Control of a Nonlinear System by Linearization

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CONTROL OF A NONLINEAR SYSTEM BY LINEARIZATION

BY

DREW DOUGLAS NELSON
B.S., Pacific Lutheran University, 1981

RESEARCH REPORT

Submitted in partial fulfillment of the requirements for the degree of Master of Science in the Graduate Studies Program of the College of Engineering University of Central Florida Orlando, Florida

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ABSTRACT

In today's linear control systems, exact solutions can be obtained by the use of Laplace Transforms in the frequency domain. In dealing with nonlinear systems, exact solutions are not always achievable. For this reason, it is necessary to linearize the system and then apply frequency response methods.

This paper shows the comparison of a nonlinear system with the linearized model of the same system. For both proportional and proportional-integral control, the response to a unit step change in the set point showed minimal difference between the linearized and nonlinear system.
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CHAPTER I
INTRODUCTION

Control Theory had its origins in the 1700s with James Watt's centrifugal governor for the speed control of a steam engine. Due to its proportional-control action, the Watt governor resulted in a static error of engine speed. To eliminate this error, an integral control action was implemented. In this instance, the integral action created an unstable condition. Without modern tools, stable solutions could only be obtained by experimentation and intuition.

In the 1900s, the advent of instruments and regulators for process and power industries created a need for theory to replace intuition in the design of control systems. The use of differential equations and the Routh-Hurwitz stability criteria became more widespread. These applications were still constrained to low-order and simple systems.

Due to World War II, a large interest in weapon position-control developed. This development spurred the subsequent development of the frequency-response and root-locus methods. These two methods form the core of
classical control theory. Basic feedback control soon included such problems as sample-data control, random-signal systems and some phenomena caused by system nonlinearities and nonlinear control. Recent developments are geared towards finding optimal control for both deterministic and stochastic systems.

In today's linear control systems, exact solutions can be obtained by using Laplace Transforms. While dealing with non-linear systems, exact solutions are not always achievable. For this reason, it is necessary to linearize the system and then apply frequency response methods. In most practical cases, the main concern is with the stability of the non-linear system. Approximate solutions and stability checks are obtained by applying Laplace Transform techniques to the linearized system.

In this paper, a comparison of a nonlinear system with its linearized model will show how both differ in response to a unit step input. This comparison should show if the linear model fails and if so, by how much. A study of how any failure can be affected will be explored.

This paper will first describe the nonlinear system. The real-life tanks and pumps will be physically described and then modeled to arrive at a nonlinear system. A
linearization will be conducted and linear control theory will be applied to arrive at an exact linear solution. A computer simulation will then numerically calculate the actual nonlinear numerical solution. A comparison of the results should show any failures of the linear model.
CHAPTER II

PHYSICAL SYSTEM

The nonlinear system to be controlled is a series of two tanks attached to a controller and pump. A diagram of the tank apparatus is depicted in Figure 2-1.

The two tanks are connected by four holes. Three of these holes are at a height of 3 cm above the base of the tank and can be closed off. Their diameters are 1.27 cm, .95 cm and .635 cm. The other hole, .317 cm in diameter, is at a height of 1.5 cm and remains open at all times. The total cross-sectional area of the four holes, $a_1$, is the cross-sectional area of orifice one. This orifice also has a discharge coefficient of $C_{D1}$.

The second tank has two input flows, $F_2$ and $F_L$. $F_2$ is the flow of liquid between the two tanks and $F_L$ is a load flow that is supplied from a source outside of the system. The output flow of the second tank, $F_0$, is through a valve 3 cm from the base of the tank. This valve is an adjustable tap that creates orifice two. Fully open, the diameter of the tap is .70 cm. Orifice two has a discharge coefficient $C_{D2}$ and cross-sectional area $a_2$. 
Figure 2-1. Tank Apparatus.
F₁, according to a voltage supplied to its motor, Vₘ. This voltage is supplied from a controller which receives a voltage input, V₁, from a depth sensor. The depth sensor transforms the variable height of tank one, H₁, into a voltage, V₁. The height of orifice two, H₃, remains fixed at 3 cm.

There are several different types of parameters in the system. Parameters such as a₁, a₂, Cᵈ₁, Cᵈ₂, H₃ and the cross-sectional area of each tank, A, (A = 200 cm² in this system) are fixed by the physical nature of the system.

Parameters such as the proportional gain, Kₐ, the reset time, T₉, and the derivative time, T₉, are input into the system. These three control parameters create which type of controller action the system will follow. This paper will only investigate two types of controller action. These two types are proportional control (Kₐ = 10, T₉ = ∞, T₉ = 0) and proportional-integral control (Kₐ = 10, T₉ = 10, T₉ = 0).

The two remaining parameters are used only in the linearized system. The pump motor constant, Gₚ, and the depth sensor constant, Gₐ, both depend on operating conditions. Gₚ is the small change in pump flow f₁ when a
small change in voltage to the motor, \( v_M \), is supplied at original voltage \( V_m \). This is an approximation to the derivative of the relationship between \( F_1 \) and \( V_1 \) evaluated at some constant \( V_m \). In much the same way, \( G_D \) is the small change in voltage to the controller, \( v_1 \), when a small change in the height of tank one, \( h_1 \), occurs at some original height \( H_1 \). The derivation of \( G_p \) begins by finding the relationship between \( V_M \) and \( F_1 \) by applying a least-squares fit polynomial approximation to actual data. From Figure 2-2 this relationship is given by:

\[
F_1 = f(V_M) = \frac{-49.176V_M^2 + 1023.8V_M - 687.28}{60} \quad \text{(eq. 2-1)}
\]

The next step is finding the rate of change by differentiating equation 2-1.

\[
\frac{dF_1}{dV_M} = \frac{-98.352V_M + 1023.8}{60} \quad \text{(eq. 2-2)}
\]

\( G_p \) is then found by evaluating equation 2-2 at the original voltage \( V_M \).

\[
G_p = \frac{f_1}{v_M} = \left| \frac{dF_1}{dV_M} \right| = \frac{-1.6392V_M + 17.063}{V_M = \bar{V}_M} \quad \text{(eq. 2-3)}
\]
Figure 2-2. Relationship of $V_M$ and $F_1$. 

**Plot Standards (y; m):**

- **Averaging and Smoothing**
  - # of Points Averaging: 0-12 (6)
  - # of Points Smoothing: 0-12 (0)

**Curve Fitting**

- Interpolation of least squares fitting (1; lin): (L)
- # of Points on Curve: 25:1877 (25)

Linear, Geometric, Exponential, or Polynomial Least Squares (LGEIP): (F)

- Degree of Polynomial (2; 6): (2)
- 0 Degree Coefficient = -687.278083
- 1 Degree Coefficient = 1023.83071
- 2 Degree Coefficient = -49.1758679

Coefficient of Determination = 0.992413849
Coefficient of Correlation = 0.993199704
Standard Error of Estimate = 115.336272
Satisfactory (y; m): (Y)
To derive $G_D$, the same procedure is followed. First, the relationship between $H_1$ and $V_1$ must be found by applying a least-squares fit polynomial approximation to actual data. From Figure 2-3, this equation is given by equation 2-4.

$$V_1 = f(H_1) = 0.00081H_1^3 - 0.02214H_1^2 + 0.47795H_1 + 1.1766$$  
(eq. 2-4)

Differentiating equation 2-4 yields equation 2-5.

$$\frac{dV_1}{dH_1} = 0.00243H_1^2 - 0.04428H_1 + 0.47795$$  
(eq. 2-5)

And then evaluating equation 2-5 at $\bar{H}_1$ gives equation 2-6.

$$G_D = \frac{V_1}{h_1} = \frac{dV_1}{dH_1} \bigg|_{H_1 = \bar{H}_1} = 0.00243\bar{H}_1^2 - 0.04428\bar{H}_1 + 0.47795$$  
(eq. 2-6)
Figure 2-3. Relationship of $H_1$ and $V_1$. 
CHAPTER III
SYSTEM MODELING

Since the system has already been described physically, it will now be described with dynamic equations. The first equations will be those that describe the time rate of change of the heights, $H_1$ and $H_2$, in terms of the flow variables, $F_1$, $F_2$, $F_L$ and $F_0$, and physical parameters. Equations for the flows, $F_0$ and $F_2$, in terms of physical parameters and height variables, $H_1$ and $H_2$, will be next derived. Finally, the steady state relationship for $H_2$ in terms of $H_1$ and $F_L$ will be found. Assuming the system is initially in steady state, this will enable the initial height of tank two to be found from the height in tank one and the load variable. For this model, orifice one shall have all but the largest hole open ($a_1 = 1.109$) and the tap on orifice two will be wide open ($a_2 = .384$). No load variable will be applied ($F_L = 0$) and tank one is initially at a height of 10 cm ($H_1 = 10$). With these two sets of equations, the goal of comparing linear and non-linear solutions can be met. The time rate of change of $H_1$ and $H_2$ is found by deriving $\frac{dH_1}{dt}$ and $\frac{dH_2}{dt}$.
The rate of change of volume of a tank is equal to the rate of volume into the tank minus the rate of volume out of the tank. This implies the rate of change in the volume of tank one ($V_1$) is described by equation 3-1.

$$\frac{dV_1}{dt} = F_1 - F_2$$

(eq. 3-1)

In the same manner the rate of change in the volume of tank two ($V_2$) is given by equation 3-2.

$$\frac{dV_2}{dt} = (F_2 + F_L) - F_0$$

(eq. 3-2)

Since the volume of a rectangular tank is the area times the height of the tank, equations 3-3 and 3-4 hold for this example.

$$V_1 = A*H_1$$

(eq. 3-3)

$$V_2 = A*H_2$$

(eq. 3-4)

Since the area $A$ is constant, equations 3-5 and 3-6 apply.
\[ \frac{dV_1}{dt} = A \frac{dH_1}{dt} \]  \hspace{1cm} (eq. 3-5)

\[ \frac{dV_2}{dt} = A \frac{dH_2}{dt} \]  \hspace{1cm} (eq. 3-6)

By substituting equation 3-5 into equation 3-1 and solving for the rate of change of height one, equation 3-7 holds.

\[ \frac{dH_1}{dt} = \frac{F_1 - F_2}{A} \]  \hspace{1cm} (eq. 3-7)

Following the same procedure, the rate of change in height two is given by substituting equation 3-6 into equation 3-2 and is shown by equation 3-8.

\[ \frac{dH_2}{dt} = \frac{F_2 + F_L - F_O}{A} \]  \hspace{1cm} (eq. 3-8)

To find \( F_1 \) and \( F_2 \) in terms of height variables and physical parameters, several steps are completed.

First, Bernoulli's Theorem states that for Figure 3-1, equations 3-9 and 3-10 hold.
WHERE

\[ P = \text{PRESSURE} \]
\[ x = \text{HEIGHT} \]
\[ r = \text{RATE OF FLUID FLOW} \]
\[ \varphi = \text{DENSITY OF FLUID} \]
\[ \alpha = \text{CROSS-SECTIONAL AREA} \]

Figure 3-1. Single Tank With Orifice.
\[ P_1 + \rho gX_1 + \frac{1}{2} \rho v_1^2 = P_2 + \rho gX_2 + \frac{1}{2} \rho v_2^2 \] (eq. 3-9)

\[ P_2 = P_1 + \rho g(X_1 - X_2) \] (eq. 3-10)

By the equation of continuity, equation 3-11 yields the relationship between rates of flow and cross-sectional areas.

\[ a_1r_1 = a_2r_2 \] (eq. 3-11)

For the situation depicted in Figure 3-2, applying the relationship in equation 3-10 yields equation 3-12.

\[ P_2 = P_1 + \rho g(X_3 - X_2) \] (eq. 3-12)

Substituting equation 3-11 into equation 3-9 yields equation 3-13.

\[ P_1 + \rho gX_1 + \frac{1}{2} \rho v_1^2 = P_1 + \rho g(X_3 - X_2) + \rho gX_2 + \frac{1}{2} \rho v_2^2 \] (eq. 3-13)

Subtracting \( P_1 \) from both sides, regrouping and dividing by the density yields equation 3-14.
Figure 3-2. Dual Connected Tanks.

\[ p_1 = p_3 \]
Equation 3-11 can be rewritten as shown in equation 3-15.

\[ r_1 = \frac{a_2}{a_1} r_2 \]  

(eq. 3-15)

Substituting the simplified equation 3-14 into equation 3-15 and solving for \( r_2^2 \) yields equation 3-16.

\[ r_2^2 = \frac{2g(X_1 - X_3)}{1 - \left( \frac{a_2}{a_1} \right)^2} \]  

(eq. 3-16)

By finding the square root of both sides of equation 3-16, \( r_2 \) is shown by equation 3-17 after some regrouping.

\[ r_2 = a_1 \frac{\sqrt{2g(X_1 - X_3)}}{a_1^2 - a_2^2} \]  

(eq. 3-17)

Since Figure 3-2 is a sharp-edged orifice and the volume flowrate through such an orifice equals the cross-sectional area of the orifice times the rate of flow
through the orifice, the volume flowrate for Figure 3-2 is
given by equation 3-18.

\[
\text{Volume} = (a_2) \sqrt{2g(X_1 - X_3)} \\
\text{Flowrate} = \frac{1}{a_1^2 - a_2^2} \\
\text{(eq. 3-18)}
\]

Therefore the tank in Figure 2-1 has a flowrate out of
the first tank given by equation 3-19.

\[
F_2 = a_1 A \frac{\sqrt{2g \sqrt{H_1 - H_2}}}{A^2 - a_1^2} \\
\text{(eq. 3-19)}
\]

Substituting the variables that describe the flowrate
out of the second tank into equation 3-18 yields equation
3-20.

\[
F_0 = a_2 A \frac{\sqrt{2g \sqrt{H_2 - H_3}}}{A^2 - a_2^2} \\
\text{(eq. 3-20)}
\]

Upon regrouping equation 3-19 and calling the second
factor the discharge coefficient $C_{D1}$, equation 3-21
describes $F_2$.

\[
F_2 = C_{D1} a_1 \sqrt{2g \sqrt{H_1 - H_2}} \\
\text{(eq. 3-21)}
\]
F₀ is found in the same manner with the second factor as the discharge coefficient C₀₂ and is calculated with equation 3-22.

\[ F₀ = C₀₂a₂\sqrt{2g \sqrt{H₂ - H₃}} \]  
(eq. 3-22)

Since \( A \gg a₁ \) and \( A \gg a₂ \) for Figure 2-1, both \( C₀₁ \) and \( C₀₂ \) approximately equal 1.

Finally, using equation 3-22, the steady state relationship for \( H₂ \) in terms of \( H₁ \) and \( F_L \) can be found. Since at steady state \( \frac{dH₂}{dt} = 0 \), equation 3-8 yields equation 3-23.

\[ 0 = \frac{dH₂}{dt} = \frac{F₂ + F_L - F₀}{A} \]  
(eq. 3-23)

Solving for \( F₀ \) in terms of \( F₂ \) and \( F_L \) gives the relationship in equation 3-24.

\[ F₀ = F₂ + F_L \]  
(eq. 3-24)

Substituting equations 3-21 and 3-22 into equation 3-24 gives the following equation.
\[ C_{D_2} \alpha_2 \sqrt{2g} \sqrt{H_2 - H_3} = C_{D_1} \alpha_1 \sqrt{2g} \sqrt{H_1 - H_2} + F_L \]  
(eq. 3-25)

To solve this let \( y = \sqrt{H_1 - H_2} \), then \( y^2 = H_1 - H_2 \) and \( H_2 = H_1 - y^2 \). Substituting these last two equations into equation 3-25 gives equation 3-26.

\[ C_{D_2} \alpha_2 \sqrt{2g} \sqrt{(H_1 - H_3) - y^2} = C_{D_1} \alpha_1 \sqrt{2gy} + F_L \]  
(eq. 3-26)

Squaring both sides of equation 3-25 yields equation 3-27.

\[ C_{D_2}^2 \alpha_2^2 2g \left[ (H_1 - H_3) - y^2 \right] = C_{D_1}^2 \alpha_1^2 2gy^2 + 2FLC_{D_1} \alpha_1 \sqrt{2gy} + F_L^2 \]  
(eq. 3-27)

Rewriting equation 3-27 as a quadratic gives equation 3-28.

\[ 0 = (C_{D_1} \alpha_1^2 + C_{D_2} \alpha_2^2) y^2 + F_L C_{D_1} \alpha_1 \sqrt{2y} + F_L^2 - C_{D_2}^2 \alpha_2^2 (H_1 - H_3) \frac{\sqrt{y}}{2g} \]  
(eq. 3-28)

Solving equation 3-28 for the unknown \( y \) and simplifying yields equation 3-29.
Equation 3-29 gives two solutions. But since by definition \( y \) is the square root of the differences in height, it must be greater than zero. The negative solution can then be disregarded and equation 3-30 gives the proper result.

\[
y = -F_L C_{D1} a_1 + C_{D2} a_2 \sqrt{(H_1 - H_3) (C_{D1}^2 a_1^2 + C_{D2}^2 a_2^2) 2g - F_L^2} / \sqrt{2g (C_{D1}^2 a_1^2 + C_{D2}^2 a_2^2)}
\]

(eq. 3-29)

Applying the definition of \( y \) and solving for \( H_2 \), the desired solution is achieved and given by equation 3-31.

\[
H_2 = H_1 - C_{D2} a_2 \sqrt{(H_1 - H_3) (C_{D1}^2 a_1^2 + C_{D2}^2 a_2^2) 2g - F_L^2 - F_L C_{D1} a_1} / \sqrt{2g (C_{D1}^2 a_1^2 + C_{D2}^2 a_2^2)}
\]

(eq. 3-31)
The non-linear system has been found in terms of physical parameters, operating parameters, control parameters and variables. To control a linearization of the system in Figure 2-1 several steps must be completed. First, a feedback loop is described as in Figure 4-1. Using this loop, the unknown characteristics, $G_D$ and $G_P$, must be found as shown back in Chapter Two. A Taylor's series will be used to find the unknown system transfer function, $G(s)$. The feedback loop will then be applied and the control height $h_1(s)$ will be found in response to the change in reference height, $h_{1R}(s)$. Laplace Transform techniques will be used in the two different types of control. For proportional only ($K_C = 10$, $T_R = \infty$, $T_D = 0$) and proportional-integral ($K_C = 10$, $T_R = 10$, $T_D = 0$) control, $h_1(t)$ and $h_2(t)$ will be found. After finding these small changes, $H_1(t)$ and $H_2(t)$ will be found.

Finding the depth constant, $G_D$, and pump constant, $G_P$ for this system begins by applying the initial conditions $H_1 = 10$ and $F_L = 0$. By substituting these values into equation 3-34, $H_2$ is 9.24 cm. Inserting this value into equation 3-21, yields $F_1$ as 42.6 cc/sec. After finding this
Figure 4-1. Feedback Loop For Linear System.
flow and placing this into equation 2-4, $V_M$ is found to be 3.9 volts. Finally from equations 2-3 and 2-6, $G_D$ is .278 volts/cm and $G_P$ is 10.67 cc/sec per volt.

To find $G(s)$, the pump flow as a function of time first must be found. For a small change in pump flow, $f_1(t)$, the total flow of fluid into tank one, $F_1(t)$, is described by equation 4-1 where $\bar{F}_1$ is some original flow.

$$F_1(t) = \bar{F}_1 + f_1(t) \quad \text{(eq. 4-1)}$$

In the same manner equations 4-2 and 4-3 describe the heights of tank one and two for small changes in height, $h_1(t)$ and $h_2(t)$, and original heights $\bar{H}_1$ and $\bar{H}_2$.

$$H_1(t) = \bar{H}_1 + h_1(t) \quad \text{(eq. 4-2)}$$

$$H_2(t) = \bar{H}_2 + h_2(t) \quad \text{(eq. 4-3)}$$

By differentiating equations 4-2 and 4-3, equations 4-4 and 4-5 are found.

$$\frac{dH_1}{dt} = \frac{dh_1}{dt} \quad \text{(eq. 4-4)}$$

$$\frac{dH_2}{dt} = \frac{dh_2}{dt} \quad \text{(eq. 4-5)}$$
By using a Taylor's series with small changes $h_1$ and $h_2$, the flow $F_2$ is given by equation 4-6.

\[
F_2 = \bar{F}_2 + \delta F_2 \left|_{H_1} \right. + \delta F_2 \left|_{H_2} \right. h_2
\]

As equation 4-7 shows, $F_0$ can also be written in a Taylor's series.

\[
F_0 = \bar{F}_0 + \delta F_0 \left|_{H_1} \right. + \delta F_0 \left|_{H_2} \right. h_2
\]

By differentiating equation 3-21 and substituting into equation 4-6, equation 4-8 is found.

\[
F_2 = \bar{F}_2 + C_{D1a1} \sqrt{\frac{g}{2(\bar{H}_1 - \bar{H}_2)}} h_1 - C_{D1a1} \sqrt{\frac{g}{2(\bar{H}_1 - \bar{H}_2)}} h_2
\]

Regrouping like terms and rewriting yields equation 4-9.

\[
F_2 = \bar{F}_2 + k_1 (h_1 - h_2)
\]

where $k_1 = C_{D1a1} \sqrt{\frac{g}{2(\bar{H}_1 - \bar{H}_2)}}$
For the tank described in Chapter Two, $k_1$ is 28.03. In the same manner, the flow out is given by equation 4-10.

\[ F_0 = F_0 + k_2 h_2 \]

where \( k_2 = \frac{C_{D2}a_2\sqrt{g}}{\sqrt{2(H_2 - H_3)}} \) \hspace{1cm} (eq. 4-10)

For the system in Chapter Two, $k_2$ is 3.41.

Taking equation 3-7 and substituting into equations 4-4, 4-1 and 4-9 yields equation 4-11.

\[ \frac{dh_1}{dt} = \frac{F_1}{A} - \frac{F_2 + k_1(h_1 - h_2)}{A} \] \hspace{1cm} (eq. 4-11)

Since the system is initially in steady state, $F_1$ is equal to $F_2$ and equation 4-11 can be rewritten as equation 4-12.

\[ \frac{dh_1}{dt} = -\frac{k_1}{A}h_1 + \frac{k_1}{A}h_2 + \frac{f_1}{A} \] \hspace{1cm} (eq. 4-12)

In the same manner, equation 3-8 can be rewritten as shown in equation 4-13 by inserting equations 4-5, 4-9 and 4-10.
\[
\frac{dh_2}{dt} = \frac{F_L}{A} + \frac{F_2}{A} + \frac{k_1 h_1}{A} - \frac{k_1 h_2}{A} - \frac{F_0}{A} + \frac{k_2 h_2}{A} \\
\text{(eq. 4-13)}
\]

Since this system is also initially in steady state and \( F_L \) remains constant throughout the problem \( F_L + F_2 = F_0 \) and equation 4-14 holds.

\[
\frac{dh_2}{dt} = \frac{k_1 h_1}{A} - \frac{k_1}{A} + \frac{k_2 h_2}{A} \\
\text{(eq. 4-14)}
\]

Rewriting equations 4-12 and 4-14 into matrix form yields equation 4-15.

\[
\begin{bmatrix}
\frac{dh_1}{dt} \\
\frac{dh_2}{dt}
\end{bmatrix} = \begin{bmatrix}
\frac{-k_1}{A} & \frac{k_1}{A} \\
\frac{k_1}{A} & \frac{-k_1 + k_2}{A}
\end{bmatrix} \begin{bmatrix}
h_1 \\
h_2
\end{bmatrix} + \begin{bmatrix}
1 \\
0
\end{bmatrix} f_1 \\
\text{(eq. 4-15)}
\]

Since the system transfer function \( G(s) \) is \( \frac{h_1(s)}{f_1(s)} \), Laplace Transforms will be needed. By transforming equation 4-15 and since \( h_1(t=0), h_2(t=0) \) and \( f_1(t=0) \) all are zero, equation 4-16 holds.
\[
\begin{bmatrix}
    h_1(s) \\
    h_2(s)
\end{bmatrix}
= \begin{bmatrix}
    \frac{-k_1}{A} & \frac{k_1}{A} \\
    \frac{k_1}{A} & \frac{-k_1 + k_2}{A}
\end{bmatrix}
\begin{bmatrix}
    h_1(s) \\
    h_2(s)
\end{bmatrix}
+ \begin{bmatrix}
    1 \\
    0
\end{bmatrix} f_1(s)
\]  
(eq. 4-16)

Regrouping, inverting and solving yields equation 4-17.

\[
\begin{bmatrix}
    h_1(s) \\
    h_2(s) \\
    f_1(s)
\end{bmatrix}
= \frac{1}{s^2 + 2k_1 + k_2 s - k_1 k_2}
\begin{bmatrix}
    s + k_1 + k_2 \\
    A
\end{bmatrix}
\begin{bmatrix}
    \frac{k_1}{A} \\
    \frac{k_1}{A^2}
\end{bmatrix}
\]  
(eq. 4-17)

Therefore the system transfer function is given by equation 4-18.

\[
G(s) = \frac{s + k_1 + k_2}{s^2 + 2k_1 + k_2 s - k_1 k_2}
\]  
(eq. 4-18)

The next step is finding the change in height \( h_1(s) \) in response to a small change in the reference height \( h_{1R}(s) \). From Figure 4-1, it can be seen that the reference voltage to reference height relationship is given by equation 4-19.

\[
v_{1R} = G_D h_{1R}(s)
\]  
(eq. 4-19)
Also for Figure 4-1, the voltage/height relationship is given by equation 4-20.

\[ v_1 = G_D h_1(s) \]  

(eq. 4-20)

By completing the loop around Figure 4-1, equation 4-21 is found.

\[ (v_{1R} - v_1)K(s)G_p G(s) = h_1(s) \]  

(eq. 4-21)

Substituting 4-19 and 4-20 into 4-21 yields equation 4-22.

\[ [G_D h_{1R}(s) - G_D h_1(s)]K(s)G_p G(s) = h_1(s) \]  

(eq. 4-22)

Solving for the first height in the frequency domain gives equation 4-23.

\[ h_1(s) = \frac{G_p G_p K(s) G(s) h_{1R}(s)}{1 + G_D G_p K(s) G(s)} \]  

(eq. 4-23)

Therefore, for the system in Chapter Two, this equation can be rewritten as seen in equation 4-24.
\[ h_1(s) = \frac{.0148K(s)(s+.157)}{s^2+.297s+.00239+.0148K(s)(s+.157)} \]  
\[ h_{1R}(s) \]

(eq. 4-24)

Now the change in height one in response to a unit step change in the reference height with proportional control \((K_C = 10, \ T_R = \infty, \ T_D = 0)\) can be found. Since it is a unit step change \(h_{1R}(s)\) is \(1/s\). And since this is proportional only control, \(K(s)\) is 10. Inserting these values into equation 4-24 yields equation 4-25.

\[ h_1(s) = \frac{.148(s+.157)}{s(s^2+.445s+.0257)} \]  
\[ (eq. 4-25) \]

\(h_1(t)\) will be the inverse Laplace Transform of equation 4-25. To find this inverse, a Heaviside expansion will be used. The roots of the denominator are \(-.0682\) and \(-.3768\), therefore:

\[ h_1(t) = .148 \mathcal{L}^{-1}[Z(s)] \]

where \[ Z(s) = \frac{s + .157}{s(s+.0682)(s+.3768)} \]  
\[ (eq. 4-26) \]

Rewriting this into a manageable form gives equation 4-27.
\[ h_1(t) = 0.148 \mathcal{L}^{-1} \left[ \frac{A}{s} + \frac{B}{s+0.0682} + \frac{C}{s+0.3768} \right] \]

where \( A \) is the limit of \( n \) as \( s \) goes to zero, 6.109, \( B \) is the limit of \( (s+0.0682)Z(s) \) as \( s \) goes to \(-0.0682, -4.219 \) and \( C \) is the limit of \( (s+0.3768)Z(s) \) as \( s \) goes to \(-0.3768, -1.89 \).  

Finally, inverting equation 4-27 gives equation 4-28.

\[ h_1(t) = (0.904 - 0.624e^{-0.0682t} - 0.28e^{-0.3768t}) \delta_1(t) \]

(eq. 4-28)

By inserting equation 4-28 into equation 4-2, the time response for the height of water in tank one is:

\[ H_1(t) = 10 + (0.904 - 0.624e^{-0.0682t} - 0.28e^{-0.377t}) \delta_1(t) \]

(eq. 4-29)

To find \( h_2(s) \) and subsequently its response to the step change as a function of time, \( H_2(t) \), equation 4-14 will be transformed by a Laplace Transform.

\[ \mathcal{L}(h_2(s)) = \frac{k_1 h_1(s) - k_1k_2 h_2(s)}{A} \]

(eq. 4-30)

Solving for \( h_2(s) \) and inserting the values for the physical system gives equation 4-31.
\[ h_2(s) = \frac{.14}{s + .157} h_1(s) \]  
\text{(eq. 4-31)}

By inserting \( h_1(s) \) as found in equation 4-25 and inverting yields equation 4-32.

\[
h_2(t) = (.148) \mathcal{L}^{-1}\left[ \frac{Y(s)}{(s+.157)s(s^2+.445s+.0257)} \right]
\]
where \( Y(s) = .14(s+.157) \)
\[
\text{(eq. 4-32)}
\]

Going ahead and expanding equation 4-32, yields equation 4-33.

\[
h_2(t) = (.148) \mathcal{L}^{-1}\left[ A \frac{1}{s} + B \frac{1}{s + .0682} + C \frac{1}{s + .3768} \right]
\]
where \( A \) is the limit of \( sY(s) \) as \( s \) goes to zero, 5.45, \( B \) is the limit of \( (s+.0682)Y(s) \) as \( s \) goes to -.0682, -6.65, and \( C \) is the limit of \( (s+.3768)Y(s) \) as \( s \) goes to -.3768, 1.20. (eq. 4-33)

Inverting equation 4-33 yields the final result for \( h_2(t) \).

\[
h_2(t) = (.806 - .984e^{-0.0682t} + .178e^{-0.3768t}) \delta_1(t)
\]
\text{(eq. 4-34)}
Putting equation 4-34 into equation 4-3 yields equation 4-35.

\[ H_2(t) = 9.24 + (0.806 - 0.984e^{-0.0682t} + 0.178e^{-0.3768t}) \delta_1(t) \]  
(eq. 4-35)

Figure 4-2 lists the results for \( H_1(t) \) and \( H_2(t) \) in response to a step change with proportional control.

To find the change in height one in response to a unit step change in the reference height with integral control, the same steps are followed. For the problem, let \( K_c = 10 \), \( T_R = 10 \) and \( T_D = 0 \). Then, as before, \( h_1R(s) \) is \( 1/s \) and since this is an integral control example: \( K(s) = 10 \left( 1 + 1/10s \right) \). Substituting these values into equation 4-24, gives equation 4-36.

\[ h_1(s) = \frac{0.148(s^2 + 0.257s + 0.0157)}{s(s^3 + 0.445s^2 + 0.0404s + 0.00232)} \]  
(eq. 4-36)

Separating equation 4-36 into a workable form yields equation 4-37.

\[ h_1(s) = \frac{1 - s^2 + 0.297s + 0.00236}{s \left( s^3 + 0.445s^2 + 0.0404s + 0.00232 \right)} \]  
(eq. 4-37)
Figure 4-2. \( H_1(t), H_2(t) \) With Proportional Control.

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<th>( H_2(t) )</th>
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</table>
By numerical methods \( s = -0.348 \) is a root of the
denominator of the second term. Therefore rewriting using
Heaviside expansion yields equation 4-38.

\[
\frac{h_1(s)}{s + 0.348} = \frac{1 - 0.214}{s + 0.348} \frac{0.786}{s^2 + 0.097s + 0.00664} 
\]  

(eq. 4-38)

Rewriting equation 4-38 as equation 4-39, allows easy
conversion through inverse Laplace Transforms.

\[
\frac{h_1(s)}{s + 0.348} = \frac{1 - 0.214}{s + 0.348} \frac{0.786}{(s + 0.0485)^2 + 0.0655^2} + \frac{0.0354}{(s + 0.0485)^2 + 0.0655^2} 
\]  

(eq. 4-39)

The final solution for the small change in height one
as function of time is given by equation 4-40.

\[
h_1(t) = (1 - 0.214e^{-0.348t} - 0.786e^{-0.0485t}\cos(0.0655t)) + 0.541e^{-0.0485t}\sin(0.0655t)\delta_1(t)
\]  

(eq. 4-40)

Substituting equation 4-40 into equation 4-2, gives the
final solution.

\[
H_1(t) = 10 + (1 - 0.214e^{-0.348t} - 0.786e^{-0.0485t}\cos(0.0655t)) + 0.541e^{-0.0485t}\sin(0.0655t)\delta_1(t)
\]  

(eq. 4-41)
Substituting equation 4-36 into equation 4-31 and then factoring yields:

\[
h_2(s) = \frac{0.0207(s + 0.157)(s + 1)}{(s + 0.157)(s + 0.348)s(s^2 + 0.097s + 0.00664)}
\]

(eq. 4-42)

Rewriting equation 4-42 by factoring and using Heaviside expansions again gives the following:

\[
h_2(s) = \frac{0.896 + 0.157}{s} - \frac{1.05}{s + 0.348} \left( \frac{s + 0.0453}{s^2 + 0.097s + 0.00664} \right)
\]

(eq. 4-43)

Writing equation 4-43 into the proper form for inverse Laplace Transforms gives:

\[
h_2(s) = \frac{0.896 + 0.157}{s} - 1.05 \left( \frac{s + 0.11}{(s + 0.0485)^2 + 0.0655^2} - \frac{0.0032}{(s + 0.0485)^2 + 0.0655^2} \right)
\]

(eq. 4-44)

Inverting equation 4-44 yields the following solution for a small change in the second tank height as a function of time.
Substituting equation 4-45 into equation 4-3 gives the final solution for the second tank height with proportional-integral control.

\[
h_2(t) = 0.896 + 0.157e^{-0.348t} - 1.05e^{-0.0485t}\cos(0.0655t) + 0.0513e^{-0.0485t}\sin(0.0655t)\] 8_1(t)
\]

(eq. 4-45)

\[
H_2(t) = 9.24 + \left[0.896 + 0.157e^{-0.348t} - 1.05e^{-0.0485t}\cos(0.0655t) + 0.0513e^{-0.0485t}\sin(0.0655t)\right]8_1(t)
\]

(eq. 4-46)

Results for \(H_1(t)\) and \(H_2(t)\) in response to a unit step change in the set point with integral control are supplied in Figure 4-3.
Figure 4-3. $H_1(t), H_2(t)$ With Proportional-Integral Control.

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10.12
CHAPTER V
NONLINEAR SYSTEM

The system of equations found in Chapter Three will be used to evaluate the nonlinear system. These differential equations will be rewritten as difference equations with step size $\Delta t$. During each $\Delta t$, the equations will be used to find the flow rates (equations 3-21 and 3-22), use these rates to find the rate of change in height (equations 3-7 and 3-8) and then calculate the new height. This cycle will be repeated $PTSP$ times and the heights will be output to the user each time segment of length $(PTSP) \times (\Delta t)$. After 80 of these print steps, the program ends. This cycle is illustrated in Figure 5-1.

The use of this program involves several steps by the user. First, the user must describe the tank settings for orifice one and two. For the purpose of this paper, orifice one is set so the 1.27 cm hole is plugged (Ans. "Y") and holes .95 cm and .625 cm are open (Ans. "N"). The tap for orifice two will be wide open (Ans. "1"). The physical parameters for this configuration are now displayed. The next entries by the user are the control parameters. The user is asked for the proportional gain, $K_c$, and the reset
FIND VOLTAGE OF TANK 1

\[ v_1(0) = f(H_1(0)) \]  
\text{eqn 2-4}

FIND ERROR

\[ e(0) = v_{1R} - v_1(0) \]

FIND NEW MOTOR VOLTAGE

\[ v_m(0) = v_m + K_C(e(0) + T_D e(0)/\Delta t) \]

FIND FLOW INTO TANK 1

\[ F_1(0) = f(v_m(0)) \]  
\text{eqn 2-1}

FIND \( \frac{dH_1(0)}{dt} \)

\[ \frac{dH_1(0)}{dt} = \frac{F_1(0) - F_1}{A} \]

FIND \( \frac{dH_2(0)}{dt} \)

\[ \frac{dH_2(0)}{dt} = 0 \]

AS \( n = 1, 2, 3... \) \[ [t_j = j \Delta t] \]

FIND NEW HEIGHT 1

\[ H_1(t_n) = H_1(t_{n-1}) + \frac{dH_1(t_{n-1}) \Delta t}{dt} \]

FIND NEW HEIGHT 2

\[ H_2(t_n) = H_1(t_n) + \frac{dH_1(t_n) \Delta t}{dt} \]

FIND FLOW BETWEEN TANKS

\[ F_2(t_n) = f(H_1(t_n), H_2(t_n)) \]  
\text{eqn 3-21}

FIND FLOW OUT OF TANK 2

\[ F_0(t_n) = f(H_2(t_n)) \]  
\text{eqn 3-22}

FIND VOLTAGE OF TANK 1

\[ v_1(t_n) = f(H_1(t_n)) \]  
\text{eqn 2-4}

FIND ERROR

\[ e(t_n) = v_{1R} - v_1(t_n) \]

FIND NEW MOTOR VOLTAGE

\[ v_m(t_n) = v_m + K_C \left( e(t_n) + \sum_{i=0}^{n} \frac{e(t_i)}{T_R} + T_D \frac{e(t_n) - e(t_{n-1})}{\Delta t} \right) \]

FIND FLOW INTO TANK 1

\[ F_1(t_n) = f(v_m(t_n)) \]

FIND \( \frac{dH_1(t_n)}{dt} \)

\[ \frac{dH_1(t_n)}{dt} = \frac{F_1(t_n) - F_2(t_n)}{A} \]  
\text{eqn 3-7}

FIND \( \frac{dH_2(t_n)}{dt} \)

\[ \frac{dH_2(t_n)}{dt} = \frac{F_2(t_n) + F_L - F_0(t_n)}{A} \]  
\text{eqn 3-8}

NEXT \( n \)

Figure 5-1. Cycle of steps for Solving the Nonlinear System.
time, $T_R$. Throughout this paper $K_C$ is ten. For proportional only control, as in the first example, $T_R = 0$. For this paper's proportional-integral control problem, example two, $T_R = 10$. The final control input derivative time, $T_D$, is input. $T_D = 0$ in both of these examples.

The initial values are then input to begin the simulation. These consist of the initial level in tank one (10 cm in both examples), the step change in the control variable (1 cm in both examples) and the load variable that describes the input into tank two (0 cc/min for these cases). Several informational values are output and the simulation begins.

A reference listing is supplied in the Appendix. Sample outputs are supplied in Figure 5-2 (Proportional Only Control) and Figure 5-3 (Proportional-Integral Control).

As a check for accuracy of this numerical solution by Euler's method, another method was attempted. The second method to be chosen was a fourth order Runge-Kutta solution. The equations that describe this solution are given as depicted in equation 5-1 for $H_1$.

$$H_1(t_0 + \Delta t) = H_1(t_0) + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$  (eq. 5-1)
RUN
CHOOSE A TANK CONFIGURATION
WHICH HOLES DO YOU WISH PLUGGED?
(Y=YES, N=NO)
1. 27 CM ?Y
. 95 CM ?N
. 635 CM?N

HOW DO YOU WANT THE TAP OF THE SECOND TANK SET ?
1. FULL OPEN
2. 3/4 OPEN
3. 1/2 OPEN
4. 1/4 OPEN

PHYSICAL PARAMETERS

H3 = 3 G = 980
CD1= 1 CD2= 1
A1 = 1.1 A2 = .38
A = 200

CONTROL PARAMETERS

ENTER KC?10
ENTER TR FOR INTEGRAL CONTROL (0 IF NONE DESIRED)?10
ENTER TD FOR DIFFERENTIAL CONTROL (0 FOR NONE)?0

INITIAL VALUES

WHAT IS YOUR INITIAL LEVEL IN TANK 1 (3 TO 25 CM)?10
INPUT THE STEP CHANGE IN THIS CONTROL VARIABLE?1
INPUT THE LOAD VARIABLE (CC/MIN)?0

VARIABLE VALUES

H1(0) = 10 H2(0) = 9.24
P1D = .2552 FL = 0
SET POINT 11
SET POINT VOLTAGE 4.83
DESIGN VOLTAGE 3.88

Figure 5-2. Results For Proportional Control Of ... The Nonlinear System.
Figure 5-2. - Continued.

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</table>
CHOOSE A TANK CONFIGURATION
WHICH HOLES DO YOU WISH PLUGGED?
(Y= YES, N= NO)
1. 27 CM? Y
.95 CM? N
.635 CM? N

HOW DO YOU WANT THE TAP OF THE SECOND TANK SET?
1. FULL OPEN
2. 3/4 OPEN
3. 1/2 OPEN
4. 1/4 OPEN

PHYSICAL PARAMETERS
H3 = 3  G = 980
CD1 = 1  CD2 = 1
A1 = 1.1  A2 = .38
A = 200

CONTROL PARAMETERS
ENTER KD= 10
ENTER TR FOR INTEGRAL CONTROL (0 IF NONE DESIRED)? 0
ENTER TD FOR DIFFERENTIAL CONTROL (0 FOR NONE)? 0

INITIAL VALUES
WHAT IS YOUR INITIAL LEVEL IN TANK 1 (3 TO 25 CM)? 10
INPUT THE STEP CHANGE IN THIS CONTROL VARIABLE? 1
INPUT THE LOAD VARIABLE (CC/ MIN)? 0

VARIABLE VALUES
H1(0) = 10  H2(0) = 9.24
F1D = 2552  FL = 0
SET POINT 11
SET POINT VOLTAGE 4.63
DESIGN VOLTAGE 3.88

Figure 5-3. Results For Proportional-Integral Control Of The Nonlinear System.
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Equation 5-2 gives the value for $H_2$.

$$H_2(t_0 + \Delta t) = H_2(t_0) + \frac{n_1 + 2n_2 + 2n_3 + n_4}{6} \quad (\text{eq. 5-2})$$

The value of $k_1$ is found from equation 5-3.

$$k_1 = \Delta t * f(H_1(t_0), H_2(t_0)) \quad (\text{eq. 5-3})$$

Where $f(H_1, H_2)$ is the time derivative of $H_1$ calculated by equation 3-7. In the same manner $n_1$ is found by equation 5-4.

$$n_1 = \Delta t * g(H_1(t_0), H_2(t_0)) \quad (\text{eq. 5-4})$$

Where $g(H_1, H_2)$ is the time rate of change of $H_2$ as found in equation 3-8. The definition of $k_2$ is shown in equation 5-5.

$$k_2 = \Delta t * f(H_1(t_0) + k_1/2, H_2(t_0) + n_1/2) \quad (\text{eq. 5-5})$$

In the same manner, $n_2$ is given by equation 5-6.

$$n_2 = \Delta t * g(H_1(t_0) + k_1/2, H_2(t_0) + n_1/2) \quad (\text{eq. 5-6})$$
k₃ is given by equation 5-7.

\[ k₃ = Δt*f(H₁(t₀)+k₂/2,H₂(t₀)+n₂/2) \]  \hspace{1cm} \text{(eq. 5-7)}

Equation 5-8 yields n₃.

\[ n₃ = Δt*g(H₁(t₀)+k₂/2,H₂(t₀)+n₂/2) \]  \hspace{1cm} \text{(eq. 5-8)}

The value of k₄ is found by equation 5-9.

\[ k₄ = Δt*f(H₁(t₀)+k₃/2,H₂(t₀)+n₃/2) \]  \hspace{1cm} \text{(eq. 5-9)}

To find n₄, equation 5-10 is used.

\[ n₄ = Δt*g(H₁(t₀)+k₃/2,H₂(t₀)+n₃/2) \]  \hspace{1cm} \text{(eq. 5-10)}

Results of the Runge-Kutta solution and a comparison to the previously calculated Euler's method are shown in Figure 5-4 for Proportional Only Control and Figure 5-5 for Proportional-Integral Control. As can be seen by these results there are no major differences in the two solutions. Since an analytic solution does not exist, this lack of a difference is an opportunity for future study of the two nonlinear solutions. But for this paper's purpose it is not required and all future references to the nonlinear system will imply the Euler's solution.
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Figure 5-4. Proportional Only Control Comparison For The Nonlinear System.
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Figure 5-5. Proportional-Integral Control Comparison For The Nonlinear System.
CHAPTER VI
COMPARISON OF RESULTS

Plotting the results of the proportional only control solutions in figures 6-1 and 6-2 show a comparison of the linear and nonlinear solutions. In this example, no noticeable difference occurred between the two systems. The nonlinear system reacts slightly slower, but at no point does it differ by more than 1/2%. In both instances the final steady state values are the same. By these results the linearization appears to hold throughout. In proportional only control, the linear model displayed no major faults.

For Proportional-Integral control, as displayed in figures 6-3 and 6-4, no major difference between the linear and nonlinear solutions appear. Once again the nonlinear system lags behind the linear model. No difference appeared in the steady state values either. From these results, the linear model appears to also hold for proportional-integral control.
Figure 6-1. Height Of Tank 1 vs Time Proportional Control Only.
Figure 6-2. Height Of Tank 2 Vs Time Proportional Control Only.
Figure 6-3. Height Of Tank 1 Vs Time Proportional-Integral Control.
Figure 6-4. Height Of Tank 2 Vs Time Proportional-Integral Control.
CHAPTER VII
SUMMARY

This report investigated the comparison of linearized and nonlinearized solutions of a nonlinear physical situation. The system was modeled through the relationships of height to voltage, voltage to pump flow, tank flow rates and orifice flows. The system was then linearized and an analytic solution was calculated using linear control theory and Laplace Transform techniques. A nonlinear system simulation found the exact solution using the model's flow rate equations as difference equations.

Results for these two different solution techniques were obtained and compared. No major differences were apparent between the linearized and nonlinearized solutions. The only perceptable difference between these solutions was a slight lag between the linear and actual nonlinear result. This lag arose in both proportional only and proportional-integral control situations. In both of these situations, the linearized and nonlinearized solutions attained the same steady state value.
In conclusion, this paper has shown that for a nonlinear system under proportional only or proportional-integral control, a linearization can be used to find an analytical solution that closely resembles the exact numerical solution. This analytical solution sufficiently replicates the exact numerical system to use modern control theory techniques for subsequent studies in the variation of proportional gain, reset time and the effects of a change in step size of the reference variable on the system.
APPENDIX

REFERENCE LISTING FOR COMPUTER MODEL OF THE NONLINEAR SYSTEM AND ASSOCIATED VARIABLE DESCRIPTIONS
H3 = HEIGHT OF ORIFICE 3 (H3)
CD1 = DISCHARGE COEFFICIENT ORIFICE 1 (C_{D1})
CD2 = DISCHARGE COEFFICIENT ORIFICE 2 (C_{D2})
G = GRAVITATIONAL CONSTANT (g)
A = AREA OF TANK (A)
A1 = CROSS SECTIONAL AREA OF ORIFICE 1 (a_1)
A2 = CROSS SECTIONAL AREA OF ORIFICE 2 (a_2)
KC = PROPORTIONAL GAIN (K_C)
TR = RESET TIME (T_R)
TD = DERIVATIVE TIME (T_D)
H1 = HEIGHT OF TANK 1 (H_1)
H2 = HEIGHT OF TANK 2 (H_2)
HSP = SET POINT HEIGHT (H_{1R})
FL = LOAD VARIABLE INTO TANK 2 (F_L)
F1D = ORIGINAL PUMP FLOW (F_1)
VD = ORIGINAL MOTOR VOLTAGE (v_m)
VS = REFERENCE VOLTAGE (v_{1R})
V = VOLTAGE OF TANK 1 (v_1)
EV = DIFFERENCE FROM REFERENCE (e)
VM = VOLTAGE TO THE MOTOR (v_m)
D1 = RATE OF CHANGE OF HEIGHT IN TANK 1 (\frac{dH_1}{dt})
D2 = RATE OF CHANGE OF HEIGHT IN TANK 2 (\frac{dH_2}{dt})

Definitions of Nonlinear System Variables With Their Linear System Equivalent in Parentheses
DT = TIME STEP
PTSP = NUMBER OF TIME STEPS PER PRINTING
O(A) = VOLTAGE-HEIGHT RELATIONSHIP eqn. 2-3
X(A) = FLOW-VOLTAGE RELATIONSHIP eqn. 2-1
A = DUMMY VARIABLE
CNF = TOTAL OF OPEN RADIUS$^2$ FOR ORIFICE 1
A$^S =$ INPUT VARIABLE TO SIGNIFY OPEN HOLE "Y" PLUGGED
"N" OPEN
B = SIGNIFIES OPENING OF TAP
C1 = CONSTANT DESCRIBING ORIFICE 1
C2 = CONSTANT DESCRIBING ORIFICE 2
HSTP = STEP CHANGE IN THE REFERENCE VARIABLE
PEV = PREVIOUS TIME STEP'S ERROR
TTE = TOTAL ERROR
T = RUNNING TIME
L = COUNTS PRINT STEPS
W = COUNTS TIME STEPS

Description of Other Variables
20 REM STEP SIZE AND PRINT STEP
30 REM
40 DT = .1
50 PSTP = 10
60 REM
70 REM VOLTAGE-HEIGHT RELATIONS
   IF FOR TANK 1
80 REM
90 DEF FN 0(A) = .00681 * A - 3
   -.02214 * A ^ 2 + .47795 * A + 1.1766
100 REM
110 REM PUMP-VOLTAGE RELATIONSH
120 REM
130 DEF FN X(A) = (- 49.175 * A ^ 2 + 1023.5 * A - 687.28) / 20
140 REM
150 REM PHYSICAL PARAMETERS
160 REM
170 H3 = 3
180 CD1 = 1
190 CD2 = 1
200 G = 980
210 A = 200
220 PRINT "CHOOSE A TANK CONFIGU-
230 RATION"
240 PRINT "WHICH HOLES DO YOU WISH FLOODED?"
250 PRINT "(Y=YES,N=NO)"
260 PRINT "1.27 CM"
270 INPUT AF
280 IF AF = "N" THEN CNF = CNF + (.127) ^ 2
290 PRINT ".95 CM"
300 INPUT AF
310 IF AF = "N" THEN CNF = CNF + (.95) ^ 2
320 PRINT ".335 CM"
330 INPUT AF
340 IF AF = "N" THEN CNF = CNF + (.335) ^ 2
350 A1 = CNF * 3.14 / 4
360 PRINT
370 PRINT "HOW DO YOU WANT THE 1
   AP OF THE SECOND TANK SET?"
380 PRINT "1. FULL OPEN"
390 PRINT "2. 3/4 OPEN"
400 PRINT "3. 1/2 OPEN"
410 PRINT "4. 1/4 OPEN"
420 INPUT B
430 IF 0 < B AND B < 5 AND B = INT (B) THEN 450
440 PRINT "THAT WAS NOT ONE OF Y
   OUR CHOICES, TRY AGAIN"
450 GOTO 370
460 A2 = (.875 - .175 * B) * 2 + 3.14 / 4
470 PRINT
480 PRINT "PHYSICAL PARAMETERS"
490 PRINT
500 PRINT "H3 = "H3,"D = "D
510 PRINT "CD1= "CD1,"CD2= "CD2
520 PRINT "A1 = " INT (A1 * 100) / 100,"A2 = " INT (A2 * 100) / 100
530 PRINT "A = "A
540 C1 = CD1 * A1 * 500 (G = 2)
550 C2 = CD2 * A2 * 500 (G = 2)
560 PRINT
570 PRINT "CONTROL PARAMETERS"
580 PRINT
590 PRINT "ENTER EC"
600 INPUT EC
610 PRINT "ENTER TR FOR INTEGRAL
   CONTROL (0 IF NONE DESIRED)"
620 INPUT TR
630 IF TR = 0 THEN TR = 999999999999
640 PRINT "ENTER TD FOR DIFFEREN-
   TIAL CONTROL (0 FOR NONE)"
650 INPUT TD
660 PRINT
670 PRINT "INITIAL VALUES"
680 PRINT
690 PRINT "WHAT IS YOUR INITIAL
   LEVEL IN TANK 1 ("H3") TO 25
   CM?"
700 INPUT H1
710 IF H3 = H1 AND H1 < 25
   THEN 740
720 PRINT "THAT IS OUT OF RANGE, TRY AGAIN"
730 GOTO 700
740 PRINT "INPUT THE STEP CHANGE IN THIS CONTROL VARIABLE";
750 INPUT HSTP
760 HSP = H1 + HSTP
770 PRINT "INPUT THE LOAD VARIABLE (CM/Min)";
780 INPUT FL
790 HE = FL / 60
800 H2 = H1 - ((C2 * SOR ((C1 + 2 + C2) / 2) * (H1 - H2)) - FL)
810 FID = C1 * SOR (H1 - H2)
820 VD = 10.4 - 1.105 * SOR (77.
830 VS = FN 0(HSP)
840 PRINT
850 PRINT "VARIABLE VALUES"
860 PRINT
870 PRINT "H1(0) = " INT (H1 * 100) / 100, "H2(0) = " INT (H2 * 100) / 100
880 PRINT "FID = " INT (FID * 600) / 100, "FL = " INT (FL * 600) / 100
890 PRINT "SET POINT " INT (HSP * 100) / 100
900 PRINT "SET POINT VOLTAGE " INT (VS * 100) / 100
910 PRINT "DESIGN VOLTAGE " INT (VD * 100) / 100
920 PRINT
930 REM SET OTHER INITIAL VALUES:
940 REM
950 V = FN 0(H1)
960 EV = VS - V
970 VM = VD + KC * (EV + TD * EV / DT)
980 PEV = 0
990 TTE = 0
1000 IF VM > 10 THEN VM = 10
1010 IF VM < 2 THEN 1030 AND F1 = 0
1020 F1 = FN X(VM)
1030 D1 = (F1 - FID) / A
1040 D2 = (F2 + FL - F0) / A
1050 T = 0
1060 PRINT "TIME", "TANK1", "TANK2"
1070 PRINT "F1", "F2", "H1", "H2"
BIBLIOGRAPHY


