Weighted Lp-stability For Localized Infinite Matrices

2009

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WEIGHTED ℓ^p-STABILITY FOR LOCALIZED INFINITE MATRICES

by

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A dissertation submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Summer Term
2008

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ABSTRACT

This dissertation originates from a classical result that the ℓ^p-stability of the convolution operator associated with a summable sequence are equivalent to each other for $1 \le p \le \infty$.

This dissertation is motivated by the recent result by C. E. Shin and Q. Sun (*Journal of Functional Analysis*, **256**(2009), 2417–2439), where the ℓ^p-stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class are proved to be equivalent to each other for different $1 \le p \le \infty$.

In the dissertation, for an infinite matrix having certain off-diagonal decay, its weighted ℓ^p-stability for different $1 \le p \le \infty$ are proved to be equivalent to each other and hence a result by Shin and Sun is generalized.

I dedicate my dissertation to my family and many friends.

ACKNOWLEDGMENTS

I wish to thank all committee members. A special thanks to Dr. Qiyu Sun, my committee chairman and advisor for his patience throughout the entire process. I would like to thank Prof. Xianliang Shi for his encouragement and support. Last but not least, I would like to thank all the people who helped me throughout my life.

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CHAPTER ONE: HISTORICAL ORIGIN

The origin of this dissertation comes from the classical result that the ℓ^p-stability of the convolution operator associated with a summable sequence are equivalent to each other for $1 \le p \le \infty$ (Theorem 1.3.1), see [2, 5, 6] and the references therein for further discussion on the stability of convolution operators.

1.1 Convolution Operator

In this section, we define the sequence space $\ell^p, 1 \le p \le \infty$, and the convolution operator C_a associated with a summable sequence a, and we also recall the conclusion that any convolution operator C_a associated with a summable sequence a is a bounded operator on ℓ^p.

Definition 1.1.1 *For $1 \le p \le \infty$, let*

$$\ell^p(\mathbb{Z}) = \left\{ c := (c(j))_{j\in\mathbb{Z}} \Big| \; \|c\|_{\ell^p} < \infty \right\}, \tag{1.1.1}$$

where

$$\|c\|_{\ell^p} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} |c(j)|^p\right)^{1/p} & if\ 1 \le p < \infty, \\ \sup_{j\in\mathbb{Z}} |c(j)| & if\ p = \infty. \end{cases} \tag{1.1.2}$$

A sequence in $\ell^p(\mathbb{Z})$ is said to be *p-summable*, while for the special case $p = 1$, a sequence in $\ell^1(\mathbb{Z})$ is said to be *summable*. We observe that a sequence belongs to $\ell^\infty(\mathbb{Z})$ if and only if it is a bounded sequence. In this dissertation, I also use ℓ^p instead of $\ell^p(\mathbb{Z})$ for brevity.

Definition 1.1.2 *Let* $a = (a(j))_{j\in\mathbb{Z}}$ *be a summable sequence. Define the* convolution operator C_a *associated with the sequence* a *by*

$$C_a : \ \ell^p(\mathbb{Z}) \ni (b(j))_{j\in\mathbb{Z}} \longmapsto \left(\sum_{k\in\mathbb{Z}} a(j-k)b(k)\right)_{j\in\mathbb{Z}} \in \ell^p(\mathbb{Z}), \tag{1.1.3}$$

where $1 \le p \le \infty$.

The convolution operator associated with a summable sequence defines a bounded operator on $\ell^p(\mathbb{Z})$ for all $1 \le p \le \infty$.

Proposition 1.1.1 *Let* $a = (a(j))_{j\in\mathbb{Z}}$ *be a summable sequence. Then the convolution operator* C_a *associated with the sequence* a *defines a bounded operator on* $\ell^p(\mathbb{Z})$ *for all* $1 \le p \le \infty$. *Furthermore its operator norm is bounded by the* ℓ^1 *norm of the sequence* a.

Proof. Take $1 \le p < \infty$ and $b = (b(j))_{j\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$. Then

$$\begin{aligned}
\|C_a b\|_{\ell^p} &= \left(\sum_{j\in\mathbb{Z}} \Big|\sum_{k\in\mathbb{Z}} a(j-k)b(k)\Big|^p\right)^{1/p} \\
&\le \left(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |a(j-k)||b(k)|^p\Big) \times \Big(\sum_{k\in\mathbb{Z}} |a(j-k)|\Big)^{p-1}\right)^{1/p} \\
&= \|a\|_{\ell^1}^{(p-1)/p} \Big(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |a(j-k)||b(k)|^p\Big)^{1/p} \\
&= \|a\|_{\ell^1}^{(p-1)/p} \left(\sum_{k\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}} |a(j-k)|\Big)|b(k)|^p\right)^{1/p} \\
&= \|a\|_{\ell^1}\|b\|_{\ell^p},
\end{aligned}$$

where we have used the Hölder inequality to obtain the inequality. This proves the conclusion for $1 \le p < \infty$.

For $p=\infty$ and $b=(b(j))_{j\in\mathbb{Z}}\in\ell^\infty(\mathbb{Z})$, it follows that

$$\begin{aligned}\|C_a b\|_{\ell^\infty} &= \sup_{j\in\mathbb{Z}}\Big|\sum_{k\in\mathbb{Z}} a(j-k)b(k)\Big| \\ &\le \sup_{k\in\mathbb{Z}}|b(k)|\times\sup_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|a(j-k)| \\ &= \|a\|_{\ell^1}\|b\|_{\ell^\infty},\end{aligned}$$

which proves the desired conclusion for $p=\infty$.

1.2 ℓ^p-stability of Convolution Operators

In this section, we define the ℓ^p-stability of a linear operator on ℓ^p, and we characterize the ℓ^p-stability for a convolution operator associated with a summable sequence.

Definition 1.2.1 *Let $1\le p\le\infty$ and T be a linear operator on $\ell^p(\mathbb{Z})$. We say that the linear operator T has* ℓ^p-stability *if there exist positive constants A and B such that*

$$A\|c\|_{\ell^p}\le\|Tc\|_{\ell^p}\le B\|c\|_{\ell^p}\quad \textit{for all } c\in\ell^p. \tag{1.2.1}$$

Clearly a linear operator T has ℓ^p-stability if and only if it is a bounded operator on ℓ^p and has bounded inverse on ℓ^p.

For any convolution operator associated with a summable sequence, its ℓ^p-stability can be characterized via the Fourier series of that summable sequence.

Proposition 1.2.1 *Let $1\le p\le\infty$, $a=(a(j))_{j\in\mathbb{Z}}$ be a summable sequence, and C_a be the convolution operator associated with the sequence a. Then C_a has ℓ^p-stability if and only if*

$$\hat{a}(\xi)\ne 0\quad \textit{for all}\ \ \xi\in\mathbb{R} \tag{1.2.2}$$

where $\hat{a}(\xi) := \sum_{j\in\mathbb{Z}} a(j)e^{-ij\xi}$.

To prove Proposition 1.2.1, we recall the following result, which is known as the classical Wiener's lemma ([10]).

Lemma 1.2.2 *If* $\hat{a}(\xi) = \sum_{j\in\mathbb{Z}} a(j)e^{-ij\xi}$ *for some summable sequence* $a := (a(j))_{j\in\mathbb{Z}}$*, and* $\hat{a}(\xi) \neq 0$ *for all* $\xi \in \mathbb{R}$*, then*

$$\frac{1}{\hat{a}(\xi)} = \sum_{j\in\mathbb{Z}} b(j)e^{-ij\xi}$$

for some summable sequence $b = (b(j))_{j\in\mathbb{Z}}$.

Now we start to prove Proposition 1.2.1, the characterization of ℓ^p-stability of a convolution operator.

Proof of Proposition 1.2.1.

$(\Longrightarrow)$ First we prove the conclusion for $p = \infty$. Suppose on the contrary that $\hat{a}(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$. Then for the bounded sequence $e_{\xi_0} = (e^{ij\xi_0})_{j\in\mathbb{Z}}$, we have that $\|e_{\xi_0}\|_{\ell^\infty} = 1$ but

$$\begin{aligned} C_a e_{\xi_0} &= \Big(\sum_{k\in\mathbb{Z}} a(j-k)e^{ik\xi_0}\Big)_{j\in\mathbb{Z}} \\ &= \Big(e^{ij\xi_0}\hat{a}(\xi_0)\Big)_{j\in\mathbb{Z}} = 0. \end{aligned} \tag{1.2.3}$$

This is a contradiction.

Now we prove the conclusion for $1 \le p < \infty$. Suppose on the contrary that $\hat{a}(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$. Let $N \ge 1$ and define the sequence $e_{N,\xi_0} = (e_{N,\xi_0}(j))_{j\in\mathbb{Z}}$ where $e_{N,\xi_0}(j) = e^{ij\xi_0}$ if $|j| \le N$ and $e_{N,\xi_0}(j) = 0$ otherwise. Then

$$\|e_{N,\xi_0}\|_{\ell^p} = (2N+1)^{1/p} \tag{1.2.4}$$

and

$$
\begin{aligned}
\|C_a e_{N,\xi_0}\|_{\ell^p} &= \Big(\sum_{j=-\infty}^{\infty}\Big|\sum_{k=-N}^{N} a(j-k)e^{-ik\xi_0}\Big|^p\Big)^{1/p}\\
&\le \Big(\sum_{|j|\ge N+N^{1/2}}\Big|\sum_{k=-N}^{N} a(j-k)e^{-ik\xi_0}\Big|^p\Big)^{1/p}\\
&\quad +\Big(\sum_{|j|\le N-N^{1/2}}\Big|\sum_{k=-N}^{N} a(j-k)e^{-ik\xi_0}\Big|^p\Big)^{1/p}\\
&\quad +\Big(\sum_{N-N^{1/2}<|j|<N+N^{1/2}}\Big|\sum_{k=-N}^{N} a(j-k)e^{-ik\xi_0}\Big|^p\Big)^{1/p}\\
&\le \Big(\sum_{|j|\ge N+N^{1/2}}\Big(\sum_{k=-N}^{N} |a(j-k)|\Big)^p\Big)^{1/p}\\
&\quad +\Big(\sum_{|j|\le N-N^{1/2}}\Big|\Big(\sum_{k=j+N}^{\infty}+\sum_{k=-\infty}^{j-N}\Big)a(k)e^{-ik\xi_0}\Big|^p\Big)^{1/p}\\
&\quad +\Big(\sum_{N-N^{1/2}<|j|<N+N^{1/2}}\Big|\sum_{k=-N}^{N} |a(j-k)|\Big|^p\Big)^{1/p}\\
&\le \Big(\sum_{|k|\ge N^{1/2}}|a(k)|\Big)^{(p-1)/p}\Big(\sum_{|j|\ge N+N^{1/2}}\sum_{k=-N}^{N}|a(j-k)|\Big)^{1/p}\\
&\quad +\Big(\sum_{|k|\ge N^{1/2}}|a(k)|\Big)^{(p-1)/p}\Big(\sum_{|j|\le N-N^{1/2}}\Big(\sum_{k=j+N}^{\infty}+\sum_{k=-\infty}^{j-N}\Big)|a(k)|\Big)^{1/p}\\
&\quad +\Big(\sum_{k\in\mathbb{Z}}|a(k)|\Big)\times\Big(\sum_{N-N^{1/2}<|j|<N+N^{1/2}}1\Big)^{1/p}\\
&\le 2(2N+1)^{1/p}\Big(\sum_{|k|\ge N^{1/2}}|a(k)|\Big)^{(p-1)/p}\Big(\sum_{k\in\mathbb{Z}}|a(k)|\Big)^{1/p}\\
&\quad +(4N^{1/2}+2)^{1/p}\Big(\sum_{k\in\mathbb{Z}}|a(k)|\Big). \qquad (1.2.5)
\end{aligned}
$$

The above two estimates, together with the assumption that $a \in \ell^1$, imply that

$$\begin{aligned}
0 &\le \limsup_{N\to\infty} \frac{\|C_a e_{N,\xi_0}\|_{\ell^p}}{\|e_{N,\xi_0}\|_{\ell^p}} \\
&\le \limsup_{N\to\infty} \Big(\sum_{|k|\ge N^{1/2}} |a(k)|\Big)^{(p-1)/p} \Big(\sum_{k\in\mathbb{Z}} |a(k)|\Big)^{1/p} \\
&\quad + \limsup_{N\to\infty} \Big(\frac{4N^{1/2}+2}{2N+1}\Big)^{1/p} \Big(\sum_{k\in\mathbb{Z}} |a(k)|\Big) \\
&= 0,
\end{aligned} \tag{1.2.6}$$

which contradicts to the ℓ^p-stability of the convolution operator C_a.

($\Longleftarrow$) By the classical Wiener's lemma (Lemma 1.2.2),

$$\frac{1}{\hat{a}(\xi)} = \sum_{j\in\mathbb{Z}} b(j) e^{-ij\xi} \tag{1.2.7}$$

for some summable sequence $b = (b(j))_{j\in\mathbb{Z}}$. Let C_b denote the convolution operator associated with the sequence b. Then C_b is a bounded operator on ℓ^p by Proposition 1.1.1. By (1.2.7), we have that

$$\begin{aligned}
1 &= \Big(\sum_{k\in\mathbb{Z}} a(k) e^{-ik\xi}\Big) \times \Big(\sum_{j\in\mathbb{Z}} b(j) e^{-ij\xi}\Big) \\
&= \sum_{n\in\mathbb{Z}} \Big(\sum_{k=-\infty}^{\infty} a(k) b(n-k)\Big) e^{-in\xi}.
\end{aligned}$$

This implies that

$$\sum_{k=-\infty}^{\infty} a(k) b(n-k) = \delta_n \tag{1.2.8}$$

where δ_n is the Kronecker delta defined by

$$\delta_n := \begin{cases} 1 & if\ n = 0, \\ 0 & if\ 0 \ne n \in \mathbb{Z}. \end{cases}$$

Thus

$$\begin{aligned} C_b C_a &= \Big(\sum_{k\in\mathbb{Z}} b(i-k)a(k-j)\Big)_{i,j\in\mathbb{Z}} \\ &= \Big(\sum_{k\in\mathbb{Z}} a(k)b((i-j)-k)\Big)_{i,j\in\mathbb{Z}} \\ &= \big(\delta_{i-j}\big)_{i,j\in\mathbb{Z}}. \end{aligned} \tag{1.2.9}$$

This shows that C_b is the inverse of the convolution operator C_a, that is,

$$C_b C_a c = c \quad for\ all\ c \in \ell^p. \tag{1.2.10}$$

By (1.2.10) and Proposition 1.1.1, we have

$$\|C_a c\|_{\ell^p} \leq \|a\|_{\ell^1} \|c\|_{\ell^p} \tag{1.2.11}$$

and

$$\|c\|_{\ell^p} = \|C_b C_a c\|_{\ell^p} \leq \|b\|_{\ell^1} \|C_a c\|_{\ell^p} \quad for\ all\ c \in \ell^p. \tag{1.2.12}$$

Then the ℓ^p-stability of the convolution operator C_a follows from (1.2.11) and (1.2.12).

1.3 Historical Origin

Now we state precisely the classical result from which this dissertation originates.

Theorem 1.3.1 *Let a be a summable sequence, and C_a be the convolution operator* (1.1.3) *associated with the sequence a. Then C_a has ℓ^p-stability for some $1 \leq p \leq \infty$ if and only if it has ℓ^q-stability for all $1 \leq q \leq \infty$.*

Proof. Clearly it suffices to prove the sufficiency. By Proposition 1.2.1 and the assumption that C_a has ℓ^p-stability, the Fourier series $\hat{a}(\xi)$ does not vanish on the whole line, which in turn implies that the convolution operator C_a has ℓ^q-stability by Proposition 1.2.1.

CHAPTER TWO: MOTIVATION

In [8], Shin and Sun gave a new proof of Theorem 1.3.1, the ℓ^p-stability of a convolution operator associated with a summable sequence for different $1 \le p \le \infty$, without using Proposition 1.2.1. Furthermore they showed that the ℓ^p-stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class are equivalent to each other (Theorem 2.3.1). This motived me to consider the equivalence of weighted ℓ^p-stability for infinite matrices in the weighted Gohberg-Baskakov-Sjöstrand class for different $1 \le p \le \infty$, see next chapter for details.

2.1 Schur Class

In this section, we introduce the Schur class of infinite matrices, define every infinite matrix in the Schur class as a bounded operator on ℓ^p, and also show that the Schur class is the smallest class of linear operators that are bounded on ℓ^p for all $1 \le p \le \infty$.

Definition 2.1.1 *Let*

$$\mathcal{A} = \left\{ A := (a(j,j'))_{j,j' \in \mathbb{Z}} \middle| \ \|A\|_{\mathcal{A}} < \infty \right\} \tag{2.1.1}$$

be the Schur class *of infinite matrices* $A = (a(j,j'))_{j,j' \in \mathbb{Z}}$, *where*

$$\|A\|_{\mathcal{A}} := \max \left(\sup_{j \in \mathbb{Z}} \sum_{j' \in \mathbb{Z}} |a(j,j')|, \ \sup_{j' \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} |a(j,j')| \right). \tag{2.1.2}$$

An infinite matrix $A = (a(j,j'))_{j,j'\in\mathbb{Z}}$ in the Schur class can be thought as a bounded operator on $\ell^p, 1 \le p \le \infty$, which is defined by

$$A:\ \ell^p \ni (c(j))_{j\in\mathbb{Z}} \longmapsto \Big(\sum_{j'\in\mathbb{Z}} a(j,j')c(j')\Big)_{j\in\mathbb{Z}} \in \ell^p. \tag{2.1.3}$$

Proposition 2.1.1 *Let $1 \le p \le \infty$. If $A = (a(i,j))_{i,j\in\mathbb{Z}}$ is an infinite matrix in the Schur class $\mathcal{A}$, then A defines a bounded operator on ℓ^p. Furthermore*

$$\|Ac\|_{\ell^p} \le \|A\|_{\mathcal{A}}\|c\|_{\ell^p} \quad for\ all\ c \in \ell^p. \tag{2.1.4}$$

Proof. Clearly it suffices to establish (2.1.4).

First we prove (2.1.4) for $p = \infty$. For any sequence $c = (c(j))_{j\in\mathbb{Z}} \in \ell^\infty$,

$$\begin{aligned} \|Ac\|_{\ell^\infty} &= \sup_{j\in\mathbb{Z}} \Big|\sum_{k\in\mathbb{Z}} a(j,k)c(k)\Big| \\ &\le \sup_{k\in\mathbb{Z}} |c(k)| \times \sup_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |a(j,k)|\Big) \\ &\le \|A\|_{\mathcal{A}}\|c\|_{\ell^\infty} \end{aligned} \tag{2.1.5}$$

and (2.1.4) for $p = \infty$ follows.

Now we prove (2.1.4) for $1 \le p < \infty$. For any sequence $c = (c(j))_{j\in\mathbb{Z}} \in \ell^p$,

$$\begin{aligned} \|Ac\|_{\ell^p} &= \Big(\sum_{j\in\mathbb{Z}} \Big|\sum_{k\in\mathbb{Z}} a(j,k)c(k)\Big|^p\Big)^{1/p} \\ &\le \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |a(j,k)|\Big)^{p-1} \times \Big(\sum_{k\in\mathbb{Z}} |a(j,k)||c(k)|^p\Big)\Big)^{1/p} \\ &\le \|A\|_{\mathcal{A}}^{(p-1)/p} \times \Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |a(j,k)||c(k)|^p\Big)^{1/p} \\ &\le \|A\|_{\mathcal{A}}\|c\|_{\ell^p}. \end{aligned} \tag{2.1.6}$$

This proves (2.1.4) for $1 \le p < \infty$ and completes the proof.

In the next result, we show that the converse of the conclusion in Proposition 2.1.1 holds, which implies that the Schur class is the *smallest* class of linear operators that are bounded on ℓ^p for all $1 \leq p \leq \infty$.

Proposition 2.1.2 *If T is a bounded linear operator bounded on ℓ^p for all $1 \leq p \leq \infty$ then $T = A$ for some infinite matrix A in the Schur class $\mathcal{A}$.*

Proof For any $k \in \mathbb{Z}$, let $e_k = (\delta_{j-k})_{j\in\mathbb{Z}}$ where δ_n is the Kronecker delta. Then $e_k \in \ell^1$ and $Te_k := (a(i,k))_{i\in\mathbb{Z}} \in \ell^1$ by the assumption. By the ℓ^1 boundedness of the operator T, we conclude that

$$\|Te_k\|_{\ell^1} = \sum_{i\in\mathbb{Z}} |a(i,k)| \leq \|T\|_1 \|e_k\|_{\ell^1} = \|T\|_1 \quad for\ all\ k \in \mathbb{Z},$$

or equivalently,

$$\sup_{k\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} |a(i,k)| \leq \|T\|_1, \tag{2.1.7}$$

where $\|T\|_p, 1 \leq p \leq \infty$, is the linear operator norm of the operator T on ℓ^p.

Let T^* be the conjugate operator of the operator T on ℓ^2. Note that

$$\langle T^* e_i, e_k \rangle = \langle e_i, Te_k \rangle = a(i,k)$$

for any $i, k \in \mathbb{Z}$. Therefore

$$T^* e_i = (a(i,j))_{j\in\mathbb{Z}} \quad for\ all\ i \in \mathbb{Z}. \tag{2.1.8}$$

By the ℓ^∞ boundedness of the operator T, and by the dual property between sequence spaces ℓ^∞ and ℓ^1, the conjugate operator T^* is a bounded operator on ℓ^1 and

$$\|T^*\|_1 = \|T\|_\infty. \tag{2.1.9}$$

Similar to the argument in establishing (2.1.7), we obtain from (2.1.8) and (2.1.9) that

$$\sup_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|a(i,j)| \le \|T^*\|_1 = \|T\|_\infty, \tag{2.1.10}$$

Therefore the infinite matrix $A = (a(i,j))_{i,j\in\mathbb{Z}}$ belongs to the Schur class by (2.1.7) and (2.1.10).

From the definition of the matrix A, we see that $Te_k = Ae_k$ for any $k \in \mathbb{Z}$. For any $c = (c(j))_{j\in\mathbb{Z}} \in \ell^p$, define $c_n = (c_n(j))_{j\in\mathbb{Z}}, n \ge 1$, where $c_n(j) = c(j)$ if $|j| \le n$ and $c_n(j) = 0$ if $|j| > n$. Then for all $n \ge 1$ we have that $c_n \in \ell^p$ (in fact, $\|c_n\|_{\ell^p} \le \|c\|_{\ell^p}$), and that

$$Tc_n = \sum_{|k|\le n} c_k Te_k = \sum_{|k|\le n} c_k Ae_k = Ac_n. \tag{2.1.11}$$

Therefore from (2.1.7), (2.1.10), and (2.1.11) it follows that

$$\langle e_k, Tc\rangle = \lim_{n\to\infty}\langle e_k, Tc_n\rangle = \lim_{n\to\infty}\langle e_k, Ac_n\rangle = \langle e_k, Ac\rangle \quad for\ all\ k \in \mathbb{Z}.$$

This proves that

$$Tc = Ac \quad for\ all\ c \in \ell^p. \tag{2.1.12}$$

Combining (2.1.7), (2.1.10) and (2.1.12) completes the proof.

2.2 Gohberg-Baskakov-Sjöstrand Class

In this section, we recall the Gohberg-Baskakov-Sjöstrand class of infinite matrices and show that for every matrix A in the Gohberg-Baskakov-Sjöstrand class, its band-truncation A_s converges as the band s tends to infinity (Proposition 2.2.2).

Definition 2.2.1 *Let*

$$\mathcal{C} = \left\{ A := (a(j,j'))_{j,j'\in\mathbb{Z}} \,\middle|\, \|A\|_{\mathcal{C}} < \infty \right\} \tag{2.2.1}$$

be the Gohberg-Baskakov-Sjöstrand class *of infinite matrices* $A = (a(j,j'))_{j,j'\in\mathbb{Z}}$*, where*

$$\|A\|_{\mathcal{C}} := \sum_{k\in\mathbb{Z}} \sup_{i-j=k} |a(i,j)|. \tag{2.2.2}$$

Any infinite matrix A in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}$ belongs to the Schur class.

Proposition 2.2.1 *If* $A \in \mathcal{C}$*, then* $A \in \mathcal{A}$*. Furthermore*

$$\|A\|_{\mathcal{A}} \le \|A\|_{\mathcal{C}} \quad for\ all \quad A \in \mathcal{C}. \tag{2.2.3}$$

Proof. Let $A = (a(i,j))_{i,j\in\mathbb{Z}} \in \mathcal{C}$ and set $r(k) = \sup_{i-j=k} |a(i,j)|$ for any $k \in \mathbb{Z}$. Then

$$\begin{aligned} \|A\|_{\mathcal{A}} &= \max\Big(\sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |a(i,j)|, \sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} |a(i,j)|\Big) \\ &\le \max\Big(\sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} r(i-j), \sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} r(i-j)\Big) \\ &= \sum_{k\in\mathbb{Z}} r(k) = \|A\|_{\mathcal{C}}. \end{aligned} \tag{2.2.4}$$

This proves (2.2.3) and completes the proof.

For any $s \ge 0$ and any infinite matrix $A = (a(i,j))_{i,j\in\mathbb{Z}}$, we define the *band-truncation matrix*

$$A_s = (a_s(i,j))_{i,j\in\mathbb{Z}} \tag{2.2.5}$$

where

$$a_s(i,j) = \begin{cases} a(i,j) & if\ |i-j| < s, \\ 0 & otherwise. \end{cases}$$

Clearly

$$\lim_{s\to\infty} a_s(i,j) = a(i,j) \ for \ all \ i,j \in \mathbb{Z}.$$

Moreover for infinite matrices A in the Gohberg-Baskakov-Sjöstrand class, A_s tends to A in the norm $\|\cdot\|_{\mathcal{C}}$ of the Gohberg-Baskakov-Sjöstrand class.

Proposition 2.2.2 *Let $A \in \mathcal{C}$ and for $s \geq 0$ define A_s as in* (2.2.5). *Then*

$$\lim_{s\to\infty} \|A - A_s\|_{\mathcal{C}} = 0. \tag{2.2.6}$$

Proof. Let $A = (a(i,j))_{i,j\in\mathbb{Z}}$ and define

$$r(k) = \sup_{i-j=k} |a(i,j)| \quad for \ all \ k \in \mathbb{Z}.$$

Then

$$\sum_{k\in\mathbb{Z}} r(k) = \|A\|_{\mathcal{C}} < \infty. \tag{2.2.7}$$

From the definition of the truncation matrix A_s, we have that

$$\begin{aligned} \|A - A_s\|_{\mathcal{C}} &= \sum_{k\in\mathbb{Z}} \sup_{i-j=k} |a(i,j) - a_s(i,j)| \\ &= \sum_{|k|\geq s} \sup_{i-j=k} |a(i,j)| = \sum_{|k|\geq s} r(k). \end{aligned} \tag{2.2.8}$$

Combining (2.2.7) and (2.2.8) proves (2.2.6).

We remark that the truncation matrix A_s may not tend to A for infinite matrices in the Schur class. The infinite matrix $(\delta_{i+j})_{i,j\in\mathbb{Z}}$ is such an infinite matrix in the Schur class.

Let

$$\psi_0(x) = \begin{cases} 1 & if \ |x| \leq 1, \\ 2 - |x| & if \ 1 < |x| < 2, \\ 0 & if \ |x| \geq 2. \end{cases} \tag{2.2.9}$$

be a cut-off function. Define the *multiplication operator* $\Psi_n^N : \ell^p \to \ell^p$ by

$$\Psi_n^N c = \Big(\psi_0\big(\frac{j-n}{N}\big)c(j)\Big)_{j\in\mathbb{Z}} \quad for\ c = (c(j))_{j\in\mathbb{Z}} \in \ell^p. \tag{2.2.10}$$

The multiplication operator Ψ_n^N is a diagonal matrix $diag(\psi_0((j-n)/N))_{j\in\mathbb{Z}}$ in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}$.

Proposition 2.2.3 *Let $A \in \mathcal{C}$, and the truncation matrices A_N and the multiplication operator Ψ_n^N be defined as in* (2.2.5) *and* (2.2.10) *respectively. Then*

$$\|A_N\Psi_n^N - \Psi_n^N A_N\|_{\mathcal{C}} \le \inf_{0\le s\le N} \|A - A_s\|_{\mathcal{C}} + \frac{s}{N}\|A\|_{\mathcal{C}} \tag{2.2.11}$$

for all $N \ge 1$ and $n \in \mathbb{Z}$.

Proof. Let $A = (a(i,j))_{i,j\in\mathbb{Z}}$ and set $r(k) = \sup_{i-j=k} |a(i,j)|$ for $k \in \mathbb{Z}$. Then

$$\begin{aligned}
\|A_N\Psi_n^N - \Psi_n^N A_N\|_{\mathcal{C}} &= \sum_{k\in\mathbb{Z}} \sup_{i-j=k} |a_N(i,j)||\psi_0((i-n)/N) - \psi_0((j-n)/N)| \\
&= \inf_{0\le s\le N} \sum_{|k|<s} \sup_{i-j=k} |a(i,j)||\psi_0((i-n)/N) - \psi_0((j-n)/N)| \\
&\quad + \sum_{N>|k|\ge s} \sup_{i-j=k} |a(i,j)||\psi_0((i-n)/N) - \psi_0((j-n)/N)| \\
&\le \inf_{0\le s\le N} \frac{1}{N} \sum_{|k|<s} |k| r(k) + \sum_{|k|\ge s} r(k) \\
&\le \inf_{0\le s\le N} \frac{s}{N}\|A\|_{\mathcal{C}} + \|A - A_s\|_{\mathcal{C}},
\end{aligned}$$

where in the first inequality we have used the following properties:

$$0 \le \psi_0(x) \le 1 \ for\ all\ x \in \mathbb{R}$$

and

$$|\psi_0(x) - \psi_0(y)| \le |x-y| \ for\ all\ x, y \in \mathbb{R}$$

for the cut-off function ψ_0.

The above property for the commutator between the multiplication operator and truncated infinite matrices in the Gohberg-Baskakov-Sjöstrand class plays essential role in [8] to establish the equivalence of the ℓ^p stability for infinite matrices in the Gohberg-Baskakov-Sjöstrand class.

2.3 ℓ^p-stability of Infinite Matrices

In this section, we restate Shin and Sun's result on the ℓ^p-stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class ([8]) and give a sketch of the proof.

Theorem 2.3.1 *Let $1 \le p, q \le \infty$ and $A \in \mathcal{C}$. If A has ℓ^q-stability, then A has ℓ^p-stability.*

This generalizes the conclusion in Theorem 1.3.1 for convolution operators, because the convolution operator C_a associated with the summable sequence $a := (a(j))_{j\in\mathbb{Z}}$ is the same as an infinite matrix $A = (a(i-j))_{i,j\in\mathbb{Z}}$ in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}$ with $\|A\|_{\mathcal{C}} = \|a\|_{\ell^1} < \infty$.

Now we give a sketch of the proof of Theorem 2.3.1 as we will follow those steps to prove the equivalence of weighted ℓ^p-stability in the next chapter.

Proof of Theorem 2.3.1. Given $1 \le p, q \le \infty$ and $N \ge 1$, define a new norm $\|\cdot\|_{p,q,N}$ on ℓ^p as follows:

$$\|c\|_{p,q,N} = \begin{cases} \left(\sum_{n\in N\mathbb{Z}} \|\Psi_n^N c\|_{\ell^q}^p\right)^{1/p} & if\ 1 \le p < \infty, \\ \sup_{n\in N\mathbb{Z}} \|\Psi_n^N c\|_{\ell^q} & if\ p = \infty, \end{cases} \tag{2.3.1}$$

for any $c \in \ell^p$, where Ψ_n^N is the multiplication operator defined in (2.2.10). For this new norm $\|\cdot\|_{p,q,N}$ on ℓ^p, the following claims hold true:

Claim 1: Given $1 \leq p \leq \infty$, there exists a positive constant C independent of $N \geq 1$ such that

$$\|c\|_{\ell^p} \leq \|c\|_{p,p,N} \leq C\|c\|_{\ell^p} \tag{2.3.2}$$

hold for all $c \in \ell^p$.

Claim 2: Given $1 \leq p, q_1, q_2 \leq \infty$, there exists a positive constant C independent of $N \geq 1$ such that

$$C^{-1}N^{\min(1/q_2-1/q_1,0)}\|c\|_{p,q_1,N} \leq \|c\|_{p,q_2,N} \leq CN^{\max(1/q_2-1/q_1,0)}\|c\|_{p,q_1,N} \tag{2.3.3}$$

for all $c \in \ell^p$.

Claim 3: Given $1 \leq p, q \leq \infty$, there exists a positive constant C independent of $N \geq 1$ such that

$$\|c\|_{p,q,N} \leq \|c\|_{p,q,6N} \leq C\|c\|_{p,q,N} \tag{2.3.4}$$

for all $c \in \ell^p$.

Claim 4: There exists a positive constant C independent of $N \geq 1$ such that

$$\|Ac\|_{p,q,N} \leq C\|A\|_{\mathcal{C}}\|c\|_{p,q,N} \ for\ all\ c \in \ell^p. \tag{2.3.5}$$

Now we can prove our desired result: the ℓ^p stability for infinite matrix A. Take any $c \in \ell^p$. By the ℓ^q-stability of the infinite matrix A, there exists an absolute positive constant C_0 (independent of $N \geq 1, n \in \mathbb{Z}$ and $c \in \ell^p$) such that

$$\|A\Psi_n^N c\|_{\ell^q} \geq C_0\|\Psi_n^N c\|_{\ell^q}. \tag{2.3.6}$$

By (2.2.3) and Proposition 2.1.1, we have

$$\|(A - A_N)\Psi_n^N c\|_{\ell^q} \le \|A - A_N\|_{\mathcal{C}} \|\Psi_n^N c\|_{\ell^q}. \tag{2.3.7}$$

By Proposition 2.2.3, we have

$$\begin{aligned} \|(A_N\Psi_n^N - \Psi_n^N A_N)c\|_{\ell^q} &= \|(A_N\Psi_n^N - \Psi_n^N A_N)\Psi_n^{6N} c\|_{\ell^q} \\ &\le \Big(\inf_{0\le s\le N} \frac{s}{N}\|A\|_{\mathcal{C}} + \|A - A_s\|_{\mathcal{C}}\Big)\|\Psi_n^{6N} c\|_{\ell^q}. \end{aligned} \tag{2.3.8}$$

Combining (2.3.6), (2.3.7) and (2.3.8), we get

$$\begin{aligned} \|\Psi_n^N A_N c\|_{\ell^q} &\ge \Big(C_0 - \|A - A_N\|_{\mathcal{C}}\Big)\|\Psi_n^N c\|_{\ell^q} \\ &\quad -\Big(\inf_{0\le s\le N} \frac{s}{N}\|A\|_{\mathcal{C}} + \|A - A_s\|_{\mathcal{C}}\Big)\|\Psi_n^{6N} c\|_{\ell^q}, \end{aligned} \tag{2.3.9}$$

for any $N \ge 1$ and $n \in \mathbb{Z}$, which implies that

$$\begin{aligned} \|A_N c\|_{p,q,N} &\ge \Big(C_0 - \|A - A_N\|_{\mathcal{C}}\Big)\|c\|_{p,q,N} \\ &\quad -\Big(\inf_{0\le s\le N} \frac{s}{N}\|A\|_{\mathcal{C}} + \|A - A_s\|_{\mathcal{C}}\Big)\|c\|_{p,q,6N}. \end{aligned} \tag{2.3.10}$$

This together with Claim 3 and Claim 4 yields

$$\|Ac\|_{p,q,N} \ge \Big(C_0 - C_1\|A - A_N\|_{\mathcal{C}} - C_2\big(\inf_{0\le s\le N} \frac{s}{N}\|A\|_{\mathcal{C}} + \|A - A_s\|_{\mathcal{C}}\big)\Big)\|c\|_{p,q,N}, \tag{2.3.11}$$

where C_1 and C_2 are positive constants independent of $N \ge 1$. Recall from Proposition 2.2.2 that

$$\lim_{s\to\infty} \|A - A_s\|_{\mathcal{C}} = 0.$$

Therefore there exists a sufficiently large integer N_0 such that

$$\begin{aligned} & C_1\|A - A_{N_0}\|_{\mathcal{C}} + C_2\Big(\inf_{0\le s\le N_0} \frac{s}{N_0}\|A\|_{\mathcal{C}} + \|A - A_s\|_{\mathcal{C}}\Big) \\ \le\ & (C_1 + C_2)\|A - A_{\sqrt{N_0}}\|_{\mathcal{C}} + C_2 N_0^{-1/2}\|A\|_{\mathcal{C}} \\ \le\ & C_0/2. \end{aligned} \tag{2.3.12}$$

Thus by (2.3.10) and (2.3.12) we have

$$\|Ac\|_{p,q,N_0} \geq \frac{C_0}{2}\|c\|_{p,q,N_0} \tag{2.3.13}$$

for any $c \in \ell^p$. Therefore the desired ℓ^p stability from from (2.3.13), Claim 1, and Claim 2.

CHAPTER THREE: WEIGHTED ℓ^P-STABILITY FOR LOCALIZED INFINITE MATRICES

In this chapter, I show that the weighted ℓ^p-stability of infinite matrices in the weighted Gohberg-Baskakov-Sjöstrand class are equivalent to each other for different $1 \le p \le \infty$. Please see Theorem 3.3.1 for the details.

3.1 Weighted ℓ^p Space

In this section, we introduce the weighted ℓ^p space of sequences with the standard norm $\|\cdot\|_{\ell^p_w}$ and its equivalent norm $\|\cdot\|_{p,q,N,w}$.

Definition 3.1.1 *A* weight *is a continuous function* $w : \mathbb{R}^d \longmapsto [1, \infty)$*. A weight* w *is said to be* submultiplicative *if there exists a positive constant* C *such that*

$$w(x + y) \le Cw(x)w(y) \quad for\ all\ x, y \in \mathbb{R}^d. \tag{3.1.1}$$

A typical example of submultiplicative weight is the power weight $(1 + |x|)^\alpha$ with $\alpha \ge 0$, which becomes the trivial weight $w \equiv 1$ when $\alpha = 0$.

We consider the infinite matrices of the form $(a(\lambda, \lambda'))_{\lambda\in\Lambda,\lambda'\in\Lambda'}$ having certain off-diagonal decay.

Definition 3.1.2 *A discrete subset* Λ *of* $\mathbb{R}^d$ *is called* relatively-separated *if*

$$R(\Lambda) = \sup_{x\in\mathbb{R}^d} \sum_{\lambda\in\Lambda} \chi_{\lambda+[0,1)^d}(x) \ < \ \infty. \tag{3.1.2}$$

As usual, an infinite matrix $A = (a(\lambda, \lambda^{'}))_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}}$ in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\Lambda, \Lambda^{'})$ defines a bounded operator from $\ell^p(\Lambda^{'})$ to $\ell^p(\Lambda)$,

$$A : l^p(\Lambda^{'}) \ni (c(\lambda^{'}))_{\lambda^{'}\in\Lambda^{'}} := c \mapsto Ac := \left(\sum_{\lambda^{'}\in\Lambda^{'}} a(\lambda, \lambda^{'})c(\lambda^{'})\right)_{\lambda\in\Lambda} \in \ell^p(\Lambda), \tag{3.1.3}$$

where $1 \le p \le \infty$.

Definition 3.1.3 *For $n \in \mathbb{Z}^d$ and $N \in \mathbb{N}$, the cut-off function is defined as*

$$\psi(x) = \min(\max(2 - ||x||_\infty, 0), 1) \quad \text{for } x \in \mathbb{R}^d, \tag{3.1.4}$$

here $\|x\|_\infty = \max_{1\le i\le d} |x_i|$ for $x = (x_1, ..., x_d) \in \mathbb{R}^d$.

Definition 3.1.4 *The multiplication operator $\Psi_n^N : \ell^p(\Lambda) \to \ell^p(\Lambda)$ is defined by*

$$\Psi_n^N c = \left(\psi(\frac{\lambda - n}{N})c(\lambda)\right)_{\lambda\in\Lambda} \quad \text{for } c = (c(\lambda))_{\lambda\in\Lambda} \in \ell^p, \tag{3.1.5}$$

where Λ is a relatively-seperated subset of $\mathbb{R}^d$.

Proposition 3.1.1 *The cut-off function $\psi(x)$ is Lipschitz continuous, that is,*

$$|\psi(x) - \psi(y)| \le ||x - y||_\infty \quad \text{for } x, y \in \mathbb{R}^d. \tag{3.1.6}$$

Proof. By Definition 3.1.4, we have

$$
\begin{aligned}
|\psi(x)-\psi(y)| &= \min(\max(2-||x||_\infty,0),1)-\min(\max(2-||y||_\infty,0),1) \\
&= \begin{cases}
|\,\|x\|_\infty-\|y\|_\infty| & if\ 1\le\|x\|_\infty\le 2,\ 1\le\|y\|_\infty\le 2; \\
|2-\|x\|_\infty| & if\ 1\le\|x\|_\infty\le 2,\ \|y\|_\infty\ge 2; \\
|2-\|y\|_\infty| & if\ 1\le\|y\|_\infty\le 2,\ \|x\|_\infty\ge 2; \\
|1-\|x\|_\infty| & if\ 1\le\|x\|_\infty\le 2,\ \|y\|_\infty\le 1; \\
|1-\|y\|_\infty| & if\ 1\le\|y\|_\infty\le 2,\ \|x\|_\infty\ge 2; \\
1 & if\ \|x\|_\infty\le 1,\ \|y\|_\infty\ge 2; \\
1 & if\ \|x\|_\infty\le 1,\ \|y\|_\infty\ge 2; \\
0 & otherwise.
\end{cases} \\
&\le \|x-y\|_\infty \quad \text{for } x,y\in\mathbb{R}^d. \qquad (3.1.7)
\end{aligned}
$$

Proposition 3.1.2 *Let* $A=(a(\lambda,\lambda^{'}))_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}}$ *be an infinite matrix in the Gohberg-Baskakov-Sjöstrand class* $\mathcal{C}(\Lambda,\Lambda^{'})$ *and* A_N *be the truncation matrix. The mutiplication operator* Ψ_n^N *is defined by* (3.1.5). *Then*

$$A_N\Psi_n^N-\Psi_n^N A_N=(A_N\Psi_n^N-\Psi_n^N A_N)\Psi_n^{6N} \qquad (3.1.8)$$

for all $N\ge 1$ *and* $n\in\mathbb{Z}^d$.

Proof. First we prove that for all $\lambda\in\Lambda$, $\lambda^{'}\in\Lambda^{'}$, we have

$$a_N(\lambda,\lambda^{'})\left(\psi\left(\frac{\lambda^{'}-n}{N}\right)-\psi\left(\frac{\lambda-n}{N}\right)\right)\left(1-\psi\left(\frac{\lambda^{'}-n}{6N}\right)\right)=0 \qquad (3.1.9)$$

(Case 1:) If $|\lambda^{'}-\lambda|\ge N$, by definition we have $a_N(\lambda,\lambda^{'})=0$.

(Case 2:) If $|\lambda^{'}-\lambda|<N$ and $|\lambda^{'}-n|\le 3N$, by definition we have $\psi\left(\frac{\lambda^{'}-n}{6N}\right)=1$. So

$1-\psi\left(\frac{\lambda'-n}{6N}\right)=0.$

(Case 3:) If $|\lambda'-\lambda|<N$ and $|\lambda'-n|>3N$, by definition we have $\psi\left(\frac{\lambda'-n}{N}\right)=0$. Since $|\lambda-n|\geq|\lambda'-n|-|\lambda'-\lambda|>3N-N=2N$, we have $\frac{|\lambda-n|}{N}>2$ which implies $\psi\left(\frac{\lambda-n}{N}\right)=0$. So we get $\psi\left(\frac{\lambda'-n}{N}\right)-\psi\left(\frac{\lambda-n}{N}\right)=0$.

Then we finish prove (3.1.9). From (3.1.9), for all $\lambda\in\Lambda$, $\lambda'\in\Lambda'$,

$$a_N(\lambda,\lambda')\left(\psi\left(\frac{\lambda'-n}{N}\right)-\psi\left(\frac{\lambda-n}{N}\right)\right)=a_N(\lambda,\lambda')\left(\psi\left(\frac{\lambda'-n}{N}\right)-\psi\left(\frac{\lambda-n}{N}\right)\right)\psi\left(\frac{\lambda'-n}{6N}\right)$$

$$\begin{aligned}&(A_N\Psi_n^N-\Psi_n^NA_N)=A_N\Psi_n^N-\Psi_n^NA_N\\ =&\left(\sum_{\lambda'\in\Lambda'}a_N(\lambda,\lambda')\psi\left(\frac{\lambda'-n}{N}\right)\right)_{\lambda\in\Lambda}-\left(\psi\left(\frac{\lambda-n}{N}\right)\sum_{\lambda'\in\Lambda'}a_N(\lambda,\lambda')\right)_{\lambda\in\Lambda}\\ =&\left(\sum_{\lambda'\in\Lambda'}a_N(\lambda,\lambda')\left(\psi\left(\frac{\lambda'-n}{N}\right)-\psi\left(\frac{\lambda-n}{N}\right)\right)\right)_{\lambda\in\Lambda}\end{aligned}\tag{3.1.10}$$

and

$$\begin{aligned}&(A_N\Psi_n^N-\Psi_n^NA_N)\psi_n^{6N}=A_N\Psi_n^N\Psi_n^{6N}-\Psi_n^NA_N\Psi_n^{6N}\\ =&\left(\sum_{\lambda'\in\Lambda'}a_N(\lambda,\lambda')\left(\psi\left(\frac{\lambda'-n}{N}\right)-\psi\left(\frac{\lambda-n}{N}\right)\right)\psi\left(\frac{\lambda'-n}{6N}\right)\right)_{\lambda\in\Lambda}.\end{aligned}\tag{3.1.11}$$

Therefore from (3.1.9), (3.1.10) and (3.1.11), we can get (3.1.8).

Proposition 3.1.3 *The cut-off function* $\psi(x)$ *is defined by* (3.1.4). *Then for all* $N\geq1$, $n\in\mathbb{Z}^d$ *and* $\lambda'\in\Lambda'$ *we have*

$$\sum_{j\in\mathbb{Z}^d}\left|\psi\left(\frac{\lambda'-n}{6N}\right)\right|\left|\psi\left(\frac{\lambda'-n}{N}-2j\right)\right|=\sum_{j\in\mathbb{Z}^d\ \ with\ \ ||j||_\infty\leq6}\left|\psi\left(\frac{\lambda'-n}{6N}\right)\right|\left|\psi\left(\frac{\lambda'-n}{N}-2j\right)\right|.\tag{3.1.12}$$

Proof. Clearly (3.1.12) is equivalent to

$$\sum_{j\in\mathbb{Z}^d \text{ with } ||j||_\infty>6} \left|\psi\left(\frac{\lambda^{'}-n}{6N}\right)\right| \left|\psi\left(\frac{\lambda^{'}-n}{N}-2j\right)\right| = 0. \tag{3.1.13}$$

By the definition of $\psi(x)$, we know that $\psi(x) = 0$ if $||x||_\infty \geq 2$. So if $||\frac{\lambda^{'}-n}{6N}||_\infty \geq 2$ which is $||\frac{\lambda^{'}-n}{N}||_\infty \geq 12$, $\psi(\frac{\lambda^{'}-n}{6N}) = 0$. If $\psi(\frac{\lambda^{'}-n}{6N}) \neq 0$, $||\frac{\lambda^{'}-n}{N}||_\infty < 12$. For $||j||_\infty > 6$, we have

$$\left\|\frac{\lambda^{'}-n}{N}-2j\right\|_\infty \geq \left| \left\|\frac{\lambda^{'}-n}{N}\right\|_\infty - ||2j||_\infty\right| = ||2j||_\infty - \left\|\frac{\lambda^{'}-n}{N}\right\|_\infty \geq 14-12 = 2.$$

So $\psi(\frac{\lambda^{'}-n}{N}-2j) = 0$. This finishes the proof of (3.1.13), and hence (3.1.12).

Definition 3.1.5 *For* $1 \leq p \leq \infty$ *and a weight* w*, let*

$$\ell^p_w(\Lambda) = \left\{\|c\|_{\ell^p_w(\Lambda)} < +\infty\right\} \tag{3.1.14}$$

be the weighted ℓ^p *space of all sequences on* Λ*, where*

$$\|c\|_{\ell^p_w(\Lambda)} := \|cw\|_{\ell^p(\Lambda)} \quad \text{for all } c \in \ell^p_w(\Lambda). \tag{3.1.15}$$

We also use ℓ^p_w instead of $\ell^p_w(\Lambda)$ for brevity. From the definition of the weighted ℓ^p space, we have that

$$\|c+d\|_{\ell^p_w} \leq \|c\|_{\ell^p_w} + \|d\|_{\ell^p_w} \quad \text{for all } c, d \in \ell^p_w. \tag{3.1.16}$$

In fact, ℓ^p_w is a Banach space for all $1 \leq p \leq \infty$ and weights w.

To establish our desired equivalence of weighted ℓ^p-stability for different $1 \leq p \leq \infty$, we need another norm $\|\cdot\|_{p,q,N,w}$ on ℓ^p_w, where

$$\|c\|_{p,q,N,w} = \begin{cases} \left(\sum_{n\in N\mathbb{Z}^d} \|\Psi^N_n c\|^p_{\ell^q_w}\right)^{1/p} & if\ 1 \leq p < \infty \\ \sup_{n\in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^q_w} & if\ p = \infty. \end{cases} \tag{3.1.17}$$

The next result shows that the new norm $\|\cdot\|_{p,q,N,w}$ is equivalent to the standard norm $\|\cdot\|_{\ell^p_w}$ on ℓ^p_w.

Proposition 3.1.4 *Let $1 \le p, q_1, q_2 \le \infty$ and w be a submultiplicative weight. Then there exists a positive constant C independent of $N \ge 1$ such that the following hold for all $c \in \ell^p_w$:*

$$2^{d/p}\|c\|_{\ell^p_w} \le \|c\|_{p,p,N,w} \le 4^{d/p}\|c\|_{\ell^p_w}; \tag{3.1.18}$$

$$\begin{aligned} C^{-1}(R(\Lambda)N^d)^{\min(1/q_2-1/q_1,0)}\|c\|_{p,q_1,N,w} &\le \|c\|_{p,q_2,N,w} \\ &\le C(R(\Lambda)N^d)^{\max(1/q_2-1/q_1,0)}\|c\|_{p,q_1,N,w}; \end{aligned} \tag{3.1.19}$$

and

$$\|c\|_{p,q,N,w} \le \|c\|_{p,q,6N,w} \le C\|c\|_{p,q,N,w}. \tag{3.1.20}$$

Proof. First we prove (3.1.18). For $p = \infty$ and $c \in \ell^\infty_w$,

$$\|c\|_{\infty,\infty,N,w} = \sup_{n\in N\mathbb{Z}^d}\sup_{\lambda\in\Lambda}\psi((\lambda-n)/N)|c(\lambda)|w(\lambda) \le \sup_{n\in N\mathbb{Z}^d}\sup_{\lambda\in\Lambda}|c(\lambda)|w(\lambda) = \|c\|_{\ell^\infty_w}$$

and

$$\begin{aligned} \|c\|_{\infty,\infty,N,w} &= \sup_{n\in N\mathbb{Z}^d}\sup_{\lambda\in\Lambda}\psi((\lambda-n)/N)|c(\lambda)|w(\lambda) \\ \ge \ & \sup_{\lambda\in\Lambda}\sup_{n\in N\mathbb{Z}^d}\chi_{[-1,1]^d}(\frac{\lambda-n}{N})|c(\lambda)|w(\lambda) = \|c\|_{\ell^\infty_w}. \end{aligned}$$

Hence the estimate in (3.1.18) follows for $p = \infty$.

For $1 \le p < \infty$ and $c \in \ell^p_w$,

$$\begin{aligned} \|c\|_{p,p,N,w} &= \Big(\sum_{n\in N\mathbb{Z}^d}\sum_{\lambda\in\Lambda}|\psi((\lambda-n)/N)|^p|c(\lambda)|^p(w(\lambda))^p\Big)^{1/p} \\ &\le \Big(\sum_{n\in N\mathbb{Z}^d}\sum_{\lambda\in\Lambda}\chi_{[-2,2)^d}((\lambda-n)/N)|c(\lambda)|^p(w(\lambda))^p\Big)^{1/p} \\ &\le 4^{d/p}\Big(\sum_{\lambda\in\Lambda}|c(\lambda)|^p(w(\lambda))^p\Big)^{1/p} = 4^{d/p}\|c\|_{\ell^p_w}, \end{aligned}$$

and

$$\begin{aligned}\|c\|_{p,p,N,w} &= \Big(\sum_{n\in N\mathbb{Z}^d}\sum_{\lambda\in\Lambda}|\psi((\lambda-n)/N)|^p|c(\lambda)|^p(w(\lambda))^p\Big)^{1/p}\\ &\ge \Big(\sum_{n\in N\mathbb{Z}^d}\sum_{\lambda\in\Lambda}\chi_{[-1,1)^d}((\lambda-n)/N)|c(\lambda)|^p(w(\lambda))^p\Big)^{1/p}\\ &= 2^{d/p}\Big(\sum_{\lambda\in\Lambda}|c(\lambda)|^p(w(\lambda))^p\Big)^{1/p} = 2^{d/p}\|c\|_{\ell^p_w}.\end{aligned}$$

Therefore the estimate in (3.1.18) follows for $1\le p<\infty$.

Now we prove (3.1.19). Clearly it suffices to prove the second inequality in (3.1.19). Let $c\in\ell^p_w$. If $q_2\ge q_1$, recalling from $\|d\|_{\ell^{q_2}_w}\le\|d\|_{\ell^{q_1}_w}$ for any $d\in\ell^{q_1}_w$ that

$$\|\Psi_n^N c\|_{\ell^{q_2}_w}\le\|\Psi_n^N c\|_{\ell^{q_1}_w} \tag{3.1.21}$$

for all $n\in N\mathbb{Z}^d$. If $q_2<q_1$ and $1\le q_1<\infty$, then

$$\begin{aligned}\|\Psi_n^N c\|_{\ell^{q_2}_w} &= \Big(\sum_{\lambda\in\Lambda}|\psi((\lambda-n)/N)|^{q_2}|c(\lambda)|^{q_2}(w(\lambda))^{q_2}\Big)^{1/q_2}\\ &\le \Big(\sum_{\lambda\in\Lambda}|\psi((\lambda-n)/N)|^{q_1}|c(\lambda)|^{q_1}(w(\lambda))^{q_1}\Big)^{1/q_1}\times\Big(\sum_{\|\lambda-n\|_\infty\le 2N}1\Big)^{1/q_2-1/q_1}\\ &\le C(R(\Lambda)N^d)^{1/q_2-1/q_1}\|\Psi_n^N c\|_{\ell^{q_1}_w}.\end{aligned} \tag{3.1.22}$$

If $q_2<q_1$ and $q_1=\infty$, then

$$\begin{aligned}\|\Psi_n^N c\|_{\ell^{q_2}_w} &= \Big(\sum_{\lambda\in\Lambda}|\psi((\lambda-n)/N)|^{q_2}|c(\lambda)|^{q_2}(w(\lambda))^{q_2}\Big)^{1/q_2}\\ &\le \sup_{\lambda\in\Lambda}|\psi((\lambda-n)/N)||c(\lambda)|w(k)\times\Big(\sum_{\|\lambda-n\|_\infty\le 2N}1\Big)^{1/q_2}\\ &\le C(R(\Lambda)N^d)^{1/q_2}\|\Psi_n^N c\|_{\ell^\infty_w}.\end{aligned} \tag{3.1.23}$$

Combining (3.1.21), (3.1.22) and (3.1.23), we obtain that

$$\|\Psi_n^N c\|_{\ell^{q_2}_w}\le C(R(\Lambda)N^d)^{\max(1/q_2-1/q_1,0)}\|\Psi_n^N c\|_{\ell^{q_1}_w}\quad \textit{for all } c\in\ell^p_w, \tag{3.1.24}$$

where $n \in N\mathbb{Z}^d$ and $1 \le N \in \mathbb{Z}$. Then the second inequality in (3.1.19) follows.

Finally we prove (3.1.20). Noting that for any $\lambda \in \Lambda, n \in N\mathbb{Z}^d$ and $N \ge 1$,

$$\psi\Big(\frac{k-n}{N}\Big) \le \psi\Big(\frac{k-n}{6N}\Big)$$

and

$$\begin{aligned}\psi\Big(\frac{\lambda-n}{6N}\Big) &\le \chi_{[-12N,12N)^d}(\lambda-n) = \sum_{j=-6}^{5} \chi_{[-N,N)^d}(\lambda-n-(2j+1)N)\\ &\le \sum_{j=-6}^{5} \psi\Big(\frac{\lambda-n-(2j+1)N}{N}\Big),\end{aligned}$$

we have

$$\|\Psi_n^N c\|_{\ell_w^q} \le \|\Psi_n^{6N} c\|_{\ell_w^q} \le \sum_{j=-6}^{5} \|\Psi_{n+(2j+1)N}^N c\|_{\ell_w^q} \tag{3.1.25}$$

for any $n \in N\mathbb{Z}^d$ and $N \ge 1$. Therefore

$$\|c\|_{p,q,N,w} \le \|c\|_{p,q,6N,w} \le 12\|c\|_{p,q,N,w}$$

and (3.1.20) is proved.

3.2 Weighted Gohberg-Baskakov-Sjöstrand Class

In this section, we introduce the weighted Gohberg-Baskakov-Sjöstrand class of infinite matrices and provide some of its properties such as norm estimate for the truncation and the commutator with the multiplication operator, and the boundedness on weighted sequence spaces.

Definition 3.2.1 *For a weight w, let*

$$\mathcal{C}_w(\Lambda, \Lambda^{'}) = \left\{ A := (a(\lambda, \lambda^{'}))_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}} \Big| \ \|A\|_{\mathcal{C}_w(\Lambda,\Lambda^{'})} < \infty \right\} \tag{3.2.1}$$

be the weighted Gohberg-Baskakov-Sjöstrand class *of infinite matrices* $A = (a(\lambda, \lambda^{'}))_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}}$*,* *where*

$$\|A\|_{\mathcal{C}_w(\Lambda,\Lambda^{'})} := \sum_{k\in\mathbb{Z}^d} w(k) \sup_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}} |a(\lambda, \lambda^{'})|\chi_{k+[0,1)^d}(\lambda - \lambda^{'}). \tag{3.2.2}$$

We use $\mathcal{C}_w$ instead of $\mathcal{C}_w(\Lambda, \Lambda^{'})$ and $\|A\|_{\mathcal{C}_w}$ instead of $\|A\|_{\mathcal{C}_w(\Lambda,\Lambda^{'})}$ for brevity. Similar to the argument in the proofs of Propositions 2.2.2 and 2.2.3, we have the following properties for the weighted Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_w$.

Definition 3.2.2 *For an infinite matrix $A = (a(\lambda, \lambda^{'}))_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}}$ and any $s \geq 0$, the truncation matrix is defined as*

$$A_s = (a_s(\lambda, \lambda^{'}))_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}} \tag{3.2.3}$$

where $a_s(\lambda, \lambda^{'}) = a(\lambda, \lambda^{'})$ if $||\lambda - \lambda^{'}||_\infty < s$ and $a_s(\lambda, \lambda^{'}) = 0$ otherwise.

Proposition 3.2.1 *Let w be a weight, $A \in \mathcal{C}_w$, and A_s, $s \geq 0$, be as in* (3.2.3)*. Then*

$$\lim_{s\to\infty} \|A - A_s\|_{\mathcal{C}_w} = 0. \tag{3.2.4}$$

Proof. Let $A = (a(\lambda, \lambda^{'}))_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}}$ and define

$$r(k) = w(k) \sup_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}} |a(\lambda, \lambda^{'})|\chi_{k+[0,1)^d}(\lambda - \lambda^{'}) \tag{3.2.5}$$

By Definition 3.2.1 we have

$$\sum_{k\in\mathbb{Z}^d} r(k) = \|A\|_{\mathcal{C}_w} < \infty, \tag{3.2.6}$$

and

$$
\begin{aligned}
\|A - A_s\|_{\mathcal{C}_w} &= \sum_{k\in\mathbb{Z}^d} w(k) \sup_{\lambda\in\Lambda,\lambda'\in\Lambda'} |a(\lambda,\lambda') - a_s(\lambda,\lambda')|\chi_{k+[0,1)^d}(\lambda-\lambda') \\
&\le \sum_{k\in\mathbb{Z}^d \text{ and } \|k\|_\infty \ge s-1} w(k) \sup_{\lambda\in\Lambda,\lambda'\in\Lambda'} |a(\lambda,\lambda')|\chi_{k+[0,1)^d}(\lambda-\lambda') \\
&= \sum_{k\in\mathbb{Z}^d \text{ and } \|k\|_\infty \ge s-1} r(k). \qquad (3.2.7)
\end{aligned}
$$

Combining (3.2.6) and (3.2.7) proves (3.2.4).

Proposition 3.2.2 *Let w be a weight, $A \in \mathcal{C}_w$, and the truncation matrices A_N and the multiplication operator Ψ_n^N be defined as in* (3.2.3) *and* (3.1.5) *respectively. Then*

$$
\|A_N\Psi_n^N - \Psi_n^N A_N\|_{\mathcal{C}_w} \le \inf_{0\le s\le N} \|A - A_s\|_{\mathcal{C}_w} + \frac{s}{N}\|A\|_{\mathcal{C}_w} \qquad (3.2.8)
$$

for all $N \ge 1$ and $n \in N\mathbb{Z}^d$.

Proof. Let $A = (a(\lambda,\lambda'))_{\lambda\in\Lambda,\lambda'\in\Lambda'}$. We have

$$
\begin{aligned}
& \|A_N\Psi_n^N - \Psi_n^N A_N\|_{\mathcal{C}_w} \\
= & \left\| \left(a(\lambda,\lambda') \left(\psi\left(\frac{\lambda'-n}{N}\right) - \psi\left(\frac{\lambda-n}{N}\right)\right)\right)_{\lambda\in\Lambda,\lambda'\in\Lambda'} \right\|_{\mathcal{C}_w} \\
\le & \inf_{0\le s\le N} \left\| \left(\left(a(\lambda,\lambda') - a_s(\lambda,\lambda')\right) \left(\psi\left(\frac{\lambda'-n}{N}\right) - \psi\left(\frac{\lambda-n}{N}\right)\right)\right)_{\lambda\in\Lambda,\lambda'\in\Lambda'} \right\|_{\mathcal{C}_w} \\
+ & \left\| \left(a_s(\lambda,\lambda') \left(\psi\left(\frac{\lambda'-n}{N}\right) - \psi\left(\frac{\lambda-n}{N}\right)\right)\right)_{\lambda\in\Lambda,\lambda'\in\Lambda'} \right\|_{\mathcal{C}_w} \\
\le & \inf_{0\le s\le N} \|A - A_s\|_{\mathcal{C}_w} + \frac{s}{N}\|A\|_{\mathcal{C}_w}. \qquad (3.2.9)
\end{aligned}
$$

In the next theorem we show that an infinite matrix in the weighted Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_w$ defines a bounded operator on ℓ^p_w.

Proposition 3.2.3 *Let $1 \le p \le \infty$ and w be a submultiplicative weight. Then any infinite matrix A in the weighted Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_w$ defines a bounded operator on ℓ^p_w. Furthermore there exists a positive constant C (independent of A) such that*

$$\|Ac\|_{\ell^p_w} \le C(R(\Lambda^{'}))^{\frac{p-1}{p}}(R(\Lambda))^{\frac{1}{p}}\|A\|_{\mathcal{C}_w}\|c\|_{\ell^p_w} \quad for\ all\ c \in \ell^p_w. \tag{3.2.10}$$

Proof. Clearly it suffices to prove (3.2.10).

Write $A = (a(\lambda, \lambda^{'}))_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}}$ and set $s(k) = \sup_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}} |a(\lambda, \lambda^{'})|\chi_{k+[0,1)^d}(\lambda - \lambda^{'})$ for $k \in \mathbb{Z}^d$. Then

$$\sum_{k\in\mathbb{Z}^d} s(k)w(k) = \|A\|_{\mathcal{C}_w} < \infty. \tag{3.2.11}$$

Let $p = \infty$ and take any $c \in \ell^\infty_w$. Then we obtain from (3.2.11) that

$$\begin{aligned}
\|Ac\|_{\ell^\infty_w} &= \sup_{\lambda\in\Lambda} w(\lambda)\Big|\sum_{\lambda^{'}\in\Lambda^{'}} a(\lambda,\lambda^{'})c(\lambda^{'})\Big| \\
&\le \sup_{\lambda\in\Lambda} w(\lambda)\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda,\lambda^{'})|\ |c(\lambda^{'})| \\
&\le C\sup_{\lambda\in\Lambda}\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda,\lambda^{'})|w(\lambda-\lambda^{'})|c(\lambda^{'})|w(\lambda^{'}) \\
&= C\|c\|_{\ell^\infty_w}\sup_{\lambda\in\Lambda}\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda,\lambda^{'})|w(\lambda-\lambda^{'}),
\end{aligned} \tag{3.2.12}$$

where C is the constant in the definition of the submultiplicative weight w. Note that

$$\begin{aligned}
&\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda,\lambda^{'})|w(\lambda-\lambda^{'}) \\
= &\sum_{k\in\mathbb{Z}^d}\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda,\lambda^{'})|w(\lambda-\lambda^{'})\chi_{k+[0,1)^d}(\lambda-\lambda^{'}) \\
\le\ &C\sum_{k\in\mathbb{Z}^d} s(k)w(k)\sum_{\lambda^{'}\in\Lambda^{'}}\chi_{k+[0,1)^d}(\lambda-\lambda^{'}) \le CR(\Lambda^{'})\|A\|_{\mathcal{C}_w},
\end{aligned} \tag{3.2.13}$$

for any $\lambda \in \Lambda$ and

$$\begin{aligned}
& \sum_{\lambda\in\Lambda} |a(\lambda,\lambda^{'})| w(\lambda-\lambda^{'}) \\
= & \sum_{k\in\mathbb{Z}^d}\sum_{\lambda\in\Lambda} |a(\lambda,\lambda^{'})| w(\lambda-\lambda^{'}) \chi_{k+[0,1)^d}(\lambda-\lambda^{'}) \\
\le & C\sum_{k\in\mathbb{Z}^d} s(k)w(k)\sum_{\lambda\in\Lambda}\chi_{k+[0,1)^d}(\lambda-\lambda^{'}) \le CR(\Lambda)\|A\|_{\mathcal{C}_w} \qquad (3.2.14)
\end{aligned}$$

for any $\lambda^{'} \in \Lambda^{'}$, where C is a positive constant independent of $\lambda \in \Lambda$ and $\lambda^{'} \in \Lambda^{'}$. Combining (3.2.12) and (3.2.13) proves (3.2.10) for $p = \infty$.

Let $1 \le p < \infty$ and take any $c \in \ell^p_w$. By (3.2.13) and (3.2.14) we have

$$\begin{aligned}
\|Ac\|^p_{\ell^p_w} = & \sum_{\lambda\in\Lambda} w(\lambda)\Big|\sum_{\lambda^{'}\in\Lambda^{'}} a(\lambda,\lambda^{'})c(\lambda^{'})\Big|^p \\
\le & \sum_{\lambda\in\Lambda} w(\lambda)\Big|\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda,\lambda^{'})||c(\lambda^{'})|\Big|^p \\
\le & C^p\sum_{\lambda\in\Lambda}\Big|\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda,\lambda^{'})|w(\lambda-\lambda^{'})|c(\lambda^{'})|w(\lambda^{'})\Big|^p \\
\le & C^p\sum_{\lambda\in\Lambda}\Big(\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda,\lambda^{'})|w(\lambda-\lambda^{'})(|c(\lambda^{'})|w(\lambda^{'}))^p\Big) \\
& \times\Big(\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda,\lambda^{'})|w(\lambda-\lambda^{'})\Big)^{p-1} \\
\le & C^p(R(\Lambda^{'}))^{p-1}\|A\|^{p-1}_{\mathcal{C}_w}\sum_{\lambda^{'}\in\Lambda^{'}}(|c(\lambda^{'})|w(\lambda^{'}))^p\sum_{\lambda\in\Lambda}|a(\lambda,\lambda^{'})|w(\lambda-\lambda^{'}) \\
= & C^p(R(\Lambda^{'}))^{p-1}R(\Lambda)\|A\|^p_{\mathcal{C}_w}\|c\|^p_{\ell^p_w}, \qquad (3.2.15)
\end{aligned}$$

where C is a positive constant independent of Λ, $\Lambda^{'}$, A and $\mathcal{C}_w$. Hence the inequality in (3.2.10) for $1 \le p < \infty$ follows.

To prove Theorem 3.3.1, we also need the following uniform boundedness result on the equivalent norm $\|\cdot\|_{p,q,N,w}$ for any infinite matrix in the weighted Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_w$.

Proposition 3.2.4 *Let $1 \leq p, q \leq \infty$, $1 \leq N \in \mathbb{Z}$, w be a submultiplicative weight. Then there exists a positive constant C (independent of A and $N \geq 1$) such that*

$$\|Ac\|_{p,q,N,w} \leq C(R(\Lambda^{'}))^{\frac{p-1}{p}}(R(\Lambda))^{\frac{1}{p}}\|A\|_{\mathcal{C}_w}\|c\|_{p,q,N,w} \quad \textit{for all } c \in \ell^p_w. \tag{3.2.16}$$

Proof. Write $A = (a(\lambda, \lambda^{'}))_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}}$ and set

$$s(k) = \sup_{\lambda\in\Lambda,\lambda^{'}\in\Lambda^{'}} |a(\lambda, \lambda^{'})|\chi_{k+[0,1)^d}(\lambda - \lambda^{'})$$

for $k \in \mathbb{Z}^d$. Then

$$\sum_{k\in\mathbb{Z}^d} s(k)w(k) = \|A\|_{\mathcal{C}_w} < \infty. \tag{3.2.17}$$

Take any $c \in \ell^p_w$. For any $n \in N\mathbb{Z}^d$ and $N \geq 1$,

$$\begin{aligned}
\|\Psi^N_n Ac\|_{\ell^\infty_w} &= \sup_{\lambda\in\Lambda} w(\lambda)\psi\Big(\frac{\lambda - n}{N}\Big)\Big|\sum_{\lambda^{'}\in\Lambda^{'}} a(\lambda, \lambda^{'})c(\lambda^{'})\Big| \\
&\leq \sup_{\lambda\in\Lambda} w(\lambda)\psi\Big(\frac{\lambda - n}{N}\Big)\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda, \lambda^{'})|\ |c(\lambda^{'})| \\
&\leq C\sum_{n^{'}\in N\mathbb{Z}^d}\sup_{\lambda\in\Lambda}\psi\left(\frac{\lambda - n}{N}\right)\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda, \lambda^{'})|w(\lambda - \lambda^{'}) \\
&\quad \times\psi\left(\frac{\lambda^{'} - n + n^{'}}{N}\right)\chi_{[-2,2)^d}\left(\frac{\lambda^{'} - n + n^{'}}{N}\right)|c(\lambda^{'})|w(\lambda^{'}) \\
&\leq C\sum_{n^{'}\in N\mathbb{Z}^d}\|\Psi^N_{n-n^{'}}c\|_{\ell^\infty_w} \\
&\quad \times\sup_{\lambda\in\Lambda}\psi\left(\frac{\lambda - n}{N}\right)\sum_{\lambda^{'}\in\Lambda^{'}} |a(\lambda, \lambda^{'})|w(\lambda - \lambda^{'})\chi_{[-2,2)^d}\left(\frac{\lambda^{'} - n + n^{'}}{N}\right) \\
&\leq C\sum_{n^{'}\in N\mathbb{Z}^d}\|\Psi^N_{n-n^{'}}c\|_{\ell^\infty_w} \\
&\quad \times\sup_{\lambda\in\Lambda}\psi\left(\frac{\lambda - n}{N}\right)\sum_{\lambda^{'}\in\Lambda^{'} \text{ and } \|\lambda-\lambda^{'}-n^{'}\|_\infty\leq 4N} |a(\lambda, \lambda^{'})|w(\lambda - \lambda^{'})
\end{aligned}$$

$$\le C \sum_{n'\in N\mathbb{Z}^d} \|\Psi^N_{n-n'} c\|_{\ell^\infty_w} \sum_{k\in\mathbb{Z}^d \text{ and } \|k-n'\|_\infty \le 6N} s(k)w(k)$$

$$\times \sup_{\lambda\in\Lambda} \psi\left(\frac{\lambda-n}{N}\right) \sum_{\lambda'\in\Lambda'} \chi_{[-4,4)^d}\left(\frac{\lambda-\lambda'-n'}{N}\right)\chi_{k+[0,1)^d}(\lambda-\lambda')$$

$$\le \quad CR(\Lambda') \sum_{n'\in N\mathbb{Z}^d} \|\Psi^N_{n-n'} c\|_{\ell^\infty_w}$$

$$\times \sum_{k\in\mathbb{Z}^d} s(k)w(k)\chi_{[-6N,6N]^d}(k-n'), \tag{3.2.18}$$

and for $1 \le q < \infty$,

$$\begin{aligned}
\|\Psi^N_n Ac\|_{\ell^q_w} \quad &= \quad \left(\sum_{\lambda\in\Lambda}\left(w(\lambda)\psi\left(\frac{\lambda-n}{N}\right)\left|\sum_{\lambda'\in\Lambda'} a(\lambda,\lambda')c(\lambda')\right|\right)^q\right)^{1/q} \\
&\le \quad C\Big(\sum_{\lambda\in\Lambda}\Big(\sum_{n'\in N\mathbb{Z}^d}\psi\left(\frac{\lambda-n}{N}\right)\sum_{\lambda'\in\Lambda'}|a(\lambda,\lambda')|w(\lambda-\lambda') \\
&\qquad \times\psi\left(\frac{\lambda'-n+n'}{N}\right)\chi_{[-2,2)^d}\left(\frac{\lambda'-n+n'}{N}\right)|c(\lambda')|w(\lambda')\Big)^q\Big)^{1/q} \\
&\le \quad C\sum_{n'\in N\mathbb{Z}^d}\Big(\sum_{\lambda\in\Lambda}\Big(\psi\Big(\frac{\lambda-n}{N}\Big)\sum_{\lambda'\in\Lambda'}|a(\lambda,\lambda')|w(\lambda-\lambda') \\
&\qquad \times\psi\left(\frac{\lambda'-n+n'}{N}\right)\chi_{[-2,2)^d}\left(\frac{\lambda'-n+n'}{N}\right)|c(\lambda')|w(\lambda')\Big)^q\Big)^{1/q} \\
&\le \quad C\sum_{n'\in N\mathbb{Z}^d}\left(\sum_{\lambda\in\Lambda}\Big(\psi\Big(\frac{\lambda-n}{N}\Big)\sum_{\lambda'\in\Lambda'}|a(\lambda,\lambda')|w(\lambda-\lambda')\chi_{[-2,2)^d}\Big(\frac{\lambda'-n+n'}{N}\Big)\Big)^{q-1}\right. \\
&\quad \times\Big(\sum_{\lambda'\in\Lambda'}|a(\lambda,\lambda')|w(\lambda-\lambda') \\
&\quad \left.\times\big(\psi\left(\frac{\lambda'-n+n'}{N}\right)\chi_{[-2,2)^d}\left(\frac{\lambda'-n+n'}{N}\right)|c(\lambda')|w(\lambda')\big)^q\Big)\right)^{1/q} \\
&\le \quad C\sum_{n'\in N\mathbb{Z}^d}\|\Psi^N_{n-n'}c\|_{\ell^q_w}\sum_{k\in\mathbb{Z}^d}s(k)w(k)\chi_{[-6N,6N]^d}(k-n') \\
&\qquad \times\Big(\sum_{\lambda\in\Lambda}\chi_{k+[0,1)^d}(\lambda-\lambda')\big(\sum_{\lambda'\in\Lambda'}\chi_{k+[0,1)^d}(\lambda-\lambda')\big)^{q-1}\Big)^{1/q} \\
&\le \quad C(R(\Lambda'))^{\frac{q-1}{q}}(R(\Lambda))^{\frac{1}{q}} \\
&\qquad \times\sum_{n'\in N\mathbb{Z}^d}\|\Psi^N_{n-n'}c\|_{\ell^q_w}\sum_{k\in\mathbb{Z}^d}s(k)w(k)\chi_{[-6N,6N]^d}(k-n'),
\end{aligned} \tag{3.2.19}$$

where C is a positive constant independent of $\mathcal{C} \in \ell^q_w$ and $N \geq 1$. Therefore for any $1 \leq q \leq \infty$ it follows from (3.2.17), (3.2.18) and (3.2.19) that

$$\begin{aligned}
&\|Ac\|_{\infty,q,N,w} = \sup_{n\in N\mathbb{Z}^d} \|\Psi_n^N Ac\|_{\ell^q_w} \\
\leq\ & C(R(\Lambda^{'}))^{\frac{q-1}{q}}(R(\Lambda))^{\frac{1}{q}} \sup_{n\in N\mathbb{Z}^d} \sum_{n'\in N\mathbb{Z}^d} \|\Psi_{n-n'}^N c\|_{\ell^q_w} \sum_{k\in\mathbb{Z}^d} s(k)w(k)\chi_{[-6N,6N]^d}(k-n^{'}) \\
\leq\ & C(R(\Lambda^{'}))^{\frac{q-1}{q}}(R(\Lambda))^{\frac{1}{q}} \|c\|_{\infty,q,N,w} \sum_{n'\in N\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} s(k)w(k)\chi_{[-6N,6N]^d}(k-n^{'}) \\
\leq\ & C(R(\Lambda^{'}))^{\frac{q-1}{q}}(R(\Lambda))^{\frac{1}{q}} \|c\|_{\infty,q,N,w} \|A\|_{\mathcal{C}_w}, \qquad (3.2.20)
\end{aligned}$$

and for $1 \leq p < \infty$,

$$\begin{aligned}
&\|Ac\|^p_{p,q,N,w} = \sum_{n\in N\mathbb{Z}^d} \|\Psi_n^N Ac\|^p_{\ell^q_w} \\
\leq\ & C\Big((R(\Lambda^{'}))^{\frac{q-1}{q}}(R(\Lambda))^{\frac{1}{q}}\Big)^p \sum_{n\in N\mathbb{Z}^d} \left(\sum_{n'\in N\mathbb{Z}^d} \|\Psi_{n-n'}^N c\|_{\ell^q_w} \sum_{k\in\mathbb{Z}^d} s(k)w(k)\chi_{[-6N,6N]^d}(k-n^{'}) \right)^p \\
\leq\ & C\Big((R(\Lambda^{'}))^{\frac{q-1}{q}}(R(\Lambda))^{\frac{1}{q}}\Big)^p \sum_{n\in N\mathbb{Z}^d} \left(\sum_{n'\in N\mathbb{Z}^d} \|\Psi_{n-n'}^N c\|^p_{\ell^q_w} \sum_{k\in\mathbb{Z}^d} s(k)w(k)\chi_{[-6N,6N]^d}(k-n^{'}) \right) \\
& \times \left(\sum_{n'\in N\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} s(k)w(k)\chi_{[-6N,6N]^d}(k-n^{'}) \right)^{p-1} \\
\leq\ & C\Big((R(\Lambda^{'}))^{\frac{q-1}{q}}(R(\Lambda))^{\frac{1}{q}}\Big)^p \|A\|^{p-1}_{\mathcal{C}_w} \|c\|^p_{p,q,N,w} \sum_{n'\in N\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} s(k)w(k)\chi_{[-6N,6N]^d}(k-n^{'}) \\
\leq\ & C\Big((R(\Lambda^{'}))^{\frac{q-1}{q}}(R(\Lambda))^{\frac{1}{q}}\Big)^p \|A\|^p_{\mathcal{C}_w} \|c\|^p_{p,q,N,w}, \qquad (3.2.21)
\end{aligned}$$

where C is a positive constant which could be different at different occurrence. Thus the conclusion (3.2.16) follows from (3.2.20) and (3.2.21).

3.3 Weighted ℓ^p-stability

In this section, we establish the equivalence of the weighted ℓ^p-stability for infinite matrices in the Gohberg-Baskakov-Sjöstrand class.

Definition 3.3.1 *Let $1 \le p \le \infty$, w be a weight, and T be a linear operator on ℓ^p_w. We say that T has ℓ^p_w-stability if there exist positive constants A and B such that*

$$A\|c\|_{\ell^p_w} \le \|Tc\|_{\ell^p_w} \le B\|c\|_{\ell^p_w} \quad \textit{for all } c \in \ell^p_w. \tag{3.3.1}$$

The following result generalizes Theorem 2.3.1 on ℓ^p-stability as the trivial weight $w \equiv 1$ is a submultiplicative weight.

Theorem 3.3.1 *Let $1 \le p, q \le \infty$, w be a submultiplicative weight, and A be an infinite matrix in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_w$. If A has ℓ^q_w-stability, then A has ℓ^p_w-stability.*

Proof. Take any $c \in \ell^q_w$. By the ℓ^q_w-stability of the infinite matrix A, there exists an absolute positive constant C_0 (independent of $N \ge 1, n \in N\mathbb{Z}^d$ and $c \in \ell^q_w$) such that

$$\|A\Psi_n^N c\|_{\ell^q_w} \ge C_0\|\Psi_n^N c\|_{\ell^q_w} \tag{3.3.2}$$

By Propositions 3.2.2 and 3.2.3, we have

$$\begin{aligned} & \|(A_N\Psi_n^N - \Psi_n^N A_N)c\|_{\ell^q_w} \\ = \ & \|(A_N\Psi_n^N - \Psi_n^N A_N)\Psi_n^{6N} c\|_{\ell^q_w} \\ \le \ & C_1(R(\Lambda'))^{\frac{q-1}{q}} R(\Lambda)^{\frac{1}{q}} \Big(\inf_{0\le s\le N} \frac{s}{N}\|A\|_{\mathcal{C}_w} + \|A - A_s\|_{\mathcal{C}_w}\Big) \|\Psi_n^{6N} c\|_{\ell^q_w}, \end{aligned} \tag{3.3.3}$$

and

$$\|(A - A_N)\Psi_n^N c\|_{\ell^q_w} \le C_1(R(\Lambda'))^{\frac{q-1}{q}} R(\Lambda)^{\frac{1}{q}} \|A - A_N\|_{\mathcal{C}_w} \|\Psi_n^N c\|_{\ell^q_w}, \tag{3.3.4}$$

where C_1 is a positive constant independent of $N \geq 1$ and $n \in N\mathbb{Z}^d$. Combining (3.3.2), (3.3.3) and (3.3.4), we get

$$\begin{aligned}
\|\Psi_n^N A_N c\|_{\ell_w^q} &\geq \Big(C_0 - C_1 (R(\Lambda'))^{\frac{q-1}{q}} R(\Lambda)^{\frac{1}{q}} \|A - A_N\|_{\mathcal{C}_w}\Big) \|\Psi_n^N c\|_{\ell_w^q} - C_1 (R(\Lambda'))^{\frac{q-1}{q}} R(\Lambda)^{\frac{1}{q}} \\
&\quad \times \Big(\inf_{0 \leq s \leq N} \frac{s}{N} \|A\|_{\mathcal{C}_w} + \|A - A_s\|_{\mathcal{C}_w}\Big) \|\Psi_n^{6N} c\|_{\ell_w^q}, \qquad (3.3.5)
\end{aligned}$$

for any $N \geq 1$ and $n \in \mathbb{Z}^d$. This implies that

$$\begin{aligned}
\|A_N c\|_{p,q,N,w} &\geq \Big(C_0 - C_1 (R(\Lambda'))^{\frac{q-1}{q}} R(\Lambda)^{\frac{1}{q}} \|A - A_N\|_{\mathcal{C}_w}\Big) \|c\|_{p,q,N,w} - C_1 (R(\Lambda'))^{\frac{q-1}{q}} R(\Lambda)^{\frac{1}{q}} \\
&\quad \times \Big(\inf_{0 \leq s \leq N} \frac{s}{N} \|A\|_{\mathcal{C}_w} + \|A - A_s\|_{\mathcal{C}_w}\Big) \|c\|_{p,q,6N,w}. \qquad (3.3.6)
\end{aligned}$$

Noting that

$$\|A - A_N\|_{\mathcal{C}_w} \leq \|A - A_s\|_{\mathcal{C}_w} \quad \textit{for all } 0 \leq s \leq N,$$

we then obtain from (3.1.20), (3.2.16) and (3.3.6) that

$$\|Ac\|_{p,q,N,w} \geq \Big(C_0 - C_2 (R(\Lambda'))^{\frac{q-1}{q}} R(\Lambda)^{\frac{1}{q}} \big(\inf_{0 \leq s \leq N} \frac{s}{N} \|A\|_{\mathcal{C}_w} + \|A - A_s\|_{\mathcal{C}_w}\big)\Big) \|c\|_{p,q,N,w}, \quad (3.3.7)$$

where C_2 is a positive constant independent of $N \geq 1$. Recalling from Proposition 3.2.1 that

$$\lim_{s \to \infty} \|A - A_s\|_{\mathcal{C}_w} = 0,$$

we can find a sufficiently large integer N_0 such that

$$\begin{aligned}
& C_2 (R(\Lambda'))^{\frac{q-1}{q}} R(\Lambda)^{\frac{1}{q}} \Big(\inf_{0 \leq s \leq N_0} \frac{s}{N_0} \|A\|_{\mathcal{C}_w} + \|A - A_s\|_{\mathcal{C}_w}\Big) \\
\leq\ & C_2 (R(\Lambda'))^{\frac{q-1}{q}} R(\Lambda)^{\frac{1}{q}} \Big(\|A - A_{\sqrt{N_0}}\|_{\mathcal{C}_w} + N_0^{-1/2} \|A\|_{\mathcal{C}_w}\Big) \\
\leq\ & C_0/2. \qquad (3.3.8)
\end{aligned}$$

Thus by (3.3.7) and (3.3.8) we have

$$\|Ac\|_{p,q,N_0,w} \geq \frac{C_0}{2} \|c\|_{p,q,N_0,w} \qquad (3.3.9)$$

for any $c \in \ell^p_w$. Combining (3.1.18), (3.1.19) and (3.3.9), we have

$$C_0(2C^2)^{-1}(R(\Lambda)N_0^d)^{-|1/q-1/p|}2^{-d/p}\|c\|_{\ell^p_w} \leq \|Ac\|_{\ell^p_w}, \tag{3.3.10}$$

where C_0 is a positive constant dependent on N_0 and C is a positive constant independent on N_0 for all $c \in \ell^p_w$. Therefore the desired ℓ^p_w-stability follows from (3.2.10) and (3.3.10).

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