted curves correspond to the band structure of the right array, while the solid lines correspond to that of the left array. Points A, B, C represent the propagation eigenvalues of the allowed surface solitons.

Figure 4.12: (a) Field profile of a hybrid in-phase/in-phase soliton, and (b) the corresponding power-eigenvalue diagram. The grey colored areas represent the bands of the structure.
Figure 4.13: Field profile of a hybrid (a) in-phase/staggered soliton, and (b) staggered/staggered soliton. The power-eigenvalue diagrams for these solutions are depicted in (c) and (d), respectively. The grey colored areas represent the bands of the structure.
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CHAPTER FIVE: RABI OPTICAL TRANSITIONS

5.1 Introduction

It is a well known fact that when an electron “drops” from the conduction band to the valence, the excess energy is given off as emitted radiation-or as a photon. This emission can take place spontaneously from zero-point quantum fluctuations or it can be induced through stimulated emission. On the other hand, light can also be absorbed in crystals, resulting into an electronic transition to a higher energy state (conduction band). What actually facilitates this process is in fact a periodic time perturbation added to the electron Hamiltonian due to the presence of an external optical field. In the absence of any phonon or defect interactions, such transitions obey selection rules and as a result they can only take place if the Bloch momentum is conserved (direct transitions). In addition, the energy must be also conserved. Therefore, the period of the harmonic perturbation should be equal to the energy difference between the two levels. In the framework of atom optics Rabi oscillations have been also considered at both the theoretical [1] and experimental front [2].

In view of the above it is natural to ask if these same processes can also be observed in the optical domain and in particular in waveguide lattices. Already, there have been suggestions of observing such transitions in photonic crystal systems using either $\chi^{(2)}$ nonlinear or ultrasonic time-space harmonic perturbations [3, 4]. It is worth noting however that so far no such all-optical transitions have ever been observed experimentally.
In this chapter we show that optical Rabi interband transitions and nonlinear energy exchange between two different gap solitons are possible in periodically modulated array structures (along the propagation direction \( z \)) [5, 6]. These transitions can occur among bands or gaps in the Bloch momentum \( k \)-space as shown in Figure 5.1. A schematic of a possible lattice structure is shown in the inset of Figure 5.1. The array is modulated in space with period \( \Lambda \), in such a way that the “energy difference” \( 2\pi/\Lambda \) spans the “energy/eigenvector”-difference between the allowed bands. This of course occurs provided that the transition selection rules are respected. Figure 5.1 shows such a possible direct transition between the first and the third band of this structure. In the case shown, this transition is allowed because of the parity of the Floquet-Bloch modes involved. We note that these transitions occur as a result of parametric mixing and thus are different from recently reported Zener tunneling effects [7, 8], as well as, from the Bragg-resonance induced transitions [9] and spatial four-wave mixing effects [10].

Figure 5.1: Allowed direct all-optical transition between the first and third band. The inset depicts a top view of a periodically modulated optical lattice.
5.2 Time dependent perturbation theory

In order to understand the dynamics of interband optical transitions we derive the corresponding coupled-mode equations. This is done by using time-dependent perturbation techniques in the paraxial equation of diffraction. In this case, wave propagation in an unperturbed periodic potential is governed by the following normalized evolution equation

\[ iU_z + U_{xx} + V_0 U = 0 \],

where \( U \) represents the optical field, \( z \) is the propagation distance, \( x \) the transverse coordinate, and \( V_0(x) \) is the periodic index potential with spatial period \( D \), \( V_0(x) = V_0(x + D) \). The general solution of this problem can be expressed as a linear superposition of Floquet-Bloch (FB) modes \( \phi_{k,n}(x) \), where \( k \) denotes the Bloch wavevector and \( n \) the band-index number. In general, \( U \) can be expressed as a superposition

\[ U(x,z) = \sum_{n=-\infty}^{\infty} \int_{-\pi/D}^{\pi/D} c_n(k) \phi_{k,n}(x) \exp[i\beta_n(k)z] \, dk \],

where \( c_n(k) \) represent FB mode occupancy coefficients and \( \beta_{k,n} \) is the corresponding propagation constant. Thus the unperturbed eigenvalue problem is given by

\[ -\beta_n(k)\phi_{k,n} + V_0(x)\phi_{k,n} = 0 \].

If on the other hand, the lattice is perturbed problem by a \( z \)-dependent periodic potential, then the system obeys:

\[ i \frac{\partial U}{\partial z} + \frac{\partial^2 U}{\partial x^2} + V(x,z)U = 0 \quad (5.1), \]

where \( V(x,z) \) is a lattice potential periodically modulated along the \( z \)-direction. The perturbed \( z \)-dependent potential \( V(x,z) \) is written as \( V(x,z) = V_0(x) + \varepsilon \cdot V'(x,z) \), where \( \varepsilon \ll 1 \) and \( V'(x,z) \) is a weak periodic modulation. Note that \( V'(x,z) \) has the same periodicity as the original lattice \( V_0(x) \), e.g. \( V'(x,z) = V'(x + D, z) \). The perturbation \( \varepsilon \) is so small to allow us to
assume that the FB modes of the perturbed lattice $V(x,z)$ at every $z$ are the same with that of the unperturbed lattice $V_0(x)$. This is one of the main approximations of time-dependent perturbation theory. The validity of this approximation is going to be tested in the last paragraph of this chapter where the results of coupled mode theory are compared with direct BPM simulations. In view of the above, the field $U(x,z)$ can be expressed as a linear superposition in the $\{\phi_{k,n}(x)\}$ FB basis of the unperturbed $V_0(x)$ lattice. More specifically, we have:

$$U(x,z) = \sum_{n=1}^{\infty} \int_{-\pi/D}^{\pi/D} c_n(k,z) \phi_{k,n}(x) \exp[i\beta_n(k)z] dk,$$

where the occupancy coefficients $c_n(k,z)$ are now $z$-dependent, since there is coupling and energy exchange between FB modes, induced by the $z$-dependence of $V(x,z)$ potential. By substituting the last expansion in Eq. (5.1) we find that the following equation must be satisfied for every $z$:

$$\sum_{n=1}^{\infty} \int_{-\pi/D}^{\pi/D} \{i \dot{c}_n(\phi_{k',m}, \phi_{k,n}) \exp[i\beta_n'(k)z] + \epsilon(\phi_{k',m}, V'(x,z) \phi_{k,n}) c_n \exp[i\beta_n(k)z]\} dk = 0$$

(5.2)

where the inner product is defined as $\langle f, g \rangle \equiv \int_{-\infty}^{+\infty} f^*(x) g(x) dx$.

### 5.3 Transition matrix element

Evidently, the coupling strength between FB modes depends on the transition matrix element $\langle \phi_{k',m}, V'(x,z) \phi_{k,n} \rangle$. The latter expression is not convenient, since the involved integral must be evaluated over the whole infinite lattice, something that is numerically impossible. Therefore the goal of this paragraph is to reduce the calculation of the transition matrix element
to just one cell, instead of the whole lattice. We will start from a finite lattice of \( 2N' + 1 \) cells and then we are going to take the limit \( N' \to \infty \). Firstly we split the integral in every individual cell:

\[
\int_{\text{finite lattice}} \phi_{kn}^* V'(x, z) \phi_{kn} \, dx = \int_{\text{finite lattice}} u_{km}^* V'(x, z) u_{kn} \exp\left[ i (k-k')x \right] \, dx =
\]

\[
= \int_{-\frac{N' D - D}{2}}^{-\frac{D}{2}} + \ldots + \int_{\frac{3D}{2}}^{\frac{D}{2}} + \int_{\frac{D}{2}}^{\frac{ND + D}{2}} u_{km}^* V'(x, z) u_{kn} e^{i\Delta x} \, dx + \ldots + \int_{\frac{D}{2}}^{\frac{N D + D}{2}} , \text{ where}
\]

\( \Delta k \equiv k - k' \). All these integrals have apparently the same integrand \( u_{km}^* V'(x, z) u_{kn} e^{i\Delta x} \) which it was omitted for convenience. Now we are going to change variables in every integral according to \( x = s - jD \), with \( j = \pm 1, \pm 2, \ldots, \pm N' \). In that way we reduce every integral to the central cell from \(-D/2\) to \(D/2\). It is:

\[
\int_{\text{finite lattice}} \phi_{kn}^* V'(x, z) \phi_{kn} \, dx = \int_{-\frac{D}{2}}^{\frac{D}{2}} u_{km}^* (s - N'D) V'(s - N'D, z) u_{kn} (s - N'D) e^{i\Delta x} e^{i\Delta ND} \, ds + \ldots +
\]

\[
+ \int_{-\frac{D}{2}}^{\frac{D}{2}} u_{km}^* (s - D) V'(s - D, z) u_{kn} (s - D) e^{i\Delta x} e^{-i\Delta MD} \, ds + \ldots + \int_{-\frac{D}{2}}^{\frac{D}{2}} u_{km}^* (s + D) V'(s + D, z) u_{kn} (s + D) e^{i\Delta x} e^{i\Delta MD} \, ds + \ldots +
\]

\[
+ \int_{-\frac{D}{2}}^{\frac{D}{2}} u_{km}^* (s + N'D) V'(s + N'D, z) u_{kn} (s + N'D) e^{i\Delta x} e^{i\Delta ND} \, ds.
\]

Given the fact that both \( \phi_{kn}(x) \) and \( V'(x, z) \) have the same period \( D \), we get the following result:

\[
\int_{\text{finite lattice}} \phi_{kn}^* V'(x, z) \phi_{kn} \, dx = \int_{-\frac{D}{2}}^{\frac{D}{2}} u_{km}^* (s) V'(x, z) u_{kn} (x) e^{i\Delta x} dx \cdot \left[ \sum_{j=-N'}^{N'} e^{i\Delta Dj} \right].
\]

But the last sum is the Dirichlet kernel of \( \Delta k D \) (see Appendix). In other words \( D_{N'}(\Delta k D) \equiv \sum_{j=-N'}^{N'} e^{i\Delta Dj} \). Therefore the transition matrix element in the finite lattice is given by the following Eq. (5.3):
\[ \int_\text{finite lattice} \phi^*_k V' \phi_{k'} dx = D_{N'} (\Delta k D) \cdot \int_{-D/2}^{D/2} \phi^*_k V' \phi_{k'} dx, \quad (5.3) \]

In order to calculate the transition matrix element in the infinite lattice we have to evaluate the limit of Eq. (5.3) as \( N' \to \infty \). Thus \( \int_{-\infty}^{+\infty} \phi^*_k V' \phi_{k'} dx = \lim_{N' \to \infty} \{ D_{N'} (\Delta k D) \} \cdot \int_{-D/2}^{D/2} \phi^*_k V' \phi_{k'} dx \). But we know from Fourier analysis (see Appendix) that the limit at infinity of the Dirichlet kernel is a series (comb) of equally spaced Dirac delta functions \( \lim_{N' \to \infty} \{ D_{N'} (x) \} = 2\pi \sum_{n=-\infty}^{+\infty} \delta (x - 2\pi n) \). This is a direct outcome of the Poisson summation formula. Therefore the inner product over the whole infinite lattice becomes:
\[ \int_{-\infty}^{+\infty} \phi^*_k V' \phi_{k'} dx = 2\pi \sum_{n=-\infty}^{+\infty} \delta (\Delta k D - 2\pi n) \cdot \int_{-D/2}^{D/2} \phi^*_k V' \phi_{k'} dx \quad (5.4). \]

Since we restrict the values of the Bloch wavenumber only in the first Brillouin zone (reduced zone scheme) \( k \in \left[-\frac{\pi}{D}, \frac{\pi}{D}\right) \). Therefore \( (-\pi/D \leq k < \pi/D) \Rightarrow (-2\pi < \Delta k D < 2\pi) \), and from the comb series of Eq. (5.3) only the central term (for \( n = 0 \)) survives. This means that:
\[ \int_{-\infty}^{+\infty} \phi^*_k V' \phi_{k'} dx = 2\pi \delta (\Delta k D) \cdot \int_{-D/2}^{D/2} \phi^*_k V' \phi_{k'} dx, \]
and since \( \delta (ak) = \delta (k)/|a| \), we finally reduced the transition matrix element calculation to one individual cell. It is:
\[ \langle \phi_{k_m}, V' (x, z) \phi_{k_n} \rangle = \frac{2\pi}{D} \int_{\text{cell}} \phi^*_k V' \phi_{k_n} dx \delta (k - k') \cdot (5.5) \]

This last result of Eq. (5.5) indicates that only direct transitions are allowed \( \Delta k = 0 \Rightarrow k = k' \), i.e., the transverse momentum \( k \) must be conserved in order for a transition to occur.
5.4 Coupled mode equations

By using the orthogonality condition \( \langle \phi_{k',m}, \phi_{k,n} \rangle = \frac{2\pi}{D} \delta_{n,m} \delta(k-k') \) between two FB modes of different wavenumber \( k \) and different bands, we can then derive the coupled mode equations describing the dynamics of Rabi-like oscillations. After substitution of Eq. (5.5) into Eq. (5.2), we get:

\[
\sum_{n=1}^{+\infty} \int_{-\pi/D}^{+\pi/D} \left\{ i \dot{c}_n \frac{2\pi}{D} \delta(k-k') \delta_{n,m} e^{i\beta_n z} + \epsilon c_n \frac{2\pi}{D} \delta(k-k') e^{i\beta_n z} \int_{cell} \phi_{k',m}^* V' \phi_{k,n} \, dx \right\} dk = 0 \Rightarrow \\
\sum_{n=1}^{+\infty} \int_{-\pi/D}^{+\pi/D} \left\{ i \dot{c}_n (k') \delta(k-k') e^{i\beta_n(k')z} \right\} + \epsilon \sum_{n=1}^{+\infty} \int_{-\pi/D}^{+\pi/D} c_n (k) \left[ \int_{cell} \phi_{k',m}^* V' \phi_{k,n} \, dx \right] \delta(k-k') e^{i\beta_n(k')z} \right\} dk = 0 \Rightarrow \\
i \dot{c}_n (k') e^{i\beta_n(k')z} + \epsilon \sum_{n=1}^{+\infty} \left\{ c_n (k') \left[ \int_{cell} \phi_{k',m}^* V' \phi_{k,n} \, dx \right] e^{i\beta_n(k')z} \right\} = 0. \]

Thus the FB mode occupancy coefficients associated with interband direct transitions, satisfy the following equation,

\[
i \dot{c}_m + \sum_{n=1}^{+\infty} A_{nm} c_n \exp\left(i\Delta\beta_{nm}z\right) = 0, \tag{5.6}
\]

where, the wavevector mismatch is \( \Delta\beta_{nm} = \beta_n(k) - \beta_m(k) \) and the overlap integral is defined as

\[
A_{nm} = \epsilon \int_{cell} \phi_{k',m}^* (x)V'(x,z) \phi_{k,n} (x) \, dx. \]

By expressing the periodic perturbation in a separable form \( V'(x,z) = V'(x)\cos\left(\frac{2\pi z}{\Lambda}\right) \) (note that \( \Lambda \) is the z-modulation period of the lattice), and dropping the resulting highly oscillating terms, one can derive the coupled mode equations that describe FB mode transitions from band \( n \) to band \( m \) and vice versa. These are:
\[
\begin{align*}
i \frac{dc_m}{dz} + M_{mn} c_m \exp \left[ -i \left( \Delta \beta_{mn} - 2\pi / \Lambda \right) z \right] &= 0 \\
i \frac{dc_m}{dz} + M_{mn} c_m \exp \left[ i \left( \Delta \beta_{mn} - 2\pi / \Lambda \right) z \right] &= 0
\end{align*}
\]  
(5.7)

where \( M_{mn} = \frac{E}{2} \int_{\text{cell}} \phi_{km}^* V'(x) \phi_{kn} dx \). From Eq. (5.7) it can be directly shown that the energy transition \( m \leftrightarrow n \) between two different FB modes is optimum under the phase matching condition \( \Delta \beta = 2\pi / \Lambda \). In this case complete power transfer will occur at a distance \( L = \pi / \left(2 |M_{mn}| \right) \).

### 5.5 FB mode transitions and nonlinear energy exchange

As an example we consider optical transitions in periodically modulated AlGaAs lattices under appropriate phase-matching conditions \( \Delta \beta = 2\pi / \Lambda \). More specifically, these processes can be realized in systems with periodicity \( \Lambda \) around \( \sim 0.15\text{mm} \). This in turn will allow several cycles (in actual 1-3 cm long samples) for these transitions to occur. The period in this array is \( D = 8\mu\text{m} \), the refractive index contrast is \( \sim 3 \times 10^{-3} \), and the waveguide width varies periodically \( w = w_0 \left[1 + \varepsilon \cos \left(2\pi z / \Lambda \right) \right] \) with \( w_0 = 4\mu\text{m} \) and \( \varepsilon = 0.05 \). For this particular design the transition strengths \( |M_{mn}| \) can be obtained as a function of the Bloch wavevector, as shown in Figure 5.2. This coefficient \( M_{mn} \) determines not only the strength of a transition but also its possibility to occur or not. In other words, it can provide us with a selection rule diagram for all the possible direct optical transitions. Of course, this diagram strongly depends on the design parameters of the particular lattice.
Figure 5.2: Selection rule diagram for direct transitions between the first three bands.

This latter figure indicates that in this structure interband direct transitions are most effective between the second and the third band. Forbidden transitions can also be identified at \( (k = 0, \pm \pi/D) \). Here we consider two examples of Rabi oscillations between FB modes. The first one deals with the most effective transition, e.g. between second and third band at \( k = -\pi/D \).

The beam evolution in the modulated lattice is shown in Figure 5.3(a) when the FB mode \( \phi_{-1/2} \) excited. The power distribution between the second and the third band is also depicted in Figure 5.3(b) as a function of propagation distance. As we can see, complete energy exchange from the second to the third band takes place after 1.2 cm of propagation. Before this point, the two involved FB modes interfere leading to many secondary power oscillations. The second example illustrates a weak FB mode transition between 1 ↔ 3 at \( k = 0 \). The beam propagation (when the FB mode \( \phi_0 \) excited), as well as, the projected power distribution to the two bands, are shown in Figure 5.4(a), (b), respectively.
Figure 5.3: (a) Intensity pattern evolution associated with a strong direct transition from the second to the third band at $k = -\pi/D$ Bloch wavenumber, and (b) corresponding total energy in the second (red line) and the third (blue line) band, respectively, as a function of propagation distance. The arrow in Figure 5.3(a) indicates the location where maximum energy exchange occurs. In this latter case the conversion efficiency is lower and the oscillation period is longer (~6 cm), as expected from the selection rule diagram of Figure 5.2. In both cases the agreement between coupled mode theory and BPM simulations is excellent.
Figure 5.4: (a) Intensity pattern evolution associated with a strong direct transition from the first to the third band at \( k = 0 \) Bloch wavenumber, and (b) corresponding total energy in the first (red line) and the third (blue line) band, respectively, as a function of propagation distance.

Another possibility is to examine whether or not nonlinear energy exchange from gap to gap can also occur in such systems. In essence, this may take place while the wave itself maintains its particle-like (soliton) nature. Given the fact that the propagation eigenvalues of lattice solitons [5, 6, 11-13] lie in the forbidden band gaps of the corresponding linear potential, these excitations are expected to occur between gap to gap. In this case the wave propagation in the \( z \)-modulated periodic potential is governed by the nonlinear Schrödinger equation:

\[
i \frac{\partial U}{\partial z} + \frac{\partial^2 U}{\partial x^2} + V(x,z)U + |U|^2 U = 0 \quad (5.8)
\]

A numerical simulation predicting such a nonlinear energy exchange from the semi-infinite gap to the second gap in a periodically modulated semiconductor waveguide array is examined, under phase-matching conditions. In particular, under linear conditions the beam diffracts, as shown in Figure 5.5(a). For higher powers and for the same excitation, the field localizes and forms a lattice soliton with a propagation eigenvalue in one of the two gaps. In every oscillation
cycle the soliton becomes wider and the energy is always nonlinearly oscillating between the semi-infinite and the second band (see Figure 5.5(b)).

Figure 5.5: (a) Diffraction dynamics in a periodically modulated lattice under wide beam excitation, and (b) nonlinear energy exchange between a soliton residing in the semi-infinite gap and a soliton in the second gap of the corresponding lattice, under the same excitation conditions.
References


CHAPTER SIX: BEAM DYNAMICS IN $\mathcal{PT}$-LATTICES

6.1 Introduction

Over the last few years a new concept has been proposed in an attempt to extend the framework of quantum mechanics into the complex domain. In 1998, Carl Bender et al have found [1], that it is in fact possible even for non-Hermitian Hamiltonians to exhibit entirely real eigenvalue spectra as long as they respect parity-time requirements or $\mathcal{PT}$ symmetry [2-4]. This fascinating result appears to be counter-intuitive since it implies that all the eigenmodes of a pseudo-Hermitian Hamiltonian [5] (bound as well as radiation states) are only associated with real eigenenergies. Another intriguing characteristic is related to spontaneous $\mathcal{PT}$ symmetry-breaking beyond which this class of systems can undergo an abrupt phase transition [1]. In particular, above this critical threshold, the system loses its $\mathcal{PT}$ property and as a result some of the eigenvalues become complex. The notion of $\mathcal{PT}$ symmetry is now extensively considered in diverse areas of physics including for example, quantum field theories [2], non-Hermitian Anderson models, complex Lie algebras, and lattice QCD theories just to mention a few [6]. It is worth mentioning, that, even before the $\mathcal{PT}$ concept was introduced, wave scattering from complex periodic potentials has been considered at both the theoretical [7] and experimental front [8].
In general, a Hamiltonian is $\mathcal{PT}$ symmetric provided that all its eigenfunctions are simultaneously eigenfunctions of $\mathcal{PT}$ operator [2]. Here the action of the parity operator $\hat{P}$ is defined by the relations $\hat{p} \to -\hat{p}$, $\hat{x} \to -\hat{x}$ while that of the time operator $\hat{T}$ by $\hat{p} \to -\hat{p}$, $\hat{x} \to \hat{x}$, $i \to -i$, where $\hat{p}$, $\hat{x}$ denote momentum and position operators, respectively. In operator form, the normalized Schrödinger evolution equation ($\hbar = m = 1$) is given by $i\frac{\partial \Psi}{\partial t} = \hat{H} \Psi$, where $\hat{H} = \hat{p}^2 / 2 + V(\hat{x})$ and $\hat{p} \to -i\partial / \partial \hat{x}$ [9]. Given that the $\hat{T}$ operation corresponds to a time reversal, i.e., $\hat{T}\hat{H} = \hat{p}^2 / 2 + V^*(\hat{x})$, then one can deduce that $\hat{H}\hat{P}\hat{T} = \hat{p}^2 / 2 + V(\hat{x})$ and $\hat{P}\hat{T}\hat{H} = \hat{p}^2 / 2 + V^*(-\hat{x})$. From the above considerations one finds that a necessary condition for a Hamiltonian to be $\mathcal{PT}$ symmetric is $V(x) = V^*(-x)$. This last relation indicates that parity-time symmetry requires that the real part of the complex potential involved must be an even function of position whereas the imaginary component should be odd.

While the implications of $\mathcal{PT}$ symmetry in the above mentioned fields are still under consideration, as we will show some of these basic concepts can be realized in optics. This can be achieved through a judicious design that involves a combination of optical gain/loss regions and the process of index guiding. Of particular importance is to explore the properties of periodic $\mathcal{PT}$ symmetric lattices as this may lead to pseudo-Hermitian synthetic materials. Quite recently, conventional optical array structures (based on real potentials) have received considerable attention and have been examined in several systems including semiconductors, glasses, quadratic and photorefractive materials and liquid crystals [10]. Given that even a single $\mathcal{PT}$ cell can exhibit unconventional features, one may naturally ask what new behavior and properties could be expected from parity-time symmetric optical lattices.
In this chapter we investigate optical beam dynamics in complex \( \mathcal{PT} \) arrays. The unusual band structure properties of these periodic systems is systematically examined in both one and two-dimensional geometries. We find that above the phase-transition point, bands can merge forming loops or closed ovals (attached to a 2D membrane) within the Brillouin zone and the Floquet-Bloch (FB) modes are substantially altered. Our analysis indicates that under wide beam excitation, interesting diffraction patterns emerge such as “double refraction” and power oscillations due to eigenfunction unfolding. We show that this dynamics is a direct outcome of mode skewness or non-orthogonality. The non-reciprocal characteristics of these \( \mathcal{PT} \) arrays are also discussed.

6.2 Bandstructure of a \( \mathcal{PT} \)-lattice

In optics, several classical processes are known to obey a Schrödinger-like equation. Perhaps the most widely known physical effects associated with this evolution equation are those of spatial diffraction and temporal dispersion [11]. Here we will primarily explore the diffraction dynamics of optical beams and waves in \( \mathcal{PT} \) symmetric potentials in the spatial domain. Along these lines, let us consider a complex parity-time potential. In this case, the complex refractive index of the system is described by \( n = n_0 + n_R(x) + i n_I(x) \), where \( n_0 \) is the background refractive index, \( n_R(x) \) is the real index profile of the lattice and \( n_I(x) \) represents the gain/loss periodic distribution of the structure (in practice \( n_0 >> n_{R,I}(x) \)). Under these conditions, the electric field envelope \( U \) of the beam obeys the paraxial equation of diffraction:

\[
\begin{align*}
    i U_z + (2 k_0 n_0)^{-1} U_{xx} + k_0 \left[ n_R(x) + i n_I(x) \right] U &= 0,
\end{align*}
\]

where \( z \) is the propagation distance, \( x \) is the
transverse coordinate, and \( k_0 = 2\pi / \lambda_0 \) with \( \lambda_0 \) being the light wavelength. We note that this latter equation is formally analogous to the Schrödinger equation. In an array arrangement \( n_{r,l}(x + D) = n_{r,l}(x) \) where \( D \) represents the lattice period. From our previous discussion, this complex potential is \( \mathcal{PT} \) symmetric provided that its real part or refractive index profile is even, i.e. \( n_R(x) = n_R(-x) \), while the imaginary component \( n_I(x) \) (that is loss or gain) is odd. From a physical perspective such \( \mathcal{PT} \) symmetric lattices can be realized in the visible and in the long wavelength regime \( (0.5\mu m < \lambda_0 < 1.6\mu m) \) using a periodic index modulation of the order of \( \Delta n_R^{\text{max}} \approx 10^{-3} \) with \( D \approx 10 - 20\mu m \) (similar to those encountered in real arrays [10]) provided that the maximum gain/loss values are approximately \( g = -\alpha \approx 30\text{cm}^{-1} \) or \( \Delta n_I^{\text{max}} \approx 5 \times 10^{-4} \). Such gain/loss coefficients can be realistically obtained from quantum well lasers or photorefractive structures through two-wave mixing [11]. By introducing the following scaled quantities, \( \xi = z/(2k_0^2) \), \( \eta = x/x_0 \), \( V(\eta) = 2k_0^2n_0x_0^2(n_R + in_I) \), (where \( x_0 \) is an arbitrary scaling factor) the normalized equation of diffraction can now be expressed in the form:

\[
i \frac{\partial U}{\partial \xi} + \frac{\partial^2 U}{\partial \eta^2} + V(\eta)U = 0 \quad . \tag{6.1}
\]

To understand the properties of a periodic \( \mathcal{PT} \) structure we must first analyze its corresponding band-structure. In particular we seek solutions of the form \( \phi_n(\eta)\exp(i\beta_n\xi) \), where \( \phi_n(\eta) \) is the \( n \)-band Floquet-Bloch mode at Bloch momentum \( k \), and \( \beta_n \) is the associated eigenvalue or propagation constant. For illustration purposes we assume the periodic \( \mathcal{PT} \) potential \( V(\eta) = A\left[\cos^2(\eta) + iV_0\sin(2\eta)\right] \), \( (A = 4) \) with period \( D = \pi x_0 \) for both real and imaginary component (shown schematically in Figure 6.1(a)).
Figure 6.1: (a) Real part (solid line) and imaginary component (dotted line) of the $\mathcal{PT}$ potential $V(\eta) = 4\left[\cos^2(\eta) + iV_0 \sin(2\eta)\right]$, (b) corresponding bandstructure for $V_0 = 0.2$ (dotted line), and $V_0 = 0.5$ (solid line), (c), (d) real and imaginary part of the double valued band for $V_0 = 0.7$, respectively, resulting from the merging of the two first bands.

We stress that the requirement $V(\eta) = V^*(-\eta)$ satisfied by this potential is a necessary but not a sufficient condition for the eigenvalue spectrum to be real. By using spectral techniques we numerically identify the $\mathcal{PT}$ threshold ($V_0^{th}$), below which all the propagation eigenvalues for every band and every Bloch wavenumber $k$ are real. Above this $\mathcal{PT}$ threshold, an abrupt phase transition occurs because of spontaneous symmetry breaking and as a result the spectrum is
partially complex. This happens in spite of the fact that $V(\eta)=V^*(-\eta)$ is still satisfied. For the particular potential considered here we find that $V_0^{th} = 0.5$. More specifically for $V_0 < 0.5$, the band-structure is entirely real while for $V_0 > 0.5$ becomes complex (starting from the lowest bands). Figure 6.1(b) depicts the first two bands of this potential for two cases, i.e. when $V_0 = 0.2$ and $0.5$. Note that below $V_0^{th}$ all the forbidden gaps are open whereas at the threshold $V_0^{th} = 0.5$ all the band gaps at the edges of the Brillouin zone close (no gaps exist at $k = \pm 1$) as shown in Figure 6.1(b). On the other hand, when $V_0$ exceeds this critical value these two same bands start to merge together and in doing so they form oval-like structures with a related complex spectrum. The real as well as the imaginary parts of such a double-valued band when $V_0 = 0.7$ are depicted in Figures 6.1(c) and Figure 6.1(d), respectively. These figures show that the propagation eigenvalues are entirely real in the double valued regions (oval R-regions) while along the overlapped sections (C-lines) happen to be complex conjugate. Some of these aspects associated with the real part of these bands were also discussed by Bender et al [12] for pseudo-hermitian periodic potentials having zero $\mathcal{PT}$ threshold (purely imaginary potentials with $V_0^{th} = 0$).

Relevant to our previous discussion is the structure and properties of the corresponding Floquet-Bloch modes for $\mathcal{PT}$ symmetric potentials. Unlike real potentials, the eigenfunctions have no zero nodes at $k = \pm 1$ (edge of the Brillouin zone) [12]. In addition, at $k = \pm 1$ in the complex conjugate part, these functions are shifted with respect to their potentials. We emphasize that the above unexpected modal structure is a direct consequence of the non-orthogonality of the related Floquet-Bloch functions. In particular, the usual orthogonality
condition \( \int_{-\infty}^{\infty} \phi_{m}^{*}(\eta) \phi_{n}(\eta) d\eta = \delta_{mn}\delta(k-k') \) (that holds in real crystals) is no longer applicable in \( \mathcal{PT} \) symmetric lattices. This skewness of the modes [13] is an inherent characteristic of \( \mathcal{PT} \) symmetric periodic potentials and has a profound effect on their algebra.

These effects are can also be considered in two-dimensional configurations provided that the optical potential satisfies \( V(\eta,\zeta) = V^\ast(-\eta,-\zeta) \) in the wave equation \( iU_{\zeta} + U_{\eta\eta} + U_{\zeta\zeta} + V(\eta,\zeta)U = 0 \). In the following examples we consider the complex \( \mathcal{PT} \) symmetric potential \( V(\eta,\zeta) = A\left[ \cos^2(\eta) + \cos^2(\zeta) + iV_0\left[ \sin(2\eta) + \sin(2\zeta) \right] \right] \), with \( A = 4 \).

Numerical analysis reveals that the threshold in this separable 2D case is again \( V_0^{th} = 0.5 \). The real part of the band structure corresponding to this potential is shown in Figures 6.2 for two cases, below and above threshold \( (V_0 = 0.45, \text{ and } V_0 = 0.6) \). Again below threshold the eigenvalue spectrum is real while at \( V_0^{th} = 0.5 \) the two bands collide at their M points at the edges of the Brillouin zone, Figure 6.2(a). On the other hand, above the phase transition point \( (V_0 = 0.6) \) the first two bands merge thus forming a two-dimensional oval double-valued surface (upon which all the propagation constants are real) attached to a 2D membrane where the complex conjugate eigenvalues reside (see Figure 6.2(b)).
Figure 6.2: 2D-bandstructures associated with $V(\eta, \zeta), A = 4$ and (a) $V_0 = 0.45$, and (b) $V_0 = 0.6$.

A process analog to double refraction takes place in such 2D pseudo-Hermitian structures ($V_0 = 0.45$) as shown in Figure 6.3(a) when the system is excited by a normally incident wide 2D Gaussian beam. As opposed to the familiar 2D discrete diffraction pattern occurring in real lattices (Figure 6.3(b) with $V_0 = 0$), in the $\mathcal{PT}$ case, two significant secondary lobes are produced only in the first quadrant. This is of course another manifestation of parity-time symmetry.

Figure 6.3: Output intensity profiles: (a) for the 2D $\mathcal{PT}$ potential $V(\eta, \zeta)$ with $V_0 = 0.45$, and (b) for the corresponding real lattice $V_0 = 0$. 

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6.3 Inner product for $PT$-lattices

In standard Hermitian optics of real potentials $(V(\eta)\in R)$, the inner product of two complex-valued functions is defined in the most general case as $\langle f, g \rangle = \int_{-\infty}^{+\infty} f^*(\eta)g(\eta)d\eta$. With respect to this inner product the orthogonality relation between two different FB modes $\phi_{kn}(\eta), \phi_{km}(\eta)$ of a periodic potential is $\int_{-\infty}^{+\infty} \phi_{kn}^*(\eta)\phi_{kn}(\eta)d\eta = \delta_{nm}\delta(k-k')$. The question that naturally arises is what the corresponding relationship is in a complex $PT$-symmetric lattice. In order to answer this question we have first to find the conjugate pairs that are associated with the corresponding Lagrangian density. More specifically, by substituting a FB mode profile $\phi_{kn}(\eta) = \exp[i\beta_n(k)\eta]$ in Eq. (6.1) we get the following linear eigenvalue problem:

$$-\beta_n(k)\phi_{kn} + \phi_{kn}^* + V(\eta)\phi_{kn} = 0 \Rightarrow \phi_{kn}^* + V(\eta)\phi_{kn} = \beta_n(k)\phi_{kn}$$

(6.2)

The Lagrangian density associated with the above equation is

$$L_1 = \frac{\partial}{\partial \eta}[\partial_\eta \phi_{kn}(\eta)]\left[\partial_\eta \phi_{kn}^*(-\eta)\right] - V(\eta)\phi_{kn}(\eta)\phi_{kn}^*(-\eta)$$

(6.3),

where $\partial_\eta \phi(\eta) = \frac{\partial \phi(\eta)}{\partial \eta}$. It is straightforward to see that the Euler-Lagrange equations

$$\frac{\partial}{\partial \eta}\left(\frac{\partial L_1}{\partial \phi_{kn}(\eta)}\right) - \frac{\partial L_1}{\partial \phi_{kn}(\eta)} = 0$$

both lead to the Eq. (6.2), under the condition of $PT$-symmetry $V(\eta) = V^*(-\eta)$. As a result the conjugate pairs of the
Eq. (6.2) are \( \phi_{kn}(\eta), \phi_{kn}^*(-\eta) \) and the corresponding inner product \( \langle f, g \rangle \equiv \int_{-\infty}^{\infty} f^*(-\eta) g(\eta) d\eta \).

Since the Floquet-Bloch theorem is valid for all periodic potentials (real and complex) every FB mode can be written as \( \phi_{kn}(\eta) = f_{kn}(\eta) \exp[ik\eta] \), where \( f_{kn}(\eta) = f_{kn}(\eta + \tilde{D}) \), where \( \tilde{D} \) represents the normalized period of the complex PT-symmetric periodic potential \( V(\eta) \). Therefore the Eq. (6.2) becomes now:

\[
f_{kn}'' + 2ikf_{kn}' + \left[ V(\eta) - k^2 \right] f_{kn} = \beta_n(k) f_{kn} \quad (6.4),
\]

with the following Lagrangian density

\[
L_2 = \left[ \frac{\partial}{\partial \eta} f_{kn}(\eta) \right] \left[ \frac{\partial}{\partial \eta} f_{-kn}^*(\eta) \right] - \left[ V(\eta) - k^2 \right] f_{kn}(\eta) f_{-kn}^*(\eta) + 2ikf_{kn}(\eta) \left[ \frac{\partial}{\partial \eta} f_{-kn}^*(\eta) \right]
\]

\[ (6.5) \]

Apparently the conjugate pairs of Eq. (6.5) are now different \( \phi_{kn}(\eta), \phi_{-kn}^*(-\eta) \) and the corresponding inner product \( \langle f_k, g_k \rangle \equiv \int_{-\infty}^{\infty} f_k^*(-\eta) g_k(\eta) d\eta \). For the conjugate variable now we don’t have only to invert the spatial coordinate \( x \), but the Bloch wavenumber \( k \) as well. Since the inner product is different now, the orthogonality conditions in a lattice will be also different and must be systematically derived. In order to do this we must start from the orthogonality in a single cell, then consider a finite lattice, and at the end deal with the infinite lattice. In all these three cases we refer always to the eigenvalues problem of Eq. (6.4).
6.4 Orthogonality in one cell

The goal of this paragraph is to derive the orthogonality condition in one individual cell of the periodic potential. This central cell is defined for \( x \in [-\bar{D}/2, \bar{D}/2] \). Let us consider two FB modes \( f_{kn}, f_{km} \) of different bands but of the same wavenumber \( k \). We also assume that the \( V(\eta) \) potential is below the phase transition point, and therefore the eigenvalue spectrum is entirely real. Then from Eq. (6.4) we get the following two equations:

\[
\begin{align*}
 f_{kn}'' + 2ikf_{kn}' + \left[V(\eta) - k^2\right] f_{kn} &= \beta_n(k) f_{kn} \tag{1} \\
 f_{km}'' - 2ikf_{km}' + \left[V(-\eta) - k^2\right] f_{km} &= \beta_m(-k) f_{km} \tag{2}
\end{align*}
\]

By multiplying the first one with \( f_{km} \) and the second one with \( f_{kn} \) and take into accounts that \( V(\eta) = V^*(-\eta) \) and \( \beta_m(k) = \beta_m(-k) \) (symmetric bandstructure), we have:

\[
\begin{align*}
 f_{kn}'' f_{km}' - 2ikf_{kn}' f_{km}' + \left[V(\eta) - k^2\right] f_{kn} f_{km}' &= \beta_n(k) f_{kn} f_{km}' \\
 f_{km}'' f_{kn}' - 2ikf_{km}' f_{kn}' + \left[V(-\eta) - k^2\right] f_{km} f_{kn}' &= \beta_m(-k) f_{km} f_{kn}'
\end{align*}
\]

By subtracting them and taking the integral over the whole cell, we get:

\[
\begin{align*}
 \left[\beta_n(k) - \beta_m(k)\right] \int_{-\bar{D}/2}^{\bar{D}/2} f_{kn} f_{km}' \, d\eta &= \int_{-\bar{D}/2}^{\bar{D}/2} \left[ f_{kn}'' f_{km}' - f_{km}'' f_{kn}' \right] \, d\eta + 4ik \int_{-\bar{D}/2}^{\bar{D}/2} \left[ f_{kn}' f_{km}' + f_{km}' f_{kn}' \right] \, d\eta \\
 \text{but} \quad \int_{-\bar{D}/2}^{\bar{D}/2} \left[ f_{kn}'' f_{km}' - f_{km}'' f_{kn}' \right] \, d\eta &= \left[ f_{kn}' f_{km}' - f_{km}' f_{kn}' \right]_{-\bar{D}/2}^{\bar{D}/2} = 0, \text{ and }
\end{align*}
\]

\[
\begin{align*}
 \int_{-\bar{D}/2}^{\bar{D}/2} \left[ f_{kn}' f_{km}' + f_{km}' f_{kn}' \right] \, d\eta &= \left[ f_{kn} f_{km} \right]_{-\bar{D}/2}^{\bar{D}/2} = 0, \text{ where we used the periodicity of } f_{kn}, \text{ which implies that } f_{kn}(-\bar{D}/2) = f_{kn}(\bar{D}/2). \text{ Therefore } \left[\beta_n(k) - \beta_m(k)\right] \int_{-\bar{D}/2}^{\bar{D}/2} f_{km}' f_{kn} \, d\eta = 0 \text{ and since }
\end{align*}
\]

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there is no degeneracy $\beta_n(k) \neq \beta_m(k)$, we finally get the orthogonality condition in one single cell of a periodic potential, which is:

$$\int_{\text{cell}} f^*_{-k_m}(-\eta) f_{k_n}(\eta) d\eta = 0 \quad (6.6)$$

The next step is the normalization of the FB modes with respect to their norm. More specifically, we have:

$$\Phi_{k_n}(\eta) \equiv \phi_{k_n}(\eta) / \sqrt{c_{k_n}}$$

where

$$c_{k_n} = \int_{-D/2}^{D/2} \phi^*_{-k_n}(-\eta) \phi_{k_n}(\eta) d\eta.$$

It is easy to see that every FB function has a unique normalization coefficient $c_{k_n}$ depending on the band index $n$ and the Bloch wavenumber $k$. In general $c_{k_n}$ are complex numbers and satisfy the symmetry relation $c_{k_n} = c^*_{-k_n}$. Based on this normalization, the Eq. (6.6) leads to the final orthonormality condition, which is:

$$\int_{\text{cell}} \Phi^*_{-k_m}(-\eta) \Phi_{k_n}(\eta) d\eta = d_{k_n} \delta_{n,m} \quad (6.7)$$

where

$$d_{k_n} = \begin{cases} 1, & \text{when } c_{k_n} \in C \text{ or } c_{k_n} > 0 \\ -1, & \text{when } c_{k_n} < 0 \end{cases},$$

and $\delta_{n,m}$ is the Kronecker delta. It has been numerically checked that $c_{k_n} \neq 0$. In other words no self-orthogonal FB modes exist below the $PT$-phase transition point [5]. Note that the FB modes of a $PT$-lattice under the conventional inner product are not orthogonal but skew. Therefore a new orthogonal basis is necessary in order to analyze the energy distribution, among the transmission bands, of an optical beam.

### 6.5 Orthogonality in a finite lattice

Let us consider a finite lattice with $N$ even number of waveguide cells. In particular we have $N'$ cells from the left, $N'$ cells from the right and one central cell. Thus $2N' + 1 = N$. We
denote the length of the lattice with \( L \) and it is \( L = N\tilde{D} \). The purpose of this paragraph is to reduce the calculation of the involved inner product to just one cell, instead of the whole finite lattice. As we did in chapter 5, firstly we split the integral in every individual cell:

\[
\int \Phi_{-k'm}^*(-\eta)\Phi_{kn}(\eta)\,d\eta = \int u_{-k'm}^*(-\eta)u_{kn}(\eta)e^{i(k-k')\eta}\,d\eta = 
\]

\[
\int_{-(N'-1)\tilde{D}-\tilde{D}/2}^{-(N'-1)\tilde{D}} + \ldots + \int_{-\tilde{D}/2}^{\tilde{D}/2} + \int_{-\tilde{D}/2}^{\tilde{D}/2} + \cdots + \int_{(N'-1)\tilde{D}+\tilde{D}/2}^{(N'-1)\tilde{D}}\left(\int_{-\tilde{D}/2}^{\tilde{D}/2}\int_{-\tilde{D}/2}^{\tilde{D}/2}u_{-k'm}^*(-\eta)u_{kn}(\eta)e^{i\Delta k s}e^{-i\Delta k N\tilde{D}}ds + \ldots + \int_{-\tilde{D}/2}^{\tilde{D}/2}\int_{-\tilde{D}/2}^{\tilde{D}/2}u_{-k'm}^*(-\eta)u_{kn}(\eta)e^{i\Delta k s}d\eta + \right)
\]

\[
\Delta k \equiv k - k' \quad \text{and according to the normalization of the 6.4 paragraph it is} \quad \Phi_{kn}(\eta) = u_{kn}(\eta)e^{i\eta k} \quad \text{. Now we are going to change variables in every integral according to}\ 
\eta = s - j\tilde{D}, \quad \text{with} \quad j = \pm 1, \pm 2, \ldots, \pm N'. \text{In that way we reduce every integral to the central cell from} -\tilde{D}/2 \text{ to } \tilde{D}/2. \text{ It is:}
\]

\[
\int \Phi_{-k'm}^*(-\eta)\Phi_{kn}(\eta)\,d\eta = \int_{-\tilde{D}/2}^{\tilde{D}/2} u_{-k'm}^*(-s + N'\tilde{D})u_{kn}(s - N'\tilde{D})e^{i\Delta k s}e^{-i\Delta k N\tilde{D}}ds + \ldots + 
\]

\[
+ \int_{-\tilde{D}/2}^{\tilde{D}/2} u_{-k'm}^*(-s + \tilde{D})u_{kn}(s - \tilde{D})e^{i\Delta k s}e^{-i\Delta k \tilde{D}}ds + \int_{-\tilde{D}/2}^{\tilde{D}/2} u_{-k'm}^*(-\eta)u_{kn}(\eta)e^{i\Delta k \eta}d\eta + 
\]

\[
+ \int_{-\tilde{D}/2}^{\tilde{D}/2} u_{-k'm}^*(-s - \tilde{D})u_{kn}(s + \tilde{D})e^{i\Delta k s}e^{i\Delta k \tilde{D}}ds + \ldots + 
\]

\[
+ \int_{-\tilde{D}/2}^{\tilde{D}/2} u_{-k'm}^*(-s - N'\tilde{D})u_{kn}(s + N'\tilde{D})e^{i\Delta k s}e^{i\Delta k N\tilde{D}}ds
\]

Given the fact that \( \phi_{kn}(\eta) \) is periodic with period \( \tilde{D} \), we get the following result:

\[
\int \Phi_{-k'm}^*(-\eta)\Phi_{kn}(\eta)\,d\eta = \int_{-\tilde{D}/2}^{\tilde{D}/2} u_{-k'm}^*(-\eta)u_{kn}(\eta)e^{i\Delta k \eta}d\eta \left[ \sum_{j=-N'}^{N'} e^{i\Delta k j\tilde{D}} \right], \quad \text{and since the sum of the geometric series is the Dirichlet kernel of} \ \Delta k \tilde{D}, \quad \text{we get:}
\]
\[
\int \Phi^*_{-k'm}(-\eta) \Phi_{k'n}(\eta) \, d\eta = D_{N'}(\Delta k\tilde{D}) \cdot \int \Phi^*_{-k'm}(-\eta) \Phi_{k'n}(\eta) \, d\eta , \quad (6.8)
\]

It is known (see Appendix) that the closed form of the Dirichlet kernel is:

\[
D_{N'}(\Delta k\tilde{D}) = \begin{cases} 
\sin \left[ \left( N' + \frac{1}{2} \right) \Delta k\tilde{D} \right] & \text{when } \Delta k\tilde{D} \neq 0, 2\pi, 4\pi, ... \text{ We apply periodic boundary conditions at the end points of the lattice so the Bloch wavenumber takes discrete values only as far as it always belong to the first Brillouin zone } k \in [-\pi/\tilde{D}, \pi/\tilde{D}) . \text{ In fact:} \\
2N' + 1, \text{ otherwise} 
\end{cases}
\]

\[
\Phi_{k,n}(-L/2) = \Phi_{k,n}(L/2) \Rightarrow \exp[ikL] = 1 \Rightarrow k_j = \frac{2\pi j}{L}, j = 0, \pm 1, \pm 2, \ldots \text{ It is easy to see now that} \\
\sin \left[ \left( N' + \frac{1}{2} \right) \Delta k\tilde{D} \right] = 0 \text{ and by combining the last expression } D_{N'}(\Delta k\tilde{D}) \text{ with Eq. (6.7) and Eq. (6.8), we get the orthonormality condition between FB modes in a finite } \mathcal{PT}-\text{lattice with periodic boundary conditions:} \\
\int \Phi^*_{-k'm}(-\eta) \Phi_{k'n}(\eta) \, d\eta = N \cdot d_{k,n} \cdot \delta_{n,m} \cdot \delta_{k,k'} \quad (6.9)
\]

**6.6 Orthogonality in an infinite lattice**

In order to derive the orthogonality condition in the infinite lattice we have to evaluate the limit of Eq. (6.8) as \( N' \to \infty \). Thus

\[
\int_{-\infty}^{+\infty} \Phi^*_{-k'm}(-\eta) \Phi_{k'n}(\eta) \, d\eta = \lim_{N' \to \infty} \left[ D_{N'}(\Delta k\tilde{D}) \right] \cdot \int_{-\tilde{D}/2}^{+\tilde{D}/2} \Phi^*_{-k'm}(-\eta) \Phi_{k'n}(\eta) \, d\eta . \text{ But we know from Fourier analysis (see Appendix) that the limit of the Dirichlet kernel is a series (comb) of equally }
\]
spaced Dirac delta functions \( \lim_{N \to \infty} \{D_N(x)\} = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n) \) as a result of the Poisson summation formula. Therefore the inner product over the whole infinite lattice becomes:

\[
\int_{-\infty}^{+\infty} \Phi_{k,m}^*(-\eta) \Phi_{k,n}(\eta) d\eta = 2\pi \sum_{n=-\infty}^{+\infty} \delta(\Delta kD - 2\pi n) \cdot \int_{-\tilde{D}/2}^{\tilde{D}/2} \Phi_{k,m}^*(-\eta) \Phi_{k,n}(\eta) d\eta \quad (6.10).
\]

Since we restrict the values of the Bloch wavenumber only in the first Brillouin zone (reduced zone scheme) \( k \in \left[-\frac{\pi}{D}, \frac{\pi}{D}\right] \). Therefore \((-\pi/D \leq k < \pi/D) \Rightarrow (-2\pi < \Delta kD < 2\pi)\), and from the comb series of Eq. (6.10) only the central term (for \( n = 0 \)) survives. This means that:

\[
\int_{-\infty}^{+\infty} \Phi_{k,m}^*(-\eta) \Phi_{k,n}(\eta) d\eta = 2\pi \delta(\Delta kD) \cdot \int_{-\tilde{D}/2}^{\tilde{D}/2} \Phi_{k,m}^*(-\eta) \Phi_{k,n}(\eta) d\eta , \text{ and since } \delta(ak) = \delta(k)/|a|,
\]

we finally reduced the inner product calculation to one individual cell:

\[
\int_{-\infty}^{+\infty} \Phi_{k,m}^*(-\eta) \Phi_{k,n}(\eta) d\eta = \frac{2\pi}{D} \cdot \left( \int_{-\tilde{D}/2}^{\tilde{D}/2} \Phi_{k,m}^*(-\eta) \Phi_{k,n}(\eta) d\eta \right) \delta(k-k').
\]

By combining this last relation with Eq. (6.7) we arrive at the orthonormality condition in an infinite \(PT\)-lattice:

\[
\int_{-\infty}^{+\infty} \Phi_{k,m}^*(-\eta) \Phi_{k,n}(\eta) d\eta = \frac{2\pi}{D} \cdot d_{k,n} \delta_{n,m} \delta(k-k') \quad (6.11)
\]

### 6.7 Projection and completeness

An arbitrary field profile \( H(\eta) \) can be expressed as a linear superposition in the new orthonormal basis \( \{\Phi_{k,n}(\eta)\} \) and the projection coefficients \( A_n(k) \) of this expansion can be uniquely determined, by applying the orthonormality condition of Eq. (6.11). More specifically
we have the following: \[ H(\eta) = \sum_{n=1}^{\infty} \int_{-\pi/D}^{\pi/D} A_n(k) \Phi_{kn}(\eta) \, dk \] and by multiplying both sides with the corresponding conjugate pair \( \Phi^*_{-k' m}(-\eta) \) we get:

\[
\int_{-\infty}^{+\infty} \Phi^*_{-k' m}(-\eta) H(\eta) \, d\eta = \sum_{n=1}^{\infty} \int_{-\pi/D}^{\pi/D} A_n(k) \left( \int_{-\infty}^{+\infty} \Phi^*_{-k' m}(-\eta) \Phi_{kn}(\eta) \, d\eta \right) \, dk \quad (6.11)
\]

\[
\int_{-\infty}^{+\infty} \Phi^*_{-k' m}(-\eta) H(\eta) \, d\eta = \sum_{n=1}^{\infty} \int_{-\pi/D}^{\pi/D} A_n(k) \left( \frac{2\pi}{D} d_{kn} \delta_{n,m} \delta(k-k') \right) \, dk \quad (6.12)
\]

and by multiplying both sides with the corresponding conjugate pair \( \Phi^*_{-k' m}(-\eta) \) we get:

\[
\int_{-\infty}^{+\infty} \Phi^*_{-k' m}(-\eta) H(\eta) \, d\eta = \sum_{n=1}^{\infty} \int_{-\pi/D}^{\pi/D} A_n(k) \left( \frac{2\pi}{D} d_{kn} \delta_{n,m} \delta(k-k') \right) \, dk \quad (6.11)
\]

\[
\int_{-\infty}^{+\infty} \Phi^*_{-k' m}(-\eta) H(\eta) \, d\eta = \sum_{n=1}^{\infty} \int_{-\pi/D}^{\pi/D} A_n(k) \left( \frac{2\pi}{D} d_{kn} \delta_{n,m} \delta(k-k') \right) \, dk \quad (6.13)
\]

The completeness of the FB mode basis is directly related to the Parseval’s identity, which now takes a completely different form than the corresponding one for a real lattice. In particular it is:

\[
H^*(-\eta) H(\eta) = \left[ \sum_{n=1}^{+\infty} \int_{-\pi/D}^{\pi/D} A_n(k) \Phi_{kn}(\eta) \, dk \right] \cdot \left[ \sum_{m=1}^{+\infty} \int_{-\pi/D}^{\pi/D} A_m(-k') \Phi^*_{-k' m}(-\eta) \, dk' \right] \Rightarrow
\]

\[
\int_{-\infty}^{+\infty} H^*(-\eta) H(\eta) \, d\eta = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \int_{-\pi/D}^{\pi/D} \int_{-\pi/D}^{\pi/D} A^*_m(-k') A_n(k) \left( \int_{-\infty}^{+\infty} \Phi^*_{-k' m}(-\eta) \Phi_{kn}(\eta) \, d\eta \right) \, dk dk' \quad (6.11)
\]

\[
\int_{-\infty}^{+\infty} H^*(-\eta) H(\eta) \, d\eta = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \int_{-\pi/D}^{\pi/D} \int_{-\pi/D}^{\pi/D} A^*_m(-k') A_n(k) \left( \frac{2\pi}{D} d_{kn} \delta_{n,m} \delta(k-k') \right) \, dk dk' \Rightarrow
\]

\[
\int_{-\infty}^{+\infty} H^*(-\eta) H(\eta) \, d\eta = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \int_{-\pi/D}^{\pi/D} \int_{-\pi/D}^{\pi/D} d_{kn} A^*_n(-k) A_n(k) \, dk
\]
It is straightforward to see from Eq. (6.12) that in general $A_n(k) \neq A_n^*(-k)$ and the Parseval’s identity in a $\mathcal{P}\mathcal{T}$-lattice is not in any sense a generalization of that we have in a real lattice. We have numerically checked the validity of the discrete version of Eq. (6.13) in a finite lattice. This indicates that the orthonormal set of FB modes under the new inner product is also complete. This in turn allows us to decompose any beam profile in a linear superposition of FB modes and thus determine the energy content of this beam in every band in the first Brillouin zone.

### 6.8 Diffraction dynamics

The most interesting aspects associated with $\mathcal{P}\mathcal{T}$ symmetric lattices are revealed during dynamic beam evolution.
Figure 6.4: Intensity evolution of a broad optical beam under normal incidence when, (a) $V_0 = 0.49$, (b) $V_0 = 0$. Figure (c) depicts the FB decomposition of the input in (a) for the first three bands (solid black-1$^{st}$, left dashed blue-2$^{nd}$, right dashed red-3$^{rd}$), and the inset shows the corresponding bandstructure. (d) Single channel excitation of this same lattice when $V_0 = 0.49$.

Figure 6.4(a) illustrates the intensity distribution during propagation when the $PT$ array $V(\eta) = A \left[ \cos^2(\eta) + i V_0 \sin(2\eta) \right]$, (with $V_0 = 0.49, A = 4$) is excited by a wide optical beam at normal incidence. Figure 6.4(b) on the other hand shows this same process in the real version of this lattice ($V_0 = 0$) under the same input conditions. These two figures indicate that there is a marked difference between these two regimes. In the $PT$ array the beam splits in two and double refraction occurs at an angle of $\sim 1^\circ$ after 3 cm of propagation when $D = 20 \mu m, g = 35 cm^{-1}, \Delta n_{R}^{\max} = 10^{-3}$. In order to explain this behavior we project the input field on the new orthonormal Floquet-Bloch basis of the complex array, and we calculate the mode occupancy coefficients $A_n(k)$ (see Eq. (6.12)) in every band $n$ and for every Bloch momentum $k$. Figure 6.4(c) depicts the $|A_n(k)|$ occupancy (among bands) corresponding to the input used in Figure 6.4(a). This result clearly shows that this distribution is asymmetric in $k$-space especially in the second and third band while in the first band is almost symmetric. This asymmetry is attributed to the skewness of the FB modes. Keeping in mind that the beam components will propagate along the gradient $\nabla_k(\beta)$, one can then explain from Figure 6.4(c) why the double refraction process occurs towards the right. Intuitively this can be understood given that the $PT$ periodic structure involves gain/loss dipoles, thus promoting energy flow from left to right. Another feature associated with Figure 6.4(a) is power oscillation. Even though this lattice is
operated below the $\mathcal{PT}$ threshold value and hence the entire spectrum is real, what is conserved here is the quasi-power [14], e.g. $Q = \int_{-\infty}^{+\infty} U(\eta, \xi) U^*(\eta, \xi) d\eta$ as opposed to the actual power itself $P = \int_{-\infty}^{+\infty} |U(\eta, \xi)|^2 d\eta$, which oscillates during propagation. These power oscillations are due the unfolding of the non-orthogonal FB modes. This unfolding process becomes even more pronounced under narrow-beam excitation conditions where secondary emissions can be observed during discrete diffraction as shown in Figure 6.4(d).

Another direct consequence of this modal “skewness” is non-reciprocity. Figure 6.5 shows beam propagation in a $\mathcal{PT}$ lattice when excited by a wide beam at $\pm \theta$ angle of incidence (in this case, $2^\circ$ degrees). Note that the two diffraction patterns are different and hence, light propagating in $\mathcal{PT}$ symmetric arrays can distinguish left from right. This is another general property of such pseudo-hermitian optical systems.

Figure 6.5: Intensity evolution of wide beams exciting a $\mathcal{PT}$ lattice at angle $\theta$ when $V_0 = 0.45, A = 4$ and (a) $\theta = 2^\circ$, (b) $\theta = -2^\circ$. 

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CHAPTER SEVEN: OPTICAL SOLITONS IN $\mathcal{PT}$-POTENTIALS

7.1 Introduction

Quantum mechanics demands that every physical observable is associated with a real spectrum and thus must be Hermitian. In the case of the Hamiltonian operator, this physical axiom not only implies real eigen-energies but also guarantees conservation of probability [1]. Yet in recent years, a series of studies by Bender and co-workers has demonstrated that even non-Hermitian Hamiltonians can exhibit entirely real spectra provided they respect parity-time ($\mathcal{PT}$) symmetry [2]. By definition, a Hamiltonian belongs to this latter class as long as it shares a common set of eigenfunctions with the $\hat{P}\hat{T}$ operator. In general the action of the parity operator $\hat{P}$ is defined by the relations $\hat{p} \rightarrow -\hat{p}, \hat{x} \rightarrow -\hat{x}$ ($\hat{p}, \hat{x}$ stand for momentum and position operators, respectively) whereas that of the time operator $\hat{T}$ by $\hat{p} \rightarrow -\hat{p}, \hat{x} \rightarrow \hat{x}, \ i \rightarrow -i$. Given the fact that the action of $\hat{T}$ leads to a time reversal, i.e., $\hat{T}\hat{H} = \hat{p}^2/2 + V^*(x)$, one finds that $\hat{P}\hat{T}\hat{H} = \hat{H}\hat{P}\hat{T} = \hat{p}^2/2 + V^*(-x) = \hat{H}$. From here we conclude that a Hamiltonian is $\mathcal{PT}$ symmetric when the following condition is satisfied $V(x) = V^*(-x)$. Therefore the real part of a $\mathcal{PT}$ complex potential must be an even function of position whereas the imaginary component should be odd. Among the most intriguing characteristics of such a pseudo-Hermitian Hamiltonian, is the existence of a critical threshold above which the system undergoes a sudden phase transition because of spontaneous $\mathcal{PT}$ symmetry breaking. In this regime the spectrum is no longer real but
instead it becomes complex. The relevance of these recent mathematical developments in quantum field theories and other areas of physics, has also been addressed in a number of studies [2-7].

Optics can provide a fertile ground where $\mathcal{PT}$ related concepts can be realized and experimentally tested. In fact, this can be achieved through a judicious inclusion of gain/loss regions in guided wave geometries [8]. Given that the complex refractive index distribution in a structure is $n(x) = n_0(x) + n_g(x) + in_l(x)$, one can deduce that $n(x)$ plays the role of the optical potential (where $x$ represents the normalized transverse coordinate). The parity-time condition implies that the index waveguiding profile $n_g(x)$ should be even in the transverse direction while the loss/gain term $n_l(x)$ must be odd. In fact, gain/loss levels of approximately $\pm 40\text{cm}^{-1}$ at wavelengths of $\approx 1\mu\text{m}$, that are typically encountered in standard quantum well semiconductor lasers or semiconductor optical amplifiers [8], will be sufficient to observe $\mathcal{PT}$ behavior. The imaginary part of the $\mathcal{PT}$ potential in such SOA arrangements can alternate between gain and loss in a diatomic waveguide lattice configuration depending on whether the input current is used above or below lasing threshold. Of interest will be to synthesize periodic systems [9] that can exhibit novel features stemming from parity-time symmetry. Even more importantly, the involvement of optical nonlinearities (quadratic, cubic, photorefractive nonlinearities etc [10]), may allow the study of such configurations under nonlinear conditions.

In this chapter we show that $\mathcal{PT}$ symmetric nonlinear lattices can support soliton solutions. These self-trapped states can be stable over a wide range of parameters in spite of the fact that gain/loss regions are present in this system. We first consider the propagation dynamics of nonlinear beams in a single $\mathcal{PT}$ waveguide cell and then we examine their behavior in a $\mathcal{PT}$
symmetric optical lattice. Both 1-D and 2-D soliton solutions are presented along with their associated transverse power-flow density. Our analysis sheds light for the first time on the interplay between nonlinearity and parity-time symmetry. Interestingly enough, even in the presence of relatively strong gain/loss effects, stationary self-trapped states (single cell and lattice) can exist with real propagation eigenvalues. This is a direct outcome of the $PT$ symmetric nature of the potentials involved. It is important to stress that our results are fundamentally different from those previously obtained within the context of complex Ginzburg-Landau (GL) systems [11].

### 7.2 Linear stability analysis

We begin our analysis by considering optical wave propagation in a self-focusing Kerr nonlinear $PT$ symmetric potential. In this case, the beam evolution is governed by the following normalized nonlinear Schrödinger-like equation,

$$i \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} + \left[ V(x) + iW(x) \right] \psi + |\psi|^2 \psi = 0 \quad (7.1)$$

where $\psi$ is proportional to the electric field envelope and $z$ is a scaled propagation distance. Based on the previous discussion, the real and the imaginary components of the $PT$ symmetric potential satisfy the following relations $V(-x) = V(x), W(-x) = -W(x)$, respectively. Physically, $V(x)$ is associated with index guiding while $W(x)$ represents the gain/loss distribution of the optical potential. Note that in the linear regime, Eq. (7.1) conserves the “quasi-power” $Q(z) = \int_{-\infty}^{\infty} \psi(x,z)\psi^*(x,z)dx$ as opposed to the actual electromagnetic
power, \( P(z) = \int_{-\infty}^{\infty} |\psi(x, z)|^2 \, dx \) [12]. In the nonlinear domain however, these quantities evolve according to:

\[
\begin{align*}
\int_{-\infty}^{\infty} [i \frac{dQ}{dz} + \int_{-\infty}^{\infty} \psi(x, z) \psi^* (-x, z) \left[ |\psi(x, z)|^2 - |\psi(-x, z)|^2 \right] \, dx = 0 \\
\int_{-\infty}^{\infty} \frac{dP}{dz} + 2 \int_{-\infty}^{\infty} W(x) |\psi(x, z)|^2 \, dx = 0.
\end{align*}
\]

Stationary soliton solutions to Eq. (7.1) are sought in the form \( \psi(x, z) = \phi(x) \exp(i\lambda z) \) where \( \phi(x) \) is the nonlinear eigenmode and \( \lambda \) is the corresponding real propagation constant. In this case \( \phi \) satisfies:

\[
\frac{d^2 \phi}{dx^2} + [V(x) + iW(x)] \phi + |\phi|^2 \phi = \lambda \phi. \tag{7.2}
\]

In order to determine the linear stability properties of such self-trapped localized modes, we consider small perturbations on the solutions of Eq. (7.1) of the form [13],

\[
\psi(x, z) = \phi(x) e^{i\lambda z} + \epsilon \left[ F(x) e^{i\sigma z} + G^*(x) e^{-i\sigma z} \right] e^{i\lambda z} \tag{7.3}
\]

where \( \epsilon \ll 1 \). Here, \( F \) and \( G \) are the perturbation eigenfunctions and \( \sigma \) indicates the growth rate of the perturbation. By linearizing Eq. (7.1) around the localized solution \( \phi(x) \) we obtain the following linear eigenvalue problem for the perturbation modes

\[
\begin{pmatrix}
\hat{L} & \phi^2 \\
-\phi^* & -\hat{L}^*
\end{pmatrix}
\begin{pmatrix} F \\ G \end{pmatrix} = \sigma
\begin{pmatrix} F \\ G \end{pmatrix} \tag{7.4}
\]

where \( \hat{L} = \frac{d^2}{dx^2} + V(x) + iW(x) + 2|\phi|^2 - \lambda \). Evidently, the \( PT \) nonlinear modes are linearly unstable if \( \sigma \) has an imaginary component; while they are stable if \( \sigma \) is real.
7.3 Solitons in $\mathcal{PT}$-potentials

Before we consider light self-trapping in complex lattices, it is important to first understand nonlinear optical beam dynamics in a single $\mathcal{PT}$ complex potential. For illustration purposes, we assume a Scarff II potential, e.g.:

$$V(x) = V_0 \sec h^2(x), \quad W(x) = W_0 \sec h(x) \tanh(x), \quad (7.5)$$

with $V_0$ and $W_0$ being the amplitudes of the real and imaginary part. Notice that the corresponding linear problem associated with the potential of Eq. (7.5) exhibits an entirely real spectrum provided that, $W_0 \leq V_0 + 1/4$ [14]. Thus for a fixed value of $V_0$, there exists a threshold for the imaginary amplitude $W_0$. Above this so-called $\mathcal{PT}$ threshold, a phase transition occurs and the spectrum enters the complex domain. Interestingly enough, even if the Scarff potential of Eq. (7.5) has crossed the phase transition point (its spectrum is complex), nonlinear states can still be found with real eigenvalues. In other words, the beam itself can alter the amplitude of the refractive index distribution through the optical nonlinearity. Thus for a given $W_0$, this new effective potential nonlinearly shifts the $\mathcal{PT}$ $V_0$ threshold and in turn allows nonlinear eigenmodes with real eigenvalues to exist. In contrast, at lower power levels the parity-time symmetry can not be nonlinearly restored and hence remains broken. A nonlinear mode of this potential corresponding to $\lambda = 0.98$, when $V_0 = 1, W_0 = 0.5$ is shown in Figure 7.1. Equation (7.2) admits an exact solution of the form $\phi = \phi_0 \sec h(x) \exp[i\mu \tan^{-1}(\sinh(x))]$, where $\mu = W_0 / 3$, $\lambda = 1$ and $\phi_0 = \sqrt{2 - V_0 + (W_0^2 / 9)}$. We next examine the stability of these nonlinear modes by numerically solving the corresponding perturbation eigenvalue problem of Eq. (7.4). To support the linear stability results we have checked the robustness of each nonlinear state using beam
propagation methods and by adding random noise on both amplitude and phase. The results of this simulation, shown in Figure 7.1 for $V_0 = 1, W_0 = 0.5$, indicate that the beam is nonlinearily stable.

Figure 7.1: Intensity evolution of a nonlinear mode in a $\mathcal{PT}$ Scarff II potential, when $\lambda = 0.98$.

The inset depicts the real (solid blue curve) and imaginary (dotted red curve) component of such an eigenmode.

To shed more light on the properties of these nonlinear solutions, we examine the quantity $S = (i/2)(\phi \phi_x^* - \phi^* \phi_x)$ associated with the transverse power flow density or Poynting vector across the beam. This energy flow arises from the non-trivial phase structure of these nonlinear modes. For the analytical solution mentioned above we find that $S = (W_0 \phi_0^2 / 3) \sec h^3(\lambda)$. Obviously, $S$ is everywhere positive in this $\mathcal{PT}$ cell, thus implying that the power always flows in one direction, i.e., from the gain toward the loss region.
We next investigate optical solitons and their dynamics in nonlinear periodic $\mathcal{PT}$ potentials. Since the general idea holds for any such complex potential, we here consider for simplicity the case:

$$V(x) = \cos^2(x), \quad W(x) = W_0 \sin(2x).$$

(7.6)

The linear properties of such a periodic potential can be understood by examining the corresponding linear problem of Eq. (7.2), i.e.,

$$\frac{d^2 \phi}{dx^2} + \left[ V(x) + iW(x) \right] \phi = \lambda \phi,$$

where $\lambda$ now represents the propagation constant in the periodic structure. Since the potentials $V(x), W(x)$ of Eq. (7.6) are $\pi$-periodic, the Floquet-Bloch theorem dictates that the eigenfunctions are of the form $\phi = \Phi_k(x) \exp(ikx)$ where $\Phi_k(x + \pi) = \Phi_k(x)$ and $k$ stands for the real Bloch momentum.

We note that in general the band structure of a complex lattice can be complex. Yet, for periodic $\mathcal{PT}$ symmetric potentials, the band diagram can be entirely real as long as the system is operated below the phase transition point (unbroken $\mathcal{PT}$ symmetry). For the particular potential of Eq. (7.6), we find that purely real bands are possible in the range $0 \leq W_0 < 1/2$. In Figure 7.2 we show the associated band structure for various values of the potential parameter $W_0$ (below and above the phase transition point $W_0 = 1/2$).
Figure 7.2: Bandstructure for the $\mathcal{PT}$ potential $V(x) = \cos^2(x) + iW_0 \sin(2x)$, when $W_0 = 0.45$ (dotted line), and $W_0 = 0.6$ (solid line).

We notice that as $W_0$ is increased the band gap becomes narrower and closes completely when crossing the critical transition value $W_0 = 1/2$. Pseudo-Hermitian periodic potentials having zero $\mathcal{PT}$ threshold were also discussed [15].

Having found the band-gap structure, we next obtain soliton solutions to Eq. (7.2) when the complex potential is given by Eq. (7.6). For $W_0 < 1/2$, we numerically construct a family of localized solutions with real eigenvalues located within the semi-infinite “energy” gap. A typical field profile of such a soliton is shown in Figure 7.3 (a).
Figure 7.3: (a) $\mathcal{PT}$ lattice ($W_0 = 0.45$) soliton field profile (real part: blue line, imaginary part: red line) for $\lambda = 0.7$. (b) Stable propagation of a $\mathcal{PT}$ lattice soliton with eigenvalue $\lambda = 1.57$. (c) Transverse power flow (solid line) of the soliton in (a) across the lattice. The dotted line represents the real part of the potential in both (a) and (c).

We next address the stability of these solutions given that these complex structures involve strong loss and gain. In general we found that the instability growth rate tends to increase with $W_0$. In addition, narrower self-trapped waves are more stable since the nonlinearity tends to further enhance the index guiding, thus perturbing the local $\mathcal{PT}$ phase transition point. To further examine the robustness of these $\mathcal{PT}$ lattice self-trapped modes, beam propagation methods were
used. Under linear conditions symmetric diffraction occurs in this periodic complex system. On the other hand, as the power is increased the beam becomes confined and propagates undistorted, thus forming a lattice soliton-in spite of any symmetry breaking perturbations. Figure 7.3(b), shows the propagation dynamics of such a soliton (for $V_0 = 1, W_0 = 0.45, \lambda = 1.57$) as a function of the propagation distance. The transverse power flow is also plotted in Figure 7.3(c). Unlike the single-cell case considered before, the power flow in this case is more involved. As indicated in Figure 7.3(c), the direction of the flow from gain to loss regions varies across the lattice. More specifically, it is positive (from left to right) in the waveguides and becomes negative (from right to left) in the space between channels. This should be physically anticipated since power transport occurs always from gain to loss domains. We would like to emphasize that the distribution of the power flow density in these self-trapped $PT$ states differs from that encountered in Ginzburg-Landau dissipative solitons [11]. More specifically, in GL systems the power flow is an anti-symmetric function of position whereas in $PT$ lattices is even, as clearly indicated in Figure 7.3(c).

Notice that it is also possible to find stationary self-trapped modes with real propagation eigenvalues even above the symmetry breaking point $W_0 = 1/2$, as shown in the inset of Figure 7.4. This is due to the fact that part of the bandstructure still remains real even above the $PT$-threshold (Figure 7.2). This family of solitons exists provided that the Fourier spectrum of these solutions (in Bloch-momentum space) is primarily contained within the region where the band is real ($\lambda$ real) located around the $k = 0$ point. Stability analysis however reveals that this latter class of lattice solitons is in fact unstable. This instability is corroborated by numerical simulations, as shown in Figure 7.4.
Figure 7.4: Intensity evolution of an unstable $PT$ soliton above the phase transition point ($W_0 = 0.6$). The inset depicts the field profile (real part/blue line, imaginary part/red line) of an unstable $PT$-soliton.

7.4 Two dimensional $PT$-solitons

Finally, we discuss the formation of $PT$ lattice solitons in two-dimensional periodic geometries. In this case, Eq. (7.1) becomes $i \frac{\partial \psi}{\partial z} + \nabla^2 \psi + [V + iW] \psi + |\psi|^2 \psi = 0$, where again the potentials $V$ and $W$ obey the $PT$ symmetry requirement, $V(-x,-y) = V(x,y)$ and $W(-x,-y) = -W(x,y)$. In Figure 7.5(a) the bandstructure corresponding to the periodic potentials $V(x,y) = \cos^2(x) + \cos^2(y)$ and $W(x,y) = W_0[\sin(2x) + \sin(2y)]$ is depicted for $W_0 = 0.3$. It is instructive to observe that the symmetry breaking level for this two-
dimensional potential is identical to the one-dimensional case ($W_0 = 0.5$). Above this phase transition point the first two bands merge together forming an oval, a double-valued surface (upon which all the propagation constants are real) attached to a 2D membrane of complex eigenvalues. A two-dimensional $\mathcal{PT}$ symmetric soliton with eigenvalues within the semi-infinite gap is shown in Figure 7.5(b). At low intensities, the nonlinearity is not strong enough and hence this beam asymmetrically diffracts in this complex lattice as shown in Figure 7.5(c). At soliton power levels however, this nonlinear wave propagates in a stable fashion. To further understand the internal structure of these self-trapped states, we plot the transverse power flow vector (Poynting vector) $\mathbf{S} = (i/2) \left[ \phi \nabla \phi^* - \phi^* \nabla \phi \right]$, as shown in Figure 7.5(d), which indicates again energy exchange among gain/loss domains.
Figure 7.5: (a) Bandstructure of a 2D-$PT$ potential when $W_0 = 0.3$. (b) The intensity profile of a $PT$-soliton when the propagation eigenvalues is $\lambda = 1.3$. (c) Linear diffraction pattern under single channel excitation (soliton input with $\lambda = 1.3$), and (d) Transverse power flow of this $PT$-soliton solution within one cell where the dark area of the background represents the waveguide area. The regions where the gain/loss is maximum are indicated by the G, L points, respectively.
References


CHAPTER EIGHT: CONCLUSIONS

In chapter 2, we have shown that optical wave propagation in discrete boundary geometries can be analyzed using the method of images. This was done by introducing fictitious sources outside the region of interest. Analytical solutions in various 1D and 2D lattice topologies have been obtained. These include for example 1D semi-infinite arrays, finite systems, and 2D angular sectors.

In chapter 3, we have shown that discrete surface solitons are possible in waveguide arrays of cubic (AlGaAs) and quadratic (LiNbO$_3$) nonlinearities. These new families of self-trapped states exist only when their power exceeds a critical power threshold. The existence and stability of such surface solitons was systematically investigated by linear stability methods and BPMs. The agreement between theory and experiment was very good in all the cases.

In chapter 4, we have theoretically demonstrated the existence of surface spatial solitons in nonlinear optical lattices. Such surface self-trapped waves can exist at the interface between two different semi-infinite 1D waveguide arrays as well as at the boundaries of 2D optical lattices. Hybrid solitons at the interface between two dissimilar semi-infinite waveguide arrays, were also examined.

In chapter 5, it was theoretically demonstrated that Rabi-type oscillations are possible in $z$-modulated periodic potentials. Such transitions can take place in optical lattices when the channels are periodically modulated along the propagation direction. Energy exchange between
two different FB modes can occur under phase matching conditions. The coupled mode
equations that govern this dynamic process were derived. In the nonlinear domain, oscillations
between two different lattice solitons were also investigated.

In chapter 6, we have demonstrated that $\mathcal{PT}$ symmetric periodic potentials can exhibit
new behavior in optics. Beam dynamics in such structures reveals that double refraction, power
oscillations and secondary emissions are possible. The existence of abrupt phase transitions, as
well as, the associated band-structure of $\mathcal{PT}$ lattices in both one and two geometries was also
examined in detail.

In chapter 7, a new class of one- and two-dimensional nonlinear self-trapped modes
residing in parity-time symmetric wells and lattices was reported. The existence, stability, and
propagation dynamics of such $\mathcal{PT}$-solitons were examined in detail.
APPENDIX:
DIRICHLET KERNELS
Twice so far in this thesis (calculation of the transition matrix element in Chapter 5 and the derivation of orthogonality of FB modes in a $PT$-lattice in Chapter 6), we had to deal with sums of the form $\sum_{m=-N'}^{N'} e^{i m x}$. Quite unexpectedly, this is nothing more than the so called Dirichlet kernel in Fourier analysis. The basic properties of this mathematical object are the scope of this Appendix.

A Dirichlet kernel is, by definition, every function of the following form:

$$D_{N'}(x) \equiv \sum_{m=-N'}^{N'} e^{i m x}, \quad x \in \mathbb{R} \quad (A.1)$$

A closed form expression to this sum can be analytically found [1].

**Proposition 1:**

$$D_{N'}(x) = \begin{cases} \sin \left( \frac{(N' + 1/2) x}{\sin(x/2)} \right), & x \neq 0, \pm 2\pi, \pm 4\pi, \ldots \\ 2N' + 1, & \text{otherwise} \end{cases} \quad (A.2)$$

**Proof:** For the case when $x = 2n\pi$, $n \in \mathbb{Z}$, we have obviously

$$\sum_{m=-N'}^{N'} e^{i (2n\pi) m} = \sum_{m=-N'}^{N'} 1 = 2N' + 1.$$

On the other hand, when $x \neq 2n\pi$, $n \in \mathbb{Z}$ the situation is less obvious. In particular, the sum is

$$\sum_{m=-N'}^{N'} e^{i m x} = \sum_{m=-N'}^{N'} \left[ \cos(mx) + i \sin(mx) \right] = 1 + \sum_{m=-N'}^{N'} \left[ \cos(mx) + i \sin(mx) \right] = 1 + 2 \sum_{m=1}^{N'} \cos(mx).$$

$$2 \cos(mx) \sin \left( \frac{x}{2} \right) = \sin \left( \frac{x}{2} + mx \right) + \sin \left( \frac{x}{2} - mx \right) = \sin \left[ \left( m + \frac{1}{2} \right) x \right] - \sin \left[ \left( m - \frac{1}{2} \right) x \right] \Rightarrow$$

$$\sin \left[ \left( m + \frac{1}{2} \right) x \right] - \sin \left[ \left( m - \frac{1}{2} \right) x \right] = 2 \cos(mx) \sin \left( \frac{x}{2} \right), \quad m = 1, 2, \ldots, N'$$

and we have for every different value of $m$, the following identities:

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\[ m = 1: \quad \sin \left[ (1 + \frac{1}{2}) x \right] - \sin \left[ \frac{x}{2} \right] = 2 \cos(x) \sin \left( \frac{x}{2} \right) \]

\[ m = 2: \quad \sin \left[ (2 + \frac{1}{2}) x \right] - \sin \left[ (1 + \frac{1}{2}) x \right] = 2 \cos(2x) \sin \left( \frac{x}{2} \right) \]

\[ m = 3: \quad \sin \left[ (3 + \frac{1}{2}) x \right] - \sin \left[ (2 + \frac{1}{2}) x \right] = 2 \cos(3x) \sin \left( \frac{x}{2} \right) \]

\[ m = N': \quad \sin \left[ (N' + \frac{1}{2}) x \right] - \sin \left[ (N' - 1 + \frac{1}{2}) x \right] = 2 \cos(N'x) \sin \left( \frac{x}{2} \right), \text{ and by adding them all we get:} \]

\[ \sin \left[ (N' + \frac{1}{2}) x \right] - \sin \left( \frac{x}{2} \right) = 2 \sin \left( \frac{x}{2} \right) \left( \cos(x) + \cos(2x) + \cos(3x) + \ldots + \cos(N'x) \right) \Rightarrow \]

\[
\sin \left[ (N' + \frac{1}{2}) x \right] = \sin \left( \frac{x}{2} \right) + 2 \sin \left( \frac{x}{2} \right) \sum_{m=1}^{N'} \cos(mx) \Rightarrow \\
\sin \left[ (N' + \frac{1}{2}) x \right] = \sin \left( \frac{x}{2} \right) \left[ 1 + 2 \sum_{m=1}^{N'} \cos(mx) \right] \Rightarrow \\
1 + 2 \sum_{m=1}^{N'} \cos(mx) = \frac{\sin \left[ (N' + \frac{1}{2}) x \right]}{\sin \left( \frac{x}{2} \right)}
\]

therefore

\[ D_{N'}(x) = \frac{\sin \left[ (N' + \frac{1}{2}) x \right]}{\sin \left( \frac{x}{2} \right)}, \text{ which completes the proof. It is easy to see from the Le Hospital rule that} \]

\[ \lim_{x \to 2\pi} \frac{\sin \left[ (N' + \frac{1}{2}) x \right]}{\sin \left( \frac{x}{2} \right)} = 2N' + 1, \text{ which means that the function} D_{N'}(x) \text{ is continuous for every} \ x \in \mathbb{R}. \]

A plot of a Dirichlet kernel for \( N' = 10 \) is depicted in the following Figure A.1. The maximum of the plotted function is \( 2N' + 1 = 21 \) and the period equals to \( 2\pi \), as expected from the Eq. (A.1) and Eq. (A.2).
Poisson summation formula: Let’s assume for a function \( f \) that \( \int_{-\infty}^{+\infty} |f(x)| \, dx < \infty \). If the series

\[
\sum_{n=-\infty}^{+\infty} f(x - 2nL)
\]

converges uniformly for \( |x| \leq L \) and \( \frac{1}{2L} \sum_{n=-\infty}^{+\infty} \left| F \left( \frac{n}{2L} \right) \right| \) also converges, then it is

\[
\sum_{n=-\infty}^{+\infty} f(x - 2nL) = \frac{1}{2L} \sum_{n=-\infty}^{+\infty} F \left( \frac{n}{2L} \right) e^{i \pi x x / L},
\]

(A.3)

where \( F(s) \equiv \int_{-L}^{L} f(x) e^{-i 2\pi x x} \, dx \) is the Fourier transform of the function \( f(x) \).

Proof:

For the proof see the reference [2].

A direct consequence of the Poisson summation formula is that the limit of the Dirichlet kernel is a series (comb) of equally spaced Dirac delta functions,

\[
\lim_{N \to \infty} \{D_N(x)\} = 2\pi \sum_{n=-\infty}^{+\infty} \delta(x - 2\pi n). \]

This can be shown as follows:
Proposition 2: \[ \sum_{n=-\infty}^{\infty} e^{inx} = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n), \quad \forall x \in \mathbb{R} \]

Proof:

From Eq. (A.3) we get \[ \sum_{n=-\infty}^{\infty} f(x - 2n\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F\left( \frac{n}{2\pi} \right) e^{inx}, \] when \( L = \pi \). By using the definition of the Fourier transform \( F(s) = \int_{-\pi}^{\pi} f(x) e^{-i2\pi x s} dx \) and the shifting property of Dirac-delta function \( \int_{-\pi}^{\pi} \delta(x-x') f(x') dx' = f(x) \), we get the following:

\[ \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \delta(x - 2n\pi - x') f(x') dx' = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} \int_{-\pi}^{\pi} f(x') e^{-i2\pi \frac{n}{2\pi} x'} dx' \Rightarrow \]

\[ \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \left\{ \delta(x-x') \delta(2n\pi - x') \right\} f(x') dx' = 0 \Rightarrow \]

\[ \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \left\{ \delta(x-x'-2n\pi) - \frac{1}{2\pi} e^{inx} \right\} f(x') dx' = 0 \Rightarrow \]

\[ \int_{-\pi}^{\pi} f(x') \left[ \sum_{n=-\infty}^{\infty} \left\{ \delta(x-x'-2n\pi) - \frac{1}{2\pi} e^{inx} \right\} \right] dx' = 0 \]

and since this relation must be satisfied from every function \( f(x) \), the integrand must be zero:

\[ \sum_{n=1}^{\infty} \left\{ \delta(x-x'-2n\pi) - \frac{1}{2\pi} e^{inx} \right\} = 0 \Rightarrow \]

\[ \sum_{n=1}^{\infty} \left\{ \delta(x-x'-2n\pi) - \frac{1}{2\pi} e^{inx} \right\} = 0 \Rightarrow \]

\[ 2\pi \sum_{n=1}^{\infty} \delta(x-x'-2n\pi) = \sum_{n=1}^{\infty} e^{inx}, \forall x \in \mathbb{R}, \text{and } x' \in [-\pi, \pi], \text{ and for } x' = 0 \text{ we finally arrive at the result:} \]

\[ 2\pi \sum_{n=1}^{\infty} \delta(x-2n\pi) = \sum_{n=1}^{\infty} e^{inx}, \forall x \in \mathbb{R}. \]
References
