The Sheffer B-type 1 Orthogonal Polynomial Sequences

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THE SHEFFER B-TYPE 1 ORTHOGONAL POLYNOMIAL SEQUENCES

by

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ABSTRACT

In 1939, I.M. Sheffer proved that every polynomial sequence belongs to one and only one *type*. Sheffer extensively developed properties of the *B-Type 0* polynomial sequences and determined which sets are also orthogonal. He subsequently generalized his classification method to the case of arbitrary *B-Type k* by constructing the generalized generating function $A(t)\exp\left[xH_1(t) + \cdots + x^{k+1}H_k(t)\right] = \sum_{n=0}^{\infty} P_n(x)t^n$, with $H_i(t) = h_{i,i}t^i + h_{i,i+1}t^{i+1} + \cdots,\ h_{1,1} \neq 0$. Although extensive research has been done on characterizing polynomial sequences, no analysis has yet been completed on sets of type one or higher ($k \geq 1$). We present a preliminary analysis of a special case of the *B-Type 1* ($k = 1$) class, which is an extension of the *B-Type 0* class, in order to determine which sets, if any, are also orthogonal sets. Lastly, we consider an extension of this research and comment on future considerations. In this work the utilization of computer algebra packages is indispensable, as computational difficulties arise in the *B-Type 1* class that are unlike those in the *B-Type 0* class.

To the memory of my father, Daniel V. Galiffa

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CHAPTER 1: INTRODUCTION

The field of mathematical analysis is one that is constantly expanding and the study of orthogonal polynomials and special functions is no exception. Richard A. Askey states in the foreword of [14]:

> *"When I started to work on orthogonal polynomials and special functions, I was told by a number of people that the subject was out-of-date, and some even said dead. They were wrong. It is alive and well. The one variable theory is far from finished..."*

We have witnessed the field of orthogonal polynomials and special functions extend itself from the cases of Hermite, Laguerre and Jacobi to the q-analogues discovered by W. Al-Salam, R. Askey, M. E. H. Ismail and others. As the extensions continue we see researchers in this field encompassing many branches of mathematics, utilizing a multitude of techniques and thusly producing a wide range of material. As a result, new theorems related to orthogonal polynomials appear not only in journals of pure mathematics but in applied and computational journals as both theoretical and practical applications seem to increase in number, see [9], [11], [17] and [14] for four of many examples.

The particular sub-branch of orthogonal polynomials and special functions that is of particular interest for this study is characterization theorems. As the name implies, characterization theorems give a very complete description of a polynomial sequence and are quite useful in both theory and physical applications, see [14] and [17]. In order to aptly motivate the presentation of our original research we briefly discuss some fundamental results in the characterizations of orthogonal polynomial sequences.

We begin with Appell polynomial sets $\{p_n\}$, which in general are defined as

$$A(t)\,e^{xt} = \sum_{m=0}^{\infty} p_m(x)\,t^m, \qquad A(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad a_0 = 1, \tag{1.1}$$

refer to [1] or [18]. An example of an Appell polynomial set are the polynomials defined as $\left\{\frac{x^n}{n!}\right\}$, which is clear since

$$e^{xt} = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n.$$

This set can readily be shown not to be orthogonal and therefore, not all Appell sets are orthogonal and it is then natural to attempt to discover which polynomial sets are Appell *and* orthogonal. To discover these polynomials we will need the following theorem that is motivated by (1.1), see [1].

Theorem 1.1. *A polynomial sequence $\{p_n\}$ is Appell if and only if*

$$(i) \qquad p_n(x) = \sum_{k=0}^{n} a_{n-k} \frac{x^k}{k!}$$

or

$$(ii) \qquad p_n'(x) = p_{n-1}(x)$$

for $n = 0, 1, 2, \ldots$

A simple example of an Appell sequence that is also orthogonal is the Hermite polynomial sequence, see [15] or [20]. Hermite polynomial sequences can be characterized by two structures; a generating function and a three-term recurrence relation. In fact, *all* polynomial sequences that are also orthogonal can be characterized by exactly one three-term recurrence relation (it may be in an unrestricted form or a monic form, which we discuss in Chapter 2), and can also be characterized by at least one generating function, see [14] or [15] for details and examples - both generating functions and three-term recurrence relations will be discussed in more detail in Chapter 2. Below we characterize the Hermite polynomial sequences, as an example of an Appell sequence that is also orthogonal.

The three-term recurrence relation for the Hermite polynomial sequence $\{H_n\}$ (in unrestricted form) is

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

A generating function for the Hermite polynomial sequence $\{H_n\}$ is

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n.$$

Now, there is one and only one type of Appell polynomial sequence that is also orthogonal and we state this result as our next theorem, see [1].

Theorem 1.2. *The only Appell polynomial sequences that are orthogonal are the Hermite polynomial sequences.*

Now, if we examine the three-term recurrence relation for the Hermite polynomial sequence $\{H_n\}$ it is clear that with only knowledge of the first two polynomials (H_0 and H_1) all subsequent polynomials can be obtained. In addition, if we examine the generating function for $\{H_n\}$ above, it is clear that the k^{th} Hermite polynomial can also be obtained by computing the coefficient of t^k on both sides of the equation. This logic is consistent for all orthogonal polynomial sequences as well.

We now turn our attention to the "classical" orthogonal polynomial sequences of Hermite, Laguerre and Jacobi and it should be noted that the Hermite and Laguerre sequences are actually special cases of the Jacobi sequences - see [14] and [15] for details. These sequences are deemed "classical" because they were among the first to be fully characterized and each of them has six properties, which we discuss below, refer to [1] and [2]. It is also important to mention that in contemporary mathematics, the term "classical" refers to any Askey-Wilson orthogonal polynomial sequences or their special and limiting cases, c.f. [15].

Property I: *All classical orthogonal polynomial sequences satisfy a Sturm-Liouville differential equation:*

$$\sigma(x)\,y''(x) + \tau(x)\,y'(x) + \lambda_n y(x) = 0$$

with $\sigma(x)$ at most quadratic and $\tau(x)$ a linear polynomial, both independent of n and λ_n is independent of x.

Property II: *All classical orthogonal polynomial sequences have derivatives which form orthogonal polynomials sequences.*

For example, the fact that the derivative of a Hermite polynomial is another Hermite polynomial can be verified by direct computation or by using their generating function, see [20].

Property III: *All classical orthogonal polynomial sequences possess a Rodrigues formula*

$$P_n(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n}\left[w(x)\,\sigma^n(x)\right] \qquad n = 0, 1, 2, \ldots$$

where $w(x)$ *is the respective weight function of the orthogonal polynomial sequence and* $\sigma(x)$ *is a polynomial in* x *independent of* n*.*

Property IV: *Given a classical orthogonal polynomial sequence the respective weight function satisfies a differential equation of the form*

$$\frac{d}{dx}\left(\sigma(x)\,w(x)\right) = \tau(x)\,w(x).$$

To illustrate this, consider the weight function of the Laguerre polynomials: $w(x) = x^{\alpha}e^{-x}$, see [18]. We should also mention that the equation above is often referred to as the Pearson differential equation.

Property V: *All classical orthogonal polynomial sequences satisfy the following differential-difference equation*

$$\pi(x)\,p_n'(x) = (\alpha_n x + \beta_n)\,p_n(x) + \gamma_n p_{n-1}(x).$$

For an example see [14].

Property VI: *All classical orthogonal polynomial sequences satisfy a non-linear equation of the form*

$$\sigma(x)\frac{d}{dx}\left[p_n(x)\,p_{n-1}(x)\right] = (\alpha_n x + \beta_n)\,p_n(x)p_{n-1}(x) + \gamma_n p_n^2(x) + \delta p_{n-1}^2(x)$$

As it turns out, each of the aforementioned six properties has a converse, so any polynomial sequence that satisfies any one of the six properties must be one of the classical orthogonal polynomial sequences, i.e. Hermite, Laguerre or Jacobi. Several authors contributed to establishing these converse results. For example, as far back as 1929 Bochner determined all polynomial solutions of the second order linear ODE of Property I in [6]. It is worth noting that Ismail generalized Bochner's work in [13].

Polynomial sequences can also be characterized by generating functions with a convolution structure [3]. Consider the Brenke polynomials [7], which are defined as follows:

$$B_n\left(x\right)=\sum_{k=0}^{n}a_{n-k}b_kx^k, \tag{1.2}$$

with $b_k\neq 0$ and $a_0=b_0=1$. Multiplying both sides of the above relationship (1.2) by t^n and summing for $n=1,2,\ldots$ we see

$$\sum_{n=0}^{\infty}B_n\left(x\right)t^n=\sum_{j=0}^{\infty}a_jt^j\sum_{k=0}^{\infty}\left(xt\right)^k$$

so we can equivalently define the Brenke polynomials in terms of their generating relation

$$\sum_{n=0}^{\infty}B_n\left(x\right)t^n=A\left(t\right)C\left(xt\right).$$

It is important to note that Chihara determined all of the polynomial sequences that are of Brenke-type and are also orthogonal in [7] and [8].

We now discuss a way that the Brenke polynomials can be generalized. Notice that if in (1.2) we replace x^k with $w_k(x)$ where,

$$w_k\left(x\right)=\prod_{j=1}^{k}\left(x-x_j\right)$$

we therefore have

$$G_n(x) = \sum_{k=0}^{n} a_{n-k} b_k w_k(x), \qquad n = 0, 1, 2, \ldots \tag{1.3}$$

This leads us to one of the most important unsolved problems in characterization theory, entitled "The Geronimus Problem" see [1] and [12]:

(The Geronimus Problem)**:** *Describe all orthogonal polynomial sequences $\{G_n\}$ that satisfy (1.3) for arbitrary sequences $\{a_n\}$ and $\{b_n\}$ and $\{x_k\}$, with $b_k \neq 0$ and $a_0 = b_0 = 1$.*

This problem was raised by Geronimus in 1947 in his paper [12]. He did not solve this problem and it currently remains unsolved in its full generality. However, Geronimus did provide some necessary and sufficient conditions on the coefficient sequences $\{a_n\}$ and $\{b_n\}$ and the sequence $\{x_k\}$. We state this result as a theorem.

Theorem 1.3 *The polynomials $\{G_n\}$ in (1.3) satisfy*

$$\frac{P_{n+1}(x)}{b_{n+1}} = [x - c_n] \frac{P_n(x)}{b_n} - \lambda_n \frac{P_{n-1}(x)}{b_{n-1}}, \qquad n \geq 1$$

if and only if $a_{k+1}(B_{n-k} - B_{n+1}) = a_1 a_k (B_n - B_{n+1}) + \frac{\lambda_n}{B_{n-1}} a_{k-1} + a_k (x_{n+1} - x_{n-k+1}),$ *for $k = 0, 1, 2, \ldots, n$, where $B_n = \frac{b_{n-1}}{b_n}$ and $B_0 = 0$.*

Now that several examples of characterization theorems have been briefly discussed, we turn our attention to the motivation of our present research. In 1939, I.M. Sheffer proved that every polynomial sequence belongs to one and only one *type*, see [19]. In essence, Sheffer actually established several abstract relationships that characterized polynomial sequences that are more general than the Appell sets. Moreover, one of his foremost results involved extensively developing properties of the *B-Type 0* polynomial sequences and determining which of these sets are also orthogonal. He subsequently generalized his classification method to the case of arbitrary *B-Type k* by constructing the generalized generating function $A(t)\exp\big[xH_1(t) + \cdots + x^{k+1}H_k(t)\big] = \sum_{n=0}^{\infty} P_n(x)t^n$, with $H_i(t) = h_{i,i}t^i + h_{i,i+1}t^{i+1} + \cdots$, $h_{1,1} \neq 0$. Our research focuses on continuing the work of Sheffer by utilizing this generating function. We now discuss the main elements of each of the chapters to follow.

In Chapter 2 we cover definitions, preliminary ideas and rudimentary theorems that will be necessary in the following chapters including a brief discussion regarding generating functions and three-term recurrence relations. We also cover Sheffer's research [19] on *B-Type 0* polynomial sequences that are also orthogonal. Lastly, we explain that in this work the utilization of computer algebra packages, like Mathematica [21], is indispensable as computational difficulties arise in the *B-Type 1* class that are unlike those in the *B-Type 0* class. We conclude this chapter with an example of the format for which the computations are displayed in this work.

Chapter 3 can be viewed as the crux of our research. Although extensive research has been done on characterizing polynomial sequences, no analysis has yet been completed on Sheffer sets classified as type one or higher ($k \geq 1$). In this chapter we present a preliminary analysis on a special case of the *B-Type 1* ($k = 1$) class, which is an extension of the *B-Type 0* class, in order to determine which, if any, sets are also orthogonal sets. Our methodology basically utilizes only the generalized generating function defined above for $k = 1$ and a corresponding three-term recurrence relation, which we develop from the generating function. From these two structures we develop conditions for the *B-Type 1* polynomial sequences to be orthogonal by constructing and attempting to solve a simultaneous system of nonlinear algebraic equations. In Chapter 4 we extend the work completed in Chapter 3 by using the methods and knowledge gleaned from Chapter 3 to consider a special case of the *B-Type 2* class.

Lastly, in Chapter 5 we formally state two problems based on the results achieved in Chapters 3 and 4. We also reflect on the previous chapters by further discussing several of the key elements of this work and commenting on preliminary approaches that were unsuccessful. Additionally, we address additional extensions of this research and discuss future considerations including how our method could be used to approach other problems and motivate future researchers.

CHAPTER 2: PRELIMINARIES

In this chapter we discuss the preliminary definitions, theorems, concepts and notational conventions that are used throughout this work. We begin with the definitions of a polynomial sequence and an orthogonal polynomial sequence from [1].

Definition 2.1. We shall always define a *polynomial sequence* as $\{P_n(x) : n = 0, 1, 2, \ldots\}$, where the degree of $P_n(x)$ is exactly n, which we denote as $\deg(P_n) = n$.

Definition 2.2. By an *orthogonal polynomial sequence* we mean a polynomial sequence orthogonal on the real line with respect to a positive measure $d\mu(x)$ supported on a subset of the real line with finitely or infinitely many points of increase so that

$$\int_{-\infty}^{\infty} P_m(x) P_n(x) \, d\mu(x) = h_n \delta_{m,n}$$

where $\delta_{m,n}$ denotes the Kronecker delta

$$\delta_{m,n} := \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$$

where the moments

$$\mu_n = \int_{-\infty}^{\infty} x^n d\mu(x)$$

are finite for all n, see [1].

We first discuss a classical result in the field of orthogonal polynomials that is essential in all of our analysis, see [1].

Theorem 2.3. *Every orthogonal polynomial sequence* $\{P_n\}$ *that satisfies Definition 2.2 must necessarily satisfy a three-term recurrence relation, which may take on one of the following forms listed below.*

(I) Orthonormal Form ($h_n = 1$):

$$xP_n(x) = a_{n+1}P_{n+1}(x) + c_nP_n(x) + a_nP_{n-1}(x), \quad a_n \neq 0, \quad a_n, c_n \in \mathbb{R} \tag{2.1}$$

(II) Monic Form (p_n)*:*

$$p_{n+1}(x) = (x - d_n)p_n(x) - \lambda_n p_{n-1}(x), \quad \lambda_n > 0, \quad d_n \in \mathbb{R} \tag{2.2}$$

(III) Unrestricted Form:

$$P_{n+1}(x) = (A_nx + B_n)P_n(x) - C_nP_{n-1}(x), \quad A_nA_{n-1}C_n > 0 \tag{2.3}$$

where $P_0(x) = 1$ and $P_{-1}(x) = 0$.

The converse of Theorem 2.3 is often entitled the Spectral Theorem- see [10] for a detailed explanation.

Theorem 2.4. *(Spectral Theorem) Any polynomial sequence* $\{P_n\}$ *that satisfies either (2.1), (2.2) or (2.3) must also be an orthogonal polynomial sequence.*

In this work, we specifically utilize the three-term recurrence relation in the unrestricted form (III) of Theorem 2.3. Additionally, we briefly cover two well-known examples of three-term recurrence relations for emphasis.

Example 2.5. The three-term recurrence relation for the Hermite polynomial sequence $\{H_n\}$ in unrestricted form is

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

and the monic form is

$$xp_n(x) = p_{n+1}(x) + \frac{n}{2}p_{n-1}(x),$$

where $p_n(x) := 2^{-n}H_n(x)$, refer to [15].

Example 2.6. The Laguerre polynomials have the unrestricted recurrence relation

$$(n+1)L_{n+1}^{(\alpha)}(x) - (2n+\alpha+1-x)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0$$

and the monic relation

$$xp_n(x) = p_{n+1}(x) + (2n+\alpha+1)p_n(x) + n(n+\alpha)p_{n-1}(x),$$

where $p_n(x) := (-1)^n n! L_n^{(\alpha)}(x)$, again refer to [15].

2.1 GENERATING FUNCTIONS

Here, we briefly discuss the standard notion of a generating function and supplement this notion with some basic examples. We begin with the definition of a generating function from

[18] and discuss some of its important subtleties.

Definition 2.7. Consider a function $F(x,t)$, which has a formal power series expansion in t where the coefficient of t^n is a function of x:

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n. \tag{2.4}$$

We say that the expansion of $F(x,t)$ generates a set $\{f_n\}_{n=0}^{\infty}$ and $F(x,t)$ is therefore the *generating function* for $\{f_n\}_{n=0}^{\infty}$.

However, it is very important to note that the series defined by the left hand side of (2.4) need not converge for the relation (2.4) to define $\{f_n\}_{n=0}^{\infty}$ and to be useful in establishing properties of those functions [18]. We have the following, also well-known, examples of generating functions, see [14] or [18].

Example 2.8. A generating function for the Jacobi polynomial sequence $\left\{P_n^{(\alpha,\beta)}\right\}$ is

$$\frac{2^{\alpha+\beta}}{R(1+R-t)^{\alpha}(1+R+t)^{\beta}} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n, \quad R = \sqrt{1-2xt+t^2}.$$

Example 2.9. The Laguerre polynomials have a generating function of the form

$$(1-t)^{-(\alpha+1)}\exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n.$$

2.2 THE SHEFFER B-TYPE 0 ORTHOGONAL POLYNOMIAL SEQUENCES

In this section we discuss one of the main results established by I. M. Sheffer in [19]. More specifically, we cover an overview of the analysis that Sheffer completed to determine which *B-Type 0* polynomial sequences are orthogonal. Our overview is essentially based on [1]. For more discussions on this aspect of Sheffer's work refer [14] and [18]. We begin with the definition of the Sheffer *B-Type 0* polynomial sequences.

Definition 2.10. A polynomial sequence $\{S_n\}$ is considered Sheffer *B-Type 0* if it satisfies the generating relation

$$A(t)e^{xu(t)} = \sum_{n=0}^{\infty} S_n(x)t^n, \tag{2.5}$$

where $A(0) = 1$, $u(0) = 0$ and $u'(0) = 1$.

In 1934 Meixner [16] endeavored to discover all orthogonal polynomial sequences that had a generating function of the form (2.5) and later in 1939 Sheffer conducted a complete analysis of *all* polynomial sets that satisfy (2.5) and also posed the same question as Meixner.

In order to determine which *B-Type 0* polynomial sequences are also orthogonal polynomial sequences, Sheffer defined a differential operator J of infinite order that commutes with the standard differential operator d/dx and acts on polynomials in the same way d/dx acts on the powers. He took the formal inverse of $u(t)$ defined by $u\left(J(t)\right) = J\left(u(t)\right) = t$ so that from (2.5) $J\left(D\right) S_n(x) = S_{n-1}(x)$, which is a degree-lowering operator analogous to the derivative. Both Meixner and Sheffer proved the following result.

Theorem 2.11. *The polynomial sequence $\{S_n\}$ as defined by (2.5) is an orthogonal polynomial sequence if and only if $\{S_n\}$ satisfies the three-term recurrence relation:*

$$S_{n+1}(x) = [x - (an+b)]S_n(x) - n(c-dn)S_{n-1}(x).$$

Sheffer then proved that Theorem 2.11 implies

$$J'(u) = 1 - ct + kt^2 = (1-\alpha t)(1-\beta t) = \frac{1}{u'(t)}$$

and

$$\frac{A'(t)}{A(t)} = \frac{\gamma t}{(1-\alpha t)(1-\beta t)},$$

where of course $c = \alpha + \beta$ and $k = \alpha\beta$. Upon considering all of the possible cases that result in the above analysis, which were based on the nature of α and β, Sheffer determined that $\{S_n\}$ must in fact be one of the classes listed below. It should be noted that all of the five classes below have been rescaled so they fit the standard contemporary form as they appear in [15].

Case 1: $\alpha = \beta = 0$

This case yields the Hermite polynomial sequences as defined in Example 2.5.

Case 2: $\alpha = \beta \neq 0$

This case results in the Laguerre polynomial sequence as defined by the generating function in Example 2.9.

Case 3: $\alpha \neq 0$, $\beta = 0$

This gives the Charlier polynomials defined as

$$e^t \left(1 - \frac{t}{a}\right)^x = \sum_{n=0}^{\infty} c_n(x; a) \frac{t^n}{n!},$$

refer to [15].

Case 4: $\alpha \neq \beta$ and $\alpha, \beta \in \mathbb{R}$

Here, we obtain the Meixner polynomial sequences, which are defined as

$$\left(1 - \frac{t}{c}\right)^x (1-t)^{-(x+\beta)} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(x; \beta, c) t^n,$$

also see [15].

Case 5: $\alpha \neq \beta$ (complex-conjugates)

In this case we achieve the Meixner-Pollaczek [15] orthogonal polynomial sequences $\{P_n\}$, which satisfy the three-term recurrence relation

$$(n+1)P_{n+1}^{(\lambda)}(x;\phi) - 2[x\sin\phi + (n+\lambda)\cos\phi]P_n^{(\lambda)}(x;\phi)$$
$$+ (n+2\lambda-1)P_{n-1}^{(\lambda)}(x;\phi) = 0.$$

Sheffer also generalized the generating relation of (2.5) to the arbitrary case of *B-Type k* as follows:

$$A(t)\exp\big[xH_1(t)+\cdots+x^{k+1}H_k(t)\big]=\sum_{n=0}^{\infty}P_n(x)t^n,$$

$$\text{with } \; H_i(t)=h_{i,i}t^i+h_{i,i+1}t^{i+1}+\cdots\,, \;\; h_{1,1}\neq 0. \tag{2.6}$$

It is clear that if we take $k=0$ in (2.6) we obtain the relation (2.5). Also, if we take $k=1$ we obtain the Sheffer *B-Type 1* polynomial sequences

$$A(t)\exp\big[xH_1(t)+x^2H_2(t)\big]=\sum_{n=0}^{\infty}P_n(x)t^n,$$

$$\text{with } \; H_i(t)=h_{i,i}t^i+h_{i,i+1}t^{i+1}+\cdots\,, \;\; h_{1,1}\neq 0, \tag{2.7}$$

and if we take $k=2$ we obtain the Sheffer *B-Type 2* polynomial sequences

$$A(t)\exp\big[xH_1(t)+x^2H_2(t)+x^3H_3(t)\big]=\sum_{n=0}^{\infty}P_n(x)t^n,$$

$$\text{with } \; H_i(t)=h_{i,i}t^i+h_{i,i+1}t^{i+1}+\cdots\,, \;\; h_{1,1}\neq 0. \tag{2.8}$$

2.3 SOME REMARKS ON THE UTILIZATION OF MATHEMATICA

In this short section we discuss the importance of the role that symbolic computer algebra packages, like Mathematica [21], play in this work and in the study of orthogonal polynomials in general. In fact, much of the computational aspects of this work were completed with Mathematica. In addition, many contemporary mathematicians have used such packages to conduct research in orthogonal polynomials and special functions. For example, in [9] Gasper demonstrates how to use symbolic computer algebra packages to derive formulas involving orthogonal polynomials and other special functions. Specifically, he derives transformation formulas for the Racah and the q-Racah polynomials (see [15] for definitions); however, he also alludes to the fact that these packages can be used in many other areas of orthogonal polynomials as well. He says,

> *"Now that several symbolic computer algebra systems such as [Mathematica] are available for various computers, it is natural for persons having access to such a system to try to have it perform the tedious manipulations needed to derive certain formulas involving orthogonal polynomials and special functions."*

In our present work, the utilization of Mathematica is not simply a convenience but a necessity, as the complexities involved in the computational aspects of this work are quite tumultuous.

Throughout this work we clearly indicate when and how Mathematica was used. For emphasis, we display the Mathematica calculations and their respective outputs with a distinctive font class that is analogous to the style used in the Mathematica notebook. In addition, we use the same conventions that Gasper used in [9] for displaying a given Mathematica input and its respective output. We demonstrate this notion with an example from Gasper's work.

Example 2.12. Gasper demonstrates a method for deriving a certain Laguerre polynomial expansion formula using Mathematica. His displays his input and respective output in a similar manner as seen below:

$$In[5] := \mathrm{p[b+1,n]p[-n,j]x^j/(p[1,n]p[1,j]p[b+1,j])}$$

$$\textit{Out[5]}{=}\frac{\mathrm{x^j p[-n,j]p[1+b,n]}}{\mathrm{p[1,j]p[1,n]p[1+b,j]}}$$

Of course, the point of emphasis in the above example is not the actual computation but the fashion in which it is presented, as this type of display is consistent for all of the Mathematica computations in this work.

CHAPTER 3: A SPECIAL CASE OF THE SHEFFER B-TYPE 1 POLYNOMIAL SEQUENCES

In endeavoring to ascertain which polynomial sequences, if any, arise from (2.7) and are also orthogonal, of course results in two considerations. The first amounts to developing an analogue of Sheffer's method as discussed in Chapter 2. This approach is tantamount to utilizing various functional relationships and producing a multitude of lemmas and theorems as Sheffer did in [19]. The second consideration is to develop a novel approach to the problem. In advocating this approach, the natural question that arises is whether or not a fundamentally simpler method than Sheffer's method can be implemented. As we show, this is the case; however, even though the overall methodology is fundamentally simpler than Sheffer's approach, an exceedingly large amount of complexity is embedded within the simplicity.

Now, as discussed in Chapter 2, a necessary and sufficient condition for a sequence of polynomials to be orthogonal is that it satisfies a three-term recurrence relation of the form (2.3). Moreover, with only knowledge of the constant and linear polynomials all subsequent polynomials in the respective sequence can be discovered solely from this relation. In addition, although computationally involved, any element of a polynomial sequence can also be discovered entirely from its generating function. These two pieces of information are well known and also indispensable. However, it is crucial to emphasize that the polynomial sequences that arise from the Sheffer *B-Type 1* class (or any polynomial sequence for that matter) can be generated from these two separate structures and by unicity these polynomials must be indiscernible. Therefore, we develop a method of obtaining the Sheffer *B-Type 1* polynomial sequences from both the generating function as defined by (2.7) and the three-term recurrence relation (2.3) in order to develop conditions for orthogonality.

3.1 LOWER-ORDER SHEFFER B-TYPE 1 POLYNOMIAL SEQUENCES OBTAINED VIA GENERATING FUNCTION

We begin by constructing a method of effectively expanding the generating function (2.7) that defines the Sheffer *B-Type 1* polynomial sequences in order to acquire the coefficients of $x^n t^n$, $x^{n-1}t^n$ and $x^{n-2}t^n$. We do this since only these coefficients and the constant and linear polynomials discovered from the Sheffer *B-Type 1* generating function (2.7) are required to obtain all of the respective subsequent polynomials. We accomplish this by expanding the generating function (2.7) using first principles and thusly acquire a sequence of polynomials that must be identical to the sequence that we obtain from the three-term recurrence relation (2.3), which will be completed in Section 3.2. By comparing these two sequences, conditions for orthogonality are established for several of the constant terms (the a's, h's and g's) in (2.7).

For our analysis of the Sheffer *B-Type 1* case, we define $H_1(t) := H(t)$ and $H_2(t) := G(t)$ in (2.7) for ease of notation. Therefore, the Sheffer *B-Type 1* polynomials sequences are defined as all of the polynomials sequences $\{P_n\}$ that satisfy

$$A(t)\exp\left[xH(t) + x^2G(t)\right] = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{3.1}$$

Clearly, the left hand side of (3.1) involves a powers series expansion of power series expansions and can be written as

$$\sum_{i=0}^{\infty} a_i t^i \sum_{j=0}^{\infty} \frac{1}{j!} \left[x \sum_{k=1}^{\infty} h_k^k t^k + x^2 \sum_{l=2}^{\infty} g_l^l t^l \right]^j = \sum_{n=0}^{\infty} P_n(x) t^n.$$

This is certainly not the most advantageous way to perceive (3.1), but it does demonstrate its complexity. We ultimately expand (3.1) in a much more practical way. However, we first discuss some initial assumptions. Namely, we take $a_0 = 1$ and $h_1 = 1$ and therefore, $A(t)$ and $H(t)$ respectively have the following structures:

$$A(t) = 1 + a_1 t + a_2 t^2 + \cdots \qquad \text{and} \qquad H(t) = t + h_2 t^2 + h_3 t^3 + \cdots .$$

Now, we expand the left hand side of (3.1) in the following *practical* manner:

$$(1 + a_1 t + a_2 t^2 + \cdots) \left[e^{xt} e^{h_2 x t^2} e^{h_3 x t^3} \cdots \right] \left[e^{g_2 x^2 t^2} e^{g_3 x^2 t^3} e^{g_4 x^2 t^4} \cdots \right]. \tag{3.2}$$

Thus, expanding (3.2) in terms of the Maclaurin series of each product, yields:

$$\sum_{m=0}^{\infty} a_m t^m \prod_{i=1}^{\infty} \left[\sum_{j=0}^{\infty} \frac{h_i^j x^j t^{ij}}{j!} \right] \prod_{k=2}^{\infty} \left[\sum_{l=0}^{\infty} \frac{g_k^l x^{2l} t^{kl}}{l!} \right]. \tag{3.3}$$

With this convention, we write out the general term in each of the products of (3.3) as

$$a_{k_0} t^{k_0} \cdot \frac{x^{k_1} t^{k_1}}{k_1!} \cdot \frac{h_2^{k_2} x^{k_2} t^{2k_2}}{k_2!} \cdot \frac{h_3^{k_3} x^{k_3} t^{3k_3}}{k_3!} \cdot \cdots \cdot \frac{g_2^{k_4} x^{2k_4} t^{2k_4}}{k_4!} \cdot \frac{g_3^{k_5} x^{2k_5} t^{3k_5}}{k_5!} \cdot \frac{g_4^{k_6} x^{2k_6} t^{4k_6}}{k_6!} \cdot \cdots , \tag{3.4}$$

where $\{k_0, k_1, k_2, \ldots\}$ are all non-negative integers. It is important to note that in the expansion (3.4) we have only explicitly written the six terms above since only these terms are needed to discover the coefficients of $x^n t^n$, $x^{n-1} t^n$ and $x^{n-2} t^n$, as we explain below.

Notice, that the sums of the x-exponents and the t-exponents of (3.4) respectively take on the form

$$k_1 + k_2 + k_3 + 2k_4 + 2k_5 + 2k_6 + \cdots = r \tag{3.5}$$

and

$$k_0 + k_1 + 2k_2 + 3k_3 + 2k_4 + 3k_5 + 4k_6 + \cdots = s. \tag{3.6}$$

That is, if we wish to discover the coefficient of $x^r t^s$ we see that the sum defined by (3.5) must add up to r and the sum defined by (3.6) must add up to s. It is clear that writing out any additional terms in (3.5) or (3.6) is superfluous since upon subtracting (3.6) from (3.5) the additional terms will not contribute to discovering the coefficients of $x^n t^n$, $x^{n-1} t^n$ and $x^{n-2} t^n$ since they will each be multiplied by a number greater than 2. In addition, by subtracting (3.6) from (3.5) we immediately see that each of the coefficients in the polynomial P_n defined by (3.1) is a *finite sum*. This logic will be the basis of the analysis involved in discovering the aforementioned coefficients $x^n t^n$, $x^{n-1} t^n$ and $x^{n-2} t^n$, which we partition into three parts.

(1.) *The coefficient of* $x^n t^n$

From subtracting (3.6) from (3.5) with $r = s = n$ we see that every k-value must be zero, except k_1 and k_4, which can take on any non-negative integer values, i.e. are free variables. Therefore, by studying (3.4) we realize that the only terms involved in the coefficient of $x^n t^n$ are $\frac{1}{k_1!}$ and $\frac{g_2^{k_4}}{k_4!}$. Hence, by combining these two pieces of information we observe that the coefficient of $x^n t^n$ must be a sum taken over all non-negative integers k_1 and k_4 such that $k_1 + 2k_4 = n$ with argument $\frac{g_2^{k_4}}{k_1!k_4!}$, as seen below

$$\sum_{k_1+2k_4=n} \frac{g_2^{k_4}}{k_1!k_4!}. \tag{3.7}$$

(2.) ***The coefficient of*** $x^{n-1}t^n$

Upon subtracting (3.6) from (3.5) with $r = n-1$ and and $s = n$ we obtain:

$$k_0 + k_2 + 2k_3 + k_5 + 2k_6 = 1. \tag{3.8}$$

Therefore, we see that $k_3 = k_6 = 0$ with k_1 and k_4 free variables, so (3.8) becomes

$$k_0 + k_2 + k_5 = 1,$$

yielding three cases.

Case 1: $k_0 = 1$, $k_2 = 0$ and $k_5 = 0$

In this case we see that (3.6) becomes

$$1 + k_1 + 2k_4 = n,$$

resulting in

$$\sum_{k_1+2k_4=n-1} \frac{a_1 g_2^{k_4}}{k_1!k_4!}.$$

Case 2: $k_2 = 1$, $k_0 = 0$ and $k_5 = 0$

Here we see that (3.6) is now

$$k_1 + 2 + 2k_4 = n,$$

yielding

$$\sum_{k_1+2k_4=n-2} \frac{h_2 g_2^{k_4}}{k_1!k_4!}.$$

Case 3: $k_5 = 1$, $k_0 = 0$ and $k_2 = 0$

Lastly, we see that for this case (3.8) becomes

$$k_1 + 2k_4 + 3 = n,$$

yielding

$$\sum_{k_1+2k_4=n-3} \frac{g_3 g_2^{k_4}}{k_1!k_4!}$$

and we now have exhausted all possibilities.

Therefore, the coefficient of $x^{n-1}t^n$ is

$$\sum_{k_1+2k_4=n-1} \frac{a_1 g_2^{k_4}}{k_1!k_4!} + \sum_{k_1+2k_4=n-2} \frac{h_2 g_2^{k_4}}{k_1!k_4!} + \sum_{k_1+2k_4=n-3} \frac{g_3 g_2^{k_4}}{k_1!k_4!}. \tag{3.9}$$

(3.) ***The coefficient of*** $x^{n-2}t^n$

Here, after subtracting (3.6) and (3.5) with $r = n-2$ and $s = n$ we obtain

$$k_0 + k_2 + 2k_3 + k_5 + 2k_6 = 2.$$

Analogous to the previous coefficient derivations, the cases involved in determining the coefficient of $x^{n-2}t^n$ are determined by all of the non-negative integer solutions to the equation above. In each case, the substitution of the these solutions (the k-values) into (3.6) with $s = n$ are written as (i) and the resulting sum as (ii). We have a total of 8 cases.

Case 1: $k_3 = 1$ and $k_0 = k_2 = k_5 = k_6 = 0$

$$(i) \qquad k_1 + 3 + 2k_4 = n$$

$$(ii)\ \sum_{k_1+2k_4=n-3} \frac{h_3 g_2^{k_4}}{k_1!k_4!}.$$

Case 2: $k_6 = 1$ and $k_0 = k_2 = k_3 = k_5 = 0$

$$(i) \qquad k_1 + 2k_4 + 4 = n$$

$$(ii) \qquad \sum_{k_1+2k_4=n-4} \frac{g_4 g_2^{k_4}}{k_1!k_4!}.$$

Case 3: $k_0 = 2$ and $k_2 = k_3 = k_5 = k_6 = 0$

$$(i) \qquad 2 + k_1 + 2k_4 = n$$

$$(ii) \qquad \sum_{k_1+2k_4=n-2} \frac{a_2 g_2^{k_4}}{k_1!k_4!}.$$

Case 4: $k_2 = 2$ and $k_0 = k_3 = k_5 = k_6 = 0$

$$(i) \qquad k_1 + 4 + 2k_4 = n$$

$$(ii) \qquad \sum_{k_1+2k_4=n-4} \frac{h_2^2 g_2^{k_4}}{2!k_1!k_4!}.$$

Case 5: $k_5 = 2$ and $k_0 = k_2 = k_3 = k_6 = 0$

$$(i) \qquad k_1 + 2k_4 + 6 = n$$

$$(ii) \qquad \sum_{k_1+2k_4=n-6} \frac{g_3^2 g_2^{k_4}}{2!k_1!k_4!}.$$

Case 6: $k_0 = k_2 = 1$ and $k_3 = k_5 = k_6 = 0$

$$(i) \qquad 1 + k_1 + 2 + 2k_4 = n$$

$$(ii) \qquad \sum_{k_1+2k_4=n-3} \frac{a_1 h_2 g_2^{k_4}}{k_1!k_4!}.$$

Case 7: $k_0 = k_5 = 1$ and $k_2 = k_3 = k_6 = 0$

$$(i) \qquad 1 + k_1 + 2k_4 + 3 = n$$

$$(ii) \qquad \sum_{k_1+2k_4=n-4} \frac{a_1 g_3 g_2^{k_4}}{k_1!k_4!}.$$

Case 8: $k_2 = k_5 = 1$ and $k_0 = k_3 = k_6 = 0$

$$(i) \qquad k_1 + 2 + 2k_4 + 3 = n$$

$$(ii) \qquad \sum_{k_1+2k_4=n-5} \frac{h_2 g_3 g_2^{k_4}}{k_1!k_4!}.$$

Hence, the coefficient of $x^{n-2}t^n$ is

$$\begin{aligned}
&\sum_{k_1+2k_4=n-3} \frac{h_3 g_2^{k_4}}{k_1!k_4!} + \sum_{k_1+2k_4=n-4} \frac{g_4 g_2^{k_4}}{k_1!k_4!} + \sum_{k_1+2k_4=n-2} \frac{a_2 g_2^{k_4}}{k_1!k_4!} \\
&+ \sum_{k_1+2k_4=n-4} \frac{h_2^2 g_2^{k_4}}{2!k_1!k_4!} + \sum_{k_1+2k_4=n-6} \frac{g_3^2 g_2^{k_4}}{2!k_1!k_4!} + \sum_{k_1+2k_4=n-3} \frac{a_1 h_2 g_2^{k_4}}{k_1!k_4!} \\
&+ \sum_{k_1+2k_4=n-4} \frac{a_1 g_3 g_2^{k_4}}{k_1!k_4!} + \sum_{k_1+2k_4=n-5} \frac{h_2 g_3 g_2^{k_4}}{k_1!k_4!}.
\end{aligned} \tag{3.10}$$

Now, notice that in (3.7), (3.9) and (3.10) each component was a sum involving the term

$$\frac{g_2^{k_4}}{k_1!k_4!}$$

and also, it can readily be shown that

$$\sum_{k_1+2k_4=n} \frac{g_2^{k_4}}{k_1!k_4!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{g_2^k}{(n-2k)!k!}.$$

Therefore, we define the following function

$$\phi_n(x) := \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{x^k}{(n-2k)!k!}$$

in order to simplify notation. Thus, taking $\phi_n(g_2) := \phi_n$ the coefficients of $x^n t^n$, $x^{n-1}t^n$ and $x^{n-2}t^n$ respectively have the following form:

$$c_{n,0} := \phi_n \tag{3.11}$$

$$c_{n,1} := a_1\phi_{n-1} + h_2\phi_{n-2} + g_3\phi_{n-3} \tag{3.12}$$

$$\begin{aligned} c_{n,2} := a_2\phi_{n-2} + (h_3 + a_1 h_2)\phi_{n-3} + \left(g_4 + \frac{h_2^2}{2!} + a_1 g_3\right)\phi_{n-4} \\ + h_2 g_3 \phi_{n-5} + \frac{g_3^2}{2!}\phi_{n-6}. \end{aligned} \tag{3.13}$$

Based on this analysis we see that $P_n(x)$ as defined by (3.1) now becomes

$$P_n(x) = c_{n,0}x^n + c_{n,1}x^{n-1} + c_{n,2}x^{n-2} + \text{L.O.T.}$$

and upon expanding the substitution of P_n above into the three-term recurrence relation (2.3) we obtain

$$\begin{aligned} & c_{n+1,0}x^{n+1} + c_{n+1,1}x^n + c_{n+1,2}x^{n-1} + \text{L.O.T.} \\ & = A_n c_{n,0}x^{n+1} + A_n c_{n,1}x^n + A_n c_{n,2}x^{n-1} + \text{L.O.T.} \\ & \quad + B_n c_{n,0}x^n + B_n c_{n,1}x^{n-1} + B_n c_{n,2}x^{n-2} + \text{L.O.T.} \\ & \quad - C_n c_{n-1,0}x^{n-1} - C_n c_{n-1,1}x^{n-2} - C_n c_{n-1,2}x^{n-3} + \text{L.O.T.}, \end{aligned}$$

where L.O.T. represents the respective lower-order terms of each polynomial.

Thus, comparing the coefficients of x^{n+1}, x^n and x^{n-1} above results in the following lower-triangular simultaneous system of linear equations:

$$\begin{bmatrix} c_{n,0} & 0 & 0 \\ c_{n,1} & c_{n,0} & 0 \\ c_{n,2} & c_{n,1} & -c_{n-1,0} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = \begin{bmatrix} c_{n+1,0} \\ c_{n+1,1} \\ c_{n+1,2} \end{bmatrix}.$$

Since the diagonal terms $c_{n,0}$ and $c_{n-1,0}$ are non-zero, solving this system via elementary methods gives the following:

$$\begin{aligned} A_n &= \frac{c_{n+1,0}}{c_{n,0}} \\ B_n &= \frac{c_{n+1,1}c_{n,0} - c_{n+1,0}c_{n,1}}{c_{n,0}^2} \\ C_n &= \frac{c_{n+1,0}(c_{n,0}c_{n,2} - c_{n,1}^2) + c_{n,0}(c_{n+1,1}c_{n,1} - c_{n+1,2}c_{n,0})}{c_{n-1,0}c_{n,0}^2}. \end{aligned} \tag{3.14}$$

We now can obtain any polynomial P_k by directly expanding (3.1). In order to discover P_k we first compute the coefficient of t^k since this coefficient must be a polynomial in x and of degree k, which is clear since when writing (3.1) as

$$A(t)\exp\big[xH(t) + x^2G(t)\big] = P_0 + P_1t + P_2t^2 + \cdots + P_kt^k + \cdots$$

it is readily seen that the coefficient of t^k in the left-hand side of (3.1) must be P_k. We first discover the polynomials $P_0, \ldots, P_5$. The constant and linear polynomials are easily obtained by 'hand' computations analogous to the method previously described for discovering the coefficients of $x^n t^n$, $x^{n-1}t^n$ and $x^{n-2}t^n$. For the constant polynomial P_0, it is immediate that $a_0 = 1$ is the coefficient of t^0. Therefore, $P_0(x) = 1$. For the linear polynomial P_1, we see from direct computation that the coefficient of t^1 is $a_1 + x$ and thus $P_1(x) = a_1 + x$.

Of course, P_2, P_3 and the latter polynomials can be calculated in the same fashion. However, as would be the case for any polynomial sequence, the computations involved become increasingly more complicated as n increases, which is certainly epitomized in the Sheffer *B-Type 1* class. Therefore, in order to determine polynomials P_k for $k \geq 2$ we utilize a Mathematica program entitled `GenPoly` where the results of which are presented as discussed in Chapter 2. We explicitly demonstrate the procedure for P_2 and P_3 below.

We first find the coefficient of t^2 by expanding the left-hand side of (3.1). Both the input and the respective output are seen below.

In[1]:= `Expand[Coefficient[`

$$\left(1+\sum_{m=1}^{10} \mathrm{a_m t^m}\right) * \prod_{j=1}^{10}\left[\sum_{i=0}^{10} \frac{\mathrm{h_j^i x^i t^{ij}}}{\mathrm{i!}}\right] * \prod_{k=2}^{10}\left[\sum_{l=0}^{10} \frac{\mathrm{g_k^l x^{2l} t^{kl}}}{\mathrm{l!}}\right], \mathrm{t^2}, \mathrm{h_1} = 1]]$$

Out[1]= $\frac{\mathrm{x^2}}{2} + \mathrm{x\,a_1} + \mathrm{a_2} + \mathrm{x^2\,g_2} + \mathrm{x\,h_2}$

Thus, we see that

$$P_2(x) = a_2 + (a_1 + h_2)x + \left(\frac{1}{2!} + g_2\right)x^2. \tag{3.15}$$

For the coefficient the of t^3 we have the following computation and respective output:

In[2]:= `Expand[Coefficient[`

$$\left(1+\sum_{m=1}^{10} \mathrm{a_m t^m}\right) * \prod_{j=1}^{10}\left[\sum_{i=0}^{10} \frac{\mathrm{h_j^i x^i t^{ji}}}{\mathrm{i!}}\right] * \prod_{k=2}^{10}\left[\sum_{l=0}^{10} \frac{\mathrm{g_k^l x^{2l} t^{kl}}}{\mathrm{l!}}\right], \mathrm{t^3}, \mathrm{h_1} = 1]]$$

$$Out[2]= \frac{x^3}{6} + \frac{x^2 a_1}{2} + x a_2 + a_3 + x^3 g_2 + x^2 a_1 g_2 + x^2 g_3 + x^2 h_2 + x a_1 h_2 + x h_3$$

Therefore, we have

$$P_3(x) = a_3 + (a_2 + a_1 h_2 + h_3)x + \left(\frac{a_1}{2!} + a_1 g_2 + g_3 + h_2\right) x^2 + \left(\frac{1}{3!} + g_2\right) x^3. \quad (3.16)$$

It is important to mention that for the higher-order polynomials (P_k, $k \geq 4$) the process is slightly adjusted since the outputs quickly become much more compecated than the ones above. For these polynomials, we first discover the coefficient of t^n and then compute each x-coefficient individually. To demonstrate this procedure we show all of the computations needed to construct P_4.

We first discover the coefficient of t^4:

In[3]:= `Expand[Coefficient[`

$$\left(1 + \sum_{m=1}^{10} a_m t^m\right) * \prod_{j=1}^{10} \left[\sum_{i=0}^{10} \frac{h_j^i x^i t^{ji}}{i!}\right] * \prod_{k=2}^{10} \left[\sum_{l=0}^{10} \frac{g_k^l x^{2l} t^{kl}}{l!}\right], t^4, h_1 = 1]]$$

$$Out[3]= \frac{x^4}{24} + \frac{x^3 a_1}{6} + \frac{x^2 a_2}{2} + x a_3 + a_4 + \frac{x^4 g_2}{2} + x^3 a_1 g_2 + x^2 a_2 g_2 + \frac{x^4 g_2^2}{2} + x^3 g_3$$
$$+ x^2 a_1 g_3 + x^2 g_4 + \frac{x^3 h_2}{2} + x^2 a_1 h_2 + x a_2 h_2 + x^3 g_2 h_2 + \frac{x^2 h_2^2}{2} + x^2 h_3 + x a_1 h_3 + x h_4$$

For simplicity, we define the the above output as follows:

$$In[4]:= \text{FOURTH} := \frac{x^4}{24} + \frac{x^3 a_1}{6} + \frac{x^2 a_2}{2} + x a_3 + a_4 + \frac{x^4 g_2}{2} + x^3 a_1 g_2 + x^2 a_2 g_2 + \frac{x^4 g_2^2}{2}$$
$$+ x^3 g_3 + x^2 a_1 g_3 + x^2 g_4 + \frac{x^3 h_2}{2} + x^2 a_1 h_2 + x a_2 h_2 + x^3 g_2 h_2 + \frac{x^2 h_2^2}{2} + x^2 h_3 + x a_1 h_3 + x h_4$$

Then we compute each coefficient separately.

For the coefficient of x^4 we have

$$In[5]:= \texttt{Coefficient[FOURTH, x}^4\texttt{]}$$

$$Out[5]= \frac{1}{24} + \frac{g_2}{2} + \frac{g_2^2}{2}$$

Then, for coefficient of x^3 we see that

$$In[6]:= \texttt{Coefficient[FOURTH, x}^3\texttt{]}$$

$$Out[6]= \frac{a_1}{6} + a_1 g_2 + \frac{h_2}{2} + g_2 h_2 + g_3$$

Next, the coefficient of x^2 is computed as

$$In[7]:= \texttt{Coefficient[FOURTH, x}^2\texttt{]}$$

$$Out[7]= \frac{a_2}{2} + a_2 g_2 + a_1 h_2 + \frac{h_2^2}{2} + h_3 + a_1 g_3 + g_4$$

For the coefficient of x we obtain

$$In[8]:= \texttt{Coefficient[FOURTH, x]}$$

$$Out[8]= a_3 + a_2 h_2 + a_1 h_3 + h_4$$

and for the constant term we achieve

$$In[9]:= \texttt{Coefficient[x*FOURTH, x]}$$

$$Out[9]= a_4$$

Hence, putting these above pieces together we have

$$\begin{aligned} P_4(x) =& a_4 + (a_3 + a_2 h_2 + a_1 h_3 + h_4)x \\ &+ \left(\frac{a_2}{2!} + a_2 g_2 + a_1 h_2 + \frac{h_2^2}{2!} + h_3 + a_1 g_3 + g_4\right) x^2 \\ &+ \left(\frac{a_1}{3!} + a_1 g_2 + \frac{h_2}{2!} + g_2 h_2 + g_3\right) x^3 + \left(\frac{1}{4!} + \frac{g_2}{2!} + \frac{g_2^2}{2!}\right) x^4. \end{aligned} \tag{3.17}$$

Using the same process as demonstrated above we also obtain an expression for P_5.

$$\begin{aligned} P_5(x) &= a_5 + (a_4 + a_3h_2 + a_2h_3 + a_1h_4 + h_5)\,x \\ &+ \left(\frac{a_3}{2!} + a_3g_2 + a_2g_2 + \frac{a_1h_2^2}{2!} + a_1h_3 + h_2h_3 + h_4 + a_2g_3 + a_1g_4 + g_5\right)x^2 \\ &+ \left(\frac{a_2}{3!} + a_2g_2 + \frac{a_1h_2}{2!} + a_1g_2h_2 + \frac{h_2^2}{2!} + \frac{h_3}{2!} + g_2h_3 + g_3h_2 + a_1g_3 + g_4\right)x^3 \\ &+ \left(\frac{a_1}{4!} + \frac{a_1g_2}{2!} + \frac{a_1g_2^2}{2!} + \frac{h_2}{3!} + g_2h_2 + \frac{g_3}{2!} + g_2g_3\right)x^4 + \left(\frac{1}{5!} + \frac{g_2}{3!} + \frac{g_2^2}{2!}\right)x^5 \quad (3.18) \end{aligned}$$

3.2 LOWER-ORDER SHEFFER B-TYPE 1 POLYNOMIAL SEQUENCES OBTAINED VIA THREE-TERM RECURRENCE RELATION

We next discover the polynomials $P_0, \ldots, P_5$ from three-term recurrence relation (2.3) with the A_n, B_n and C_n as defined in (3.14). To accomplish this we utilize a Mathematica program entitled `ThreeTerm`. We first define $c_{n,0}$, $c_{n,1}$ and $c_{n,2}$, as established in (3.11), (3.12) and (3.13), respectively.

In[1]:= $\mathrm{c_0[n\_]} := \displaystyle\sum_{\mathrm{k=0}}^{\mathrm{Floor[n/2]}} \frac{\mathrm{g_2^k}}{\mathrm{(n-2k)!k!}}$

In[2]:= $\mathrm{c_1[n\_]} := \mathrm{a_1} * \displaystyle\sum_{\mathrm{k=0}}^{\mathrm{Floor[(n-1)/2]}} \frac{\mathrm{g_2^k}}{\mathrm{(n-1-2k)!k!}}$
$\quad + \mathrm{h_2} * \displaystyle\sum_{\mathrm{k=0}}^{\mathrm{Floor[(n-2)/2]}} \frac{\mathrm{g_2^k}}{\mathrm{(n-2-2k)!k!}} + \mathrm{g_3} * \sum_{\mathrm{k=0}}^{\mathrm{Floor[(n-3)/2]}} \frac{\mathrm{g_2^k}}{\mathrm{(n-3-2k)!k!}}$

$$\textit{In[3]:=}\ \mathrm{c_2[n\_] := a_2 * \sum_{k=0}^{Floor[(n-2)/2]} \frac{g_2^k}{(n-2-2k)!k!}}$$

$$\mathrm{+(h_3 + a_1 * h_2) * \sum_{k=0}^{Floor[(n-3)/2]} \frac{g_2^k}{(n-3-2k)!k!}}$$

$$\mathrm{+ \left(g_4 + \frac{h_2^2}{2!} + a_1 * g_3\right) * \sum_{k=0}^{Floor[(n-4)/2]} \frac{g_2^k}{(n-4-2k)!k!}}$$

$$\mathrm{+h_2 * g_3 * \sum_{k=0}^{Floor[(n-5)/2]} \frac{g_2^k}{(n-5-2k)!k!}}$$

$$\mathrm{+\frac{g_3^2}{2!} * \sum_{k=0}^{Floor[(n-6)/2]} \frac{g_2^k}{(n-6-2k)!k!}}$$

Then we define the A_n, B_n and C_n, as derived in (3.14).

$$\textit{In[4]:=}\ \mathrm{A[n\_] := \frac{c_0[n+1]}{c_0[n]}}$$

$$\textit{In[5]:=}\ \mathrm{B[n\_] := \frac{c_1[n+1] * c_0[n] - c_0[n+1]c_1[n]}{c_0[n]^2}}$$

$$\textit{In[6]:=}\ \mathrm{C[n\_] := \frac{1}{c_0[n-1] * c_0[n]^2} \left(c_0[n+1] * \left(c_0[n] * c_2[n] + c_1[n]^2\right)\right.}$$

$$\mathrm{\left. +c_0[n] * (c_1[n+1] * c_1[n] - c_2[n+1] * c_0[n])\right)}$$

Lastly, in accordance with the previous section, we assign the constant and linear polynomials as seen below

$$\mathrm{P_0 := 1 \ \text{ and } \ P_1 := a_1 + x}$$

Thus, we can now produce any polynomial defined by (3.1) of degree greater than one. As an example of the process, we achieve P_2 by separately computing the quadratic, linear and constant terms and then amalgamating the results, as seen below.

In[7]:= `Together[Coefficient[` $\mathrm{x} * \mathrm{A}[1] * \mathrm{P}_1 + \mathrm{B}[1] * \mathrm{P}_1 - \mathrm{C}[1] * \mathrm{P}_0$, x^2]]

Out[7]= $\frac{1}{2!} + \mathrm{g}_2$

In[8]:= `Together[Coefficient[` $\mathrm{x} * \mathrm{A}[1] * \mathrm{P}_1 + \mathrm{B}[1] * \mathrm{P}_1 - \mathrm{C}[1] * \mathrm{P}_0$, x]]

Out[8]= $\mathrm{a}_1 + \mathrm{h}_2$

In[9]:= `Together[Coefficient[` $\mathrm{x} * (\mathrm{x} * \mathrm{A}[1] * \mathrm{P}_1 + \mathrm{B}[1] * \mathrm{P}_1 - \mathrm{C}[1] * \mathrm{P}_0)$, x]]

Out[9]= a_2

Therefore,

$$P_2(x) = a_2 + (a_1 + h_2)x + \left(\frac{1}{2!} + g_2\right) x^2,$$

which is equal to (3.15).

Continuing in the same manner, we discover that the cubic, quadratic and linear terms of P_3 as obtained in `ThreeTerm` coincide exactly with those in (3.16). However, the constant term is quite different as seen below:

In[10]:= `Together[Coefficient[` $\mathrm{x} * (\mathrm{x} * \mathrm{A}[2] * \mathrm{P}_2 + \mathrm{B}[2] * \mathrm{P}_2 - \mathrm{C}[2] * \mathrm{P}_1)$, x]]

Out[10]= $\frac{1}{3(1+2\mathrm{g}_2)^2}(-\mathrm{a}_1^3 + 3\mathrm{a}_1\mathrm{a}_2 + 4\mathrm{a}_1\mathrm{a}_2\mathrm{g}_2 - 12\mathrm{a}_1^3\mathrm{g}_2^2 + 12\mathrm{a}_1\mathrm{a}_2\mathrm{g}_2^2 -$

$2\mathrm{a}_1^2\mathrm{h}_2 + 4\mathrm{a}_2\mathrm{h}_2 + 12\mathrm{a}_1^2\mathrm{g}_2\mathrm{h}_2 - 4\mathrm{a}_1\mathrm{h}_2^2 + 3\mathrm{a}_1\mathrm{h}_3 + 12\mathrm{a}_1\mathrm{g}_2\mathrm{h}_3 + 12\mathrm{a}_1\mathrm{g}_2^2\mathrm{h}_3)$

Therefore, we have developed a relationship for the constant term a_3, since for $\{P_n\}$ as defined by (3.1) to be orthogonal it must be that

$$\begin{aligned} a_3 =& \frac{1}{3(1+2g_2)^2}(-a_1^3 + 3a_1a_2 + 4a_1a_2g_2 - 12a_1^3g_2^2 + 12a_1a_2g_2^2 \\ & - 2a_1^2h_2 + 4a_2h_2 + 12a_1^2g_2h_2 - 4a_1h_2^2 + 3a_1h_3 + 12a_1g_2h_3 + 12a_1g_2^2h_3). \end{aligned} \quad (3.19)$$

For P_4 we discover that the fourth-degree term, the cubic-term and the quadratic-term are identical to those in (3.17). However, the linear and constant-terms are dissimilar to the ones in (3.17). The linear-term is

In[11]:= `Together[Coefficient[` $x * A[3] * P_3 + B[3] * P_3 - C[3] * P_2$, x`]]`

$$
\begin{aligned}
Out[11]= \frac{1}{4\,(1+2g_2)\,(1+6g_2)^2}\big(&-a_1^3+3a_1a_2+a_3-8a_1^3g_2+28a_1a_2g_2+20a_3g_2\\
&48a_1^3g_2^2+120a_1a_2g_2^2+120a_3g_2^2+144a_1a_2g_2^3+240a_3g_2^3+144a_1^3g_2^4\\
&-144a_1a_2g_2^4+144a_3g_2^4-2a_1^2h_2+8a_2h_2+12a_1^2g_2h_2+56a_2g_2h_2\\
&-72a_1^2g_2^2h_2+192a_2g_2^2h_2-144a_1^2g_2^3h_2+288a_2g_2^3h_2-4a_1h_2^2\\
&+48a_1g_2h_2^2+48a_1g_2^2h_2^2-8h_2^3+7a_1h_3+68a_1g_2h_3+168a_1g_2^2h_3\\
&-144a_1g_2^3h_3-144a_1g_2^4h_3+12h_2h_3+96g_2h_2h_3+240g_2^2h_2h_3\\
&12a_1^2g_3+18a_2g_3-24a_1^2g_2g_3+108a_2g_2g_3+144a_1^2g_2^2g_3+72a_2g_2^2g_3\\
&+288a_1^2g_2^3g_3-144a_2g_2^3g_3-36a_1g_3^2-144a_1g_2g_3^2+144a_1g_2^2g_3^2+8a_1g_4\\
&+96a_1g_2g_4+288a_1g_2^2g_4-42a_1g_3h_2-108a_1g_2g_3h_2+216a_1g_2^2g_3h_2\\
&+144a_1g_2^3g_3h_2-36g_3^2h_2-144g_2g_3^2h_2+144g_2^2g_3^2h_2+8g_4h_2+96g_2g_4h_2\\
&+288g_2^2g_4h_2-192g_2g_3h_2^2+18g_3h_3+108g_2g_3h_3+72g_2^2g_3h_3-144g_2^3g_3h_3\big)
\end{aligned}
$$

The P_4 constant-term computation is as follows:

In[12]:= `Together[Coefficient[` $\mathrm{x} * (\mathrm{x} * \mathrm{A}[3] * \mathrm{P}_3 + \mathrm{B}[3] * \mathrm{P}_3 - \mathrm{C}[3] * \mathrm{P}_2)$, x]]

$$Out[12]= \frac{1}{4\,(1+2\mathrm{g}_2)\,(1+6\mathrm{g}_2)^2}\big(-\mathrm{a}_1^2\mathrm{a}_2 + 2\mathrm{a}_2^2 + \mathrm{a}_1\mathrm{a}_3 - 8\mathrm{a}_1^2\mathrm{a}_2\mathrm{g}_2 + 20\mathrm{a}_2^2\mathrm{g}_2$$

$$+8\mathrm{a}_1\mathrm{a}_3\mathrm{g}_2 - 48\mathrm{a}_1^2\mathrm{a}_2\mathrm{g}_2^2 + 72\mathrm{a}_2^2\mathrm{g}_2^2 + 48\mathrm{a}_1\mathrm{a}_3\mathrm{g}_2^2 + 144\mathrm{a}_2^2\mathrm{g}_2^3 + 144\mathrm{a}_1^2\mathrm{a}_2\mathrm{g}_2^4$$

$$-144\mathrm{a}_1\mathrm{a}_3\mathrm{g}_2^4 - 2\mathrm{a}_1\mathrm{a}_2\mathrm{h}_2 + 6\mathrm{a}_3\mathrm{h}_2 + 12\mathrm{a}_1\mathrm{a}_2\mathrm{g}_2\mathrm{h}_2 + 36\mathrm{a}_3\mathrm{g}_2\mathrm{h}_2 - 72\mathrm{a}_1\mathrm{a}_2\mathrm{g}_2^2\mathrm{h}_2$$

$$+120\mathrm{a}_3\mathrm{g}_2^2\mathrm{h}_2 - 144\mathrm{a}_1\mathrm{a}_2\mathrm{g}_2^3\mathrm{h}_2 + 144\mathrm{a}_3\mathrm{g}_2^3\mathrm{h}_2 - 8\mathrm{a}_2\mathrm{h}_2^2 + 6\mathrm{a}_2\mathrm{h}_3 + 60\mathrm{a}_2\mathrm{g}_2\mathrm{h}_3$$

$$+120\mathrm{a}_2\mathrm{g}_2^2\mathrm{h}_3 - 144\mathrm{a}_2\mathrm{g}_2^3\mathrm{h}_3 - 12\mathrm{a}_1\mathrm{a}_2\mathrm{g}_3 + 18\mathrm{a}_3\mathrm{g}_3 - 24\mathrm{a}_1\mathrm{a}_2\mathrm{g}_2\mathrm{g}_3$$

$$+108\mathrm{a}_3\mathrm{g}_2\mathrm{g}_3 + 144\mathrm{a}_1\mathrm{a}_2\mathrm{g}_2^2\mathrm{g}_3 + 72\mathrm{a}_3\mathrm{g}_2^2\mathrm{g}_3 + 288\mathrm{a}_1\mathrm{a}_2\mathrm{g}_2^3\mathrm{g}_3 - 144\mathrm{a}_3\mathrm{g}_2^3\mathrm{g}_3$$

$$-36\mathrm{a}_2\mathrm{g}_3^2 - 144\mathrm{a}_2\mathrm{g}_2\mathrm{g}_3^2 + 144\mathrm{a}_2\mathrm{g}_2^2\mathrm{g}_3^2 + 8\mathrm{a}_2\mathrm{g}_4 + 96\mathrm{a}_2\mathrm{g}_2\mathrm{g}_4$$

$$-48\mathrm{a}_2\mathrm{g}_3\mathrm{h}_2 - 192\mathrm{a}_2\mathrm{g}_2\mathrm{g}_3\mathrm{h}_2 + 288\mathrm{a}_2\mathrm{g}_2^2\mathrm{g}_4\mathrm{h}_2\big)$$

Thus, equating the linear and constant-terms above with the linear and constant-terms of (3.17) we discover that the linear-term comparison is

$$
\begin{aligned}
a_3 + {} & a_2 h_2 + a_1 h_3 + h_4 = \\
& \frac{1}{4\left(1+2g_2\right)\left(1+6g_2\right)^2}\left(-a_1^3 + 3a_1a_2 + a_3 - 8a_1^3g_2 + 28a_1a_2g_2 + 20a_3g_2\right. \\
& 48a_1^3g_2^2 + 120a_1a_2g_2^2 + 120a_3g_2^2 + 144a_1a_2g_2^3 + 240a_3g_2^3 + 144a_1^3g_2^4 \\
& -144a_1a_2g_2^4 + 144a_3g_2^4 - 2a_1^2h_2 + 8a_2h_2 + 12a_1^2g_2h_2 + 56a_2g_2h_2 \\
& -72a_1^2g_2^2h_2 + 192a_2g_2^2h_2 - 144a_1^2g_2^3h_2 + 288a_2g_2^3h_2 - 4a_1h_2^2 \\
& +48a_1g_2h_2^2 + 48a_1g_2^2h_2^2 - 8h_2^3 + 7a_1h_3 + 68a_1g_2h_3 + 168a_1g_2^2h_3 \\
& -144a_1g_2^3h_3 - 144a_1g_2^4h_3 + 12h_2h_3 + 96g_2h_2h_3 + 240g_2^2h_2h_3 \\
& 12a_1^2g_3 + 18a_2g_3 - 24a_1^2g_2g_3 + 108a_2g_2g_3 + 144a_1^2g_2^2g_3 + 72a_2g_2^2g_3 \\
& +288a_1^2g_2^3g_3 - 144a_2g_2^3g_3 - 36a_1g_3^2 - 144a_1g_2g_3^2 + 144a_1g_2^2g_3^2 + 8a_1g_4 \\
& +96a_1g_2g_4 + 288a_1g_2^2g_4 - 42a_1g_3h_2 - 108a_1g_2g_3h_2 + 216a_1g_2^2g_3h_2 \\
& +144a_1g_2^3g_3h_2 - 36g_3^2h_2 - 144g_2g_3^2h_2 + 144g_2^2g_3^2h_2 + 8g_4h_2 + 96g_2g_4h_2 \\
& \left. +288g_2^2g_4h_2 - 192g_2g_3h_2^2 + 18g_3h_3 + 108g_2g_3h_3 + 72g_2^2g_3h_3 - 144g_2^3g_3h_3\right) \qquad (3.20)
\end{aligned}
$$

and the constant-term comparison is

$$
\begin{aligned}
a_4 =& \frac{1}{4\left(1+2g_2\right)\left(1+6g_2\right)^2}\left(-a_1^2a_2+2a_2^2+a_1a_3-8a_1^2a_2g_2+20a_2^2g_2\right. \\
&+8a_1a_3g_2-48a_1^2a_2g_2^2+72a_2^2g_2^2+48a_1a_3g_2^2+144a_2^2g_2^3+144a_1^2a_2g_2^4 \\
&-144a_1a_3g_2^4-2a_1a_2h_2+6a_3h_2+12a_1a_2g_2h_2+36a_3g_2h_2-72a_1a_2g_2^2h_2 \\
&+120a_3g_2^2h_2-144a_1a_2g_2^3h_2+144a_3g_2^3h_2-8a_2h_2^2+6a_2h_3+60a_2g_2h_3 \\
&+120a_2g_2^2h_3-144a_2g_2^3h_3-12a_1a_2g_3+18a_3g_3-24a_1a_2g_2g_3+108a_3g_2g_3 \\
&+144a_1a_2g_2^2g_3+72a_3g_2^2g_3+288a_1a_2g_2^3g_3-144a_3g_2^3g_3-36a_2g_3^2-144a_2g_2g_3^2 \\
&\left.+144a_2g_2^2g_3^2+8a_2g_4+96a_2g_2g_4-48a_2g_3h_2-192a_2g_2g_3h_2+288a_2g_2^2g_4h_2\right).
\end{aligned}
\tag{3.21}
$$

Now, continuing this process for P_5 we obtain three additional relationships for the P_5 quadratic, linear and constant-terms. The P_5 quadratic-term is computed to be as follows:

In[13]:= `Together[Coefficient[` $x * A[4] * P_4 + B[4] * P_4 - C[4] * P_3$, x^2`]]`

$$
\begin{aligned}
\textit{Out[13]}= &\left(-a_1^3+3a_1a_2+2a_3-24a_1^3g_2+76a_1a_2g_2+76a_3g_2-260a_1^3g_2^2\right. \\
&+788a_1a_2g_2^2+1008a_3g_2^2-1152a_1^3g_2^3+3744a_1a_2g_2^3+5664a_3g_2^3-2160a_1^3g_2^4 \\
&+7056a_1a_2g_2^4+12960a_3g_2^4-5760a_1^3g_2^5+8640a_1a_2g_2^5+8640a_3g_2^5-8640a_1^3g_2^6 \\
&+8640a_1a_2g_2^6-2a_1^2h_2+14a_2h_2-4a_1^2g_2h_2+332a_2g_2h_2-144a_1^2g_2^2h_2 \\
&+3216a_2g_2^2h_2-288a_1^2g_2^3h_2+14112a_2g_2^3h_2+4320a_1^2g_2^4h_2+21600a_2g_2^4h_2
\end{aligned}
$$

$$+8640a_1^2g_2^5h_2+8640a_2g_2^5h_2+a_1h_2^2+214a_1g_2h_2^2+1464a_1g_2^2h_2^2+4176a_1g_2^3h_2^2$$

$$+6480a_1g_2^4h_2^2+4320a_1g_2^5h_2^2-16h_2^3+192g_2^2h_2^3+13a_1h_3+336a_1g_2h_3+3204a_1g_2^2h_3$$

$$+13440a_1g_2^3h_3+28080a_1g_2^4h_3+34560a_1g_2^5h_3+8640a_1g_2^6h_3+34h_2h_3+684g_2h_2h_3$$

$$+4848g_2^2h_2h_3+12960g_2^3h_2h_3+12960g_2^4h_2h_3+8640g_2^5h_2h_3+2h_4+76g_2h_4$$

$$+1008g_2^2h_4+5664g_2^3h_4+12960g_2^4h_4+8640g_2^5h_4$$

$$-18a_1^2g_3+40a_2g_3-180a_1^2g_2g_3+744a_2g_2g_3-720a_1^2g_2^2g_3$$

$$+4896a_2g_2^2g_3-3744a_1^2g_2^3g_3+14976a_2g_2^3g_3-18720a_1^2g_2^4g_3+23040a_2g_2^4g_3$$

$$-25920a_1^2g_2^5g_3+17280a_2g_2^5g_3-180a_1g_3^2-1728a_1g_2g_3^2-3744a_1g_2^2g_3^2$$

$$-11520a_1g_2^3g_3^2-25920a_1g_2^4g_3^2-432g_3^3-4320g_2g_3^3-8640g_2^2g_3^3$$

$$-17280g_2^3g_3^3+26a_1g_4+620a_1g_2g_4+4656a_1g_2^2g_4+10656a_1g_2^3g_4$$

$$+7200a_1g_2^4g_4+8640a_1g_2^5g_4+120g_3g_4+2208g_2g_3g_4+12096g_2^2g_3g_4$$

$$+17280g_2^3g_3g_4+17280g_2^4g_3g_4-252a_1g_2g_3h_2+2256a_1g_2^2g_3h_2+7200a_1g_2^3g_3h_2$$

$$-4320a_1g_2^4g_3h_2-8640a_1g_2^5g_3h_2-684g_3^2h_2-6624g_2g_3^2h_2-12960g_2^2g_3^2h_2$$

$$-17280g_2^3g_3^2h_2-8640g_2^4g_3^2h_2+64g_4h_2+1344g_2g_4h_2+8448g_2^2g_4h_2$$

$$+11520g_2^3g_4h_2-240g_3h_2^2-1872g_2g_3h_2^2-2304g_2^2g_3h_2^2+90g_3h_3$$

$$+1548g_2g_3h_3+8496g_2^2g_3h_3+18720g_2^3g_3h_3+30240g_2^4g_3h_3+8640g_2^5g_3h_3\Big)$$

$$\Big/\left(10\left(1+6g_2\right)\left(1+12g_2+12g_2^2\right)^2\right)$$

The P_5 linear-term is

In[14]:= $\texttt{Together}[\texttt{Coefficient}[\ x*A[4]*P_4+B[4]*P_4-C[4]*P_3,\ x]]$

$$
\begin{aligned}
\textit{Out[14]=}\ \big(&-a_1^2a_2+2a_2^2+a_1a_3+a_4-22a_1^2a_2g_2+48a_2^2g_2+22a_1a_3g_2+38a_4g_2\\
&-216a_1^2a_2g_2^2+432a_2^2g_2^2+216a_1a_3g_2^2+504a_4g_2^2-720a_1^2a_2g_2^3\\
&+1728a_2^2g_2^3+720a_1a_3g_2^3+2832a_4g_2^3-720a_1^2a_2g_2^4+1440a_2^2g_2^4\\
&+720a_1a_3g_2^4+6480a_4g_2^4-4320a_1^2a_2g_2^5+4320a_1a_3g_2^5+4320a_4g_2^5\\
&-a_1^3h_2+a_1a_2h_2+8a_3h_2-22a_1^3g_2h_2+70a_1a_2g_2h_2\\
&+144a_3g_2h_2-216a_1^3g_2^2h_2+504a_1a_2g_2^2h_2+1056a_3g_2^2h_2-720a_1^3g_2^3h_2\\
&+2448a_1a_2g_2^3h_2+2880a_3g_2^3h_2-720a_1^3g_2^4h_2+6480a_1a_2g_2^4h_2-4320a_1^3g_2^5h_2\\
&+4320a_1a_2g_2^5h_2-2a_1^2h_2^2-4a_2h_2^2+72a_2g_2h_2^2-144a_1^2g_2^2h_2^2+624a_2g_2^2h_2^2\\
&+1440a_2g_2^3h_2^2+4320a_1^2g_2^4h_2^2-12a_1h_2^3-72a_1g_2h_2^3-432a_1g_2^2h_2^3\\
&-1440a_1g_2^3h_2^3+11a_2h_3+246a_2g_2h_3+1800a_2g_2^2h_3+5328a_2g_2^3h_3\\
&+7920a_2g_2^4h_3+4320a_2g_2^5h_3+15a_1h_2h_3+342a_1g_2h_2h_3+2280a_1g_2^2h_2h_3\\
&+6480a_1g_2^3h_2h_3+10800a_1g_2^4h_2h_3+4320a_1g_2^5h_2h_3-12h_2^2h_3-72g_2h_2^2h_3\\
&-432g_2^2h_2^2h_3-1440g_2^3h_2^2h_3+9h_3^2+198g_2h_3^2+1368g_2^2h_3^2\\
&+3600g_2^3h_3^2+6480g_2^4h_3^2+4320g_2^5h_3^2+a_1h_4+22a_1g_2h_4\\
&+216a_1g_2^2h_4+720a_1g_2^3h_4+720a_1g_2^4h_4+4320a_1g_2^5h_4+8h_2h_4\\
&-18a_1a_2g_3+36a_3g_3-144a_1a_2g_2g_3+576a_3g_2g_3-432a_1a_2g_2^2g_3\\
&+2880a_3g_2^2g_3-2880a_1a_2g_2^3g_3+5760a_3g_2^3g_3-12960a_1a_2g_2^4g_3
\end{aligned}
$$

$+8640a_3g_2^4g_3 - 216a_2g_3^2 - 2160a_2g_2g_3^2 - 4320a_2g_2^2g_3^2 - 8640a_2g_2^3g_3^2$

$+24a_2g_4 + 528a_2g_2g_4 + 3168a_2g_2^2g_4 + 2880a_2g_2^3g_4 - 18a_1^2g_3h_2 - 90a_2g_3h_2$

$-144a_1^2g_2g_3h_2 - 576a_2g_2g_3h_2 - 432a_1^2g_2^2g_3h_2 + 720a_2g_2^2g_3h_2 - 2880a_1^2g_2^3g_3h_2$

$+5760a_2g_2^3g_3h_2 - 4320g_2^4g_3h_2 - 12960a_1^2g_2^4g_3h_2 + 4320a_2g_2^4g_3h_2 - 216a_1g_3^2h_2$

$-2160a_1g_2g_3^2h_2 - 4320a_1g_2^2g_3^2h_2 - 8640a_1g_2^3g_3^2h_2 + 24a_1g_4h_2 + 528a_1g_2g_4h_2$

$+3168a_1g_2^2g_4h_2 + 2880a_1g_2^3g_4h_2 - 1152a_1g_2g_3h_2^2 - 2160a_1g_2^2g_3h_2^2 - 4320a_1g_2^4g_3h_2^2$

$+18a_1g_3h_3 + 432a_1g_2g_3h_3 + 2448a_1g_2^2g_3h_3 + 2880a_1g_2^3g_3h_3 - 4320a_1g_2^4g_3h_3$

$-216g_3^2h_3 - 2160g_2g_3^2h_3 - 4320g_2^2g_3^2h_3 - 8640g_2^3g_3^2h_3 + 24g_4h_3$

$+528g_2g_4h_3 + 3168g_2^2g_4h_3 + 2880g_2^3g_4h_3 - 126g_3h_2h_3 - 1152g_2g_3h_2h_3$

$-2160g_2^2g_3h_2h_3 + 36g_3h_4 + 576g_2g_3h_4 + 2880g_2^2g_3h_4 + 5760g_2^3g_3h_4 + 8640g_2^4g_3h_4$

$+144g_2h_2h_4 + 1056g_2^2h_2h_4 + 2880g_2^3h_2h_4) \,/\, \left(5\ (1+6g_2)\left(1+12g_2+12g_2^2\right)^2\right)$

and the constant-term is

In[15]:= `Together[Coefficient[` $x*(x*A[4]*P_4 + B[4]*P_4 - C[4]*P_3)$, x]]

Out[15]= $(-a_1^2a_3 + 2a_2a_3 + a_1a_4 - 22a_1^2a_3g_2 + 48a_2a_3g_2$

$+22a_1a_4g_2 - 216a_1^2a_3g_2^2 + 432a_2a_3g_2^2 + 216a_1a_4g_2^2 - 720a_1^2a_3g_2^3$

$+728a_2a_3g_2^3 + 720a_1a_4g_2^3 - 720a_1^2a_3g_2^4 + 1440a_2a_3g_2^4 + 720a_1a_4g_2^4$

$-4320a_1^2a_3g_2^5 + 4320a_1a_4g_2^5 - 2a_1a_3h_2 + 8a_4h_2 + 144a_4g_2h_2$

$-144\mathrm{a_1a_3g_2^2h_2} + 1056\mathrm{a_4g_2^2h_2} + 2880\mathrm{a_4g_2^3h_2} + 4320\mathrm{a_1a_3g_2^4h_2} - 12\mathrm{a_3h_2^2}$

$-72\mathrm{a_3g_2h_2^2} - 432\mathrm{a_3g_2^2h_2^2} - 1440\mathrm{a_3g_2^3h_2^2} + 9\mathrm{a_3h_3} + 198\mathrm{a_3g_2h_3}$

$+1368\mathrm{a_3g_2^2h_3} + 3600\mathrm{a_3g_2^3h_3} + 6480\mathrm{a_3g_2^4h_3}$

$-18\mathrm{a_1a_3g_3} + 36\mathrm{a_4g_3} - 144\mathrm{a_1a_3g_2g_3} + 576\mathrm{a_4g_2g_3} - 432\mathrm{a_1a_3g_2^2g_3}$

$+2880\mathrm{a_4g_2^2g_3} - 2880\mathrm{a_1a_3g_2^3g_3} + 5760\mathrm{a_4g_2^3g_3} - 12960\mathrm{a_1a_3g_2^4g_3} + 8640\mathrm{a_4g_2^4g_3}$

$-216\mathrm{a_3g_3^2} - 2160\mathrm{a_3g_2g_3^2} - 4320\mathrm{a_3g_2^2g_3^2} - 8640\mathrm{a_3g_2^3g_3^2} + 24\mathrm{a_3g_4}$

$+528\mathrm{a_3g_2g_4} + 3168\mathrm{a_3g_2^2g_4} + 2880\mathrm{a_3g_2^3g_4} - 126\mathrm{a_3g_3h_2} - 1152\mathrm{a_3g_2g_3h_2}$

$-2160\mathrm{a_3g_2^2g_3h_2} - 4320\mathrm{a_3g_2^4g_3h_2}.$

$+4320\mathrm{a_3g_2^5h_3}\big) \,/\, \Big(5\left(1+6\mathrm{g_2}\right)\left(1+12\mathrm{g_2}+12\mathrm{g_2^2}\right)^2\Big).$

This yields three new relations. First, we have the P_5 quadratic-term comparison:

$$\frac{a_3}{2!} + a_3g_2 + a_2g_2 + \frac{a_1h_2^2}{2!} + a_1h_3 + h_2h_3 + h_4 =$$

$$\big(-a_1^3 + 3a_1a_2 + 2a_3 - 24a_1^3g_2 + 76a_1a_2g_2 + 76a_3g_2 - 260a_1^3g_2^2$$

$$+788a_1a_2g_2^2 + 1008a_3g_2^2 - 1152a_1^3g_2^3 + 3744a_1a_2g_2^3 + 5664a_3g_2^3 - 2160a_1^3g_2^4$$

$$+7056a_1a_2g_2^4 + 12960a_3g_2^4 - 5760a_1^3g_2^5 + 8640a_1a_2g_2^5 + 8640a_3g_2^5 - 8640a_1^3g_2^6$$

$$+8640a_1a_2g_2^6 - 2a_1^2h_2 + 14a_2h_2 - 4a_1^2g_2h_2 + 332a_2g_2h_2 - 144a_1^2g_2^2h_2$$

$$+3216a_2g_2^2h_2 - 288a_1^2g_2^3h_2 + 14112a_2g_2^3h_2 + 4320a_1^2g_2^4h_2 + 21600a_2g_2^4h_2$$

$$+8640a_1^2g_2^5h_2 + 8640a_2g_2^5h_2 + a_1h_2^2 + 214a_1g_2h_2^2 + 1464a_1g_2^2h_2^2 + 4176a_1g_2^3h_2^2$$

$$+6480a_1g_2^4h_2^2 + 4320a_1g_2^5h_2^2 - 16h_2^3 + 192g_2^2h_2^3 + 13a_1h_3 + 336a_1g_2h_3 + 3204a_1g_2^2h_3$$

$$+13440a_1g_2^3h_3+28080a_1g_2^4h_3+34560a_1g_2^5h_3+8640a_1g_2^6h_3+34h_2h_3+684g_2h_2h_3$$

$$+4848g_2^2h_2h_3+12960g_2^3h_2h_3+12960g_2^4h_2h_3+8640g_2^5h_2h_3+2h_4+76g_2h_4$$

$$+1008g_2^2h_4+5664g_2^3h_4+12960g_2^4h_4+8640g_2^5h_4$$

$$-18a_1^2g_3+40a_2g_3-180a_1^2g_2g_3+744a_2g_2g_3-720a_1^2g_2^2g_3$$

$$+4896a_2g_2^2g_3-3744a_1^2g_2^3g_3+14976a_2g_2^3g_3-18720a_1^2g_2^4g_3+23040a_2g_2^4g_3$$

$$-25920a_1^2g_2^5g_3+17280a_2g_2^5g_3-180a_1g_3^2-1728a_1g_2g_3^2-3744a_1g_2^2g_3^2$$

$$-11520a_1g_2^3g_3^2-25920a_1g_2^4g_3^2-432g_3^3-4320g_2g_3^3-8640g_2^2g_3^3$$

$$-17280g_2^3g_3^3+26a_1g_4+620a_1g_2g_4+4656a_1g_2^2g_4+10656a_1g_2^3g_4$$

$$+7200a_1g_2^4g_4+8640a_1g_2^5g_4+120g_3g_4+2208g_2g_3g_4+12096g_2^2g_3g_4$$

$$+17280g_2^3g_3g_4+17280g_2^4g_3g_4-252a_1g_2g_3h_2+2256a_1g_2^2g_3h_2+7200a_1g_2^3g_3h_2$$

$$-4320a_1g_2^4g_3h_2-8640a_1g_2^5g_3h_2-684g_3^2h_2-6624g_2g_3^2h_2-12960g_2^2g_3^2h_2$$

$$-17280g_2^3g_3^2h_2-8640g_2^4g_3^2h_2+64g_4h_2+1344g_2g_4h_2+8448g_2^2g_4h_2$$

$$+11520g_2^3g_4h_2-240g_3h_2^2-1872g_2g_3h_2^2-2304g_2^2g_3h_2^2+90g_3h_3$$

$$+1548g_2g_3h_3+8496g_2^2g_3h_3+18720g_2^3g_3h_3+30240g_2^4g_3h_3+8640g_2^5g_3h_3\big)$$

$$/\left(10\left(1+6g_2\right)\left(1+12g_2+12g_2^2\right)^2\right). \tag{3.22}$$

Next, we have the P_5 linear-term comparison

$$a_4 + a_3h_2 + a_2h_3 + a_1h_4 + h_5 =$$

$$\big(-a_1^2a_2 + 2a_2^2 + a_1a_3 + a_4 - 22a_1^2a_2g_2 + 48a_2^2g_2 + 22a_1a_3g_2 + 38a_4g_2$$

$$-216a_1^2a_2g_2^2 + 432a_2^2g_2^2 + 216a_1a_3g_2^2 + 504a_4g_2^2 - 720a_1^2a_2g_2^3$$

$$+1728a_2^2g_2^3 + 720a_1a_3g_2^3 + 2832a_4g_2^3 - 720a_1^2a_2g_2^4 + 1440a_2^2g_2^4$$

$$+720a_1a_3g_2^4 + 6480a_4g_2^4 - 4320a_1^2a_2g_2^5 + 4320a_1a_3g_2^5 + 4320a_4g_2^5$$

$$-a_1^3h_2 + a_1a_2h_2 + 8a_3h_2 - 22a_1^3g_2h_2 + 70a_1a_2g_2h_2$$

$$+144a_3g_2h_2 - 216a_1^3g_2^2h_2 + 504a_1a_2g_2^2h_2 + 1056a_3g_2^2h_2 - 720a_1^3g_2^3h_2$$

$$+2448a_1a_2g_2^3h_2 + 2880a_3g_2^3h_2 - 720a_1^3g_2^4h_2 + 6480a_1a_2g_2^4h_2 - 4320a_1^3g_2^5h_2$$

$$+4320a_1a_2g_2^5h_2 - 2a_1^2h_2^2 - 4a_2h_2^2 + 72a_2g_2h_2^2 - 144a_1^2g_2^2h_2^2 + 624a_2g_2^2h_2^2$$

$$+1440a_2g_2^3h_2^2 + 4320a_1^2g_2^4h_2^2 - 12a_1h_2^3 - 72a_1g_2h_2^3 - 432a_1g_2^2h_2^3$$

$$-1440a_1g_2^3h_2^3 + 11a_2h_3 + 246a_2g_2h_3 + 1800a_2g_2^2h_3 + 5328a_2g_2^3h_3$$

$$+7920a_2g_2^4h_3 + 4320a_2g_2^5h_3 + 15a_1h_2h_3 + 342a_1g_2h_2h_3 + 2280a_1g_2^2h_2h_3$$

$$+6480a_1g_2^3h_2h_3 + 10800a_1g_2^4h_2h_3 + 4320a_1g_2^5h_2h_3 - 12h_2^2h_3 - 72g_2h_2^2h_3$$

$$-432g_2^2h_2^2h_3 - 1440g_2^3h_2^2h_3 + 9h_3^2 + 198g_2h_3^2 + 1368g_2^2h_3^2$$

$$+3600g_2^3h_3^2 + 6480g_2^4h_3^2 + 4320g_2^5h_3^2 + a_1h_4 + 22a_1g_2h_4$$

$$+216a_1g_2^2h_4 + 720a_1g_2^3h_4 + 720a_1g_2^4h_4 + 4320a_1g_2^5h_4 + 8h_2h_4$$

$$-18a_1a_2g_3 + 36a_3g_3 - 144a_1a_2g_2g_3 + 576a_3g_2g_3 - 432a_1a_2g_2^2g_3$$

$$+2880a_3g_2^2g_3 - 2880a_1a_2g_2^3g_3 + 5760a_3g_2^3g_3 - 12960a_1a_2g_2^4g_3 + 8640a_3g_2^4g_3$$

$$-216a_2g_3^2 - 2160a_2g_2g_3^2 - 4320a_2g_2^2g_3^2 - 8640a_2g_2^3g_3^2 + 24a_2g_4$$

$$+528a_2g_2g_4 + 3168a_2g_2^2g_4 + 2880a_2g_2^3g_4 - 18a_1^2g_3h_2 - 90a_2g_3h_2$$

$$-144a_1^2g_2g_3h_2 - 576a_2g_2g_3h_2 - 432a_1^2g_2^2g_3h_2 + 720a_2g_2^2g_3h_2 - 2880a_1^2g_2^3g_3h_2$$

$$+5760a_2g_2^3g_3h_2 - 4320g_2^4g_3h_2 - 12960a_1^2g_2^4g_3h_2 + 4320a_2g_2^4g_3h_2 - 216a_1g_3^2h_2$$

$$-2160a_1g_2g_3^2h_2 - 4320a_1g_2^2g_3^2h_2 - 8640a_1g_2^3g_3^2h_2 + 24a_1g_4h_2 + 528a_1g_2g_4h_2$$

$$+3168a_1g_2^2g_4h_2 + 2880a_1g_2^3g_4h_2 - 1152a_1g_2g_3h_2^2 - 2160a_1g_2^2g_3h_2^2 - 4320a_1g_2^4g_3h_2^2$$

$$+18a_1g_3h_3 + 432a_1g_2g_3h_3 + 2448a_1g_2^2g_3h_3 + 2880a_1g_2^3g_3h_3 - 4320a_1g_2^4g_3h_3$$

$$-216g_3^2h_3 - 2160g_2g_3^2h_3 - 4320g_2^2g_3^2h_3 - 8640g_2^3g_3^2h_3 + 24g_4h_3$$

$$+528g_2g_4h_3 + 3168g_2^2g_4h_3 + 2880g_2^3g_4h_3 - 126g_3h_2h_3 - 1152g_2g_3h_2h_3$$

$$-2160g_2^2g_3h_2h_3 + 36g_3h_4 + 576g_2g_3h_4 + 2880g_2^2g_3h_4 + 5760g_2^3g_3h_4$$

$$+8640g_2^4g_3h_4 + 144g_2h_2h_4 + 1056g_2^2h_2h_4$$

$$+2880g_2^3h_2h_4\big) \,/\, \left(5\left(1+6g_2\right)\left(1+12g_2+12g_2^2\right)^2\right) \tag{3.23}$$

and finally, the P_5 constant-term comparison is

$$
\begin{aligned}
a_5 =&\big(-a_1^2a_3 + 2a_2a_3 + a_1a_4 - 22a_1^2a_3g_2 + 48a_2a_3g_2\\
&\big(-a_1^2a_3 + 2a_2a_3 + a_1a_4 - 22a_1^2a_3g_2 + 48a_2a_3g_2\\
&+22a_1a_4g_2 - 216a_1^2a_3g_2^2 + 432a_2a_3g_2^2 + 216a_1a_4g_2^2 - 720a_1^2a_3g_2^3\\
&+728a_2a_3g_2^3 + 720a_1a_4g_2^3 - 720a_1^2a_3g_2^4 + 1440a_2a_3g_2^4 + 720a_1a_4g_2^4\\
&-4320a_1^2a_3g_2^5 + 4320a_1a_4g_2^5 - 2a_1a_3h_2 + 8a_4h_2 + 144a_4g_2h_2\\
&-144a_1a_3g_2^2h_2 + 1056a_4g_2^2h_2 + 2880a_4g_2^3h_2 + 4320a_1a_3g_2^4h_2 - 12a_3h_2^2\\
&-72a_3g_2h_2^2 - 432a_3g_2^2h_2^2 - 1440a_3g_2^3h_2^2 + 9a_3h_3 + 198a_3g_2h_3\\
&+1368a_3g_2^2h_3 + 3600a_3g_2^3h_3 + 6480a_3g_2^4h_3\\
&-18a_1a_3g_3 + 36a_4g_3 - 144a_1a_3g_2g_3 + 576a_4g_2g_3 - 432a_1a_3g_2^2g_3\\
&+2880a_4g_2^2g_3 - 2880a_1a_3g_2^3g_3 + 5760a_4g_2^3g_3 - 12960a_1a_3g_2^4g_3 + 8640a_4g_2^4g_3\\
&-216a_3g_3^2 - 2160a_3g_2g_3^2 - 4320a_3g_2^2g_3^2 - 8640a_3g_2^3g_3^2 + 24a_3g_4\\
&+528a_3g_2g_4 + 3168a_3g_2^2g_4 + 2880a_3g_2^3g_4 - 126a_3g_3h_2 - 1152a_3g_2g_3h_2\\
&-2160a_3g_2^2g_3h_2 - 4320a_3g_2^4g_3h_2\\
&+4320a_3g_2^5h_3\big) \,/\, \Big(5\left(1+6g_2\right)\left(1+12g_2+12g_2^2\right)^2\Big). \qquad (3.24)
\end{aligned}
$$

Notice, that when comparing the above polynomials that resulted from the generating function (3.1) with those that arose from the three-term recurrence relation (2.3) the x^n, x^{n-1} and x^{n-2} coefficients for $n = 3, 4, 5$ were always identical and the remaining coefficients were dissimilar. For example, for the P_3 comparisons we observe that the x^3, x^2 and x-coefficients from the polynomial P_3 in (3.16) obtained from the generating function (3.1) were the same as those that arose from computing P_3 from the three-term recurrence relation (2.3); however the constant terms were different. This pattern continued for the P_4 and P_5 comparisons and will continue for all other higher-order comparisons as well. This is apparent because the coefficients x^n, x^{n-1} and x^{n-2} obtained via three-term recurrence relation (2.3) are dependent on A_n, B_n and C_n in (3.14), which were derived from the generating function, and the lower-order coefficients x^{n-k} for $k = 3, 4, \ldots, n-1$ are not. For example, when comparing the coefficients of P_6 we also establish relationships for the constant, linear, quadratic and cubic terms, and as mentioned above, this pattern continues for all higher-order polynomials as well.

Now that these relations have been discovered it is necessary to analyze them in order to make inferences on their nature. However, we first note the complexity involved in the above relations. Namely, if our approach is to be effective, we must make some additional assumptions that simplify calculations and reduce our problem to a manageable format, the details of which are discussed next.

3.3 SOME COMMENTS ON THE COMPLEXITY OF THE SHEFFER B-TYPE 1 CLASS

We first discuss the ramifications of the complexity of the *B-Type 1* polynomials $P_0, \ldots, P_5$, which were discovered in Section 3.1 and the comparisons obtained in Section 3.2. We emphasize that the coefficients $c_{n,0}$, $c_{n,1}$ and $c_{n,2}$ as respectively defined in (3.11), (3.12) and (3.13) are *sums* involving the g_2-term and other multiplicative constants that grow arbitrary large as n increases. In fact, it can readily be shown that all of the coefficients of a given polynomial from the set $\{P_0, P_1, \ldots, P_5\}$, and all higher-order polynomials as well, are also sums that grow arbitrary large as n increases. The notable exceptions to this are the constant terms, since for a given polynomial P_k the constant term is a_k. This is in stark contrast to the *B-Type 0* polynomials and is a crucial observation because the *B-Type 0* polynomials have coefficients that have a $fixed$ structure.

To elaborate on this notion we take $G(t) \equiv 0$ in (3.1) and obtain

$$A(t)\exp[xH(t)] = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{3.25}$$

which is the generating function for the Sheffer *B-Type 0* polynomials. Also, we can obtain the coefficients of x^n, x^{n-1} and x^{n-2} for these polynomials by evaluating each of the expressions in (3.11), (3.12) and (3.13) by assigning $g_2 = g_3 = g_4 = 0$ since $G(t) \equiv 0$. We then see that (3.11), (3.12) and (3.13) then respectively become

$$\breve{c}_{n,0} := \frac{1}{n!} \tag{3.26}$$

$$\breve{c}_{n,1} := \frac{a_1}{(n-1)!} + \frac{h_2}{(n-2)!} \tag{3.27}$$

$$\breve{c}_{n,2} := \frac{a_2}{(n-2)!} + \frac{a_1h_2 + h_3}{(n-3)!} + \frac{h_2^2}{2!(n-4)!}, \tag{3.28}$$

where we have labeled these coefficients $\breve{c}_{n,0}$, $\breve{c}_{n,1}$ and $\breve{c}_{n,2}$ to distinguish them form the *B-Type 1* coefficients. It is now clear that for any n-value we have exactly one term in (3.26), at most two terms in (3.27) and at most three terms in (3.28). Whereas (3.11) has exactly $\lfloor n/2 \rfloor + 1$ terms, (3.12) at most $\lfloor (n-1)/2 \rfloor + \lfloor (n-2)/2 \rfloor + \lfloor (n-3)/2 \rfloor + 3$ terms and (3.13) at most$\lfloor (n-2)/2 \rfloor + \lfloor (n-3)/2 \rfloor + \lfloor (n-4)/2 \rfloor + \lfloor (n-5)/2 \rfloor + \lfloor (n-6)/2 \rfloor + 5$

terms. Therefore, we obtain expressions for A_n, B_n and C_n (which we label $\breve{A}_n$, $\breve{B}_n$ and $\breve{C}_n$) in the three-term recurrence (2.3) using (3.26), (3.27) and (3.28) above:

$$\begin{aligned}
\breve{A}_n &:= \frac{1}{n+1} \\
\breve{B}_n &:= \frac{a_1 + 2nh_2}{n+1} \\
\breve{C}_n &:= \frac{1}{n+1}(a_1^2 - 2a_2 + 2a_1h_2 - 4h_2^2 + 3h_3 + (4h_2^2 - 3h_3)n).
\end{aligned}$$

Without any computations, it is intuitively obvious that the A_n, B_n and C_n in the *B-Type 1* class will *not* be as simple to work with. For example, the expression for A_n was obtained using Mathematica and is as follows:

`Together[A[n]]`

$$\frac{4\text{Gamma}[1+\text{n}]\text{HypergeometricU}[-\frac{1}{2}-\frac{\text{n}}{2},\frac{1}{2},-\frac{1}{4\text{g}_2}]}{\text{Gamma}[1+\text{n}]\text{HypergeometricU}[\frac{1}{2}-\frac{\text{n}}{2},\frac{3}{2},-\frac{1}{4\text{g}_2}]},$$

where

$$\text{Gamma[z]} := \Gamma(n) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad \text{Re}(z) > 0$$

and

$$\text{HypergeometricU[a, b, z]} := {}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}.$$

Proceeding a wealth of computing time, expressions were eventually obtained for B_n and C_n as well. However, these expressions are so obtrusive that to display them would be entirely disadvantageous - the actual size of the general expression for C_n exceeds two pages. Nonetheless, we can omit displaying explicit forms for B_n and C_n with impunity because our point of emphasis has been demonstrated as evidenced by the A_n term alone.

The second aspect we address is the exceedingly cumbersome expressions that arise when comparing respective coefficients from both the generating function (3.1) and the three-term recurrence relation (2.3) of a given *B-Type 1* polynomial as done in the previous section. In particular, we see a wealth of g_3 and g_4 terms in the comparisons developed in section 3.2 - for a paradigm example, consider the P_5 linear-term comparison. Therefore, the P_4 and P_5 comparisons of the coefficients obtained from the generating function (3.1) and the three-term recurrence relation (2.3) (and any subsequent comparisons for that matter) would be exceedingly more manageable if we additionally take $g_i = 0, \forall i \geq 3$.

Now, there are some very important structures embedded in the comparisons themselves and we address these next. We first turn our attention to the comparison of the coefficients of the constant-term in the polynomial P_3 that was derived in (3.19), which we rewrite below.

$$a_3 = \frac{1}{3(1+2g_2)^2}(-a_1^3 + 3a_1a_2 + 4a_1a_2g_2 - 12a_1^3g_2^2 + 12a_1a_2g_2^2 -$$
$$2a_1^2h_2 + 4a_2h_2 + 12a_1^2g_2h_2 - 4a_1h_2^2 + 3a_1h_3 + 12a_1g_2h_3 + 12a_1g_2^2h_3).$$

As seen above, and in all of the comparisons we established, g_2 is one of the most abundant terms, which is intuitively obvious since g_2 is in the argument of each sum in (3.11), (3.12) and (3.13). Therefore, the natural consideration that emerges is whether or not an assumption can be made on g_2 that will reduce the complexity of (3.19). We also mention that all simplifying assumptions that are additional to the original suppositions $a_0 = 1$ and $h_1 = 1$ must be restricted to alterations of the $G(t)$ terms, as varying the terms of $H(t)$ or $A(t)$ would not reduce (3.1) to the *B-Type 0* class when we take $G(t) \equiv 0$.

Now, several assumptions on the g_2-term were attempted before an appropriate choice was made. Below is an example of a particular choice for g_2 that could *not* be made, but led to a beneficial discovery nonetheless.

Notice that (3.19) can be set equal to zero, thusly taking on the form

$$\begin{aligned}&-a_1^3+3a_1a_2+4a_1a_2g_2-12a_1^3g_2^2+12a_1a_2g_2^2-2a_1^2h_2+4a_2h_2\\&+12a_1^2g_2h_2-4a_1h_2^2+3a_1h_3+12a_1g_2h_3+12a_1g_2^2h_3-3a_3(1+2g_2)^2=0.\end{aligned}$$

Then, the choice of $g_2=-1/2$ would reduce the above to the following format:

$$-a_1^3+a_1a_2-4a_1^2h_2+a_2h_2-a_1h_2^2=0.$$

However, the choice of $g_2=-1/2$ results in (3.15) becoming

$$P_2(x)=a_2+(a_1+h_2)x$$

since $g_2=-1/2$ is a zero of (3.11) for $n=2$. Therefore, this choice is not permissible as it violates Definition 2.1. Moreover, with this choice of g_2 there would never be a polynomial of degree 2 in the sequence $\{P_n\}$ as defined by (3.1). In addition, g_2 obviously cannot be a solution to $\phi_n=0$.

Nonetheless, we now consider the choice of $g_2=1/2$. This particular choice reduces (3.19) to the following form:

$$-a_1^3+2a_1a_2-3a_3+a_1^2h_2+a_2h_2-a_1h_2^2+3a_1h_3=0.$$

This results in a modified expression for a_3:

$$a_3=\frac{1}{3}(-a_1^3+2a_1a_2+a_1^2h_2+a_2h_2-a_1h_2^2+3a_1h_3). \tag{3.29}$$

Thus, in weighing (3.29) against (3.19) the discrepancies in the levels of complexity are apparent. However, there is still an unanswered question. What does the choice of $g_2 = 1/2$ do to the other comparisons that were discovered in Section 3.2? To answer this inquiry, we refer to the later comparisons as developed in Section 3.2 and substitute $g_2 = 1/2$ for each case, starting with the constant and linear-term comparisons of P_4 and continuing through the constant, linear and quadratic-term comparisons of P_5.

With the choice of $g_2 = 1/2$ the P_4 constant-term comparison (3.21) becomes

$$a_4 = \frac{1}{16}(-a_1^2a_2 + 6a_2^2 + a_1a_3 - 4a_1a_2h_2 + 9a_3h_2 - a_2h_2^2 + 6a_2h_3) \tag{3.30}$$

and the P_4 linear-term comparison (3.20) simplifies to

$$\begin{aligned} &-a_1^3 + 7a_1a_2 - 6a_3 - 4a_1^2h_2 - a_2h_2 + 4a_1h_2^2 - h_2^2 - 9a_1h_3 \\ &\quad + 15h_2h_3 - 16h_4 = 0. \end{aligned}$$

Then, for the P_5 constant-term comparison we see that (3.24) takes the form

$$\begin{aligned} a_5 = &\frac{1}{250}(-42a_1^2a_3 + 55a_2a_3 + 42a_1a_4 + 29a_1a_3h_2 + 88a_4h_2 \\ &- 42a_3h_2^2 + 180a_3h_3), \end{aligned} \tag{3.31}$$

the P_5 linear-term comparison (3.23) becomes

$$\begin{aligned} &-42a_1^2a_2 + 55a_2^2 + 42a_1a_3 - 120a_4 - 42a_1^3h_2 + 126a_1a_2h_2 - 162a_3h_2 \\ &+ 29a_1^2h_2^2 + 46a_2h_2^2 - 42a_1h_2^3 - 15a_2h_3 + 297a_1h_2h_3 - 42h_2^2h_3 + 180h_3^2 \\ &- 208a_1h_4 + 88h_2h_4 - 250h_5 = 0 \end{aligned} \tag{3.32}$$

and the P_5 quadratic-term comparison (3.22) is now

$$\begin{aligned}&-42a_1^3+97a_1a_2-120a_3+29a_1^2h_2+23a_2h_2-29a_1h_2^2+2h_2^3\\&+102a_1h_3+18h_2h_3-120h_4=0\end{aligned} \tag{3.33}$$

In all of the comparisons above, the reduction in complexity is evident.

Now that the above comparisons have been established, we see some very important potential patterns. First, we notice that the highest h-term in the expression for a_3 as defined by (3.29) is h_3. This is also the case for the expressions for a_4 in (3.21) and a_5 in (3.24). Second, we observe that the highest h-term in the P_4 linear comparison (3.20) is h_4 and the highest h-term in the P_5 linear comparison (3.25) is h_5. These observations are important because they provide information on how to develop necessary conditions that the terms a_1, a_2, a_3 and h_2 must satisfy in order for the *B-Type 1* polynomials as defined by (3.1), with $g_2 = 1/2$ and $g_i = 0, \forall i \geq 3$, to be orthogonal. We discuss this further.

With the assignment of $g_2 = 1/2$ and $g_i = 0, \forall i \geq 3$, we first discover an expression for h_3 by utilizing the relation for a_3 in (3.29), which results in

$$h_3 = \frac{1}{3a_1}(a_1^3 + 3a_3 - 2a_1a_2 - a_1^2h_2 - a_2h_2 + a_1h_2^2). \tag{3.34}$$

Therefore, we have written h_3 in terms of only a_1, a_2, a_3 and h_2. Now notice that the P_4 linear comparison (3.20) can be solved for h_4 as follows, with a_3 replaced by (3.29)

$$h_4 = \frac{1}{16}(a_1^3 + 3a_1a_2 - 6a_1^2h_2 - 3a_2h_2 + 6a_1h_2^2 - h_2^3 - 15a_1h_3 + 15h_2h_3)$$

and the right-hand side of the above equation involves the terms a_1, a_2, h_2 and h_3. However, we can then substitute (3.34) and obtain an expression for h_4 that involves only a_1, a_2, a_3

and h_2, as written below

$$h_4 = \frac{1}{16a_1}\Big(a_1^3 + 3a_1a_2 - 6a_1^2h_2 - 3a_2h_2 + 6a_1h_2^2 - h_2^3$$
$$+ 5\left(-a_1^3 + 2a_1a_2 - 3a_3 + a_1^2h_2 + a_2h_2 - a_1h_2^2\right) \tag{3.35}$$
$$-\frac{5h_2}{a_1}\left(-a_1^3 + 2a_1a_2 - 3a_3 + a_1^2h_2 + a_2h_2 - a_1h_2^2\right)\Big).$$

Also, observe that the P_5 linear-term comparison (3.23) can be solved for h_5:

$$h_5 = \frac{1}{250}(-42a_1^2a_2 + 55a_2^2 + 42a_1a_3 - 120a_4 - 42a_1^3h_2 + 126a_1a_6h_2$$
$$- 162a_3h_2 + 29a_1^2h_2 + 46a_2h_2^2 - 42a_1h_2^3 - 15a_2h_3 + 297a_1h_2h_3$$
$$- 42h_2^2h_3 + 180h_3^2 - 208a_1h_4 + 88h_2h_4)$$

and the right-hand side of the above equation involves only the terms a_1, a_2, a_3, a_4, h_2, h_3 and h_4. We then substitute in (3.34) and (3.35), with a_4 replaced by (3.30), which results in

$$h_5 = \frac{1}{250}\Bigg[\frac{-23a_1^4}{2} - \frac{23a_1^2a_2}{2} + 10a_2^2 + 46a_1^3h_2 + \frac{29a_1a_2h_2}{2}$$
$$- 59a_1^2h_2^2 - 23a_2h_2^2 + \frac{69a_1h_2^3}{2}$$
$$- \frac{23}{2}a_1\left(-a_1^3 + 2a_1a_2 - 3a_3 + a_1^2h_2 + a_2h_2 - a_1h_2^2\right)$$
$$+ \frac{20}{a_1}a_2\left(-a_1^3 + 2a_1a_2 - 3a_3 + a_1^2h_2 + a_2h_2 - a_1h_2^2\right)$$
$$- \frac{45}{2}h_2\left(-a_1^3 + 2a_1a_2 - 3a_3 + a_1^2h_2 + a_2h_2 - a_1h_2^2\right)$$
$$+ \frac{14h_2^2}{a_1}\left(-a_1^3 + 2a_1a_2 - 3a_3 + a_1^2h_2 + a_2h_2 - a_1h_2^2\right)$$
$$+ \frac{20}{a_1^2}\left(-a_1^3 + 2a_1a_2 - 3a_3 + a_1^2h_2 + a_2h_2 - a_1h_2^2\right)^2$$
$$- 13a_1\big(a_1^3 + 3a_1a_2 - 6a_1^2h_2 - 3a_2h_2 + 6a_1h_2^2 - h_2^3$$

$$+\frac{20}{a_1^2}\left(-a_1^3+2a_1a_2-3a_3+a_1^2h_2+a_2h_2-a_1h_2^2\right)^2$$

$$-13a_1\left(a_1^3+3a_1a_2-6a_1^2h_2-3a_2h_2+6a_1h_2^2-h_2^3\right.$$

$$+5\left(a_1^3+2a_1a_2-3a_3+a_1^2h_2+a_2h_2-a_1h_2^2\right)$$

$$-\frac{5h_2}{a_1}\left(-a_1^3+2a_1a_2-3a_3+a_1^2h_2+a_2h_2-a_1h_2^2\right)$$

$$+\frac{11}{2}h_2\left(a_1^3+3a_1a_2-6a_1^2h_2-3a_2h_2+6a_1h_2^2-h_2^3\right.$$

$$+5\left(-a_1^3+2a_1a_2-3a_3+a_1^2h_2+a_2h_2-a_1h_2^2\right)\Big)$$

$$\left.-\frac{5h_2}{a_1}\left(-a_1^3+2a_1a_2-3a_3+a_1^2h_2+a_2h_2-a_1h_2^2\right)\right]. \tag{3.36}$$

Thus, (3.36) now only involves a_1, a_2, a_3 and h_2. Lastly, upon substituting (3.34) and (3.35) into (3.33) and using some algebraic manipulations we see that the P_5 quadratic-term comparison becomes

$$\frac{1}{2a_1}\left(a_1-h_2\right)\left(44a_1^3-14a_1^2h_2-63\left(a_2h_2-3a_3\right)+a_1\left(44h_2^2-137a_2\right)\right)=0. \tag{3.37}$$

In the above analysis we were able to reduce (3.20) and (3.23) to expressions for h_4 and h_5 that involve only a_1, a_2, a_3 and h_2. In addition, in (3.37) we obtained an algebraic equation that related all of the terms a_1, a_2, a_3 and h_2. Therefore, if the polynomials P_6, P_7 and P_8 behave in the same fashion as P_3, P_4 and P_5 we can construct a system of four simultaneous nonlinear equations with unknowns a_1, a_2, a_3 and h_2. Solving this system will yield the conditions that a_1, a_2, a_3 and h_2 must satisfy in order for the *B-Type 1* polynomials as defined by (3.1), with $g_2 = 1/2$ and $g_i = 0, \forall i \geq 3$, to be orthogonal as was mentioned earlier.

Based on the analysis that led to (3.37), we now derive comparisons between the generating function (3.1) and the three-term recurrence relation (2.3) for the P_6 cubic term, the P_7 fourth-degree term and the P_8 fifth-degree term. That is, from our previous analysis it appears that the highest h-term in each of the comparisons for the P_6 cubic term, the P_7 fourth-degree term and the P_8 fifth-degree will be h_4. We first construct the polynomials P_6 and P_7 using the same Mathematica program `Genfunction` and the methodology described in Section 3.2. These computations are as follows.

The P_6 polynomial that results from expanding (3.1) accordingly is

$$\begin{aligned} P_6(x) =& a_6 + (a_5 + a_4h_2 + a_3h_3 + a_2h_4 + a_1h_5 + h_6)x \\ &+ \left(a_4 + a_3h_2 + \frac{a_2h_2^2}{2} + a_2h_3 + a_1h_2h_3 + \frac{h_3^2}{2} + a_1h_4 + h_2h_4 + h_5\right)x^2 \\ &+ \frac{1}{6}(4a_3 + 6a_2h_2 + 3a_1h_2^2 + h_2^3 + 6a_1h_3 + 6h_2h_3 + 6h_4)x^3 \\ &+ \frac{1}{12}(5a_2 + 8a_1h_2 + 6h_2^2 + 8h_3)x^4 \\ &+ \frac{1}{60}(13a_1 + 25h_2)x^5 + \frac{19}{180}x^6 \end{aligned} \tag{3.38}$$

and the P_7 polynomial that results from (3.1) is discovered to be

$$\begin{aligned}P_7(x) =& a_7 + (a_6 + a_5h_2 + a_4h_3 + a_3h_4 + a_2h_5 + a_1h_6 + h_7)x \\ &+ \left(a_5 + a_4h_2 + \frac{a_3h_2^2}{2} + a_3h_3 + a_2h_2h_3 + \frac{a_1h_3^2}{2} + a_2h_4\right. \\ &\left.+a_1h_2h_4 + h_3h_4 + a_1h_5 + h_2h_5 + h_6\right) x^2 \\ &+ \left(\frac{2a_4}{3} + a_3h_2 + \frac{a_2h_2^2}{2} + \frac{a_1h_2^3}{6} + a_2h_3 + a_1h_2h_3 + \frac{h_2^2h_3}{2}\right. \\ &\left.+ \frac{h_3^2}{2} + a_1h_4 + h_2h_4 + h_5\right) x^3 \\ &+ \left(\frac{5a_3}{12} + \frac{2a_2h_2}{3} + \frac{a_1h_2^2}{2} + \frac{h_2^3}{6} + \frac{2a_1h_3}{3} + h_2h_3 + \frac{2h_4}{3}\right) x^4 \\ &+ \left(\frac{13a_2}{60} + \frac{5a_1h_2}{12} + \frac{h_2^2}{3} + \frac{5h_3}{12}\right) x^5 \\ &+ \left(\frac{19a_1}{180} + \frac{13h_2}{60}\right) x^6 + \frac{29}{630}x^7 \end{aligned} \tag{3.39}$$

Now that all of our simplifying assumptions have been finalized and all of our essential computations have been completed, we restate our current problem.

The generating function (3.1) now becomes

$$A(t)\exp\left[xH(t) + \frac{1}{2}x^2t^2\right] = \sum_{n=0}^{\infty} P_n(x)t^n \tag{3.40}$$

and (3.11), (3.12) and (3.13) now have the form

$$c_{n,0} := \phi_n(1/2)$$

$$c_{n,1} := a_1\phi_{n-1}(1/2) + h_2\phi_{n-2}(1/2)$$

$$c_{n,2} := a_2\phi_{n-2}(1/2) + (a_1h_2 + h_3)\phi_{n-3}(1/2) + \frac{h_2^2}{2!}\phi_{n-4}(1/2).$$

By using the same method as conducted in Section 3.2 we compare coefficients of x^k from the generating function (3.40) and the three-term recurrence relation (2.3) and thusly have the following results for the P_6 cubic-term comparison, the P_7 fourth-degree term comparison and the P_8 fifth-degree term comparison. We first state each respective comparison and then substitute the expressions for h_3 and h_4, as discovered in (3.34) and (3.35), as necessary.

The P_6 cubic-term comparison is

$$\frac{1}{4a_1}(-320a_1^3 + 1438a_1a_2 - 1365a_3 - 478a_1^2h_2 - 73a_2h_2 + 478a_1h_2^2$$
$$- 135h_2^3 - 1236a_1h_3 + 2601h_2h_3 - 3900h_4) = 0.$$

Upon substituting (3.34) and (3.35) into the above expression we obtain

$$3\,(a_1 - h_2)\left(324a_1^3 + 92a_1^2h_2 - 469\,(a_2h_2 - 3a_3)\right.$$
$$\left.+a_1\left(-1209a_2 + 324h_2^2\right)\right) = 0. \tag{3.41}$$

The P_7 fourth-degree term comparison becomes

$$- 17225a_1^3 + 43730a_1a_2 - 49647a_3 + 7945a_1^2h_2 + 5917a_2h_2 - 7945a_1h_2^2$$
$$+ 676h_2^3 + 32139a_1h_3 + 17508h_2h_3 - 90896h_4 = 0.$$

After substituting (3.34) and (3.35) into the above expression we have

$$\frac{3}{a_1}\,(a_1 - h_2)\left(5404a_1^3 - 1148a_1^2h_2 - 7523\,(-3a_3 + a_2h_2)\right.$$
$$\left.+a_1\left(-17183a_2 + 5404h_2^2\right)\right) = 0. \tag{3.42}$$

Lastly, the P_8 fifth-degree term comparison is as follows

$$- 477085440a_1^3 + 1796404480a_1a_2 - 177307392a_3 - 36514816a_1^2h_2$$
$$- 2333056a_2h_2 + 36514816a_1h_2^2 - 11393920h_2^3 - 102887808a_1h_3$$
$$+ 280195200h_2h_3 - 558581760h_4 = 0.$$

From substituting in (3.34) and (3.35) above we observe that

$$\frac{1280}{a_1}\left(a_1-h_2\right)\left(45032a_1^3+7168a_1^2h_2-63405\left(a_2h_2-3a_3\right)\right.$$
$$\left.+a_1\left(45032h_2^2-160637a_2\right)\right)=0. \tag{3.43}$$

Hence, (3.37), (3.41), (3.42) and (3.43) result in a simultaneous system of nonlinear algebraic equations in the variables a_1, a_2, a_3 and h_2 as seen below.

EQUATION 1:

$$\frac{1}{2a_1}\left(a_1-h_2\right)\left(44a_1^3-14a_1^2h_2-63\left(a_2h_2-3a_3\right)\right.$$
$$\left.+a_1\left(44h_2^2-137a_2\right)\right)=0 \tag{3.44}$$

EQUATION 2:

$$\frac{3}{4a_1}\left(a_1-h_2\right)\left(324a_1^3+92a_1^2h_2-469\left(a_2h_2-3a_3\right)\right.$$
$$\left.+a_1\left(324h_2^2-1209a_2\right)\right)=0 \tag{3.45}$$

EQUATION 3:

$$\frac{3}{a_1}\left(a_1-h_2\right)\left(5404a_1^3-1148a_1^2h_2-7523\left(a_2h_2-3a_3\right)\right.$$
$$\left.+a_1\left(5404h_2^2-17183a_2\right)\right)=0 \tag{3.46}$$

EQUATION 4:

$$\frac{1}{a_1}1280\left(a_1-h_2\right)\left(45032a_1^3+7168a_1^2h_2-63405\left(a_2h_2-3a_3\right)\right.$$
$$\left.+a_1\left(45032h_2^2-160637a_2\right)\right)=0 \tag{3.47}$$

It is interesting that (3.44), (3.45), (3.46) and (3.47) all have the same structure. That is, they all fit the format

$$\frac{d_1}{a_1}\left(a_1 - h_2\right)\left(d_2 a_1^3 + d_3 a_1^2 h_2 + d_4\left(a_2 h_2 - 3a_3\right) + a_1\left(d_5 h_2^2 - d_6 a_2\right)\right) = 0 \tag{3.48}$$

for certain integer constants $d_1, d_2, \ldots, d_6$.

Now that the above system is established, our goal is to find *all* of its solutions, either explicit or implicit, in the variables a_1, a_2, a_3 and h_2. Upon determining all of the system's solutions we obtain restrictions on the variables a_1, a_2, a_3 and h_2 that must be satisfied in order for the polynomial sequences defined by (3.40) to be orthogonal. Namely, if all of the restrictions lead to contradictions, then no orthogonal polynomial sequences can exist and if at least one of the restrictions does not lead to a contradiction, then orthogonal polynomial sequences will arise.

Determining all of the solutions to the above system is a daunting task and we again utilize Mathematica, which has the ability to simultaneously solve nonlinear systems of algebraic equations. The command we use here is entitled `Solve`, which yields only *generic* solutions, i.e. conditions on the variables that one explicitly solves for, and not on any other parameters in the system - refer to [21] for more details.

We first note that the factor $(a_1 - h_2)$ appears in all of the equations (3.44), (3.45), (3.46) and (3.47) and that omitting this factor from each of them results in a second system, which is as follows:

EQUATION 1A:

$$\frac{1}{2a_1}\left(44a_1^3 - 14a_1^2h_2 - 63\left(a_2h_2 - 3a_3\right)\right.$$
$$\left. + a_1\left(44h_2^2 - 137a_2\right)\right) = 0 \tag{3.49}$$

EQUATION 2A:

$$\frac{3}{4a_1}\left(324a_1^3 + 92a_1^2h_2 - 469\left(a_2h_2 - 3a_3\right) + \right.$$
$$\left. a_1\left(324h_2^2 - 1209a_2\right)\right) = 0 \tag{3.50}$$

EQUATION 3A:

$$\frac{3}{a_1}\left(5404a_1^3 - 1148a_1^2h_2 - 7523\left(a_2h_2 - 3a_3\right)\right.$$
$$\left. + a_1\left(5404h_2^2 - 17183a_2\right)\right) = 0 \tag{3.51}$$

EQUATION 4A:

$$\frac{1280}{a_1}\left(45032a_1^3 + 7168a_1^2h_2 - 63405\left(a_2h_2 - 3a_3\right)\right.$$
$$\left. + a_1\left(45032h_2^2 - 160637a_2\right)\right) = 0 \tag{3.52}$$

The reminder of this section amounts to analyzing the solution sets to the system defined by (3.49), (3.50), (3.51) and (3.52) that arise upon implementing the `Solve` command.

Now, even though the entire system comprising (3.49), (3.50), (3.51) and (3.52) can be solved by Mathematica it is imperative to explain how this is accomplished, as there are some subtleties.

First, we respectively assign the left-hand side of each of (3.49), (3.50), (3.51) and (3.52) as {`E1,...,E4`}, then construct the following:

In[1]:=`Solve[{E1==0, E2==0, E3==0, E4==0},{`a_1, a_2, a_3, h_2`}]`

Out[1]=$\left\{\left\{\mathrm{a}_3 \to \frac{2}{3}\mathrm{a}_1^3, \mathrm{a}_2 \to \mathrm{a}_1^2, \mathrm{h}_2 \to \mathrm{a}_1\right\}, \left\{\mathrm{a}_3 \to \frac{2}{3}\mathrm{a}_1^3, \mathrm{a}_2 \to \mathrm{a}_1^2, \mathrm{h}_2 \to \mathrm{a}_1\right\}\right\}$

Solve::svars : Equations may not give solutions for all "solve" variables.

This prompts Mathemtica to simultaneously solve the above system for each of the variables a_1, a_2, a_3 and h_2. In this case we see that Mathematica outputs the preface; "*Equations may not give solutions for all "solve" variables*" indicating that Mathematica did not necessarily solve the system. In fact, if Mathematica does not display this preface, then the output can be interpreted as definitive. Also, any combination of these variables can be treated as parameters and then Mathematica can attempt to find implicit solutions. For example, we can solve the above system for a_1 and treat a_2, a_3 and h_2 as parameters as seen below:

In[2]:=`Solve[{E1==0, E2==0, E3==0, E4==0},{`a_1`}]`

Out[2]={}

For this particular case, Mathematica outputs "{}", which indicates that there does not exist any solutions to the system for the variable a_1. Therefore, to more accurately determine whether or not the system defined by (3.49), (3.50), (3.51) and (3.52) has any solutions we must thusly exhaust *all* possible parameter selections and there are of course a total of $\sum_{n=1}^{4} \mathrm{C}(4, n) = 15$ choices. As it turns out, only the following choices yield non-null outputs:

In[3]:=`Solve[{E1==0, E2==0, E3==0, E4==0},{`$\mathrm{a}_2, \mathrm{a}_3, \mathrm{h}_2$`}]`

Out[3]=$\left\{\left\{\mathrm{a}_3 \to \frac{2}{3}\mathrm{a}_1^3, \mathrm{a}_2 \to \mathrm{a}_1^2, \mathrm{h}_2 \to \mathrm{a}_1\right\}\right\}$

In[4]:=`Solve[{E1==0, E2==0, E3==0, E4==0},{`$\mathrm{a}_1, \mathrm{a}_3, \mathrm{h}_2$`}]`

Out[4]=$\left\{\left\{\mathrm{a}_3 \to -\frac{2}{3}\mathrm{a}_2^{3/2}, \mathrm{h}_2 \to -\sqrt{\mathrm{a}_2}, \mathrm{a}_1 \to -\sqrt{\mathrm{a}_2}\right\}, \left\{\mathrm{a}_3 \to \frac{2}{3}\mathrm{a}_2^{3/2}, \mathrm{h}_2 \to \sqrt{\mathrm{a}_2}, \mathrm{a}_1 \to \sqrt{\mathrm{a}_2}\right\}\right\}$

In[5]:=`Solve[{E1==0, E2==0, E3==0, E4==0},{`$\mathrm{a}_1, \mathrm{a}_2, \mathrm{h}_2$`}]`

Out[5]=$\left\{\left\{\mathrm{a}_2 \to \left(-\frac{3}{2}\right)^{2/3} \mathrm{a}_3^{2/3}, \mathrm{h}_2 \to -\left(-\frac{3}{2}\right)^{1/3} \mathrm{a}_3^{1/3}, \mathrm{a}_1 \to -\left(-\frac{3}{2}\right)^{1/3} \mathrm{a}_3^{1/3}\right\},\right.$

$\left\{\mathrm{a}_2 \to \left(\frac{3}{2}\right)^{2/3} \mathrm{a}_3{}^{2/3}, \mathrm{h}_2 \to \left(\frac{3}{2}\right)^{1/3} \mathrm{a}_3{}^{1/3}, \mathrm{a}_1 \to \left(\frac{3}{2}\right)^{1/3} \mathrm{a}_3{}^{1/3}\right\},$

$\left.\left\{\mathrm{a}_2 \to -(-1)^{1/3} \left(\frac{3}{2}\right)^{2/3} \mathrm{a}_3^{2/3}, \mathrm{h}_2 \to (-1)^{2/3} \left(\frac{3}{2}\right)^{1/3} \mathrm{a}_3^{1/3}, \mathrm{a}_1 \to (-1)^{2/3} \left(\frac{3}{2}\right)^{1/3} \mathrm{a}_3^{1/3}\right\}\right\}$

In[6]:=`Solve[{E1==0, E2==0, E3==0, E4==0},{`$\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3$`}]`

Out[6]=$\left\{\left\{\mathrm{a}_3 \to \frac{2}{3}\mathrm{a}_1^3, \mathrm{a}_2 \to \mathrm{a}_1^2, \mathrm{h}_2 \to \mathrm{a}_1\right\}\right\}$

Each of the above implicit solution sets each has a multiplicity of 2. Mathematica actually outputs each of the above solution sets twice to indicate this in the same fashion it solves the simple quadratic equation $x^2 - 2x + 1 = 0$ as:

In[6]:=`Solve[`$\mathrm{x}^2 - 2\mathrm{x} + 1 == 0$`,{`$\mathrm{x}$`}]`

Out[6]=$\{\{x \to 1\}, \{x \to 1\}\}$

We only write our solution sets once to avoid redundancies. Also, all of these solutions can be back-substituted into Mathematica as an additional verification.

Now notice, that we only obtained a non-null solution set when we implicitly solved for three of the variables a_1, a_2, a_3 and h_2. In particular, when we solved for a_2, a_3 and h_2 we obtained the same solution set as when we solved for a_1, a_2 and a_3, i.e. when either a_1 or h_2 was omitted as a variable. When we solved for a_1, a_3 and h_2, or a_1, a_2 and h_2 we achieved different solution sets and it is worth mentioning that these solution sets are very similar in structure to the set that resulted when implicitly solving for a_2, a_3 and h_2 or a_1, a_2 and a_3. For example, the solution set $\left\{a_3 = -\frac{2}{3}a_2^{3/2},\ h_2 = -\sqrt{a_2},\ a_1 = -\sqrt{a_2}\right\}$ can readily be shown to be equivalent to $\left\{a_3 = -\frac{2}{3}a_1^3,\ a_2 = a_1^2,\ h_2 = a_1;\ h_2 < 0 \text{ and } a_1 < 0\right\}$. The other relations can be arranged in a similar manor.

Although exceedingly difficult, the solutions obtained above can be derived without computer algebra. For example, we derive the solution set $\left\{a_3 = \frac{2}{3}a_1^3,\ a_2 = a_1^2,\ h_2 = a_1\right\}$. We first solve equation (3.49) for a_2, which yields:

$$a_2 = \frac{44a_1^3 + 189a_3 - 14a_1^2h_2 + 44a_1h_2^2}{137a_1 + 63h_2}.$$

Substituting this expression into the equation (3.50) gives

$$\frac{-13212a_1^3 - 53613a_3 + 43959a_1^2h_2 + 5331a_1h_2^2 - 336h_2^3}{274a_1 + 126h_2} = 0$$

and solving this equation for a_3 we obtain

$$a_3 = \frac{-4404a_1^3 + 14653a_1^2h_2 + 1777a_1h_2^2 - 112h_2^3}{17871}.$$

We next substitute these expressions for a_2 and a_3 into (3.51) resulting in

$$\frac{2954496\,(a_1-h_2)^2}{5957}=0,$$

which clearly has the solution $h_2 = a_1$ with a multiplicity of 2. Lastly, upon taking the expression derived for a_2, a_3 and h_2 above we see that the fourth equation is satisfied. Moreover, with a_2, a_3 and h_2 as defined above and a_1 taken as a free variable we see $a_2 = a_1^2$, $a_3 = \frac{2}{3}a_1^3$ and $h_2 = a_1$. The other relationships can be established in a similar way.

3.4 NECESSARY CONDITIONS FOR ORTHOGONALITY

From Section 3.3 we achieved restrictions on the a and h-terms that must be satisfied in order for the polynomial sequence defined by (3.40) to be orthogonal via the Mathematica command `Solve`. We thusly examine each of these solution sets in order to determine which, if any, orthogonal sets arise. We begin with the solution set $\left\{h_2 = a_1,\ \ a_2 = a_1^2,\ \ a_3 = \frac{2}{3}a_1^3\right\}$.

We first note that if we expand C_n from (3.14) for $n = 1$ we have

$$C_1 = a_1 h_2 - a_2. \tag{3.53}$$

Then, using only the assumptions $a_1 = h_2$ and $a_2 = a_1^2$ of the solution set above, (3.53) clearly becomes zero. However, this contradicts (2.3) as we originally required $A_n A_{n-1} C_n > 0$. Thus, it is impossible for $\{P_n\}$ as defined by (3.40) to be an orthogonal polynomial sequence with respect to the restrictions $\left\{h_2 = a_1,\ \ a_2 = a_1^2,\ \ a_3 = \frac{2}{3}a_1^3\right\}$. Moreover, each of the other solution sets obtained in Section 3.3 also result in (3.53) equaling zero, which can be readily verified.

We next consider the solution set $\{h_2 = a_1\}$ to the unaltered system defined by (3.44), (3.45), (3.46) and (3.47) and eventually determine what additional relationships must be established along with this assignment. In order to accomplish this, we construct another simultaneous system of equations. That is, in this case the assignment $h_2 = a_1$ must be true for every polynomial defined by (3.40) and upon substituting $h_2 = a_1$ in the comparisons that have been discovered, we obtain new conditions on the other a-terms that must be satisfied in order for the polynomial sequences defined by (3.40) to be orthogonal. The system we construct comprises the P_7 cubic-term comparison, the P_8 fourth-degree-term comparison and the P_9 fifth-degree-term comparison, which we list below.

By using the same methodology as in the previous section we discover that upon comparing the P_9 fifth-degree coefficient from the generating function (3.40) with the P_9 fifth-degree coefficient from the three-term recurrence relation (2.3) we achieve

$$\begin{aligned}
&-2355761408a_1^2a_2 + 4000450688a_2^2 + 2355761408a_1a_3 - 8468200448a_4 \\
&-4530310400a_1^3h_2 + 12934556928a_1a_2h_2 - 14068309248a_3h_2 + 1367446400a_1^2h_2^2 \\
&+1601051392a_2h_2^2 - 1760236800a_1h_2^3 + 79585280h_2^4 - 262058112a_2h_3 \\
&+15976406400a_1h_2h_3 + 4466400000h_2^2h_3 + 9558190080h_3^2 - 19256371200a_1h_4 \\
&-23084631040h_2h_4 - 23786681600h_5 = 0. \qquad (3.54)
\end{aligned}$$

In the same fashion we obtain an equation for the P_8 fourth-degree comparison

$$
\begin{aligned}
&-18349440a_1^2a_2+50743040a_2^2+18349440a_1a_3-111716352a_4-29359104a_1^3h_2\\
&+96503808a_1a_2h_2-83940480a_3h_2-22470656a_1^2h_2^2-2333056a_2h_2^2+23741952a_1h_2^3\\
&-8200704h_2^4+20868864a_2h_3-30389760a_1h_2h_3+186828288h_2^2h_3-7082496h_3^2\\
&-170940416a_1h_4-275924992h_2h_4-200299520h_5=0
\end{aligned}
\tag{3.55}
$$

and also for the P_7 cubic-term comparison

$$
\begin{aligned}
&-13780a_1^2a_2+21204a_2^2+13780a_1a_3-45448a_4-20670a_1^3h_2\\
&+58832a_1a_2h_2-67626a_3h_2+9534a_1^2h_2^2+11834a_2h_2^2-12666a_1h_2^3\\
&+546h_2^4-2964a_2h_3+98910a_1h_2h_3+3348h_2^2h_3+58995h_3^2\\
&-90480a_1h_4-45864h_2h_4-111150h_5=0.
\end{aligned}
\tag{3.56}
$$

We now take the equation (3.54) and substitute $h_2 = a_1$ and the expressions for h_3, h_4 and h_5 as respectively defined in (3.34), (3.35) and (3.36), then we substitute the expression for a_4 as in (3.30) with $g_2 = 1/2$ and after a wealth of algebra we eventually obtain

$$
\begin{aligned}
-\frac{48896}{a_1^2}\big(&35270a_1^6-170298a_1^4a_2+203820a_1^2a_2^2+149637a_1^3a_3\\
&-356013a_1a_2a_3+154782a_3^2\big)=0.
\end{aligned}
\tag{3.57}
$$

Next, for the equation defined by (3.55) we again substitute the expressions for h_3, h_4 and h_5 as respectively defined in (3.34), (3.35) and (3.36) and then substitute the expression for

a_4 as in (3.30) with $g_2 = 1/2$ and ultimately achieve

$$-\frac{44544}{5a_1^2}\left(2805a_1^6 - 15182a_1^4a_2 + 19925a_1^2a_2^2 + 14358a_1^3a_3 \right.$$
$$\left. -37002a_1a_2a_3 + 16983a_3^2\right) = 0. \tag{3.58}$$

Finally, for the equation defined by (3.56) we repeat the same procedure that that was used to obtain (3.58) and (3.59) and after some algebraic manipulations we obtain

$$-\frac{57}{a_1^2}\left(85a_1^6 - 412a_1^4a_2 + 491a_1^2a_2^2 + 363a_1^3a_3 - 855a_1a_2a_3 + 369a_3^2\right) = 0. \tag{3.59}$$

We now solve the simultaneous nonlinear comprising (3.57), (3.58) and (3.59) in the variables a_1, a_2 and a_3 by again utilizing the Mathematica command `Solve`. In this case, we assign the left-hand sides of (3.54), (3.56) and (3.57) as `S1,S2` and `S3` respectively and obtain the following outputs.

In[1]:=`Solve[{S1==0, S2==0, S3==0},{`$\mathrm{a}_2, \mathrm{a}_3$`}]`

Out[1]=$\left\{\left\{\mathrm{a}_3 \rightarrow \frac{2\mathrm{a}_1^3}{3}, \mathrm{a}_2 \rightarrow \mathrm{a}_1^2\right\}\right\}$

In[2]:=`Solve[{S1==0, S2==0, S3==0},{`$\mathrm{a}_1, \mathrm{a}_3$`}]`

Out[2]=$\left\{\left\{\mathrm{a}_3 \rightarrow -\frac{2}{3}\mathrm{a}_1^{3/2}, \mathrm{a}_1 \rightarrow -\sqrt{\mathrm{a}_2}\right\}\right\}$

$\left\{\left\{\mathrm{a}_3 \rightarrow \frac{2}{3}\mathrm{a}_1^{3/2}, \mathrm{a}_1 \rightarrow \sqrt{\mathrm{a}_2}\right\}\right\}$

In[3]:=`Solve[{S1==0, S2==0, S3==0},{`$\mathrm{a_1,a_2}$`}]`

$$\textit{Out[3]}=\left\{\mathrm{a_2}\to\left(-\tfrac{3}{2}\right)^{2/3}\mathrm{a_3^{2/3}},\mathrm{a_1}\to\left(-\tfrac{3}{2}\right)^{1/3}\mathrm{a_3^{1/3}}\right\}$$

$$\left\{\mathrm{a_2}\to\left(\tfrac{3}{2}\right)^{2/3}\mathrm{a_3^{2/3}},\mathrm{a_1}\to\left(\tfrac{3}{2}\right)^{1/3}\mathrm{a_3^{1/3}}\right\}$$

$$\left\{\mathrm{a_2}\to(-1)^{2/3}\left(\tfrac{3}{2}\right)^{2/3}\mathrm{a_3^{2/3}},\mathrm{a_1}\to(-1)^{2/3}\left(\tfrac{3}{2}\right)^{1/3}\mathrm{a_3^{1/3}}\right\}$$

Each of the above solution sets has a multiplicity of 2. In addition, all other outputs yield null solution sets and the output with respect to the variable selection $\{a_1, a_2, a_3\}$ includes the preface *"Equations may not give solutions for all "solve" variables".*

These solutions can be obtained without the use of computer algebra as well. As an example we derive the first solution set above. We begin by solving (3.57) for a_2 yielding

$$a_2 = \frac{170298a_1^3 + 2\sqrt{61620801}a_1^3 + 356013a_3 - 3\sqrt{61620801}a_3}{407640a_1}$$

or

$$a_2 = \frac{170298a_1^3 - 2\sqrt{61620801}a_1^3 + 356013a_3 + 3\sqrt{61620801}a_3}{407640a_1}.$$

Substituting the top branch into (3.58) and using a wealth of algebraic manipulations, we eventually discover that the only solution for a_3 is $\left\{a_3 = \frac{2}{3}a_1^3\right\}$. Putting these newly obtained relationships for a_2 and a_3 into the left-hand side of (3.59) we see that the result is zero and the equation (3.59) is satisfied.

Now notice that with a_2 and a_3 as defined above we see that a_2 reduces to $a_2 = a_1^2$. Hence, with respect to the top branch above of the solution of (3.57) for a_2, the only solution set to the simultaneous nonlinear system defined by (3.57), (3.58) and (3.59) is $\left\{a_3 = \frac{2}{3}a_1^3,\ a_2 = a_1^2\right\}$.

In addition, if we take the bottom branch of the solution of (3.57) for a_2 as shown above and use the same method that was used for the top branch, we discover that the only solution set is again $\left\{a_3 = \frac{2}{3}a_1^3,\ \ a_2 = a_1^2\right\}$.

Thus, with the assumption that $a_1 = h_2$ we additionally achieve $\left\{a_3 = \frac{2}{3}a_1^3,\ \ a_2 = a_1^2\right\}$. This is, of course, equivalent to the solution set $\left\{h_2 = a_1,\ \ a_2 = a_1^2,\ \ a_3 = \frac{2}{3}a_1^3\right\}$, which led to a contradiction. The other solution sets also lead to contradictions, as any of the restrictions resulting from these sets coupled with the assumption $a_1 = h_2$ make (3.53) zero. We summarize as follows.

As originally discussed in Section 3.1, the polynomial sequences that result from (3.40) are identical to the ones that results from (2.3) if and only if $\{P_n\}$ as defined by (3.40) is orthogonal. Now, Section 3.3 establishes several conditions that the terms in some of the coefficients of the lower-order polynomials of the sequence $\{P_n\}$ as defined by (3.40) must satisfy for orthogonality. From the previous analysis, we have shown that all of these restrictions established thus far lead to a contradiction. Hence, we have the following summarizing statement.

Result 3.1. *The simultaneous nonlinear algebraic system defined by (3.44), (3.45), (3.46) and (3.47) has an implicit solution set* $\{h_2 = a_1\}$, which leads to a contradiction. *When the factor* $(a_1 - h_2)$ *is omitted, the system defined by (3.49), (3.50), (3.51) and (3.52) emerges, which has the following implicit solution sets:*

$$\left\{h_2 = a_1,\;\; a_2 = a_1^2,\;\; a_3 = \frac{2}{3}a_1^3\right\}$$

when solved for a_1 or h_2,

$$\left\{a_3 = \pm\frac{2}{3}a_2^{3/2},\;\; h_2 = \pm\sqrt{a_2},\;\; a_1 = \pm\sqrt{a_2}\right\}$$

when solved for a_2, and

$$\left\{a_2 = \left(\pm\frac{3}{2}\right)^{2/3} a_3^{2/3},\;\; h_2 = \pm\left(\pm\frac{3}{2}\right)^{1/3} a_3^{1/3},\;\; a_1 = \pm\left(\pm\frac{3}{2}\right)^{1/3} a_3^{1/3}\right\}$$

and

$$\left\{a_2 = -(-1)^{1/3}\left(\frac{3}{2}\right)^{2/3} a_3^{2/3},\;\; h_2 = (-1)^{2/3}\left(\frac{3}{2}\right)^{1/3} a_3^{1/3},\;\; a_1 = (-1)^{2/3}\left(\frac{3}{2}\right)^{2/3} a_3^{1/3}\right\}$$

when solved for a_3. Each of these solution sets has a multiplicity of 2 and all of the solutions (restrictions) obtained thus far have resulted in no orthogonal polynomial sequences.

Remark: It is important to again mention that for the derivations of the h-terms in (3.35) and (3.36) we substituted the a_3 and a_4-values as necessary. Namely, for the h_4-term we substituted the a_3-value in (3.29) and for the h_5-term we substituted the a_4-value in (3.30), whcih is defined in terms of a_3. The h_3 term did not involve any additional substitutions. We comment on this further in Chapter 5.

CHAPTER 4: THE SHEFFER B-TYPE 2 ORTHOGONAL POLYNOMIAL SEQUENCES

In this chapter we address an extension to the research previously completed in Chapter 3 that amounts to analyzing a special case of (2.8). We again attempt to discover which, if any, orthogonal polynomial sequences arise. Throughout this chapter we heavily rely on the techniques and knowledge that was gained from Chapter 3.

We begin with modifying (2.8) by assuming that $H_2(t) \equiv 0$, which results in

$$A(t)\exp\Big[xH_1(t) + x^3\breve{G}(t)\Big] = \sum_{n=0}^{\infty} Q_n(x)t^n, \quad h_{1,1} \neq 0 \tag{4.1}$$

where we have assigned $H_3 := \breve{G}$ and $\breve{G}(t) = g_3t^3 + g_4t^4 + \cdots$ to be fairly consistent with the notation of Chapter 3. Also, we initially assign $a_0 = h_1 = 1$ as was done in (3.1).

Now, notice that taking $\breve{G}(t) \equiv 0$ reduces the generating function (4.1) to the Sheffer *B-Type 0* generating function. It is also important to observe that (4.1) is structurally similar to (3.1). In fact, the only differences are the x^3 - term in (4.1) as opposed to the x^2 - term in (3.1) and the fact that $\breve{G}(t) = \breve{g}_3t^3 + \breve{g}_4t^4 + \cdots$ whereas $G(t) = g_2t^2 + g_3t^3 + \cdots$. Therefore, we attempt to complete analysis on (4.1) that is entirely analogous to the analysis that was completed on (3.1) in Chapter 3.

As was done in (3.4), we write (4.1) as

$$\sum_{m=0}^{\infty} a_m t^m \prod_{i=1}^{\infty}\left[\sum_{j=0}^{\infty} \frac{h_i^j x^j t^{ij}}{j!}\right] \prod_{k=3}^{\infty}\left[\sum_{l=0}^{\infty} \frac{g_k^l x^{3l} t^{kl}}{l!}\right].$$

Writing out the general term for each of the products above results in

$$a_{l_0}t^{l_0}\cdot\frac{x^{l_1}t^{l_1}}{l_1!}\cdot\frac{h_2^{l_2}x^{l_2}t^{2l_2}}{l_2!}\cdot\frac{h_3^{l_3}x^{l_3}t^{3l_3}}{l_3!}\cdot\cdots\\ \cdot\frac{\breve{g}_3^{l_4}x^{3l_4}t^{3l_4}}{l_4!}\cdot\frac{\breve{g}_4^{l_5}x^{3l_5}t^{4l_5}}{l_5!}\cdot\frac{\breve{g}_5^{k_6}x^{3l_6}t^{5l_6}}{l_6!}\cdot\cdots, \tag{4.2}$$

where $\{l_0, l_1, l_2, \ldots\}$ are all non-negative integers. For this case, the sums of the x-exponents and the t-exponents of (4.2) respectively take on the form

$$l_1 + l_2 + l_3 + 2l_4 + 2l_5 + 2l_6 + \cdots = r \tag{4.3}$$

and

$$l_0 + l_1 + 2l_2 + 3l_3 + 3l_4 + 4l_5 + 5l_6 + \cdots = s. \tag{4.4}$$

Then, using the same methodology as in Chapter 3 to discover the coefficients of x^nt^n, $x^{n-1}t^n$ and $x^{n-2}t^n$ as respectively defined as (3.7), (3.9) and (3.10) we obtain the following.

(1.) *The coefficient of* x^nt^n:

$$\sum_{l_1+3l_4=n}\frac{\breve{g}_3^{l_4}}{l_1!l_4!} \tag{4.5}$$

(2.) *The coefficient of* $x^{n-1}t^n$:

$$\sum_{l_1+3l_4=n-1}\frac{a_1\breve{g}_3^{l_4}}{l_1!l_4!}+\sum_{l_1+3l_4=n-2}\frac{h_2\breve{g}_3^{l_4}}{l_1!l_4!}+\sum_{l_1+3l_4=n-4}\frac{\breve{g}_4\breve{g}_3^{l_4}}{l_1!l_4!} \tag{4.6}$$

(3.) ***The coefficient of*** $x^{n-2}t^n$**:**

$$\sum_{l_1+3l_4=n-2} \frac{a_2\breve{g}_3^{l_5}}{l_1!l_4!} + \sum_{l_1+3l_4=n-3} \frac{(a_1h_2+h_3)\breve{g}_3^{l_4}}{l_1!l_4!} + \sum_{l_1+3l_4=n-4} \frac{h_2^2\breve{g}_3^{l_4}}{2!l_1!l_4!}$$
$$+ \sum_{l_1+3l_4=n-5} \frac{(\breve{g}_5+a_1\breve{g}_4)\breve{g}_3^{l_4}}{l_1!l_4!} + \sum_{l_1+3l_4=n-6} \frac{h_2\breve{g}_4\breve{g}_3^{l_4}}{l_1!l_4!} + \sum_{l_1+3l_4=n-8} + \frac{\breve{g}_4^2\breve{g}_3^{l_4}}{2!l_1!l_4!} \tag{4.7}$$

We see that each of the terms above involve the expression

$$\sum_{l_1+3l_4=n} \frac{\breve{g}_3^{l_4}}{l_1!l_4!}$$

and that

$$\sum_{k_1+3k_4=n} \frac{\breve{g}_3^{l_4}}{l_1!l_4!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\breve{g}_3^k}{(n-3k)!k!}.$$

Therefore, we define

$$\psi_n(x) := \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{x^k}{(n-3k)!k!}.$$

Thus, taking $\psi_n(\breve{g}_3) := \psi_n$ the coefficients of x^nt^n, $x^{n-1}t^n$ and $x^{n-2}t^n$ become:

$$d_{n,0} := \psi_n \tag{4.8}$$

$$d_{n,1} := a_1\psi_{n-1} + h_2\psi_{n-2} + \breve{g}_4\psi_{n-4} \tag{4.9}$$

$$d_{n,2} := a_2\psi_{n-2} + (a_1h_2+h_3)\psi_{n-3} + \frac{h_2^2}{2!}\psi_{n-4} + (\breve{g}_5+a_1\breve{g}_4)\psi_{n-5}$$
$$+ h_2\breve{g}_4\psi_{n-6} + \frac{\breve{g}_4^2}{2!}\psi_{n-8}. \tag{4.10}$$

Based on this analysis we see that $Q_n(x)$ as defined by (4.1) has the form

$$Q_n(x) = d_{n,0}x^n + d_{n,1}x^{n-1} + d_{n,2}x^{n-2} + \text{L.O.T.}$$

We next write the unrestricted three-term recurrence relation of (2.3) as

$$Q_{n+1}(x) = (\breve{A}_n x + \breve{B}_n)Q_n(x) - \breve{C}_n Q_{n-1}(x), \tag{4.11}$$

where $P_0(x) = 1,\;\; P_{-1}(x) = 0,\;$ and $\;\breve{A}_n \breve{A}_{n-1} \breve{C}_n > 0$. Expanding the substitution of Q_n into (4.11) we achieve

$$\begin{aligned}
&d_{n+1,0}x^{n+1} + d_{n+1,1}x^n + d_{n+1,2}x^{n-1} + \text{L.O.T.} \\
&= \breve{A}_n d_{n,0}x^{n+1} + \breve{A}_n d_{n,1}x^n + \breve{A}_n d_{n,2}x^{n-1} + \text{L.O.T.} \\
&\quad + \breve{B}_n d_{n,0}x^n + \breve{B}_n d_{n,1}x^{n-1} + \breve{B}_n d_{n,2}x^{n-2} + \text{L.O.T.} \\
&\quad - \breve{C}_n d_{n-1,0}x^{n-1} - \breve{C}_n d_{n-1,1}x^{n-2} - \breve{C}_n d_{n-1,2}x^{n-3} + \text{L.O.T.}
\end{aligned}$$

Thus, comparing the coeffiecents of x^{n+1}, x^n and x^{n-1} above results in the following lower-triangular simultaneous system of linear equations:

$$\begin{bmatrix} d_{n,0} & 0 & 0 \\ d_{n,1} & d_{n,0} & 0 \\ d_{n,2} & d_{n,1} & -d_{n-1,0} \end{bmatrix} \begin{bmatrix} \breve{A}_n \\ \breve{B}_n \\ \breve{C}_n \end{bmatrix} = \begin{bmatrix} d_{n+1,0} \\ d_{n+1,1} \\ d_{n+1,2} \end{bmatrix}.$$

Since the diagonal terms $d_{n,0}$ and $d_{n-1,0}$ are non-zero, solving this system via elementary methods yields the following:

$$\begin{aligned}
\breve{A}_n &= \frac{d_{n+1,0}}{d_{n,0}} \\
\breve{B}_n &= \frac{d_{n+1,1}d_{n,0} - d_{n+1,0}d_{n,1}}{d_{n,0}^2} \\
\breve{C}_n &= \frac{d_{n+1,0}(d_{n,0}d_{n,2} - d_{n,1}^2) + d_{n,0}(d_{n+1,1}d_{n,1} - d_{n+1,2}d_{n,0})}{d_{n-1,0}d_{n,0}^2}.
\end{aligned} \tag{4.12}$$

We can now obtain *any* polynomial in the sequence $\{Q_n\}$ from either the generating function (4.1) or the three-term recurrence relation (4.11) with $\breve{A}$, $\breve{B}$ and $\breve{C}$ as defined in (4.12). From there, we can develop enough comparisons to achieve conditions for the polynomial sequence $\{Q_n\}$ as defined by (4.1) to be orthogonal, as was done in Chapter 3. However, our work here is minimized since the analysis from Chapter 3 gave us many insights regarding the complexities of the *B-Type 1* class and thusly which simplifying assumptions to make. Based on this reasoning, (4.1) would be exceedingly more manageable if we additionally take $g_i = 0, \forall i \geq 4$. Chapter 3 also gives us strong intuition as to which comparisons are best suited for our analysis. Namely, we should only need to establish comparisons for the Q_5 quadratic-term, the Q_6 cubic-term, the Q_7 fourth-degree term and the Q_8 fifth-degree term. Lastly, as we took $g_2 = 1/2$ in the *B-Type 1* case, we take $\breve{g}_3 = 1/3$ for the *B-Type 2* case. It is important to note that we discovered through much experimentation, in the same fashion as Chapter 3, that in fact $\breve{g}_2 = 1/3$ appears to be the *best* choice for reducing the complexities of the *B-Type 2* class. These additional simplifications result in (4.8), (4.9) and (4.10) becoming

$$d_{n,0} := \psi_n(1/3) \tag{4.13}$$

$$d_{n,1} := a_1\psi_{n-1}(1/3) + h_2\psi_{n-2}(1/3) \tag{4.14}$$

$$d_{n,2} := a_2\psi_{n-2}(1/3) + (a_1h_2 + h_3)\psi_{n-3}(1/3) + \frac{h_2^2}{2!}\psi_{n-4}(1/3) \tag{4.15}$$

and (4.1) transforming into:

$$A(t)\exp\left[xH_1(t) + \frac{1}{3}x^3t^3\right] = \sum_{n=0}^{\infty} Q_n(x)t^n, \quad h_{1,1} \neq 0. \tag{4.16}$$

To develop each of these comparisons we use the same methodology that was used in Chapter 3. In fact, we only need to slightly modify our program `GenPoly` by replacing x^2 with x^3 and $k = 3$ as instead of $k = 2$. To avoid redundancies, we only display complete details of the Q_5 quadratic-term comparison and display only the final format for all subsequent comparisons.

For the Q_5 quadratic-term comparison we first expand (4.1) using `GenPoly`, which gives us:

In[1]:= `Expand[Coefficient[`

$$\left(1+\sum_{m=1}^{10} a_m t^m\right) * \prod_{j=1}^{10}\left[\sum_{i=0}^{10} \frac{h_j^i x^i t^{ji}}{i!}\right] * \prod_{k=3}^{10}\left[\sum_{l=0}^{10} \frac{g_k^l x^{3l} t^{kl}}{l!}\right], t^2, h_1 = 1]]$$

$$\textit{Out[1]=}\ \frac{x^5}{120} + \frac{x^4 a_1}{24} + \frac{x^3 a_2}{6} + \frac{x^2 a_3}{2} + x a_4 + a_5 + \frac{x^5 g_3}{2} + x^4 a_1 g_3 + x^3 a_2 g_3 + x^4 g_4$$
$$+ x^3 a_1 g_4 + x^3 g_5 + \frac{x^4 h_2}{6} + \frac{x^3 a_1 h_2}{2} + x^2 a_2 h_2 + x a_3 h_2 + x^4 g_3 h_2 + \frac{x^3 {h_2}^2}{2} + \frac{x^2 a_1 {h_2}^2}{2}$$
$$+ \frac{x^3 h_3}{2} + x^2 a_1 h_3 + x a_2 h_3 + x^2 h_2 h_3 + x^2 h_4 + x a_1 h_4 + x h_5$$

It is clear that the coefficient of x^2 is

$$\frac{a_3}{2} + a_2 h_2 + \frac{a_1 h_2^2}{2} + a_1 h_3 + h_2 h_3 + h_4. \tag{4.17}$$

We next discover the Q_5 quadratic coefficient via the three-term recurrence relation (4.11) by first modifying our program `ThreeTerm`. We first define $d_{n,0}$, $d_{n,1}$ and $d_{n,2}$, as established in (4.13), (4.14) and (4.15), respectively.

$$\textit{In[1]:=}\ d_0[n\_] := \sum_{k=0}^{Floor[n/2]} \frac{g_3^k}{(n-3k)!k!}$$

$$\textit{In[2]:=}\ \mathrm{d_1[n\_] := a_1 * \sum_{k=0}^{Floor[(n-1)/2]} \frac{g_3^k}{(n-1-3k)!k!} + h_2 * \sum_{k=0}^{Floor[(n-2)/2]} \frac{g_3^k}{(n-2-3k)!k!}}$$

$$\textit{In[3]:=}\ \mathrm{d_2[n\_] := a_2 * \sum_{k=0}^{Floor[(n-2)/2]} \frac{g_3^k}{(n-2-3k)!k!}}$$
$$\mathrm{+(a_1h_2+h_3) * \sum_{k=0}^{Floor[(n-3)/2]} \frac{g_3^k}{(n-3-3k)!k!} + \frac{h_2^2}{2!} * \sum_{k=0}^{Floor[(n-4)/2]} \frac{g_3^k}{(n-4-3k)!k!}}$$

Then we define the $\breve{A}_n$, $\breve{B}_n$ and $\breve{C}_n$, as derived in (4.12).

$$\textit{In[4]:=}\ \mathrm{A[n\_] := \frac{d_0[1+n]}{d_0[n]}}$$

$$\textit{In[5]:=}\ \mathrm{B[n\_] := \frac{-(d_0[1+n]*d_1[n]) + d_0[n]*d_1[1+n]}{d_0[n]^2}}$$

$$\textit{In[6]:=}\ \mathrm{C[n\_] := \frac{1}{d_0[-1+n]*d_0[n]^2}\left(d_0[1+n]*\left(-d_1[n]^2+d_0[n]*c_2[n]\right)\right.}$$
$$\mathrm{\left.+c_0[n]*(d_1[n]*d_1[1+n]-d_0[n]*d_2[1+n])\right)}$$

Lastly, in accordance with the methodology of Chapter 3, we assign the constant and linear polynomials as $\mathrm{Q_0} := 1$ and $\mathrm{Q_1} := \mathrm{a_1} + \mathrm{x}$.

Now, the Q_5 quadratic coefficient from the three-term recurrence relation of (4.11) is

$$\frac{1}{10\,(1+6g_3)\,(1+24g_3)^2}\big(-a_1^3+3a_1a_2+2a_3+18a_1^3g_3+180a_3g_3-1296a_1^3{g_3}^2$$
$$+1296a_1a_2g_3^2+3888a_3g_3^2-8640a_1^3{g_3}^3+25920a_1a_2g_3^3+17280a_3g_3^3-2a_1^2h_2$$
$$+14a_2h_2+360a_1^2g_3h_2+144a_2g_3h_2-6480a_1^2g_3^2h_2+6048a_2g_3^2h_2-17280a_1^2g_3^3h_2$$
$$+69120a_2g_3^3h_2+60g_5h_2+2880g_3g_5h_2+34560g_3^2g_5h_2+a_1h_2^2+1386a_1g_3h_2^2$$
$$-11664a_1g_3^2h_2^2+8640a_1g_3^3h_2^2-16h_2^3+1584g_3h_2^3-6480g_3^2h_2^3+17280g_3^3h_2^3$$
$$+13a_1h_3+360a_1g_3h_3+6480a_1g_3^2h_3+34560a_1g_3^3h_3+34h_2h_3+288g_3h_2h_3$$
$$+4320g_3^2h_2h_3+34560g_3^3h_2h_3+2h_4+180g_3h_4+3888g_3^2h_4+17280g_3^3h_4\big) \tag{4.18}$$

Therefore, setting (4.17) and (4.18) equal to each other and assigning $\breve{g}_3 = 1/3$ we obtain

$$-459a_1^3+1107a_1a_2-81a_3-1242a_1^2h_2+864a_2h_2$$
$$-1728a_1h_2^2+432h_2^3-297a_1h_3-540h_2h_3-1296h_4=0. \tag{4.19}$$

We can now construct a simultaneous system of non-linear algebraic equations in the variables a_1, a_2, a_3 and h_2 as we did in Chapter 3. Also like Chapter 3 we first must develop expressions for h_3, h_4 and h_5 that involve only a_1, a_2, a_3 and h_2. Since our method for obtaining these new expressions is the same as the *B-Type 1* case, we omit the all of the details and discuss only the pertinent elements of the method to again avoid unnecessary repetitiveness. We first set $g_3 = 1/2$ and then achieve expressions for the h-terms.

In order to derive an expression for h_3, we first utilize the Q_3 linear-term comparison to arrive at

$$h_3 := -\frac{1}{a_1}\left(a_1^3 - a_1 a_2 - a_3 + 2\,a_1^2 h_2\right). \tag{4.20}$$

To establish an expression for h_4, we employ the Q_4 linear-term comparison and then substitute in (4.20) to attain

$$\begin{aligned} h_4 :=& \frac{1}{12}\left(-3{a_1}^3 - 3a_1a_2 - 3a_3 + 6a_1^2h_2 - 24a_2h_2 + 24a_1h_2^2 + 24h_2^3\right.\\ &\left.+3\left(a_1^3 - a_1a_2 - a_3 + 2a_1^2h_2\right)\right). \end{aligned} \tag{4.21}$$

Lastly, to attain an expression for h_5, we make use of the Q_5 linear-term comparison and then substitute in (4.20) and (4.21) to achieve

$$\begin{aligned} h_5 := &\left(-459a_1^2a_2 + 648a_2^2 + 459a_1a_3 - 648a_4 - 459a_1^3h_2 - 135a_1a_2h_2 - 351a_3h_2\right.\\ &- 1242a_1^2h_2^2 + 648a_2h_2^2 - 216a_1h_2^3 + \frac{486a_2\left(a_1^3 - a_1a_2 - a_3 + 2a_1^2h_2\right)}{a_1}\\ &+ 297h_2\left(a_1^3 - a_1a_2 - a_3 + 2a_1^2h_2\right) + \frac{216h_2^2\left(a_1^3 - a_1a_2 - a_3 + 2a_1^2h_2\right)}{a_1}\\ &+ \frac{81\left(a_1^3 - a_1\,a_2 - a_3 + 2a_1^2\,h_2\right)^2}{a_1^2} - 63a_1\left(-3a_1^3 - 3a_1a_2 - 3a_3 + 6a_1^2h_2\right.\\ &\left.-24a_2h_2 + 24a_1h_2^2 + 24h_2^3 + 3\left(a_1^3 - a_1a_2 - a_3 + 2a_1^2h_2\right)\right)\\ &+ 72h_2\left(-3a_1^3 - 3a_1a_2 - 3a_3 + 6a_1^2h_2 - 24a_2h_2 + 24a_1h_2^2 + 24h_2^3\right.\\ &\left.\left.+3\left(a_1^3 - a_1a_2 - a_3 + 2a_1^2h_2\right)\right)\right)/1215. \end{aligned} \tag{4.22}$$

We mention here that unlike the derivations of the h-terms in Chapter 3, we do not substitute the a-terms. We do this in order to compare and contrast the types of systems that can be obtained.

Now, upon substituting (4.20) and (4.21) into (4.19) and using some algebraic manipulations we have

$$\frac{-54}{a_1}\left(3a_1^4 - 27a_1^2a_2 - 5a_1a_3 + 26a_1^3h_2 - 54a_1a_2h_2 \right.$$
$$\left. +10a_3h_2 + 60a_1^2h_2^2 + 40a_1h_2^3\right) = 0. \tag{4.23}$$

Next, we consider the Q_6 cubic-term comparison. Following the same procedure as with the Q_5 quadratic-term comparison we ultimately arrive at

$$\frac{-486}{a_1}\left(\left(a_1^4 - a_1^2a_2 + 17a_1a_3 - 18a_1^3h_2 - 34a_1a_2h_2\right.\right.$$
$$\left.\left. -10a_3h_2 - 12a_1^2h_2^2 + 8a_1h_2^3\right)\right) = 0. \tag{4.24}$$

Next, we consider the Q_7 fourth-degree comparison, again following the same procedure as with the Q_5 quadratic-term comparison and the Q_6 cubic-term comparison resulting in

$$\frac{8748}{a_1}\left(\left(47a_1^4 - 119a_1^2a_2 - 5a_1a_3 + 322a_1^3h_2 - 294a_1a_2h_2\right.\right.$$
$$\left.\left. +138a_3h_2 + 908a_1^2h_2^2 + 648a_1h_2^3\right)\right) = 0. \tag{4.25}$$

Lastly, we use the Q_8 fifth-degree comparison, following the same procedure above and discover

$$\frac{-43740}{a_1}\left(\left(4491a_1^4 - 33663a_1^2a_2 - 8109a_1a_3 + 42354a_1^3h_2 - 67878a_1a_2h_2\right.\right.$$
$$\left.\left. +11826a_3h_2 + 92876a_1^2h_2^2 + 50952a_1h_2^3\right)\right) = 0. \tag{4.26}$$

Now, (4.23) through (4.26) is our desired simultaneous system of non-linear algebraic equations in the variables a_1, a_2, a_3 and h_2. To solve this system, we again make use of Mathematica. In our Mathematica program we initially respectively assign the left-hand side of equation (4.23), (4.24), (4.25) and (4.26) as `E1,...,E4`. For example, our first assignment appears as follows:

In[1]:=`E1:=`$\frac{-54}{\mathrm{a}_1}\left(3\mathrm{a}_1^4 - 27\mathrm{a}_1^2\mathrm{a}_2 - 5\mathrm{a}_1\mathrm{a}_3 + 26\mathrm{a}_1^3\mathrm{h}_2 - 54\mathrm{a}_1\mathrm{a}_2\mathrm{h}_2 + 10\mathrm{a}_3\mathrm{h}_2 + 60\mathrm{a}_1^2\mathrm{h}_2^2 + 40\mathrm{a}_1\mathrm{h}_2^3\right)$

We then simultaneously solve the above system using the `Solve` command as discussed in Chapter 3. Also as in Chapter 3, we must exhaust all $\sum_{n=1}^{4} \mathrm{C}(4, n) = 15$ variable choices in order to most efficiently attempt to determine all of the solutions to the system defined by (4.23), (4.24), (4.25) and (4.26).

As it turns out, each of the 15 choices results in a null solution set {}. To illustrate this, we display two of our results since displaying the remaining 13 choices is unnecessary.

In[2]:=`Solve[{E1==0, E2==0, E3==0, E4==0},{`$\mathrm{a}_2, \mathrm{h}_2$`}]`

Out[2]:={}

In[14]:=`Solve[{E1==0, E2==0, E3==0, E4==0},{`a_3`}]`

Out[14]:={}

We summarize this chapter with the following statement.

Result 4.1. *The simultaneous nonlinear algebraic system defined by (4.23), (4.24), (4.25) and (4.26) has no solutions with respect to the Mathematica* `Solve` *command. Therefore, no orthogonal polynomial sequences can thus far be determined.*

CHAPTER 5: EXTENSIONS AND CONCLUSIONS

5.1 FUTURE SHEFFER CHARACTERIZATION PROBLEMS

As we discussed in Chapters 1 and 2, Sheffer conducted a complete analysis on the *B-Type 0* polynomial sequences. One of his foremost results was determining which of these polynomial sequences are also orthogonal (at this time orthogonal sets were entitled Tchebycheff sets) and Sheffer of course commented on the arbitrary class of *B-Type k*. Also in Sheffer's paper was an elaborate development of the *A-Type 0* polynomial sequences and he proved that a polynomial sequence is *B-Type 0* if and only if it is *A-Type 0*. The crux of his work focused on an aesthetic development of many of the properties of the *A-Type 0* polynomial sequences and therefore the *B-Type 0* polynomial sequences which, as discussed in Chapter 2, depended on arbitrary degree-lowering operators but also depended on several functional relationships as well. Now, as it turned out the *A-Type 0* polynomial sequences (and therefore the *B-Type 0* polynomial sequences) that are also orthogonal are the familiar cases covered in Chapter 2 that have since been studied and utilized quite extensively - especially the cases of Hermite and Laguerre. It is also important to mention that Sheffer also developed an additional classification entitled *C-Type* and in [18] Rainville discusses another case of the Sheffer classification entitled *σ-Type*.

However, Sheffer only briefly mentions the definition of higher type and does not conduct any analysis on specific cases that are of higher type, like *B-Type 1* or *B-Type 2*. Because of Sheffer's aesthetic development of many of the properties of the *B-Type 0* polynomial sequences and his successful discovery of the sequences that were also orthogonal, it is quite natural to inquire whether or not the *B-Type 1* class or higher classes also possess such pleasing properties or more interestingly, yield any *new* orthogonal polynomials. Quite simply, the *B-Type 1* and/or *B-Type 2* polynomial sequences either contain orthogonal polynomial sequences or they do not. If it is shown that they do, a very important discovery is clearly made since a new orthogonal polynomial sequence would foster several new questions regarding; the three-term recurrence relation, the generating function(s), the orthogonality relation and its respective measure, the asymptotics and other considerations as well. This would be a new sequence to add to [15] and would potentially aid mathematicians and other scientists in applications as well. Now then, how do we deal with the second situation? Well, if the *B-Type 1* and/or *B-Type 2* polynomial sequences do not contain orthogonal polynomial sequences we of course cannot establish any of the aforementioned characterizations. However, we have in turn answered a very important question nonetheless and have tacitly directed future researchers to consider other sequences for new orthogonal polynomials.

In our work we analyzed *special cases* of the *B-Type 1* and *B-Type 2* classes as we discovered that our aspirations of conducting a complete analysis of the general *B-Type 1* class, and even a special case of it, were premature as we discuss several additional aspects that need to be addressed in Section 5.2. Similar conclusions are made on the *B-Type 2* class in Section 5.3. Presently, we cannot definitively assess whether or not any orthogonal polynomial sequences $\{P_n\}$ or $\{Q_n\}$ as respectively defined by (3.40) and (4.16) exist and based on the research conducted in Chapters 3 and 4 we thusly pose the following problems:

Problem 5.1 *Determine all orthogonal polynomial sequences* $\{P_n\}$ *that satisfy the relation*

$$A(t)\exp\left[xH(t)+\frac{1}{2}x^2\right]=\sum_{n=0}^{\infty}P_n(x)t^n$$

where

$$A(t)=\sum_{j=0}^{\infty}a_jt^j,\ \ a_0\neq 0 \quad \textit{and} \quad H(t)=\sum_{k=1}^{\infty}h_kt^k, \quad h_1\neq 0$$

and

Problem 5.2 *Determine all orthogonal polynomial sequences* $\{Q_n\}$ *that satisfy the relation*

$$A(t)\exp\left[xH(t)+\frac{1}{3}x^3\right]=\sum_{n=0}^{\infty}Q_n(x)t^n$$

where

$$A(t)=\sum_{j=0}^{\infty}a_jt^j,\ \ a_0\neq 0 \quad \textit{and} \quad H(t)=\sum_{k=1}^{\infty}h_kt^k, \quad h_1\neq 0.$$

Exactly how these problems may be approached is further addressed in the following section.

5.2 FUTURE CONSIDERATIONS FOR OUR SPECIAL CASES OF THE SHEFFER B-TYPE 1 AND B-TYPE 2 POLYNOMIAL SEQUENCES

In this section we address in detail what future research must be completed in order to definitively solve Problem 5.1 and Problem 5.2. We begin with our special case of the Sheffer *B-Type 1* polynomial sequences as analyzed in Chapter 3 and conclude by supplementing the results of Chapter 4.

5.2.1 FUTURE RESEARCH FOR OUR SPECIAL CASE OF THE SHEFFER B-TYPE 1 CLASS

To recap, in Chapter 3 we achieved a simultaneous system of nonlinear algebraic equations in the variables a_1, a_2, a_3 and h_2, which resulted from comparing the polynomial sequences that evolved from the generating function (3.40) and the polynomial sequences that evolved the three-term recurrence relation (2.3), with A_n, B_n and C_n as defined by (3.14). Of course solving this system yields conditions on the a_1, a_2, a_3 and h_2 terms that must be satisfied for the polynomial sequence $\{P_n\}$ as defined by (3.40) to be orthogonal. If all of the solutions to this system are obtained, one can determine whether or not orthogonal polynomial sequences arise. However, one must obtain *all* of the solutions to this system before anything definitive can be asserted. Currently, we have only determined a certain number of these aforementioned solutions, which resulted in contradictions and thusly, no orthogonal polynomial sequences.

Now, as previously discussed in Chapters 3 and 4, Mathematica has the ability to simultaneously solve nonlinear systems of algebraic equations. Moreover, it essentially has *two* commands that accomplish this. The commands are `Solve`, which we implemented in both Chapters 3 and 4, and `Reduce` and both of these commands function a little differently. The important difference between the `Solve` command and the `Reduce` command is that `Reduce` yields *all* the possible solutions to a system of equations, whereas the `Solve` command yields only *generic* solutions, i.e. conditions on the variables that one explicitly solves for, and not on any other parameters in the system. In addition, the `Reduce` command also disregards multiplicities of solutions as it always displays each solution set only once - refer to [21] for more details on these commands. To illustrate these differences, we consider the solution to the general quadratic equation implementing both the `Solve` and `Reduce` commands as seen below:

In[1]:=`Solve[`$\{a*x^2+b*x+c==0\}$`,`$\{x\}$`]`

Out[1]=$\left\{\left\{x\to\frac{-b-\sqrt{b^2-4ac}}{2a}\right\},\left\{x\to\frac{-b+\sqrt{b^2-4ac}}{2a}\right\}\right\}$

In[2]:=`Reduce[`$\{a*x^2+b*x+c==0\}$`,`$\{x\}$`]`

Out[2]=$x=\frac{-b-\sqrt{b^2-4ac}}{2a}$ && $a\neq0$ || $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ && $a\neq0$

$a=0$ && $b=0$ && $c=0$ || $a=0$ && $x=-\left(\frac{c}{b}\right)$ && $b\neq0$

We see that the `Solve` commands solves the equation with respect to x and disregards any restrictions on the other term, e.g. $a\neq0$ and the `Reduce` command considers all possible scenarios, e.g. the linear solution when $a = 0$. Therefore, to aid in assuring that we determine all of the solutions to the system we should implement both the `Solve` command and the `Reduce` command to the system defined by (3.44), (3.45), (3.46) and (3.47) and the system defined by (3.49), (3.50), (3.51) and (3.52).

Thus far we utilized the `Solve` command on the system defined by (3.49), (3.50), (3.51) and (3.52) in Chapter 3. Below we display the solutions obtained by Mathematica using the `Reduce` command on this system, again exhausting all $\sum_{n=1}^{4} \mathrm{C}(4,n) = 15$ possible parameter selections. We note that && represents the "and" operator and || represents the "or" operator as in C programming.

In[1]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},{`$\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{h}_2$`}]`

Out[1]=$\mathrm{a}_2 == \mathrm{a}_1^2 \;\&\&\; \mathrm{a}_3 == \frac{2\mathrm{a}_1^3}{3} \;\&\&\; \mathrm{h}_2 == \mathrm{a}_1 \;\&\&\; \mathrm{a}_1 \neq 0$

In[2]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},{`$\mathrm{a}_2, \mathrm{a}_3, \mathrm{h}_2$`}]`

Out[2]=$\mathrm{a}_2 == \mathrm{a}_1^2 \;\&\&\; \mathrm{a}_3 == \frac{2\mathrm{a}_1^3}{3} \;\&\&\; \mathrm{h}_2 == \mathrm{a}_1 \;\&\&\; \mathrm{a}_1 \neq 0$

In[3]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},{`$\mathrm{a}_1, \mathrm{a}_3, \mathrm{h}_2$`}]`

Out[3]=$\mathrm{a}_1 = -\sqrt{\mathrm{a}_2} \;\&\&\; \mathrm{a}_3 = \frac{-2\,\mathrm{a}_2^{3/2}}{3} \;\&\&\; \mathrm{h}_2 = -\sqrt{\mathrm{a}_2} \;\&\&\; \mathrm{a}_2 \neq 0 \;||$

$\mathrm{a}_1 = \sqrt{\mathrm{a}_2} \;\&\&\; \mathrm{a}_3 = \frac{2\,\mathrm{a}_2^{3/2}}{3} \;\&\&\; \mathrm{h}_2 = \sqrt{\mathrm{a}_2} \;\&\&\; \mathrm{a}_2 \neq 0$

In[4]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_1, a_2, h_2\}$`]`

Out[4]=$a_1 = -\left(\left(-\left(\frac{3}{2}\right)\right)^{\frac{1}{3}} a_3^{\frac{1}{3}}\right) \;\&\&\; a_2 = \left(-\left(\frac{3}{2}\right)\right)^{\frac{2}{3}} a_3^{\frac{2}{3}}$

$\&\&\; h_2 = -\left(\left(-\left(\frac{3}{2}\right)\right)^{\frac{1}{3}} a_3^{\frac{1}{3}}\right) \;\&\&\; a_3 \neq 0 \;||$

$a_1 = \left(\frac{3}{2}\right)^{\frac{1}{3}} a_3^{\frac{1}{3}} \;\&\&\; a_2 = \left(\frac{3}{2}\right)^{\frac{2}{3}} a_3^{\frac{2}{3}} \;\&\&\; h_2 = \left(\frac{3}{2}\right)^{\frac{1}{3}} a_3^{\frac{1}{3}} \;\&\&\; a_3 \neq 0 \;||$

$a_1 = (-1)^{\frac{2}{3}} \left(\frac{3}{2}\right)^{\frac{1}{3}} a_3^{\frac{1}{3}} \;\&\&\; a_2 = -\left((-1)^{\frac{1}{3}} \left(\frac{3}{2}\right)^{\frac{2}{3}} {a_3}^{\frac{2}{3}}\right)$

$\&\&\; h_2 = (-1)^{\frac{2}{3}} \left(\frac{3}{2}\right)^{\frac{1}{3}} {a_3}^{\frac{1}{3}} \;\&\&\; a_3 \neq 0$

In[5]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_1, a_2, a_3\}$`]`

Out[5]=$a_1 = h_2 \;\&\&\; a_2 = h_2^2 \;\&\&\; a_3 = \frac{2\, h_2^3}{3} \;\&\&\; h_2 \neq 0$

In[6]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_1, a_2\}$`]`

Out[6]=$a_1 = \frac{3\, a_3 + h_2^3}{3 h_2^2} \;\&\&\; a_2 = \frac{3\, a_3 + h_2^3}{3\, h_2} \;\&\&\; 9\, a_3^2 - 12\, a_3\, h_2^3 = -4\, h_2^6 \;\&\&\; h_2 \neq 0$

In[7]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_1, a_3\}$`]`

Out[7]=$a_1 = \frac{a_2}{h_2} \;\&\&\; a_3 = \frac{-\left(h_2\left(-3\, a_2 + h_2^2\right)\right)}{3} \;\&\&\; {a_2}^2 - 2\, a_2\, h_2^2 = -h_2^4 \;\&\&\; h_2 \neq 0$

In[8]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_1, h_2\}$`]`

Out[8]=$a_1 = \frac{3\, a_3 - \sqrt{-4\, a_2^3 + 9\, a_3^2}}{2\, a_2} \;\&\&\; 9\, a_3^2 = 4\, a_2^3 \;\&\&\; h_2 = \frac{-2\, a_2^2}{-3\, a_3 + \sqrt{-4\, a_2^3 + 9\, a_3^2}} \;\&\&\; a_2 \neq 0 \;||$

$a_1 = \frac{3\, a_3 + \sqrt{-4\, a_2^3 + 9\, a_3^2}}{2\, a_2} \;\&\&\; 9\, a_3^2 = 4\, a_2^3 \;\&\&\; h_2 = \frac{2\, a_2^2}{3\, a_3 + \sqrt{-4\, a_2^3 + 9\, a_3^2}} \;\&\&\; a_2 \neq 0$

In[9]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_2, a_3\}$`]`

Out[9]=$a_2 = a_1\,h_2$ && $a_3 = \frac{-\left(a_1^2\,(a_1 - 3\,h_2)\right)}{3}$ && $-2\,a_1\,h_2 + h_2^2 = -a_1^2$ && $a_1 \neq 0$

In[10]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_2, h_2\}$`]`

Out[10]=$a_2 = \frac{{a_1}^3 + 3\,a_3}{3\,a_1}$ && $-12\,a_1^3\,a_3 + 9\,a_3^2 = -4\,a_1^6$ && $h_2 = \frac{a_1^3 + 3\,a_3}{3\,a_1^2}$ && $a_1 \neq 0$

In[11]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_3, h_2\}$`]`

Out[11]=$-2\,{a_1}^2\,a_2 + a_2^2 = -a_1^4$ && $a_3 = \frac{-\left(a_1\,\left(a_1^2 - 3\,a_2\right)\right)}{3}$ && $h_2 = \frac{a_2}{a_1}$ && $a_1 \neq 0$

In[12]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_1\}$`]`

Out[12]=$a_1 = \frac{a_2}{h_2}$ && $3\,a_3 = h_2\,\left(3\,a_2 - h_2^2\right)$ && $a_2^2 - 2\,a_2\,h_2^2 = -h_2^4$ && $h_2 \neq 0$

In[13]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_2\}$`]`

Out[13]=$a_2 = a_1\,h_2$ && $3\,a_3 = a_1^2\,\left(-a_1 + 3\,h_2\right)$ && $-2\,a_1\,h_2 + h_2^2 = -a_1^2$ && $a_1 \neq 0$

In[14]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{a_3\}$`]`

Out[14]=$a_2 = a_1\,h_2$ && $a_3 = \frac{-\left(a_1^2\,(a_1 - 3\,h_2)\right)}{3}$ && $-2\,a_1\,h_2 + h_2^2 = -a_1^2$ && $a_1 \neq 0$

In[15]:=`Reduce[{E1==0, E2==0, E3==0, E4==0},`$\{h_2\}$`]`

Out[15]=$2\,a_1^2\,a_2 + a_2^2 = -a_1^4$ && $3\,a_3 = a_1\,\left(-a_1^2 + 3\,a_2\right)$ && $h_2 = \frac{a_2}{a_1}$ && $a_1 \neq 0$

We first observe that when we use the `Reduce` command for variable selections $\{a_1, a_2, a_3, h_2\}$, $\{a_2, a_3, h_2\}$, $\{a_1, a_3, h_2\}$, $\{a_1, a_2, h_2\}$ and $\{a_1, a_2, a_3\}$ we obtained essentially the same solutions as with the `Solve` command. However, the remaining solutions above were not obtained when using the `Solve` command. In fact, the solve command outputted "{}" for each of these cases. It is also very important to mention that these new relationships do not result in contradictions in the same way the relationships acquired using the `Solve` command did in Sections 3.3 and 3.4.

We now reconsider the unaltered system defined by (3.44), (3.45), (3.46) and (3.47). In Chapter 3, we originally only considered the obvious solution $a_1 = h_2$ to this system. To determine the ramifications to this solution set we constructed an additional system of equations. Thus far only the solution sets that resulted from using the `Solve` command were considered. To determine additional solutions we must again implement the `Solve` and `Reduce` commands in the same fashion we did above. We begin by displaying the following inputs and respective outputs below, which are the only ones that do not output the preface "*Equations may not give solutions for all "solve" variables*" that was discussed in Section 3.4.

In[4]:=`Solve[`$\{(\mathrm{a}_1 - \mathrm{h}_2) * \mathrm{E1} == 0, (\mathrm{a}_1 - \mathrm{h}_2) * \mathrm{E2} == 0, (\mathrm{a}_1 - \mathrm{h}_2) * \mathrm{E3} == 0,$

$(\mathrm{a}_1 - \mathrm{h}_2) * \mathrm{E4} == 0\}$`,`$\{\mathrm{a}_2, \mathrm{a}_3\}$`]`

Out[4]={}

In[2]:=`Solve[`$\{(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E1}==0,(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E2}==0,(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E3}==0,$

$(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E4}==0\},\{\mathrm{a}_1\}$`]`

Out[2]=$\{\{\mathrm{a}_1\rightarrow\mathrm{h}_2\}\}$

In[3]:=`Solve[`$\{(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E1}==0,(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E2}==0,(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E3}==0,$

$(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E4}==0\},\{\mathrm{a}_2\}$`]`

Out[3]=$\{\}$

In[4]:=`Solve[`$\{(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E1}==0,(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E2}==0,(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E3}==0,$

$(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E4}==0\},\{\mathrm{a}_3\}$`]`

Out[4]=$\{\}$

In[5]:=`Solve[`$\{(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E1}==0,(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E2}==0,(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E3}==0,$

$(\mathrm{a}_1-\mathrm{h}_2)*\mathrm{E4}==0\},\{\mathrm{h}_2\}$`]`

Out[5]=$\{\{\mathrm{a}_1\rightarrow\mathrm{h}_2\}\}$

From this we have learned that there exists no solution to this system, either implicitly or explicitly, for the variables a_2 or a_3 or the combination $\{a_2, a_3\}$. We also learned that if we try to solve the system for either a_1 or h_2, we determine that $a_1 = h_2$.

The remaining solution sets all include the preface "*Equations may not give solutions for all "solve" variables*" and we essentially observe a repeat of the solution sets obtained for the system (3.49), (3.50), (3.51) and (3.52) when using the `Solve` and `Reduce` commands, with the only difference being that the multiplicities are displayed as well - not inclusive of the variable choice $\{a_1, h_2\}$. For this case there is a drastic difference that occurs when we solve for the variable selections $\{a_1, h_2\}$, i.e. when we endeavor to discover the outputs of the following:

In[1]:=`Solve[{`$(\mathrm{a}_1 - \mathrm{h}_2) * \mathrm{E1} == 0, (\mathrm{a}_1 - \mathrm{h}_2) * \mathrm{E2} == 0, (\mathrm{a}_1 - \mathrm{h}_2) * \mathrm{E3} == 0,$
$(\mathrm{a}_1 - \mathrm{h}_2) * \mathrm{E4} == 0$`},{`$\mathrm{a}_1, \mathrm{h}_2$`}]`

In fact, the outputs for these variable choices are so complicated that to display them would take five pages. To illustrate these complications, consider one of these outputs, which is listed below:

$$\mathrm{a}_1 = \frac{200\,2^{\frac{1}{3}}\,\mathrm{a}_2}{\left(-27944028\,\mathrm{a}_3 + \sqrt{-350113536000000\,\mathrm{a}_2^3 + 780868700864784\,\mathrm{a}_3^2}\right)^{\frac{1}{3}}} + \frac{\left(-27944028\,\mathrm{a}_3 + \sqrt{-350113536000000\,\mathrm{a}_2^3 + 780868700864784\,\mathrm{a}_3^2}\right)^{\frac{1}{3}}}{222\,2^{\frac{1}{3}}} \quad \&\&$$
$$\mathrm{h}_2 = \frac{200\,2^{\frac{1}{3}}\,\mathrm{a}_2}{\left(-27944028\,\mathrm{a}_3 + \sqrt{-350113536000000\,\mathrm{a}_2^3 + 780868700864784\,\mathrm{a}_3^2}\right)^{\frac{1}{3}}} + \frac{\left(-27944028\,\mathrm{a}_3 + \sqrt{-350113536000000\,{\mathrm{a}_2}^3 + 780868700864784\,{\mathrm{a}_3}^2}\right)^{\frac{1}{3}}}{222\,2^{\frac{1}{3}}}$$

It is imperative to note that when using the `Reduce` command we obtained essentially the same solutions as previously discovered using the `Solve` command and for the variable choice $\{a_1, h_2\}$ we achieved the same solutions with some additional restrictions.

Based on this additional experimentation it is clear what needs to be completed in order to determine which, if any, orthogonal polynomial sequences arise from (3.40). First off, all of these additional solution sets need to be tested to determine if they lead to contradictions or result in new orthogonal polynomial sequences, although this may prove to be difficult - especially when we solve the system defined by (3.44), (3.45), (3.46) and (3.47) for variable choices $\{a_1, h_2\}$, as addressed previously. From there, all of the solutions would need to be determined in a *rigorous* fashion and then tested accordingly.

Lastly, we mention some novel approaches to solving the system defined by (3.44), (3.45), (3.46) and (3.47) that were not yet examined. First, it may prove to be advantageous to consider a change-of-variables approach that would subsequently reduce the system to a more manageable format. Second, would be to consider a geometric approach by considering each equation as a surface in space and determining all of the intersection regions. In any case, we chose using the Mathematica approach to solving the system to be consistent with the ways the lower-order polynomials defined by (3.40), subsequent comparisons and simultaneous systems were obtained. Lastly, it will be very beneficial to construct (and analyze accordingly) a simultaneous system *without* substituting the a-values into the h-terms as was done in Chapter 3 and mentioned in the Remark.

5.2.2 FUTURE RESEARCH FOR OUR SPECIAL CASE OF THE SHEFFER B-TYPE 2 CLASS

In this short section we discuss some extensions of the research that was completed in Chapter 4. As in Chapter 3, we ultimately developed a simultaneous system of nonlinear algebraic equations in the variables a_1, a_2, a_3 and h_2 as defined by (4.23), (4.24), (4.25) and (4.26). This system resulted from comparing the polynomial sequences that evolved form the generating function (4.16) and the polynomial sequences that evolved from the three-term recurrence relation (2.3), with $\breve{A}_n$, $\breve{B}_n$ and $\breve{C}_n$ as defined by (4.12). Of course solving this system yields necessary conditions on the a_1, a_2, a_3 and h_2 terms that must be satisfied for the polynomial sequences $\{Q_n\}$ defined by (4.16) to be orthogonal.

In Chapter 4 we again utilized the Mathematica `Solve` command as was done in Chapter 3. In Result 4.1 we concluded that there are no solutions to this system with respect to this command. Interestingly enough, upon utilizing the `Reduce` command we obtain the same results as with the `Solve` command. To demonstrate this we display two of these calculations as follows.

In[6]:=`Reduce[`$\{\mathrm{E1} == 0, \mathrm{E2} == 0, \mathrm{E3} == 0, \mathrm{E4} == 0\}$`,`$\{\mathrm{a_1}, \mathrm{a_2}\}$`]`

Out[6]=`False`

In[10]:=`Reduce[`$\{\mathrm{E1} == 0, \mathrm{E2} == 0, \mathrm{E3} == 0, \mathrm{E4} == 0\}$`,`$\{\mathrm{a_2}, \mathrm{h_2}\}$`]`

Out[10]=`False`

We see that when using the `Reduce` command Mathematica outputs "`False`" as opposed to a {} when using the `Solve` command. In conclusion, it is not exactly definite that no solutions exist for this system, as all of the solutions would need to be determined in a *rigorous* fashion and then tested before any logical conclusions can be drawn.

Also, in Chapter 4 we constructed a simultaneous system *without* substituting the a-values as was done in Chapter 3. This emphasizes the fact that the most important aspect of our research is the template that can be utilized to develop a system of equations that will lead to conditions for orthogonality. In addition, each equation in the system that resulted in Chapter 4 had an a_1-term in the denominator. Omitting this term and analyzing the solutions to the modified system would be beneficial as it may give insights into additional relationships to study.

5.3 OTHER APPROACHES TO THE ANALYSIS OF THE SHEFFER B-TYPE 1 AND B-TYPE 2 POLYNOMIAL SEQUENCES

As discussed in Chapter 1, although extensive work has been completed on characterization theorems, little to no work has been done on the cases of higher Sheffer type classification. Because of this, the goal of this work was to conduct a preliminary analysis on the *B-Type 1* class and subsequently the *B-Type 2* class. As we talked about in Chapter 3, this amounts to developing an analogue of Sheffer's method, as discussed in Section 2.2, or considering a novel approach to the problem. We of course adopted the latter consideration and the reasons for this need to be elaborated on.

In [19] Sheffer determined which *B-Type 0* sets are also orthogonal by using the *B-Type 0* three-term recurrence relation and several other functional relationships. The *B-Type 0* three-term recurrence relation for an arbitrary polynomial sequence $\{Q_n\}$ is as follows - we use the same notation that Sheffer used in [19]:

$$Q_n(x) = (x + \lambda_n)Q_{n-1}(x) + \mu_n Q_{n-2}(x), \;\; n = 1, 2, \ldots$$

with $\lambda_n, \mu_n \in \mathbb{R} \;\; \mu_n \neq 0, \;\; n > 1$. What is important here is the fact that Sheffer proved that the n-dependent constants λ_n and μ_n have the following structure:

$$\lambda_n = \alpha + bn \;\text{ and }\; \mu_n = (n-1)(c + dn).$$

Namely, λ_n is linear in n and μ_n is at most quadratic in n. By substituting $e_n q_n$ for Q_n above and replacing n with $n+1$ we attain a difference equation of the form

$$e_{n+1}q_{n+1} = e_n q_n x + (\alpha + b + bn)e_n - (cn + dn^2 + dn)e_{n-1}q_{n-1}, \;\; n = 1, 2, \ldots$$

We next choose $\{e_n\}$ as $\{n!\}$ so μ_n is linear in n. After making this substitution and dividing both sides of the relationship by $n!$ we see that

$$(n+1)q_{n+1} = xq_n + (\alpha + b + bn)q_n - (a + dn)q_{n-1},$$

where $a = c + d$. Now, multiplying both sides of the relation by t^n and summing from $n = 0, 1, \ldots$ we obtain a partial differential equation of the form:

$$\frac{\partial F}{\partial t} = \frac{(x + \alpha + b - (a+d)t)F}{(1 - bt + dt^2)},$$

where $F := \sum_{n=0}^{\infty} q_n t^n$. This equation can be solved by using partial fraction decomposition and the solutions lead to the five classes of orthogonal polynomial sequences as addressed in Chapter 2, which are all of the sequences that are *B-Type 0* and also orthogonal.

This method of transforming a three-term recurrence relation into a partial differential equation was not used by Sheffer in [19]; however it is often utilized in the study of orthogonal polynomials, see [14] for several examples. As evidenced in Chapter 3, the C_n term is quite complicated and using the method outlined above would therefore be difficult for the *B-Type 1* class. This fact coupled with the natural computational complexities may be one of the reasons that Sheffer and subsequent researchers have not addressed the higher *B-Type* classes further.

5.4 FUTURE CONSIDERATIONS

We conclude this work with some remarks on future research of the Sheffer classifications. We begin by discussing how our method provides a framework for future characterization problems to be studied.

5.4.1 THE UTILITY OF OUR ANALYSIS

Our method that was used in Chapters 3 and 4 was elementary in the sense that we only made use of Sheffer's generalized generating function for $k = 1$ and $k = 2$ and the resulting three-term recurrence relations. As we have seen, although the overall idea behind our method was simplistic, a wealth of complexities were embedded in its implementation. Nonetheless, what we have accomplished is the establishment of a procedure that can be used to attain new information on unsolved problems in characterization theory. Namely, we have shown that with only the knowledge of a generating function, the respective three-term recurrence and the aid of computer algebra systems, like Mathematica, one can determine conditions for orthogonality.

By following the procedure that is outlined in Chapter 3, one can discover new knowledge regarding the Sheffer *B-Type 3*, *B-Type 4* and higher classes, as well as the aforementioned *C-Type* and σ*-Type* classes. Moreover, it would be very interesting to consider using our method to try to gain insights on the unsolved problems of Chapter 2, like the Geronimus Problem, for example.

There is another interesting extension to consider as well. Recall the Brenke polynomial sequences $\{B_n\}$ as defined by (1.2). These polynomials have a generating function of the form

$$\sum_{n=0}^{\infty} B_n\left(x\right)t^n = A\left(t\right)C\left(xt\right),$$

with C defined as

$$C(w) := \sum_{n=0}^{\infty} c_n w^n.$$

Taking the right-hand side of this generating relation and multiplying it by $\exp[\beta x t^2]$ we obtain

$$A(t)C(xt)\exp[\beta x t^2]. \tag{5.1}$$

The question then arises as to which polynomial sequences satisfy (5.1) and are also orthogonal. We pose this formally.

Problem 5.3 *Which, if any, orthogonal polynomial sequences* $\left\{\tilde{B}_n\right\}$ *satisfy the relationship*

$$\sum_{n=0}^{\infty} \tilde{B}_n\,(x)\,t^n = A(t)C(xt)\exp[\beta x t^2], \;\; A(t) = \sum_{j=0}^{\infty} a_k t^k, \;\; a_0 = 1. \tag{5.2}$$

Now observe that the right-hand side of (5.2) can be expanded as follows:

$$\sum_{j=0}^{\infty} a_j t^j \sum_{k=0}^{\infty} c_k x^k t^k \sum_{l=0}^{\infty} \frac{\beta^l x^l t^{2l}}{l!}.$$

This is structure is entirely analogous to the expansions of the *B-Type 1* class and the *B-Type 2* class as studied in Chapters 3 and 4 respectively and is actually even simpler as there are no products. Thus, one can conduct analysis on Problem 5.3 using the method outlined in this work.

5.4.2 CHARACTERIZING THE GENERAL B-TYPE 1 AND B-TYPE 2 POLYNOMIAL SEQUENCES

Exactly how to characterize the general *B-Type 1* and *B-Type 2* classes in the most efficient and elegant way is still in its infancy. In theory, one could use the method outlined in this work *without* any simplifying assumptions and attempt to obtain conditions for orthogonality. However, as was discovered in Chapters 3 and 4, we needed to implore several simplifying assumptions to manipulate the problem into a manageable format. Moreover, even with these assumptions the complexities were quite great, as many of the relationships that were developed took up more than a page - consider the P_5 linear and quadratic comparisons in Chapter 3 for example.

It is therefore easy to imagine just how complicated the details involved with using this method will become in the general cases. Due to these difficulties, one would have to rely on powerful computers in order to use the computer algebra packages, like Mathematica, that are necessary for this method to be successful and would also have difficulties in managing and organizing the lengthy computations that will result.

Also of interest is the possibility of approaching the general *B-Type 1* and *B-Type 2* classes from an analytical approach without any expansions of the generating function or the three-term recurrence relation analogous to Sheffer's work on the *B-Type 0* class in [19]. As we have previously addressed, this approach may very well be extremely difficult or even impossible as we observed the discrepancies between the *B-Type 0* class and the classes of *B-Type 1* and *B-Type 2* throughout Chapters 3 and 4 and in Section 5.3. In any case, we hope that our analysis completed on the special cases of the *B-Type 1* and *B-Type 2* classes will encourage researchers to further consider studying these cases by not only modifying our method, but also considering alternative approaches, whatever they may be.

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