A Model for Scattering in Dense Clouds

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A MODEL FOR SCATTERING IN DENSE CLOUDS

BY

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B.S., University of Central Florida, 1981

THESIS

Submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering in the Graduate Studies Program of the College of Engineering University of Central Florida Orlando, Florida

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ABSTRACT

Light is almost always detected by its interaction with matter. One of these interaction phenomena is the scattering of light by small particles. A model is developed that estimates the amount of energy that is scattered towards a detector from a beam given the locations of the source, detector and particle. This collection of particles is allowed to be very dense so that a photon scattered from the beam can be scattered several times before leaving the scattering medium. By considering the single-scatter component and multiple-scatter component separately, the model retains the characteristics of both types.
INTRODUCTION

Light is rarely detected by direct observation of a source, but rather seen as reflections from objects in the light's path. Another process by which light is seen is scattering, or the interaction of light with small particles causing incident energy to be spread in all directions. This is the phenomenon observed when looking at the blue sky, clouds or at laser beams propagating in the atmosphere. Of particular interest is the latter case in which the presence of lasers is detected by observing scatter from atmospheric aerosols.

Many factors determine the amount of scatter a beam will experience traversing a path in the atmosphere, such as temperature, humidity, altitude, etc., most of which mainly affect the size distribution and density of aerosol particles. This is particularly true if the scattering medium is a haze or fog, in which case the aerosol particles are approximately spherical water droplets in random motion. If we know the amount of scatter a beam will experience and the directions in which the energy will scatter, we can determine the amount of the beam's energy that will reach a detector at a location away from the beam's path. If we try to apply this idea to scattering in very dense media, such as thick water clouds, another problem arises. Not only
will the beam undergo scattering within the cloud, but even this scattered energy can be subsequently scattered—a phenomenon known as "multiple scattering".

The model developed in this thesis will be most accurate in the visible-light and near infrared regions of the electromagnetic spectrum where there is negligible absorption by water vapor. It will also model longer wavelengths if absorption bands are avoided or if a correction factor for this absorption is included. These regions comprise a band from about .2 microns to 10 microns, encompassing the wavelengths of most lasers. Also, since most geometries require the beam and scattered light to pass through a substantial amount of atmosphere, wavelengths outside this range will be severely attenuated by atmospheric absorption.

We will attempt to model the cloud as a reflector and determine its equivalent reflectivity based on incident angle, scattering angle, distance and scattering profile of the cloud. Figure 1 illustrates a typical geometry for which we must find the amount of beam energy that ultimately finds its way to the receiver. A source inclined $\Theta_B$ from the ground directs a well collimated laser beam at a cloud. A detector inclined $\Theta_A$ from the ground captures scattered radiation from the cloud.
Single Scattering from Particles

Scattering by particles within a cloud can be determined if the size distribution, density and directional scattering pattern, also called the "phase function", of the constituent particles are known. Integrating the product of these functions over all sizes will yield a composite phase function for the cloud of particles. Unfortunately, the phase function is not easily integrated while the size distribution is never known exactly.

H. C. van de Hulst has derived the phase function for spherical particles as a function of wavelength, particle radius and index of refraction, for an incident plane wave. This function is given in Appendix A, while the complete derivation and some special cases can be found in van de Hulst's book. ²

D. Diermendjian has numerically computed phase functions for some aerosols by reducing van de Hulst's equations to a simpler, recursive form and assuming a particle size distribution used by many authors for atmospheric aerosols. ³ Diermendjian's book contains numerical tables for selected wavelengths and aerosol size distributions. ⁴ The functional simplifications are included in
Appendix A; the particle size distributions are discussed in Appendix B. In Table I are some values of the phase function for a wavelength of 1.61 microns and a size distribution described by:

\[ n(r) = 2.373 \times r^6 e^{-1.5r} \]  

The right column has been normalized so all energy radiating into a sphere around the scatterer is equal to the energy scattered at the scatterer. Figure 2 is a plot of this phase function.

**Multiple Scattering**

The theory described thus far has one serious drawback: it is only valid for optical depths that produce predominantly single scatter. Optical depth is the ratio of physical depth to the mean distance between scatterers. If the light has a high probability of being scattered more than once before leaving the scattering medium, the previously described model is not accurate.

Scatter from dense clouds has been characterized by both theoretical and empirical models. Some models consider only small-angle scatter to determine transmission of the signal passing through the cloud. Other models concentrate on backscatter, for use in lidar (Light Detecting And Ranging) applications. Many models assume scattering by the particles within the cloud has no preferred direction of scattering (uniform scattering).
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<th>( \text{Normalized to } \frac{1}{4\pi} )</th>
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</tbody>
</table>
Figure 2. Normalized Phase Function for C.1 Cloud for $\lambda=1.61$
Single-scatter models tend to ignore the increase in scattered light due to multiple scatter, while multiple scatter models fail to account for special single scatter phenomena such as glories and cloudbows.

Equation 2 will be used to express the energy returned from the cloud by considering its apparent reflectance.

\[ P_C(\theta_A, \theta_B, R) = I_{ss} + a P_{ms} \]  \hspace{1cm} (2)

Figure 3 illustrates a typical geometry. The cloud is a non-absorbing collection of spherical water droplets with a known size distribution. The source is monochromatic and its wavelength is outside all atmospheric absorption bands. Effects such as change in polarization and spectral line broadening are not considered.

The following method could be used to determine the unknown values in Equation 2. First, the amount of single scatter detected at a particular ground location could be calculated for the actual field of view. This is the \( I_{ss} \) component. We could then let the field of view grow to infinite size yielding the total amount of single scatter to a given ground location. Summing received energy over all ground area would yield the total amount of single scatter from the cloud. All energy not single scattered would be assumed to be multiple scattered. The ratio of multiple scatter to total scatter is the parameter \( a \) in Equation 2.
Figure 3. Variables for General Scatter Solution
The angular dependence of the multiple scatter, $P_{ms}$, would then be determined.
SCATTERING MODEL

The model below described will compute the scattered energy at a receiver under the following conditions:

(1) Non-absorbing scattering medium.

(2) Very dense scatterer population such that multiple scattering is the dominant phenomenon affecting incident light.

(3) Polarization effects are negligible.

(4) The optical thickness of the scattering region is very large.

(5) Scatter angle is approximately constant.

Single Scatter Component

Figure 3 illustrates a beam entering a cloud and a detector whose field of view includes a section of the beam propagating within it. The energy received at the detector from a small scattering element is given by:

\[ I_\Delta = e^{-\gamma(d_1+\Delta x)} (1-e^{-\gamma\Delta x}) P(\theta) e^{-\gamma d_2} \left( \frac{1}{R} \right)^2 \]

The first term is the scatter from the beam before reaching the scattering element, \( \Delta x \). The second term is the fraction of light incident on the scattering element that is scattered in all directions. Term 3 is the single scatter phase function, describing the amount of scattered light in
direction $\theta$. Term 4 is the fraction of energy scattered from $\Delta x$ that reaches the cloud boundary in the direction of the detector. The fifth term relates the area of the detector to the solid angle subtended by it.

Since $\Delta x$ is very small, a simplification in the second term can be made by using a series expansion on the exponential function.

$$e^{-\gamma \Delta x} \approx 1 + \gamma \Delta x + \frac{(\gamma \Delta x)^2}{2} + \cdots$$

(4)

The second term of Equation 3 becomes

$$1 - e^{-\gamma \Delta x} \approx \gamma \Delta x.$$ 

(5)

Putting these results into Equation 3 and integrating over the length of the beam will give the energy at the receiver. Equation 6 gives this function.

$$I = \int_0^L \frac{\gamma P(\Theta)e^{-\gamma(d_1+d_2+x)}}{R_2^2} \ dx$$

(6)

A first order approximation can be made if all of the energy scattered over the length of the beam within the field of view is concentrated at the center. Figure 4 illustrates the geometry and variables used. The energy received at the detector is

$$I = \frac{P(\Theta)e^{-\gamma(d_1'+d_2')}}{R_2^2} \sigma.$$ 

(7)

$\sigma$ is the total energy scattered along the section of the beam that is within the field of view, estimated by:

$$\sigma = \int_0^L e^{-\gamma x} \gamma \ dx.$$ 

(8)
Figure 4. Variables for First Order Approximation
\[ \sigma = 1 - e^{-\gamma L} \approx \gamma L \quad (9) \]

Putting these results into Equation 7 yields

\[ I_1 = \frac{e^{-\gamma(d_1 + d_2')}}{R_2^2} P(\Theta) \gamma L. \quad (10) \]

\[ d'_1 = \frac{R}{2 \sin \theta_B} \left[ \frac{2}{\cot(\theta_A + \theta_F) + \cot \theta_B} + \frac{1}{\cot \theta_A + \cot \theta_B} - \frac{2h}{R} \right] \quad (11) \]

\[ d'_2 = \frac{R}{2 \sin \theta_A} \left[ \frac{2}{\cot(\theta_A + \theta_F) + \cot \theta_B} + \frac{1}{\cot \theta_A + \cot \theta_B} - \frac{2h}{R} \right] \quad (12) \]

\[ R'_2 = \frac{R}{2 \sin \theta_A} \left[ \frac{2}{\cot(\theta_B + \theta_F) + \cot \theta_B} - \frac{1}{\cot \theta_A + \cot \theta_B} \right] \quad (13) \]

Details are given in Appendix C.

Since the medium under consideration has a large extinction coefficient, \( \gamma \), the first-order approximation does not remain accurate for even moderate path lengths. Restricting the field of view to a small angle, implies a small scattering angle as well. Using Figure 3, the following equation describes the energy received:

\[ I_2 = \int_0^L \frac{e^{-\gamma(d_1 + d_2 + x)}}{R_2^2} P(\Theta) \quad (14) \]

\[ d_2 = w(x + d_1) \quad (15) \]

\[ R_2 = w(x + R_1) \quad (16) \]

\[ I_2 = \frac{\gamma P(\Theta)}{(wR_1)^2} e^{-\gamma (w+1)} \int_0^L \frac{e^{-\gamma (w+1)x}}{(x/R_1 + 1)^2} \, dx. \quad (17) \]
\[ I_2 = \frac{\gamma P(\theta)}{w^2 R_1} e^{\gamma(w+1)(d_1-R_1)} \{ E_2[\gamma(w+1)R_i] \} \]
- \[ \frac{R_i}{R_1+L} E [\gamma(w+1)(L+R_1)] \] (18)

\[ d_1 = \frac{1}{\sin \theta_B} \left( \frac{R}{\cot \theta_A + \cot \theta_B} - h \right) \] (19)

\[ R_1 = \frac{R}{\sin \theta_B (\cot \theta_A + \cot \theta_B)} \] (20)

\[ L = \frac{R}{\sin \theta_B} \left[ \frac{1}{\cot(\theta_A + \theta_F) + \cot \theta_B} - \frac{1}{\cot \theta_A + \cot \theta_B} \right] \] (21)

\[ w = \frac{\sin \theta_B}{\sin \theta_A}. \] (22)

\[ E_2(x) \text{ is derivable from the exponential integral function by:} \]

\[ E_2(x) = xEi(x) + e^{-x}. \]

Complete derivation for Equations 14 through 22 are in Appendix C.

For illustration, a comparison of these two methods will be made for the following cases:

Case 1
\[ \gamma = .5/km = 5 \times 10^{-9}/m \]
\[ \theta_B = 45^\circ \]
\[ \theta_A = 35^\circ \]
\[ h = 2000m \]
\[ \theta_F = 2^\circ \]
\[ R = 5000m \]
\[ I_1 = 3.78 \times 10^{-9} \times I_0 P(\theta) \]
\[ I_2 = 4.00 \times 10^{-9} \times I_0 P(\theta) \]

Case 2
\[ \gamma = 11.18/km = 1.118 \times 10^{-2} \]
\[ \theta_B = 45^\circ \]
\[ \theta_A = 58^\circ \]
\[ h = 2000m \]
\[ \theta_F = 10^\circ \]
\[ R = 5000m \]
\[ I_1 = 6.97 \times 10^{-24} \times I_0 P(\theta) \]
\[ I_2 = 1.10 \times 10^{-21} \times I_0 P(\theta) \]
To compute the total amount of light leaving the cloud after only one scatter, it would be necessary to find the amount of light that exits the cloud after being scattered at any point on the beam, then to integrate over the length of the beam. Since finding the total amount of light that leaves the cloud from a point source without being scattered requires integration over the cloud surface, the phase function, $P(\theta)$, will be involved in three integrals. This would be an extremely difficult task at best. To get an upper bound, the assumption will be made that the cloud contains isotropic scatterers to approximate the amount of light that leaves the cloud after one scatter. For most geometries, this number will be on the high side, tending to enhance the side and backscatter and reduce the forward scatter.

A small section of the beam is viewed as a point source, radiating its scattered light into $4\pi$ steradians isotropically, as illustrated in Figure 5. The amount of light reaching the cloud surface is given by

$$I = \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\sigma}{4\pi} e^{-\gamma S} \frac{1}{S^2} \sin\theta \, dA$$

(23)

$$= \frac{\sigma}{2} E_{2}(\gamma h).$$

(24)

The expression for $\sigma$ will simply be the scatter at a point along the beam. If we consider a beam entering the cloud at an angle $\theta_B$ (equivalent to $\theta_B$ in Figure 3), and the scattering point at a distance $R$ from the cloud boundary
Figure 5. Integration Variable for Determining Ratio of Single Scatter to Total Scatter
along the beam, we get:

$$\sigma = e^{-\gamma R} \gamma dR.$$  \hspace{1cm} (25)

Writing this as a function of $h$, we get

$$\sigma = \frac{\gamma}{\sin \theta_B} e^{-\frac{\gamma h}{\sin \theta_B}}.$$ \hspace{1cm} (26)

Inserting these results into Equation (24), and integrating over the entire beam yields

$$I_{\text{tss}} = \int_0^\infty \frac{\gamma}{2 \sin \theta_B} e^{-\frac{\gamma h}{\sin \theta_B}} E_2(\gamma h) dh$$

$$= \frac{1}{2} \left[ 1 - \sin \theta_B \ln \left( 1 + \frac{1}{\sin \theta_B} \right) \right] \quad \theta_B < \frac{\pi}{2}. \hspace{1cm} (27)$$

The fact that we are integrating from zero to infinity may seem to violate the small field-of-view criteria set forth in the conditions of the single scatter approximation. Most of the energy, however, has been scattered within small cloud depths so the integral to infinity adds little. The details of the derivation of Equation (28) are included in Appendix C.

**Multiple Scatter Component**

The multiple scatter phase function has been estimated as uniform in the infrared and Lambertian, or cosine related, in the visible by Bauer.\textsuperscript{10} These are discussed in Appendix D, along with the exponential

$$P(\theta_c) = \frac{1}{2 \pi E_2(1)} e^{-\frac{1}{\cos \theta_c}}.$$ \hspace{1cm} (29)

This function is derived by considering a single
scattering element. This is the multiple scatter component, so the light being scattered by this element can be incident from any (or several) directions. Since this is a uniformly random variable, the orientation of the particles scattering phase function with respect to the cloud boundary will also be a uniformly random variable. The "average" particle can, therefore, be assumed to be an isotropic scatterer.

Referring to Figure 6, the ratio of scatter at any angle to scatter normal to the cloud's surface ($\theta_c = 0$) is given by

$$P(\theta_c) = \frac{e^{-Yh}}{e^{-Yh}} \cos \theta_c.$$  \hspace{1cm} (30)

If we assume the amount of multiple-scatter energy decreases exponentially with optical depth, we can concentrate all this energy at an optical depth of 1, as dictated by finding the weighted average of the depth (centroid) as follows:

$$\int_0^\infty xe^{-Yh} e^{-Yh} = \frac{1}{Y}.$$ \hspace{1cm} (31)

Letting $h = \frac{1}{Y}$ in Equation 30 yields

$$P(\theta_c) = Ce^{-\frac{1}{\cos \theta_c}}.$$ \hspace{1cm} (32)

The constant, $C$, is determined in Appendix D to yield Equation 29.
Figure 6. Random Scatter from a Differential Scattering Element
Since this "average" particle typifies all particles in the cloud, regardless of the number of scatters the light has undergone prior to hitting the scatterer, this same phase function (Equation 29) applies to the entire cloud.

This function closely resembles the Lambertian reflector as shown by comparing their plots in Figure 7. In fact, since true clouds have irregular boundaries, as opposed to the planar boundary assumed, scattering at large angles is likely to be greater than predicted since not all of the path will be inside the cloud. Figure 8 illustrates a beam exiting the cloud at an angle approaching 90 degrees. Note that some of the scattered beam's path is not in the cloud as would have been the case under the plane boundary assumption.

Complete Equation

Evaluation of Equation 18 under the assumption that the field of view encompasses the area of the cloud where a majority of the scattering occurs implies letting \( d_1 \) be 0 and \( L \) approach infinity. This yields

\[
I_{ss} = \frac{YP(\theta)}{w^2R_1} e^{Y(w+1)R} E_2[Y(w+1)R_1].
\]  

(33)

Since this is the actual amount of light received, no normalization is necessary. Equation 28 is the total amount of single-scattered light that exits the cloud, implying that the rest is multiple scattered. Subtracting this
Figure 7. Comparison of Lambertian Reflectors and Multiple Scatter Component.
from 1 yields
\[= \frac{1}{2}[1 + \sin \theta_B \ln (1 + \frac{1}{\sin \theta_B})] \quad 0 < \theta_B < \frac{\pi}{2}, \] (34)
which is a in Equation 2. Combining Equations 29, 33 and 34 yields:
\[I = \frac{P(\Theta)}{w^2 R_1} e^{Y(w+1)R_1} E_2[Y(w+1)R_1] \]
\[+ \frac{1}{4\pi E_2(1)}[1 + \sin \theta_B \ln (1 + \frac{1}{\sin \theta_B})e^{-\frac{1}{\cos \theta_c}}. \] (35)

This is not just the single scatter phase function multiplied by a constant added to the multiple scatter phase function multiplied by a constant. \(\Theta\) is the angle between the incident beam and exiting energy, while \(\Theta_c\) is the angle between the cloud normal and the exiting light. There is an angular offset between the two functions that is a function of the beam angle.

A further reduction in complexity can be made by applying Equation 28 to the single scatter phase function as follows:

\[I = (1-a)P_{ss}(\Theta) + a P_{ms}(\Theta_c) \] (36)
where
\[a = \frac{1}{2}[1 + \sin \theta_B \ln (1 + \frac{1}{\sin \theta_B})] \] (37)
\[P_{ss}(\Theta) = \text{single scatter phase function} \] (38)
\[P_{ms}(\Theta_c) = \frac{1}{2\pi E_2(1)} e^{-\frac{1}{\cos \Theta_c}}. \] (39)

Finally, we can see that the magnitude of the single scatter phase function for most geometries will be very small (see
Table I). For rough approximations, when the characteristics of the cloud are unknown, we can estimate the cloud as a non-ideal Lambertian reflector (the amount not received dependent upon the beam angle) of the form

\[ I \approx \frac{1}{2\pi E_2(1)} \left[ 1 + \sin\theta_B \ln(1 + \frac{1}{\sin\theta_B}) \right] \cos\theta_c. \]  

(40)
A model has been described that will compute the amount of energy received at a detector by utilizing the scattering from dense clouds. The restrictions are stated in the model. For practical purposes a further assumption is made that the van de Hulst equations have been solved for the single-scatter case, as is the case for water clouds, and some others.

First, the single-scatter component was computed. This equation (Equation 18) can be used alone if predominantly single scatter is expected, such as atmospheric aerosols and thin clouds, if all other assumptions apply. If the angle of the detector is such that the section of the beam in the field of view is not totally within the scattering medium, modifications, such as computing an effective field of view, must be made. Any losses in the path traversed before and after the scattering medium can be accounted for by multiplying the model answers by transmission along this path.

Next, the total amount of single scatter that escapes the cloud is computed. The assumption of uniform scattering must be applied here to make the problem tractable. For most geometries, this will overestimate the amount of single scatter escaping the cloud since single-scatter functions
are heavily forward-lobed. Even so, this approximation admits that only 15% of a normally incident beam escapes as single scatter. The actual amount will be even smaller. Equation 28 is a closed form approximation for the fraction of incident light escaping the cloud as single scatter in all directions. The rest is multiple-scatter under the no-absorption assumption.

A scattering phase function for the multiple scattering is then determined by considering a typical scatterer and its phase function, and applying this to the entire cloud. The scatter is again considered isotropic. This time the assumption is more valid, however, since the scattering function orientation becomes random after a few scatters.

It is interesting to consider the depth of cloud necessary to scatter a significant portion of incident light. Consider that the C.1 clouds have an extinction coefficient of 17.58 at 1.61 microns. This corresponds to a loss of 90% of the incident energy within 140 meters of cloud. While much of this is forward scattered, typical cloud depth is on the order of several kilometers.

The subject of absorption must be considered for wavelengths above about 2 microns. This can be estimated by determining the "average" photon path and applying the absorption coefficient to this path length. The "reflectivity" of the cloud can then be reduced by this amount.
Appendix A

Single Scatter Equations

The following is a collection of formulas guiding the scattering of electromagnetic radiation by spherical particles, as shown by van de Hulst\(^2\) and simplifications made by Diermendjian\(^4\) for numerical computation of these functions and subsequent integration for collections of particles of differing size.

\[
I = \frac{I_0(i_1 + i_2)}{2k^2r^2} \quad \text{(A-1)}
\]

\(i_1\) and \(i_2\) represent the exiting wave's two components

\[
k = \frac{2\pi}{\lambda}
\]

\(r\) is the distance from particle

\[
i_1 = |S_1(\theta)|^2 \quad i_2 = |S_2(\theta)|^2
\]

\[
S_1(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \{a_n\pi_n(\cos\theta) + b_n\tau_n(\cos\theta)\} \quad \text{(A-2)}
\]

\[
S_2(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \{b_n\pi_n(\cos\theta) + a_n\tau_n(\cos\theta)\} \quad \text{(A-3)}
\]

\[
\pi_n(\cos\theta) = \frac{1}{\sin\theta} p_n^{(1)}(\cos\theta) = \frac{d\pi_n(\cos\theta)}{d(\cos\theta)} \quad \text{(A-4)}
\]

\[
\tau_n(\cos\theta) = \cos\theta \cdot \pi_n(\cos\theta) - \sin^2(\theta) \frac{d\pi_n(\cos\theta)}{d(\cos\theta)}
\]

\[
= \frac{d}{d\cos\theta} p_n^{(1)}(\cos\theta) \quad \text{(A-5)}
\]
$P_n^i$ is the associated Legendre polynomials

$$a_n = \frac{\psi_n(y)\psi_n(x) - m\psi_n(y)\psi'_n(x)}{\psi_n(y)\zeta(x) - m\psi_n(y)\zeta'(x)}$$  \hspace{1cm} (A-6)$$

$$b_n = \frac{m\psi_n(y)\psi_n(x) - \psi_n(y)\psi'_n(x)}{m\psi_n(y)\zeta(x) - \psi_n(y)\zeta'(x)}$$  \hspace{1cm} (A-7)$$

$$x = \frac{2\pi a}{\lambda} \quad a = \text{particle radius}$$  \hspace{1cm} (A-8)$$

$$y = mka \quad m = \text{medium index of refraction}$$  \hspace{1cm} (A-9)$$

$$\psi_n(z) = zj_n(z) = \frac{\pi z}{2} J_{n+\frac{1}{2}}(z)$$  \hspace{1cm} (A-10)$$

$$\zeta_n(z) = z\bar{h}^{(2)}_n(z) = \frac{\pi z}{2} H_{n+\frac{1}{2}}(z).$$  \hspace{1cm} (A-11)$$

The following simplifications and recursive relationships were used by Diermendjian in his book:

$$\pi_n(\theta) = \cos \theta \frac{2n-1}{n-1} \pi_{n-1}(\theta) - \frac{1}{n-1} \pi_{n-2}(\theta)$$  \hspace{1cm} (A-12)$$

$$\pi_0(\theta) = 0$$  \hspace{1cm} (A-13)$$

$$\pi_1(\theta) = 1$$  \hspace{1cm} (A-14)$$

$$\pi_2(\theta) = 3 \cos \theta$$  \hspace{1cm} (A-15)$$

$$\tau_n(\theta) = \cos \theta [\pi_n(\theta) - \pi_{n-2}(\theta)] - (2n-1)\sin^2 \theta \pi_{n-1}(\theta) + \tau_{n-2}(\theta)$$  \hspace{1cm} (A-16)$$

$$\tau_0(\theta) = 0$$  \hspace{1cm} (A-17)$$

$$\tau_1(\theta) = 1$$  \hspace{1cm} (A-18)$$

$$\tau_2(\theta) = 3 \cos(2\theta)$$  \hspace{1cm} (A-19)$$

$$a_n(m, x) = \left[\frac{A_n(y)}{m} + \frac{n}{x}\right] \text{Re} \left\{ \frac{\omega_n(x)}{m} \right\} - \text{Re} \left\{ \frac{\omega_{n-1}(x)}{m} \right\}$$  \hspace{1cm} (A-20)$$
\[ b_n(m, x) = \frac{(mA_n(y) + \frac{n}{x}) \text{Re}\{\omega_n(x)\} - \text{Re}\{\omega_{n-1}(x)\}}{(mA_n(y) + \frac{n}{x}) \omega_n(x) - \omega_{n-1}(x)} \]  

(A-21)

\[ \omega_n(x) = \frac{2n-1}{x} \omega_{n-1}(x) - \omega_{n-2}(x) \]  

(A-22)

\[ \omega_0(x) = \sqrt{\frac{\pi x}{2}} [J_{\frac{1}{2}}(x) + iJ_{-\frac{1}{2}}(x)] \]  

(A-23)

\[ = \sin x - i \cos x \]

\[ \omega_{-1}(x) = \sqrt{\frac{\pi x}{2}} [J_{-\frac{1}{2}}(x) = iJ_{-\frac{1}{2}}(x)] \]  

(A-24)

\[ = \cos x - i \sin x \]

\[ A_n(y) = \frac{J_{n-\frac{1}{2}}(y) - \frac{n}{y} J_{n+\frac{1}{2}}(y)}{J_{n+\frac{1}{2}}(y)} = -\frac{n}{y} + \frac{J_{n-\frac{1}{2}}(y)}{J_{n+\frac{1}{2}}(y)} \]  

(A-25)

\[ = -\frac{n}{y} + [\frac{n}{y} - A_{n-1}(y)] \]  

(A-26)

\[ A_0(y) = \cot y \]  

(A-27)

Figure A-1 is a phase function for a single particle with an index of refraction of 1.33 and \( x = 10 \).

Figure A-1. Fluctuations in the \( i_1 \) and \( i_2 \) components of Equation A-1 for a single particle. ²
Appendix B
Aerosol Size Distribution

The size distributions of aerosol occurring in nature are not well known. First, the assumption that particles are spherical already oversimplifies the problem. Ice has a hexagonal shape with a correspondingly different scattering phase function, dust particles and other suspended solids are irregularly shaped, and even "spherical" water droplets change shape due to turbulence. Yet, if clouds and hazes are considered, the majority of scatterers are water droplets very nearly spherically shaped. The distribution of particle radii are found to have the form

$$n(r) = ar^\alpha e^{-br^\gamma} \quad 0 \leq r < \infty.$$  \hspace{1cm} (B-1)

This function is used by many authors and has been verified with experimental data.\(^1\), \(^3\), \(^4\), \(^5\), \(^6\) The following has been determined about this function:

$$N = \int n(r) \, dr$$  \hspace{1cm} (B-2)

$$= \int_0^\infty ar^\alpha e^{-br^\gamma} \, dr$$  \hspace{1cm} (B-3)

$$= \frac{a}{\gamma} b^{-\frac{\alpha+1}{\gamma}} \Gamma\left(\frac{\alpha+1}{\gamma}\right)$$  \hspace{1cm} (B-4)

where \(\Gamma (z)\) is the gamma function defined by:

$$\Gamma (z) = \int_0^\infty t^{z-1} e^{-t} \, dt.$$  \hspace{1cm} (B-5)
The derivative of this function, useful for finding critical points, is

\[
\frac{d}{dr} n(r) = a r^{a-1} (a - br \gamma) e^{-br \gamma}
\]

yielding \(0\), \(\infty\) and \(r_c\) as critical points where \(r_c\), the mode radius, is given by

\[
r_c = \gamma \sqrt{\frac{\alpha}{\gamma b}}.
\]

For thick cumulus clouds, the closest model for which Diermendjian has computed phase function tables is the C.1 model. Figure B-1 is a summary of the parameters describing this model, Figure B-2 is a plot of this particle size distribution.

\[
a = 2.373
\]
\[
b = 1.5
\]
\[
\gamma = 1.0
\]
\[
\alpha = 6
\]
\[
n(r) = 2.373 r^6 e^{-1.5r}
\]
\[
N = 100 \text{ particles/cm}^3
\]
\[
r_c = 4\mu m
\]
\[
n(r_c) = 24.1 \text{ particles/\mu cm}
\]
\[
\beta_{ex} = 11.18 \@ 10
\]
\[
\beta_{ex} = 16.73 \@ .70
\]

FIGURE B-1
Figure B-2. Particle Size Distribution for C.1 Cloud
Appendix C
Model Derivation

The following details the derivation of single-scatter irradiance received by a detector. Refer to Figure 3 for location of variables in the equations.

The assumption made is that the scattering angle is constant, therefore, triangles ABC and ADE are similar. The following is evident from this observation:

\[ d_2 = w(x + d_1) \]  
\[ R_2 = w(x + R_1) \]

rewriting Equation 16

\[ I = \int_0^L y e^{-Y(d_1 + d_2 + x)} \frac{P(\Theta)}{R_2} \, dx \]

substituting C-1 and C-2 into C-3 yields

\[ I = \frac{YP(\Theta)}{w^2R_1^2} e^{-Y(w+1)d_1} \int_0^L \frac{e^{-Y(w+1)x}}{(x/R_1 + 1)^2} \, dx \]

rewriting as

\[ I = A \int_0^L e^{-ax} \frac{dx}{(bx+1)} \]

where

\[ A = \frac{YP(\Theta)}{w^2R_1^2} e^{-Y(w+1)d_1} \]
\[ a = Y(w+1) \]
\[ b = \frac{1}{R_1} \]
Substituting $y = bx + 1, \, x = \frac{y-1}{b}, \, dx = \frac{1}{b} \, dy$

$$I = \frac{A}{b} \int_1^{bL+1} e^{-\frac{a(y-1)}{b}} \, dy$$

$$= \frac{A}{b} \int_1^{bL+1} e^{-\frac{a}{b}y} \, dy \tag{C-9}$$

rewriting as

$$I = B \int_1^{L_2} e^{-cy} \frac{1}{y^2} \, dy \tag{C-10}$$

where

$$B = \frac{A}{b} e^b = \frac{YP(\Sigma)}{K^2 R_1} e^{\gamma(w+1)(R_1-d_1)} \tag{C-11}$$

$$c = \frac{a}{b} = \gamma(w+1)R_1 \tag{C-12}$$

$$L_2 = bL+1 = \frac{L}{R_1} + 1 = \frac{R_1+L}{R_1}. \tag{C-13}$$

Breaking this into two integrals:

$$I = B \left[ \int_1^\infty e^{-cy} \frac{1}{y^2} \, dy - \int_1^{L_2} e^{-cy} \frac{1}{y^2} \, dy \right] \tag{C-14}$$

In the second integral, the following substitution is made: $t = \frac{y}{L_2}, \, y = L_2 t, \, dy = L_2 dt$

which yields

$$I = B \left[ \int_1^\infty e^{-cy} \frac{1}{y^2} \, dy - \int_1^{\infty} e^{-cL_2 t} \left( \frac{1}{L_2 t} \right)^2 L_2 \, dt \right] \tag{C-15}$$

$$= B \left[ \int_1^\infty e^{-cy} \frac{1}{y^2} \, dy - \frac{1}{L_2} \int_1^\infty e^{-cL_2 t} \frac{1}{t^2} \, dt \right]. \tag{C-16}$$

From Abramowitz and Stegen\textsuperscript{12}, pg. 228, Equation 5.1.4

$$\int_1^\infty e^{-ax} \frac{1}{x^2} \, dx = E_2(a). \tag{C-17}$$
We find

\[ I = B \left[ E_2(c) - \frac{1}{L^2} E_2(cL_2) \right] \tag{C-19} \]

Substituting values

\[ I = \frac{YP(\Theta)}{WR_1} e^{Y(w+1)(R_1-d_1)} \left\{ E_2[Y(w+1)R_1] \right. \]

\[ - \frac{R_1}{L+R_1} E_2[Y(w+1)(R_1+L)] \}. \tag{C-20} \]

Equation C-11 can be integrated directly in terms of the exponential integral function as follows:

\[ I = B \int_1^{L^2} \frac{e^{-cy}}{y^2} dy \tag{C-21} \]

\[ = B \left[ \frac{e^{-cy}}{y} - cEi(-cy) \right]_{1}^{L^2} \tag{C-22} \]

\[ = B \left\{ e^{-c} - \frac{1}{L^2} e^{-cL^2} + c[Ei(-c) - Ei(-cL_2)] \right\}. \tag{C-23} \]

Equation C-22 comes from Gradshteyn and Ryzhik \textsuperscript{13} pg. 93, Equation 3.325.2.

Though Equation C-23 uses the popular exponential integral function, Equation C-19 is in a simpler form. The following relationship can be derived between the exponential integral and \( E_2(x) \):

\[ Ei(-x) = -E_1(x) \tag{C-24} \]

\[ = \frac{1}{x} \left[ E_2(x) - e^{-x} \right]. \tag{C-25} \]

Figure C-1 is a plot of the exponential-integral function and \( E_2(x) \).

The necessary parameters, \( K, R_1, d_1 \) and \( L \) can now be determined from the given variables, \( \Theta_F, \Theta_B, Y, h, \) and \( R \) by simple geometrical considerations.
Figure C-1. Exponential Integral Functions

Ei(x) and E2(x)
To compute $K$, the ratio of the sides of the triangle:

$$w = \frac{d_2}{d_1 + x} = \frac{hc}{\sin\theta_A} = \frac{\sin\theta_B}{\sin\theta_A} \quad \text{(C-26)}$$

$R_1$ and $d_1$ are found as follows:

$$R_1 = \frac{hc}{\sin\theta_B}$$

$$d_1 = \frac{hc}{\sin\theta_B}$$

$$R_1 = \frac{h + hc}{\sin\theta_B}$$

$$R_1 = \frac{R}{\sin(\theta_A + \theta_B)} \frac{\sin(\theta_A + \theta_B)}{R} - h \quad \text{(C-28)}$$

$$L$$ is determined as follows:

$$h + h_L = \frac{R}{\cot(\theta_A + \theta_B) + \cot\theta_B} \quad \text{(C-32)}$$

$$R_1 + L = \frac{h + h_L}{\sin\theta_B}$$

$$L = \frac{R}{\sin\theta_B} \left[ \frac{1}{\cot(\theta_A + \theta_B) + \cot\theta_B} - \frac{1}{\cot\theta_A + \cot\theta_B} \right] \quad \text{(C-34)}$$
These values also correspond to the variable in the first-order approximation as follows:

\[
\begin{align*}
    d_1' &= \frac{R}{2 \sin \theta_B} \left[ \frac{2}{\cot(\theta_A + \theta_F) + \cot \theta_B} + \frac{1}{\cot \theta_A + \cot \theta_B} - \frac{2h}{R} \right] \quad (C-35) \\
    d_2' &= \frac{R}{2 \sin \theta_A} \left[ \frac{2}{\cot(\theta_A + \theta_F) + \cot \theta_B} + \frac{1}{\cot \theta_A + \cot \theta_B} - \frac{2h}{R} \right] \quad (C-36) \\
    R_2' &= \frac{R}{2 \sin \theta_A} \left[ \frac{2}{\cot(\theta_B + \theta_F) + \cot \theta_B} + \frac{1}{\cot \theta_A + \cot \theta_B} \right] \cdot \quad (C-37)
\end{align*}
\]

If \( d_1 = 0 \) and \( L = \infty \), which implies that all single scatter radiation is within the field-of-view, and keep "\( \infty \)" small enough for \( P(\theta) \) to be constant.

\[
I = \frac{\gamma P(\theta)}{\sqrt[2]{\omega R_1}} e^{\gamma(\omega+1)R_1} E_2[\gamma(\omega+1)R_1]. \quad (C-38)
\]

Equation C-38 could be integrated for all ground points by writing all variables in terms of ground coordinates to determine the total amount of single scatter emitted from the cloud. Something would have to be assumed about the phase function, \( P(\theta) \), before integrating since the scattering angle can no longer be assumed constant. The standard assumption is that of an isotropic radiator, making the phase function independent of angle. This will give values close to those expected with other phase functions if the integration interval is large enough.
The isotropic scatterer assumption will be made here. Rather than integrate Equation C-38, the beam will be broken into an infinite number of small isotropic radiators. The amount of scatter radiation that exits the cloud from point scatterer will then be computed as a function of position in the beam. These will then be summed (integrated) to yield the total single-scatter radiation to exit the cloud.

Figure 5 shows an isotropic radiator of magnitude \( \sigma \) located a distance \( h \) from a planar surface. Circular coordinates will be used on this surface to compute the total energy incident on the planar surface below. The following equation computes this energy:

\[
I_p = \int_0^\infty \int_0^{2\pi} \frac{\sigma}{4\pi} \frac{e^{-YS}}{S^2} \sin \theta \, dA
\]  
(C-39)

\[
\sin \theta = \frac{h}{S}
\]  
(C-40)

\[
dA = r \, dr \, d\phi \quad r = \sqrt{S^2 - h^2}
\]

\[
\int_0^{2\pi} \int_0^{\sqrt{S^2 - h^2}} \frac{S}{\sqrt{S^2 - h^2}} \, dS \, d\phi
\]

\[
= S \, dS \, d\phi
\]  
(C-41)

\[
I_p = \frac{\sigma h}{4\pi} \int_h^\infty \int_0^{2\pi} \frac{e^{-YS}}{S^2} \, d\phi \, dS
\]  
(C-42)

\[
= \frac{\sigma h}{2} \int_h^\infty \frac{e^{-YS}}{S^2} \, dS
\]  
(C-43)
Substituting \( y = \frac{S}{h} \), \( L = hy \), \( dL = h \, dy \) yields:

\[
I_p = \frac{\sigma}{2} \int_1^\infty \frac{e^{-\gamma hy}}{y^2} \, dy
\]

\[
= \frac{\sigma}{2} E_2(\gamma h).
\]

(C-44)

(C-45)

Note that if \( h \) were small, we would expect the exiting energy to approach \( \frac{\sigma}{2} \) since there would be little loss in the medium. As expected, \( E_2(0) = 1 \).

If we assume that the point source is actually a differential scattering element along the beam, we must relate the magnitude to the height above the cloud boundary. If we let the beam enter the cloud at \( \theta_B \) and locate the scattering element at a distance \( R \) from the cloud boundary along the beam, we get:

\[
\sigma = e^{-\gamma R} \frac{h}{\sin \theta_B} \, dR
\]

\[
= e^{-\gamma \left( \frac{h}{\sin \theta_B} \right)} \gamma \frac{dh}{\sin \theta_B}
\]

\[
= \frac{\gamma}{\sin \theta_B} e^{-\left( \frac{\gamma}{\sin \theta_B} \right)h}.
\]

(C-46)

(C-47)

(C-48)

Putting this into Equation C-45 and integrating over the entire beam:

\[
I_{ss} = \int_0^\infty \frac{\gamma}{2 \sin \theta_B} e^{-\frac{\gamma}{\sin \theta_B} h} E_2(\gamma h) \, dh.
\]

(C-49)

Substituting \( y = \gamma h \)

\[
I_{ss} = \int_0^\infty \frac{1}{2 \sin \theta_B} e^{-\frac{y}{\sin \theta_B}} E_2(y) \, dy
\]

(C-50)
\[ I_{ss} = \frac{a}{2} \int_{0}^{\infty} e^{-ay} E_2(y) \, dy \]  
\[ a = \frac{1}{\sin \theta_B} \]  

From Abramowitz and Stegen, we find:

\[ \int_{0}^{\infty} e^{-at} E_2(t) \, dt = \frac{-1}{a^2} [\ln(1+a) - a] \]  

pg. 230, 5.1.34

So,

\[ I_{ss} = \frac{-1}{2a} [\ln(a+1)-a] \]  
\[ = \frac{1}{2} [1-a \ln(1+a)] \]  
\[ = \frac{1}{2} [1 - \sin \theta_B \ln(1 + \frac{1}{\sin \theta_B})] \quad 0 < \theta_B < \frac{\pi}{2} \]

The following limits can be computed

\[ I_{ss} \big|_{x=0} = \frac{1}{2} [1 - \ln(1 + \frac{1}{x})] = \frac{1}{2} \]  
\[ I_{ss} \big|_{x=\frac{\pi}{2}} = \frac{1}{2} [1 - \ln(2)] = 0.1534 \]

Figure C-2 is a plot of single scatter to total scatter as a function of \( \theta_B \).
Figure C-2. Functional Relationship Between Beam Angle and Single-Scatter Fraction
Appendix D
Phase Function Normalization

The following describes the method for normalizing a phase function for use in Equation 35, such that the energy reaching a hemisphere at any distance from the source will be one, as required by conservation of energy theorems.

Consider a point source radiating into a hemisphere as shown in Figure D-1. At any point in the shaded annulus, the radiance will be the same. If we find this radiance and multiply by the area of the annulus, the result is the irradiance on the annulus. Integrating this over the range of $\theta$, will yield the total radiance. This must be 1. From Figure D-2, the width of the annulus is given by

$$W = 2R \tan \frac{d\theta}{2}.$$  \hspace{1cm} (D-1)

Since $d$ is small, this can be reduced to

$$W = R \, d\theta.$$ \hspace{1cm} (D-2)

The radius of the annulus is given by

$$r = R \sin \theta.$$ \hspace{1cm} (D-3)

The area is then

$$a_a = 2\pi R^2 \sin \theta \, d\theta.$$ \hspace{1cm} (D-4)

The irradiance at any point on the annulus is

$$\frac{MP(\theta)}{R^2}.$$ \hspace{1cm} (D-5)

The total irradiance must be one, yielding the equation
Figure D-1. (a) Irradiance into a Hemisphere
(b) Annulus Width
\[
\int_0^{\frac{\pi}{2}} 2\pi M \frac{P(\theta)}{R^2} (R^2 \sin \theta) \, d\theta = 1 \quad \text{(D-6)}
\]

\[
2\pi M \int_0^{\frac{\pi}{2}} P(\theta) \sin \theta \, d\theta = 1. \quad \text{(D-7)}
\]

For the isotropic radiator, \( P(\theta) = 1. \) From Equation D-7

\[
2\pi M \int_0^{\frac{\pi}{2}} \sin \theta \, d\theta = 1 \quad \text{(D-8)}
\]

\[
2\pi M [-\cos \theta]_0^{\frac{\pi}{2}} = 1 \quad \text{(D-9)}
\]

\[
M = \frac{1}{2\pi}. \quad \text{(D-10)}
\]

For the Lambertian reflector, \( P(\theta) = \cos \theta. \) This approximates the multiple scatter in clouds.

Putting this into Equation D-7,

\[
2\pi M \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta \, d\theta = 1 \quad \text{(D-11)}
\]

\[
2\pi M \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin(2\theta) \, d\theta = 1 \quad \text{(D-12)}
\]

\[
2\pi M [-\frac{1}{2} \cos(2\theta)]_0^{\frac{\pi}{2}} = 1 \quad \text{(D-13)}
\]

\[
\pi M = 1 \quad \text{(D-14)}
\]

\[
M = \frac{1}{\pi}. \quad \text{(D-15)}
\]

If we let \( P(\theta) = e^{-\frac{1}{\cos \theta}}, \) Equation D-7 yields

\[
2\pi M \int_0^{\frac{\pi}{2}} e^{-\frac{1}{\cos \theta}} \sin \theta \, d\theta = 1. \quad \text{(D-16)}
\]

Letting \( x = \frac{1}{\cos \theta}, \theta = \arccos \left(\frac{1}{x}\right) \)

\[
d\theta = \frac{1}{x^2 \sqrt{1-(\frac{1}{x})^2}} \, dx \quad \text{and} \quad \sin \theta = \sqrt{1-(\frac{1}{x})^2} \quad \text{(D-17)}
\]

\[
2\pi M \int_1^{\infty} e^{-x} \sqrt{1-(\frac{1}{x})^2} \left[ \frac{1}{x^2 \sqrt{1-(\frac{1}{x})^2}} \right] \, dx = 1 \quad \text{(D-18)}
\]
\[ 2\pi M \int_{1}^{\infty} \frac{e^{-x}}{x^2} \, dx = 1 \quad (D-19) \]

\[ 2\pi M E_2(1) = 1 \]

\[ M = \frac{1}{2\pi E_2(1)} \quad (D-20) \]

From tables, \( E_2(1) = .148495 \). This phase function becomes

\[ P(0) = \frac{1}{2\pi E_2(1)} e^{-\frac{1}{\cos(\theta)}} \quad (D-21) \]

\[ = 1.0718 e^{-\frac{1}{\cos(\theta)}} \quad (D-22) \]
REFERENCES


